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MODELLING ROAD TRAFFIC ASSIGNMENT AS A DAY-TO-DAY DYNAMIC, DETEMINISTIC PROCESS: A UNIFIED APPROACH TO DISCRETE AND CONTINUOUS TIME MODELS

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Abstract – We consider the modelling of road transport systems as a day-to-day dynamic, deterministic process. The main contribution is to present a unified treatment of discrete-time and continuous-time approaches, these two classes of approach having been developed in two parallel streams of research which have had little connection made between them. In doing so, we aim to clarify the usefulness of these alternative approaches. We pay particular attention to: the specification of such models; the conditions which characterise the various forms of emergent behaviour; and the relationship between the model assumptions and real-world phenomena. The proposed framework is heavily focused, in the first instance, on a probabilistic approach to user choice modelling, though we also review and analyse the limiting case of deterministic choice model.

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1. Introduction

The focussed study of day-to-day dynamics has only recently begun to gain momentum, even though there is a long history of papers that have at least considered in passing the issue of dynamic adaptation in transportation networks. In his seminal paper, Wardrop (1952) alluded to the role of such dynamics when providing a justification for the introduction of the equilibrium concept, by stating:

'It may be assumed that traffic will tend to settle down into an equilibrium situation.'

Indeed, in the monograph that motivated the whole research field of traffic network equilibrium analysis, Beckmann et al (1956) reported a study of 'stability' noting that

'In our model we shall assume simply that those road users who do not just continue in their previous choices will choose their routes and the number of trips by road on the basis of the traffic conditions that prevailed in the preceding periods.'

They went on to illustrate with numerical examples some potential characteristic patterns of inter-period adjustment. Much later Smith (1979), in his landmark paper for equilibrium analysis, posited several mechanisms for dynamic adjustment such as:

'Consider a single driver who has travelled at least once today. He may use the same routes tomorrow. However if he does change a route then he must change to a route which today was cheaper than the one he actually used today.'

before considering a weaker version of this rule in order to formally establish conditions for a stable equilibrium.

Horowitz (1984) was the first to explicitly describe day-to-day dynamics, proposing models for a two-link transportation network derived from discrete-time non-linear dynamical system theory. These models are now better known as deterministic process models. The proposed models were also used to analyse the stability of stochastic equilibrium.

But perhaps more than anyone, it was Cascetta (1987, 1989) who truly was the catalyst for work in this field to begin, proposing models derived from stochastic process theory. In previous work it had generally been the case that the focus was on 'equilibrium' as the model, and the dynamic process was defined primarily to justify the plausibility of equilibrium as a predictor. The significance of Cascetta's work was that he effectively reversed these roles; for the first time, it was the *dynamic process* that was the primary entity of interest, it was 'the model', and any interest in notions of equilibrium was justified only to gain an insight into features of the dynamic process.

Since this early work, the field has grown—albeit rather slowly—to potentially encompass a rather wide range of approaches, including deterministic and stochastic processes, discrete- and continuous-time models, and deterministic and probabilistic choice models.

To better understand the nature of time in day-to-day dynamics, or what might be called more correctly (and more generally) epoch-to-epoch dynamics, the following quote is useful to have in mind:

> '*... epochs can have either a "chronological" interpretation as successive reference* periods of similar characteristics (e.g. the a.m. peak period of successive working days) *or they can be defined as "fictitious" moments in which users acquire awareness* of *path attributes and make their choices'.* (Cascetta, 1989)

'Time' in the sense described above is thus ontologically discrete, but still we may choose to represent it as continuous as an approximation. This can often be convenient for obtaining analytical/theoretical results, which may be easier to establish in the continuous-time case. On the other hand, for numerical solution or computer simulation discrete-time is more convenient, and so even if the model is specified originally in continuous-time, it will typically be discretised for computational purposes at least. Although it may not seem such a great distinction, it turns out that the properties of the resulting non-linear systems can be rather different in the two cases, some qualitative phenomena evident in one and not in another.

In the present paper we consider the modelling of road transport systems as a deterministic process (stochastic process models are widely discussed in Watling & Cantarella, 2013a, 2013b). The main contribution is to present a unified treatment of discrete-time and continuous-time approaches, these two classes of approach having been developed in two parallel streams of research which have had little connection made between them. In doing so, we aim to clarify the usefulness of these alternative approaches. We pay particular attention to: the specification of such models; the conditions which characterise the various forms of emergent behaviour; and the relationship between the model assumptions and real-world phenomena. The proposed framework is heavily focused on a probabilistic approach to user choice modelling, though we also discuss the limiting case of deterministic choice.

The paper is structured as follows. In section 2 we describe the fundamental essence of a day-to-day dynamic model, and go on to describe the various alternative generic forms of such a process. Section 3 first introduces a brief summary of basic definitions, notations and equations commonly used in literature, together with fixed-point models for equilibrium assignment; then it describes two simple dynamic process models based on exponential smoothing, which however simple are useful to develop considerations about fixed-point stability. Section 4 reviews more general (discrete-time) deterministic process models, together with tools, theorems and methods that may be used to analyse them, and connections to existing equilibrium approach. Section 5 reviews and analyses some continuous time deterministic process models based on Wardropian user path choice behaviour. Section 6 analyses the strengths or weaknesses of these approaches, looking to potential applications that are particularly suited to dynamic process models. In doing so, we identify open issues and research questions for future investigation.

2. General deterministic process models

With the comments above in mind, then, let us begin to define the components of our specification. Two sets will play an important part in the specification, namely the time-space and the state-space. The *time-space* is denoted by $\mathcal{T}$ and defines the set of time-instants over which we wish to consider the process. For a continuous-time process we thus would have $\mathcal{T} \subseteq \mathbb{R}$, and typically taking time 0 as a start-point would have $\mathcal{T} = [0, \infty)$ (which we shall denote $\mathbb{R}_+$ for brevity). For a discrete-time process we typically define the discrete time points over a constant mesh, and so it is natural to index time by the non-negative integers $\mathcal{T} = \{t \in \mathbb{Z} \ \& \ t \geq 0\}$, which we shall denote by the natural number symbol $\mathbb{N}$ for brevity.

The process itself will be described through the *state vector* $\mathbf{x}(t)$ at time $t \in \mathcal{T}$, and the *state-space* $\mathcal{S}$ is the constant, time-independent 'universe' to which any such state must belong, i.e. $\mathbf{x}(t) \in \mathcal{S}$ for any $t \in \mathcal{T}$. The state vector itself is defined variously according to particular day-to-day models; it might be the flows on the arcs (or links) or on the paths (or routes) of a network, but may alternatively be the travel times, or may be a compound vector containing several different kinds of 'entities'. The details are important, and as we introduce each model later it will be a key first issue to specify what the state-vector actually represents. A common element is that $\mathbf{x}(t)$ is a *sufficient* description in two respects:

A1 if we know $\mathbf{x}(t)$ then we know (or can infer) everything we might want to know from the model for the purposes of design or evaluation; and

A2 $\mathbf{x}(t)$ contains sufficient information of the history of the process up to time t that we are able subsequently to write down a mathematical law/model to forecast all future modelled states.

So assumption A1 essentially says that, as the process evolves, we ensure that we record in the state vector all of the relevant variables that we may need at the end of the modelling process. Assumption A2 is a key theoretical one, and means that we can appeal to a whole range of theoretical results to establish properties of the processes concerned. Technically, this is known as the *Markov property*; often it is described as a 'memoryless' property, but actually this term is potentially confusing in the present context, since a key element of many day-to-day dynamic models is the way in which the 'memory' or 'learning process' of the travellers is represented, and (as we shall see) the Markov property certainly does *not* require that the travellers are memoryless in the sense of only reacting to their most recent temporal experiences. Rather, the Markov property is better understood as a requirement on the way we construct the process and define the state variable; namely, we aim to define these in such a way that the relevant 'memories' of travellers up to time t are encoded in $\mathbf{x}(t)$, we do not need to know additionally about $\mathbf{x}(s)$ for $s \in \mathcal{T}$ where $s < t$. In other words, systems with long finite memory may still be formulated as Markovian systems, with a duly specification of the system state $\mathbf{x}$. However, this construction process requires some craft and may be non-trivial, and we cannot guarantee to be able to achieve it for all learning models. In particular, it should be noted that we cannot simply define a component of $\mathbf{x}(t)$ to be the complete history of the process since time t = 0, since then our state-variable will be ever-expanding in time, and we violate the requirement to have a constant, time-independent state-space $\mathcal{S}$.

According to the non-linear dynamic system theory, in the case of a **continuous time**, $\mathcal{T} = \mathbb{R}_+$, it is natural to define the deterministic process through differential equations in which the state is described as a continuous-only variable. Thus, a **time-homogeneous, Markovian, continuous time deterministic system with continuous state-space** is expressed as (with a dot denoting derivative with respect to time):

$$\dot{\mathbf{x}}(t) = \boldsymbol{\psi}(\mathbf{x}(t)) \qquad (t \in \mathbb{R}_+;\ \mathbf{x}(t) \in \mathcal{S}) \tag{2.1}$$

for some time-independent function $\boldsymbol{\psi} : \mathcal{S} \times \mathbb{R}_+ \rightarrow \mathcal{S}$, called *transition function*.

Observing the system each Δt unit of time, we have the case of a **discrete time** the state-space $\mathcal{S}$ may be continuous, discrete or mixed. Thus we may write:

$$\mathbf{x}(t) = \boldsymbol{\varphi}(\mathbf{x}(t - \Delta t), \Delta t) \qquad (t \in \mathbb{R}_+;\ \Delta t \in \mathbb{R}_+;\ \mathbf{x}(t) \in \mathcal{S})$$

for some time-independent function $\boldsymbol{\varphi} : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{S}$, still called *transition function*. In this case, it is worthwhile to consider an integer time index $k = t / \Delta t \in \mathbb{N}$, thus, we may write:

$$\mathbf{x}(k) = \boldsymbol{\varphi}(\mathbf{x}(k-1)) \qquad (k \in \mathbb{N};\ \mathbf{x}(k) \in \mathcal{S})$$

where the meaning of $\mathbf{x}(k) = \mathbf{x}(t = k \cdot \Delta t)$ and $\boldsymbol{\varphi}(\mathbf{x})$ have slightly changed with respect to the above equation. Anyhow, in the following we will use notation t instead of k and $\mathbf{x}^t$ for $\mathbf{x}(t)$ as commonly used, reserving $\mathbf{x}(t)$ for **continuous time** systems only. Thus a **time-homogeneous, Markovian, discrete-time deterministic process** may be specified as:

$$\mathbf{x}^t = \boldsymbol{\varphi}(\mathbf{x}^{t-1}) \qquad (t \in \mathbb{N};\ \mathbf{x}^t \in \mathcal{S}) \tag{2.2}$$

where today state, $\mathbf{x}^t$, depends on yesterday state, $\mathbf{x}^{t-1}$.. As already noted, any time-discrete system with long finite memory may still be formulated as a Markovian system, with a duly specification of the system state $\mathbf{x}$, to include (finite) memory of the past states.

In some case it may occurs that today state also depends on itself. This condition may occur for instance when the system state is the result of aggregation and/or averaging over sub-periods of the day/epoch t, or at idealized systems which cannot exist in the real-world but act as 'benchmarks', such as idealized traveller information systems where the ITS and/or travellers can see perfectly into the future. Model (2.2) becomes:

$$\mathbf{x}^t = \boldsymbol{\varphi}(\mathbf{x}^{t-1}, \mathbf{x}^t) \qquad (t \in \mathbb{N};\ \mathbf{x}^t \in \mathcal{S}) \tag{2.3}$$

Quite often the following approach may be followed to express model (2.3) as (2.2).

The *implicit function method*. Let us express equation (2.3) as $\mathbf{y} = \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})$, assuming $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})$ it yields $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Thus, if the hypotheses of the implicit function theorem hold [briefly: if $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is continuously differentiable and the Jacobian matrix $\nabla_{\mathbf{y}}\ \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{I} - \nabla_{\mathbf{y}}\ \boldsymbol{\varphi}(\mathbf{x}, \mathbf{y})$ is invertible, then there exists a unique continuously differentiable function $\mathbf{y} = \mathbf{g}(\mathbf{x})$, from an open set X to an open set Y, such that for any given $\mathbf{x} \in X$, $\mathbf{y} = \mathbf{g}(\mathbf{x}) \in Y$ is a solution in $\mathbf{y}$ to $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$], then the following equation, formally consistent with (2.2), can be obtained for a properly defined function $\mathbf{g}(\cdot)$:

$$\mathbf{x}^t = \mathbf{g}(\mathbf{x}^{t-1}) \qquad (t \in \mathbb{N};\ \mathbf{x}^t \in \mathcal{S})$$

where $\nabla_{\mathbf{x}}\ \mathbf{g}(\mathbf{x}) = -(\nabla_{\mathbf{y}}\ \mathbf{f}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{g}(\mathbf{x})})^{-1} \cdot (\nabla_{\mathbf{x}}\ \mathbf{f}(\mathbf{x}, \mathbf{y})|_{\mathbf{y}=\mathbf{g}(\mathbf{x})})$. This expression of Jacobian is remarkably useful when analyzing the evolution over time close to a fixed-point state $\mathbf{x}^* = \mathbf{x}^t = \mathbf{x}^{t-1}$, that is $\mathbf{x}^* = \boldsymbol{\varphi}(\mathbf{x}^*, \mathbf{x}^*)$, since it does not require to know function $\mathbf{g}(\cdot)$.

If the transition function in equation (2.3) is separable with respect the two arguments another approach is also available as described below. This approach, applied in Bifulco et. (2013), can be proved a particular instance of the previous one. Anyhow it is outlined below for comparison's purpose.

The *inverse function method*. If the transition function $\varphi(\cdot, \cdot)$ in equation (2.3) is separable with respect the two arguments: $\varphi(\mathbf{x}^{t-1}, \mathbf{x}^{t}) = \varphi_1(\mathbf{x}^{t-1}) + \varphi_2(\mathbf{x}^{t})$, equation (2.3) may be rewritten as:

$$\mathbf{x}^{t} - \varphi_2(\mathbf{x}^{t}) = \varphi_1(\mathbf{x}^{t-1}) \tag{2.4}$$

Let us express equation (2.4) as $\mathbf{y} - \varphi_2(\mathbf{y}) = \varphi_1(\mathbf{x})$, assuming $\mathbf{f}(\mathbf{y}) = \mathbf{y} - \varphi_2(\mathbf{y})$ it yields $\mathbf{f}(\mathbf{y}) = \varphi_1(\mathbf{x})$. Thus, if the hypotheses of the global inverse function theorem hold [briefly: if $\mathbf{f}(\mathbf{y})$ is continuously differentiable and the Jacobian matrix $\nabla_{\mathbf{y}}\, \mathbf{f}(\mathbf{y}) = \mathbf{I} - \nabla_{\mathbf{y}}\, \varphi_2(\mathbf{y})$ is invertible in an open set Y, then there exists a unique continuously differentiable inverse function $\mathbf{h}(\mathbf{z}) = \mathbf{f}^{-1}(\mathbf{z})$ for $\mathbf{z} \in \mathbf{f}(Y)$, where $\mathbf{f}(Y)$ denotes the image of the set Y], then the following equation, formally consistent with (2.2), can be obtained for a properly defined function $\mathbf{h}(\cdot)$:

$$\mathbf{x}^{t} = \mathbf{h}(\varphi_1(\mathbf{x}^{t-1}))$$

where, since the Jacobian matrix of function $\mathbf{h}(\mathbf{z})$ is $\nabla_{\mathbf{z}}\, \mathbf{h}(\mathbf{z}) = (\mathbf{I} - \nabla_{\mathbf{y}}\, \varphi_2(\mathbf{y})|_{\mathbf{y}=\mathbf{h}(\mathbf{z})})^{-1}$, the Jacobian matrix of function $\mathbf{h}(\varphi_1(\mathbf{x}))$ is $\nabla_{\mathbf{x}}\, \mathbf{h}(\varphi_1(\mathbf{x})) = (\, \mathbf{I} - \nabla_{\mathbf{y}}\, \varphi_2(\mathbf{y})|_{\mathbf{y}=\varphi_1(\mathbf{x})})^{-1} \cdot (\nabla_{\mathbf{x}}\, \varphi_1(\mathbf{x}))$. [The same result may be obtained by applying the implicit function method expressing equation (2.4) as: $-(\mathbf{y} - \varphi_2(\mathbf{y})) + \varphi_1(\mathbf{x}) = 0$.] This expression of Jacobian is remarkably useful when analyzing the evolution over time close to a fixed-point state $\mathbf{x}^* = \mathbf{x}^{t} = \mathbf{x}^{t-1}$, that is $\mathbf{x}^* - \varphi_2(\mathbf{x}^*) = \varphi_1(\mathbf{x}^*)$, since it does not require to know the inverse function $\mathbf{h}(\cdot)$.

Relationship between continuous- and discrete-time deterministic process models can be exploited as follows. Without any loss of generality, let function $\boldsymbol{\psi}$ in equation (2.1) be specified as $\boldsymbol{\psi}(\mathbf{x}) = \gamma\, (\boldsymbol{\eta}(\mathbf{x}) - \mathbf{x})$ for a duly defined function $\boldsymbol{\eta}(\mathbf{x})$ and a strictly positive scale parameter $\gamma > 0$, then any continuous-time deterministic process (2.1) may also be specified as:

$$\dot{\mathbf{x}}(t) = \gamma\, (\boldsymbol{\eta}(\mathbf{x}(t)) - \mathbf{x}(t)) \qquad \gamma > 0 \qquad (t \in \mathbb{R}_+;\ \mathbf{x}^{t} \in \mathcal{S})$$

In small interval Δt the above equation can be expressed as :

$$\mathbf{x}(t + \Delta t) - \mathbf{x}(t) = \Delta t\, \gamma\, (\boldsymbol{\eta}(\mathbf{x}(t)) - \mathbf{x}(t)) + o(\Delta t) \qquad (t \in \mathbb{R}_+;\ \mathbf{x}^{t} \in \mathcal{S})$$

Thus the continuous time deterministic process may be approximated as:

$$\mathbf{x}(t + \Delta t) \cong \Delta t\, \gamma\, (\boldsymbol{\eta}(\mathbf{x}(t)) - \mathbf{x}(t)) + \mathbf{x}(t) \qquad (t \in \mathbb{R}_+;\ \mathbf{x}^{t} \in \mathcal{S})$$

The above finite difference equation may be considered as a discrete-time deterministic process based on exponential smoothing filter with $\beta = \Delta t\, \gamma > 0$, $\varphi(\mathbf{x}^{t}) = \beta\, (\boldsymbol{\eta}(\mathbf{x}^{t}) - \mathbf{x}^{t}) + \mathbf{x}^{t}$

$$\mathbf{x}^{t} = \beta\, \boldsymbol{\eta}(\mathbf{x}^{t-1}) + (1 - \beta)\, \mathbf{x}^{t-1} \qquad (t \in \mathbb{N};\ \mathbf{x}^{t} \in \mathcal{S})$$

or

$$\mathbf{x}^{t} - \mathbf{x}^{t-1} = \beta\, (\boldsymbol{\eta}(\mathbf{x}^{t-1}) - \mathbf{x}^{t-1}) \qquad (t \in \mathbb{N};\ \mathbf{x}^{t} \in \mathcal{S})$$

3. Simple deterministic process models for traffic assignment

In this section a discrete- and a continuous-time simple deterministic process models based on exponential smoothing are presented. These models are quite simple, so as to carry out explicit stability analysis, but also rather effective since they allow singling out the role of each of the main parameters of the system. Moreover, they are consistent with user equilibrium with probabilistic choice functions, and may be formulated with respect to arc or path variables leading to consistent evolutions over time and same stability conditions. Some basic definitions and notations are first introduced.

3.1 Basic definitions, notations and equations

In this section, main definitions, notations and equations for travel demand assignment to transportation networks are reviewed. Demand flows are assumed constant and one transportation mode is considered, hence path choice is the only user choice behaviour affected by network performances, or more properly by congestion.

User travelling between the same origin-destination pair with common behavioural features are grouped into a user class i, with a set of (elementary) available paths (assumed non-empty and finite) K_i. Let

$d_i \geq 0$ be the demand flow for user class i,
$\mathbf{p}_{[i]} \geq \mathbf{0}$ be the vector of path choice probabilities for user class i, with $\mathbf{1}^T \mathbf{p}_{[i]} = 1$;
$\mathbf{h}_{[i]}$ be the vector of path flows for user class i;
$\mathbf{v}_{[i]}$ be the vector of path systematic utilities for user class i;
$\mathbf{w}_{[i]}$ be the vector of path costs for user class i.

Demand conservation for user class i can be expressed as:

$$\mathbf{h}_{[i]} = d_i \, \mathbf{p}_{[i]} \qquad \forall i \tag{3.1}$$

It assures that flows of all paths connecting the user class i sum up to demand flow d_i.

Path choice behaviour can be modelled through a random utility model assuming that each user of class i associates to each path k in set K_i a value of *perceived utility* U_k, modelled by a random variable with mean v_k, and chooses the maximum perceived utility path. When the perceived utility co-variance matrix is non singular, *probabilistic* choice models are obtained, leading to a *probabilistic path choice function* for each user class i:

$$\mathbf{p}_{[i]} = \mathbf{p}_{[i]}(\mathbf{v}_{[i]}; \theta) \qquad \forall i \tag{3.2}$$

where $\theta \geq 0$ is the *dispersion parameter* related to the perceived utility standard deviation; this parameter models several source of uncertainty regarding both the users and the modeller. It also plays the role of utility scale parameter. In the following it is assumed common to all users. In multi-user assignment it (and possibly the choice function) may vary with the user class i. The choice function may well include others parameters, not explicitly introduced for simplicity's sake

The choice function is continuous and continuously differentiable for all usually adopted probabilistic choice models. If parameters of perceived utility distribution do not depend on path systematic utility values the resulting choice model is called *invariant*, and the path choice function is monotone increasing with respect to systematic utility with symmetric (semi-definite positive) Jacobian (Cantarella, 1997); in this case path choice probabilities depend on differences between systematic utility values only.

The systematic utility values depend on the corresponding path costs through the *path utility function*, generally through an affine transformation. In the following, for notation simplicity the scale parameter (which it assumed included in the dispersion parameter θ), and the constant are not explicitly shown leading to:

$$\mathbf{v}_{[i]} = -\mathbf{w}_{[i]} \qquad \forall i \tag{3.3}$$

Transportation supply (in within-day steady-state regime) is usually modelled through a network with a transportation cost c_a and a flow f_a associated to each arc a. (Node costs can be considered by duly modifying the graph). Let

c be the vector of arc costs, with entries c_a;
$\mathbf{B}_{[i]}$ be the arc-path incidence matrix for user class i, with entries $b_{ak} = 1$ if arc a belongs to path k, $b_{ak} = 0$ otherwise;
f be the vector of arc flows, with entries f_a.

The *arc path cost consistency* is expressed by (omitting path specific cost for simplicity):

$$\mathbf{w}_{[i]} = \mathbf{B}_{[i]}^{\mathrm{T}}\,\mathbf{c} \qquad \forall i \tag{3.4}$$

Moreover, the *arc path flows consistency* is expressed by (omitting base flows for simplicity):

$$\mathbf{f} = \Sigma_i\ \mathbf{B}_{[i]}\ \mathbf{h}_{[i]} \tag{3.5}$$

Let n be the number of arcs, arc flow vectors belong to the *feasible arc flow set*:

$$S_f = \{\mathbf{f} = \textstyle\sum_i d_i\, \mathbf{B}_{[i]}\, \mathbf{p}_{[i]} : \mathbf{p}_{[i]} \geq \mathbf{0},\ \mathbf{1}^{\mathrm{T}}\,\mathbf{p}_{[i]} = 1\ \forall i\} \subseteq R^n_+$$

which is non-empty (if the network is connected), compact (since closed and bounded), convex. By combining equations (3.1-5) the *arc flow* (vector) *function* can be defined:

$$\mathbf{f}(\mathbf{c}; \mathbf{d}, \theta) = \textstyle\sum_i d_i\, \mathbf{B}_{[i]}\, \mathbf{p}_{[i]}(-\mathbf{B}_{[i]}^{\mathrm{T}}\,\mathbf{c}; \theta) \in S_f \tag{3.6}$$

where $\mathbf{d} \geq \mathbf{0}$ is the vector of demand flows d_i. It is worth noting that the arc flow function is homogenous of degree 1 with respect to demand flows:

$$\kappa \cdot \mathbf{f}(\mathbf{c}; \mathbf{d}, \theta) = \mathbf{f}(\mathbf{c}; \kappa \cdot \mathbf{d}, \theta) \qquad \forall \kappa > 0$$

The arc flow function is continuous and continuously differentiable for usually adopted probabilistic choice functions. For invariant probabilistic choice functions it is also monotone decreasing with respect to path cost (as it also occurs for the Wardrop choice function), and has a symmetric (negative semi-definite) Jacobian matrix: $\mathbf{J_f}(\mathbf{c})$.

Congestion is simulated assuming that arc costs depend on arc flows, through the *arc cost* (vector) *function*, with non-negative values for mathematical convenience:

$$\mathbf{c} = \mathbf{c}(\mathbf{f}; \boldsymbol{\mu}) \geq \mathbf{0} \qquad \forall \mathbf{f} \in S_f \tag{3.7}$$

where $\boldsymbol{\mu}$ is the vector of all relevant parameters, such as capacity, sensitivity to congestion, … .

In the following, the arc cost function will be assumed continuous and continuously differentiable with respect to arc flows, **f**, with Jacobian matrix $\mathbf{J_c}(\mathbf{f})$.

The *user equilibrium* (UE) assignment searches for mutually consistent arc flows and costs, as introduced by Wardrop (1952) with deterministic path choice behaviour. Wardrop (or deterministic) path choice behaviour model is not dealt with here, since it leads to a path choice point-to-set map. It may be considered the limit case of any probabilistic choice model when dispersion goes to zero. User equilibrium with probabilistic path choice functions was introduced by Daganzo and Sheffi (1977), who called it *stochastic user equilibrium* (SUE). [For a comparison between UE and SUE see the appendix] In this case, equilibrium assignment can effectively be expressed by fixed-point models given by the arc cost function and the arc flow function:

$$\mathbf{c}^* = \mathbf{c}(\mathbf{f}^*; \boldsymbol{\mu}) \in \mathbf{c}(S_f) \subseteq R^n_+ \tag{3.8a}$$

$$\mathbf{f}^* = \mathbf{f}(\mathbf{c}^*; \mathbf{d}, \theta) \in S_f \subseteq R^n_+ \tag{3.8b}$$

Other equivalent models can be formulated with respect to path variables. An equivalent formulation with respect to flows (or costs) only is often used in literature (Cantarella, 1997; a different one in Daganzo, 1983), which can be obtained by explicitly including equation (3.8a) into equation (3.8b):

$$\mathbf{f}^* = \mathbf{f}(\mathbf{c}(\mathbf{f}^*; \boldsymbol{\mu}); \mathbf{d}, \theta) \in S_f \subseteq R^n_+$$

or vice versa

$$\mathbf{c}^* = \mathbf{c}(\mathbf{f}(\mathbf{c}^*; \mathbf{d}, \theta); \boldsymbol{\mu}) \in \mathbf{c}(S_f) \subseteq R^n_+ \tag{3.9}$$

Existence is guaranteed if both the arc cost function and the arc flow function are continuous (and the network is connected), through Brouwer theorem.

For monotone decreasing arc flow function, as for invariant probabilistic path choice functions (Cantarella, 1997), if the arc cost function is monotone strictly increasing uniqueness is guaranteed; if the Jacobian matrices of both the arc flow function, $\mathbf{f}(\mathbf{c})$, and the arc cost function, $\mathbf{c}(\mathbf{f})$, are well-defined, uniqueness is guaranteed by positive definite $\mathbf{J}_\mathbf{c}(\mathbf{f})$ and negative semi-definite $\mathbf{J}_\mathbf{f}(\mathbf{c})$. Uniqueness conditions can be weakened for strictly positive invariant probabilistic path choice functions only requiring that arc cost function is monotone increasing (but not necessarily strictly monotone). Anyhow uniqueness of arc flows also guarantees uniqueness of arc costs as well as path flows and costs. Weaker (sufficient) conditions for uniqueness have been recently derived (a comprehensive review of uniqueness conditions is in Cantarella et al., 2010); a full discussion of this topic is out the scope of this paper, it suffices mentioning that monotonicity of the arc cost function is not needed to assure uniqueness.

3.2 A simple discrete-time deterministic process model

A deterministic process model based on exponential smoothing is presented below. This model is quite simple, so as to carry out explicit stability analysis, but also rather effective since it allow singling out the role of each of the main parameters of the system. Moreover, its fixed-point states are equivalent to user equilibrium with probabilistic choice functions (as defined by 3.9), and it may be formulated with respect to arc or path variables leading to consistent evolutions over time and same stability conditions. Let

$\mathbf{x}^t$ be the vector of arc forecasted costs at day *t*, which are the values of costs that affect today user choice behaviour;

$\mathbf{f}^t$ be the vector of arc flows at day *t*;

$\mathbf{c}(\mathbf{f}^t; \boldsymbol{\mu})$ be the vector of arc costs occurred at day *t*.

The arc forecasted costs generally depends on yesterday actual and forecasted costs, through the *cost updating recursive equation*:

$$\mathbf{x}^{t} = \beta\, \mathbf{c}(\mathbf{f}^{t-1}; \boldsymbol{\mu}) + (1-\beta)\, \mathbf{x}^{t-1} \in \mathbf{c}(S_f) \qquad \text{with } \mathbf{x}^{0} = \mathbf{c}(\mathbf{f}^{0}; \boldsymbol{\mu}),\ \mathbf{f}^{0} \in S_f \tag{3.10}$$

where $\beta \in\ ]0,1]$ is the weight given to yesterday actual costs when forecasting today costs, assumed time invariant and common to all users. Dispersion among users is modelled through perceived utility distribution with respect to adopted path choice functions.

Remembering equation (3.6) : $\mathbf{f}^{t-1} = \mathbf{f}(\mathbf{x}^{t-1}; \mathbf{d}, \theta) \in S_f$, we get:

$$\mathbf{x}^{t} = \beta\, \mathbf{c}(\mathbf{f}(\mathbf{x}^{t-1}; \mathbf{d}, \theta); \boldsymbol{\mu}) + (1-\beta)\, \mathbf{x}^{t-1} \qquad \text{with } \mathbf{x}^{0} = \mathbf{c}(\mathbf{f}^{0}; \boldsymbol{\mu}),\ \mathbf{f}^{0} \in S_f \tag{3.11}$$

implying (see end of section 2):

$$\mathbf{x}^{t} - \mathbf{x}^{t-1} = \beta\, (\mathbf{c}(\mathbf{f}(\mathbf{x}^{t-1}; \mathbf{d}, \theta); \boldsymbol{\mu}) - \mathbf{x}^{t-1})$$
$$\mathbf{x}^{t} - \mathbf{x}^{t-1} = \beta\, \boldsymbol{\psi}(\mathbf{x}^{t-1}; \mathbf{d}, \theta, \boldsymbol{\mu}) \qquad \text{with } \boldsymbol{\psi}(\mathbf{x}; \mathbf{d}, \theta, \boldsymbol{\mu}) = \mathbf{c}(\mathbf{f}(\mathbf{x}; \mathbf{d}, \theta); \boldsymbol{\mu}) - \mathbf{x}$$

Equation (3.11) tries to model how each user make forecasts mixing own experience, experience shared with other users, as well as any other source of information. Forecasted costs provided by equation (3.11) are a convex combination of costs occurred on all the previous days until day t = 0, with weights β, $\beta\cdot(1-\beta)$, $\beta\cdot(1-\beta)^2$, …, , respectively. The weight given to any of the previous days becomes rather small after some days, for instance with $\beta = 0.64$, it is 0.5% after 6 days, and with $\beta = 0.42$, after 9 days. (According to some results for a moving average filter, the length of user memory seems rather short generally including few days only.)

The recursive equation (3.11) defines a (discrete-time) deterministic process model, for demand assignment to a transportation network. The state at day t is defined by the vectors of arc forecasted costs, $\mathbf{x}^t$. On one hand the proposed DP (3.11) is a rather simple model of complex user behaviours; on the other hand, it allows for theoretical analysis of fixed-point stability described in next sub-section 3.4. DP (3.1) is suitable for large scale application (through brute force approach), since the computer resources needed to run the deterministic process model (3.1) are comparable to those for most solution algorithms for equilibrium model (3.8).

Some system states of DP (3.11) are worth of in-depth analysis, in particular fixed-points, where the evolution over time of the 'system stops': $\mathbf{x}^t = \mathbf{x}^{t-1} = \mathbf{x}^*$. This condition combined with equations (3.11) yields:

$$\mathbf{x}^* = \mathbf{c}(\mathbf{f}(\mathbf{x}^*; \mathbf{d}, \theta); \boldsymbol{\mu}) \in \mathbf{c}(S_f) \tag{3.12}$$

A fixed-point state described by above equations (3.12) is equivalent to the user equilibrium, as defined by equations (3.9) or (3.8); hence, the already discussed equilibrium existence and uniqueness conditions still apply. Thus, definition, existence and uniqueness of fixed-point states of DP (3.11) depend on cost function parameters $\boldsymbol{\mu}$ (including arc capacities), dispersion parameter θ (and possibly other parameters of the choice function), and demand flows $\mathbf{d}$, but are not affected by the values of updating parameter β.

A deterministic process may evolve towards fixed-points as well as other kind of attractors, such periodic, quasi-periodic, and a-periodic (chaotic) attractors (attractors may be indentified through Lyapunov exponents, as shown in Cantarella & Velonà, 2003), or may not converge at all, as it can be observed by running it with different values of parameters (and/or starting states). Thus even if a fixed-point state exists and is unique the system may not converge towards it. Conditions for fixed-point (local) stability can be used to check whether the fixed-point is an attractor, without running the underlying deterministic process model, as discussed in sub-section 3.4.

We conclude by remarking that an *equivalent* model with respect to path costs is given below, using the same state variable even though with a different meaning:

$$\mathbf{x}_{[i]}^{t} = \beta\, \mathbf{B}_{[i]}^{\mathrm{T}}\, \mathbf{c}(\Sigma_j\, d_j\, \mathbf{B}_{[j]}\, \mathbf{p}_{[j]}(-\mathbf{x}_{[j]}^{t-1};\, \theta)\ ;\, \boldsymbol{\mu}) + (1-\beta)\, \mathbf{x}_{[i]}^{t-1} \qquad \forall i \tag{3.12}$$

where $\mathbf{x}_{[i]}^{t}$ is the vector of path forecasted costs for user class i at day t, which are the values of costs that affect today user choice behaviour.

This equivalence is something that is not so remarkable for probabilistic choice models, but which breaks down in the limit case of deterministic choice, leading to a range of new issues that are special to such limit cases – we deal with this issue later, in section 5.

3.3 A simple continuous-time deterministic process model

In order to understand the relation between, and implications of, the discrete- and continuous-time formulations, it is convenient to examine essentially the same underlying model for both cases. However, if the aim were to be representative of the existing body of literature on this topic, then this would be somewhat difficult to achieve, primarily since existing analyses of continuous-time systems have almost exclusively examined dynamics with respect to the deterministic / Wardrop user equilibrium model, where the analyses of discrete-time systems has focused on dynamics about a probabilistic user equilibrium. An exception is the work of Watling (1999), who considered both discrete- and continuous-time models for the same system; this paper therefore provides a useful bridge, even though it is not representative of the main bodies of work (especially on continuous-time systems).

Following Watling (1999), then, let us begin by considering the simple discrete-time model given by (3.11), which by defining:

$$\boldsymbol{\psi}(\mathbf{x};\, \mathbf{d}, \theta, \boldsymbol{\mu}) = \mathbf{c}(\mathbf{f}(\mathbf{x};\, \mathbf{d}, \theta);\, \boldsymbol{\mu}) - \mathbf{x} \tag{3.13}$$

we showed to be expressible in the form:

$$\mathbf{x}^{t} - \mathbf{x}^{t-1} = \beta\, \boldsymbol{\psi}(\mathbf{x}^{t-1};\, \mathbf{d}, \theta, \boldsymbol{\mu})\ . \tag{3.14}$$

This gives the adjustment to the process in a time increment of length 1, and thus it would be logical to assume that, in a fractional time increment of length Δt the process would adjust (from time t to $t+\Delta t$) by an amount $\beta\, \boldsymbol{\psi}(\mathbf{x}^{t-1};\, \mathbf{d}, \theta, \boldsymbol{\mu})\, \Delta t + o(\Delta t)$, i.e.:

$$\mathbf{x}(t + \Delta t) - \mathbf{x}(t) = \beta\, \boldsymbol{\psi}(\mathbf{x}^{t};\, \mathbf{d}, \theta, \boldsymbol{\mu})\, \Delta t + o(\Delta t) \qquad (t \in \mathbb{R}_{+})\ .$$

Taking the limit as $\Delta t \to 0$ then yields the standard form of a continuous time process, as a differential equation:

$$\dot{\mathbf{x}}(t) = \beta\, \boldsymbol{\psi}(\mathbf{x}(t)) \qquad (t \in \mathbb{R}_{+};\, \mathbf{x}(t) \in \mathcal{S})\ . \tag{3.15}$$

As in the discrete-time case, several kinds of system evolution may emerge from (3.15). One kind of system behaviour of special interest emerges from the counterpart to a 'system stop' as considered for the discrete-time model, namely those feasible points at which $\dot{\mathbf{x}}(t) = 0$. That is to say, the fixed points which are solutions to:

$$\boldsymbol{\psi}(\mathbf{x}^{*}) = \mathbf{0} \qquad (\mathbf{x}^{*} \in \mathcal{S})\ . \tag{3.16}$$

From (3.13) and (3.16), it is trivial to see that the fixed points of the continuous-time system coincide exactly with those of the discrete-time system, as given by (3.12), and that these fixed points coincide with the probabilistic user equilibria (3.8)/(3.9). Thus we have a bridge between discrete-time processes, continuous-time processes and equilibrium models.

Furthermore, just as noted in the discrete-time case, entirely equivalent representations exist of the above-defined model for the continuous-time case. Watling (1999) considered two such alternative possibilities. In the first, the path costs rather than arc costs were used as state variables, yielding a continuous-time system equivalent to (3.12) following the same construction logic as above. In the second, under the assumption that the path choice model (3.2) is given by a regular random utility model, an equivalent system may be developed in which, for each OD movement, we choose one 'reference path', and then consider the difference in all other path costs relative to the cost of the reference path. It is worth emphasising a key distinction with the work on deterministic / Wardrop user equilibrium models (as reviewed in section 5), where there has been a significant recent interest in distinguishing the properties of arc-based and path-based models; in the case of the present model, no such distinction occurs, the two representations are entirely equivalent. The advantage to the analyst of having such alternative representations is that some may be more amenable to deducing theoretical properties than others; Watling (1999), for example, made considerable use of the formulation in terms of path cost differences.

3.4 Stability analysis for a simple discrete-time deterministic process model

As noted above provided that exactly one fixed-point exists the system evolves towards it only if it is stable. Local stability conditions, from discrete-itme non-linear dunamic system theory, are based on a spectral analysis of the Jacobian matrix of a DP. Let

n be the state dimension (say the number of arcs);
$\mathbf{J}(\mathbf{x}^t)$ be the $(n \times n)$ Jacobian matrix of the DP at point $(\mathbf{x}^t)$;
λ_a be one of the n (not necessarily distinct) eigenvalues of matrix $\mathbf{J}$ (omitting dependence on $\mathbf{x}^t$).

Conditions for the (local) stability of a fixed-point $(\mathbf{x}^*)$ of a DP requires that the spectral radius ρ^*, that is the maximum modulus $|\lambda_a^*|$ among all the eigenvalues, of the Jacobian of the DP (at the fixed-point, $\mathbf{J}(\mathbf{x}^*)$, is less than one:

$$\rho^* = \max_a \{ | \lambda_a^* | \} < 1 \tag{3.17}$$

[It is noteworthy that, according to (3.17), null eigenvalues are not relevant to assess stability conditions. Moreover, the Jacobian matrices based on path variables have the same non-null eigenvalues of the Jacobian matrices with respect to arc variables, thus the very same results are obtained if path variables are considered.]

Applying some results of matrix algebra, Jacobian matrix $\mathbf{J}(\mathbf{x}^t)$ and its eigenvalues λ_a may be expressed highlighting the role of updating parameter β, which greatly affects the fixed-point stability:

$$\mathbf{J}(\mathbf{x}^t) = (1 - \beta)\,\mathbf{I_n} + \beta\,\mathbf{J}_c(\mathbf{f}(\mathbf{x}^t)) \cdot \mathbf{J}_f(\mathbf{x}^t) \tag{3.18}$$
$$\lambda_a = (1 - \beta) + \beta\,\omega_a = 1 + \beta\,(\omega_a - 1) \qquad \forall a = 1, \ldots, n \tag{3.19}$$

where $\omega_a = \omega_a(\mathbf{x}^t)$ is one of n eigenvalues of matrix $\mathbf{J}_c(\mathbf{f}(\mathbf{x}^t)) \cdot \mathbf{J}_f(\mathbf{x}^t)$.

Equation (3.19) also allows us to compute the determinant of the Jacobian of DP (3.11) since $| \mathbf{J} | = \prod_a \lambda_a$. Its absolute value may be out of the range [0,1[, thus the system DP may not be *dissipative*, that is it may not converge to an attractor (this issue may deserve further analysis).

The (local) stability conditions (3.17) may be expressed with respect to the eigenvalues ω_a^* of matrix $\mathbf{J}(\mathbf{x}^*)$ and updating parameter β using equation (3.19):

$$((\mathrm{Re}(\omega_a^*) - 1) + 1/\beta)^2 + \mathrm{Im}(\omega_a^*)^2 < 1/\beta^2 \qquad \forall a = 1, \dots, n \qquad (3.20)$$

The stability region for eigenvalues ω_a^* is the inside of a circle on the Argand plan with radius $(1/\beta)$ and center at $((1 - 1/\beta), 0)$. The lower the value of parameter β, the greater the area of stability region is.

The stability region is located between the two points on real axis $((1 - 2/\beta), 0)$ and $(1, 0)$. Thus, if there exist at least an eigenvalue ω_a^* that have real part greater than one, $\mathrm{Re}(\omega_a^*) > 1$, the fixed-point is always non-stable whatever the values of updating parameter β; in this case multiple fixed-points can be found. Vice versa if all the eigenvalues ω_a^* have real less than one, $\mathrm{Re}(\omega_a^*) < 1$, or $\max_a \{\mathrm{Re}(\omega_a^*) - 1\} < 0$, there always a small enough value of parameter β such that the fixed-point is stable.

Condition (3.20) allows us to clearly distinguish the role of updating parameter β, which only affects the size of the stability region, and that of all the other parameters, which only affects the eigenvalues ω_a^*. Hence, the effect of any change of updating parameter β can be analyzed without re-computing the eigenvalues ω_a^*.

Condition (3.20) is to be verified at the fixed-point only. On the other hand, it only assures local stability: that is there is an *attraction domain* of the fixed-point state such that from any starting state in the attraction domain the system converges towards the fixed-point, otherwise the system may converge to other fixed-points, other kinds of attractors, or may not converge at all. Generally the attraction domain is only a subset of the state space. Thus, global stability conditions are still an open issue. (Bie & Lo (2010) addresses the relevant issues of attraction domain definition and analysis.)

If the arc flow function has a symmetric negative semi-definite Jacobian, $\mathbf{J_f}(\mathbf{x})$, as it occurs for invariant choice functions, the stability condition (3.20) can be further exploited as described below.

- If the Jacobian, $\mathbf{J_c}(\mathbf{f}^t)$, of the arc cost function is positive semi-definite (for real vectors at least) then it can be proved that all the eigenvalues of matrix $\mathbf{J}(\mathbf{x}^t, \mathbf{f}^t)$ have non-positive real part, $\mathrm{Re}(\omega_a) \le 0$.

- If the Jacobian $\mathbf{J_c}(\mathbf{f}(\mathbf{c}))$ of arc cost function is symmetric, it can be proved that matrix $\mathbf{J}(\mathbf{x}^t)$ has only real eigenvalues, $\omega_a^* = \mathrm{Re}(\omega_a^*)$, thus the stability conditions (3.20) becomes:

$$\omega_a^* \in](1 - 2/\beta), 1[\qquad \forall a = 1, \dots, n \qquad (3.21)$$

- Thus, if the Jacobian, $\mathbf{J_c}(\mathbf{f}^t)$, of the arc cost function is both symmetric and positive semi-definite then all the eigenvalues of matrix $\mathbf{J}(\mathbf{x}^t, \mathbf{f}^t)$ are non-positive real numbers, $\omega_a^* = \mathrm{Re}(\omega_a) \le 0$; in this case the stability condition (3.21) becomes:

$$\max_a \{| \omega_a^* |\} < -1 + 2/\beta \qquad (3.22)$$

Equation (3.22) gives an upper bound for the absolute values of the eigenvalues ω_a^*. Function $(-1 + 2/\beta)$ always gets values greater than or equal to 1 for β in the range]0,1], and goes to infinity as β goes to zero. The value of the function is to be considered an input data, since it depends on updating parameter β, which is an input data resulting from the calibration of the model or the design scenario. On the other hand knowing all the eigenvalues ω_a^* equations (3.22) provides an upper bound for parameter β to assure stability:

$$0 < \beta < \beta_{max} = 2 / (\max_a \{|\omega_a^*|\} + 1)$$

3.5 Stability analysis for a simple continuous-time deterministic process model

In section 3.4, we noted how stability properties of fixed points of the discrete-time system given by (3.14), based on (3.13), are greatly affected by the value of the parameter β. Therefore any such result on stability effectively amounts to a network-specific test as to whether some particular value of β will lead to stable or unstable system behaviour. In the corresponding continuous-time system, given by (3.15) based on (3.13), a rather different picture emerges.

In continuous time systems, the local stability of a fixed point may again be determined by analysis of the eigenvalues of the system Jacobian of (3.15), evaluated at the fixed point state. From inspection, it is easy to see that if $\mathbf{J}(\mathbf{x}^*)$ is the Jacobian of the discrete-time system (3.14) at fixed point $\mathbf{x}^*$, then the Jacobian of the continuous-time system (3.15) at point $\mathbf{x}^*$is given by $\mathbf{J}(\mathbf{x}^*) - \mathbf{I}$. Thus, if the eigenvalues of $\mathbf{J}(\mathbf{x}^*)$ are $\{\lambda_a^* : a = 1,2,\ldots,n\}$, then those of $\mathbf{J}(\mathbf{x}^*) - \mathbf{I}$ are $\{\lambda_a^* - 1 : a = 1,2,\ldots,n\}$. For the continuous-time system, the corresponding necessary and sufficient condition for local stability is that the real parts of any such eigenvalues are all negative, which we may thus write as:

$$\max_a \{ \mathrm{Re}(\lambda_a^* - 1) \} < 0 \,. \tag{3.23}$$

However, using (3.23), we note that (cfr 3.19):

$$\lambda_a^* - 1 = (1 - \beta) + \beta\, \omega_a^* - 1 = \beta\, (\omega_a^* - 1) \qquad (a = 1, \ldots, n) \tag{3.24}$$

and since $\beta > 0$, condition (3.23) is equivalent to:

$$\max_a \{ \mathrm{Re}(\omega_a^* - 1) \} < 0 \,. \tag{3.25}$$

The significance of (3.25) is that ω_a^* is independent of β; in contrast to the deterministic time system, the adjustment parameter β plays no part in determining stability of the fixed point. This illustrates that the two ‘stability’ properties of continuous and discrete time systems are not interchangeable, and they may be referring to quite different phenomena.

Due to the fact that neither the fixed points nor the stability properties of the continuous-time system depend on β, a natural simplification is then to consider system (3.15) in the special case of $\beta = 1$:

$$\dot{\mathbf{x}}(t) = \boldsymbol{\psi}(\mathbf{x}(t)) \qquad (t \in \mathbb{R}_+;\ \mathbf{x}(t) \in \mathcal{S}) \,. \tag{3.26}$$

We can then deduce an interesting relationship between stability of fixed points of (3.26) and the stability of fixed points of (3.14), when we have the extra ‘degree of freedom’ of choice of the value of β in the latter system.

In particular, suppose the fixed point $\mathbf{x}^*$is stable with respect to the continuous time system (3.26). By inspection, if $\mathbf{J}(\mathbf{x}^*)$ is the Jacobian of the discrete-time system (3.14) at fixed point $\mathbf{x}^*$, then the Jacobian of the continuous-time system (3.15) at point $\mathbf{x}^*$is given by $\beta^{-1}(\mathbf{J}(\mathbf{x}^*) - \mathbf{I}) = (\mathbf{J_f}\, \mathbf{J_c} - \mathbf{I})$. Then, if the eigenvalues of $\mathbf{J}(\mathbf{x}^*)$ are $\{\lambda_a^* : a = 1,2,\ldots,n\}$, then those of $\beta^{-1}(\mathbf{J}(\mathbf{x}^*) - \mathbf{I}) = (\mathbf{J}_c(\mathbf{f}(\mathbf{x}^*)) \cdot \mathbf{J}_f(\mathbf{x}^*) - \mathbf{I})$ are $\{\gamma_a^* = \beta^{-1}(\lambda_a^* - 1) = \omega_a^* - 1 , a = 1,2,\ldots,n\}$. Thus $\lambda_a^* = 1 + \beta\gamma_a^*$ ($a = 1,2,\ldots,n$). Now, the stability of $\mathbf{x}^*$ with respect to (3.26) implies that $\max_a \{\mathrm{Re}(\gamma_a^*)\} < 0$, and so may write:

$$\gamma_a^* = \mathrm{Re}(\gamma_a^*) + i\,\mathrm{Im}(\gamma_a^*) \qquad (a = 1,2,\ldots,n).$$

Hence (for $a = 1,2,\ldots,n$):

$$|\lambda_a^*|^2 = |1 + \beta\,\gamma_a^*|^2 = |(1 + \beta\,\mathrm{Re}(\gamma_a^*)) + i\,\beta\,\mathrm{Im}(\gamma_a^*)|^2 = (1+ \beta\,\mathrm{Re}(\gamma_a^*))^2 + \beta^2\,\mathrm{Im}(\gamma_a^*)^2 =$$
$$=1+ 2\,\beta\,\mathrm{Re}(\gamma_a^*) + \beta^2\,(\mathrm{Re}(\gamma_a^*)^2 + \mathrm{Im}(\gamma_a^*)^2)$$

which (for $\beta > 0$) is less than 1 if and only if:

$$2\,\beta\,\mathrm{Re}(\gamma_a^*) + \beta^2\,((\mathrm{Re}(\gamma_a^*)^2 + \mathrm{Im}(\gamma_a^*)^2) < 0 \;\Leftrightarrow$$
$$\Leftrightarrow\; 0 < \beta < 2|\mathrm{Re}(\gamma_a^*)| \,/\, (\mathrm{Re}(\gamma_a^*)^2 + \mathrm{Im}(\gamma_a^*)^2)$$

Thus the discrete time system (3.14) is stable for:

$$0 < \beta < \beta_{\max} = \min_a \{2|\mathrm{Re}(\gamma_a^*)|/(\mathrm{Re}(\gamma_a^*)^2 + \mathrm{Im}(\gamma_a^*)^2)\} \;.$$

That is to say, if $\mathbf{x}^*$ is stable for the continuous-time system (3.26), $\max_a \{\mathrm{Re}(\omega_a^*)-1\} < 0$, then it is also stable for the discrete-time system (3.14) for sufficiently small values of the learning parameter β (as already noted in sub-section 3.4).

The converse is also true; this is readily seen by proving the contrapositive statement, namely that if $\mathbf{x}^*$ is *un*stable for (3.26), then for any value of β it is also *un*stable for (3.14). From the proof above, this can be seen by noting that if $\mathbf{x}^*$ is *un*stable for (3.26) then there exists some eigenvalue γ_a^* where $\mathrm{Re}(\gamma_a^*) < 0$, and for this eigenvalue it must be that $|\lambda_a^*| > 1$; this implies that $\mathbf{x}^*$ is *un*stable for (3.26), since it is both a necessary and sufficient condition that all eigenvalues of the relevant Jacobian are inside the unit circle.

Therefore, while stability in discrete-time is a somewhat different property to that in continuous-time, there are useful relations between the two that may be exploited. Having said this, for the particular simple family of models we are presently considering, there have yet to be any general conditions (paralleling the discrete-time case) that establish stability of the continuous-time system (3.26). On the other hand, we may test for stability of any continuous-time system in any particular numerical setting by using tools such as the Routh-Hurwitz criterion; see Watling (1999) for such an illustration.

As was noted at the start of section 3.3, our decision to analyse the particular dynamical model considered here was based on a desire to compare properties of discrete- and continuous-time systems, and in this sense we have chosen what we believe to be the most suitable example for illustrative purposes. On the other hand, virtually all analyses of continuous-time network problems have been performed with respect to dynamical processes related to the deterministic user equilibrium model, and in this case a series of general stability results exist. We review these papers in section 5; some quite distinct issues arise in what is effectively a limit case, as the variance in drivers' perceptual differences/errors tends to zero, and for this reason we believe it more suitable to treat these in a separate way.

Before leaving this section, we should also note that, just as for the discrete-time case, the stability analysis above requires only that the relevant conditions be verified at a fixed point, and as a result they assure only local stability, namely stability with respect to systems that are initialised within some attraction domain about the fixed point. As we also discuss in section 5, rather more is known about *global* stability properties in the limit case of deterministic choice models, even if for rather idealized cases and/or ad hoc specifications.

4. Extended discrete time models and research perspectives

In this section some extended discrete-time deterministic process are briefly discussed together with some research perspectives. The general specification of a deterministic process requires the explicit modelling of

- user learning and forecasting: how users forecast the level of service that they will experience today, from experience and other sources of information;
- user habit: how users make a choice today, possibly repeating yesterday choice to avoid the effort needed to take a decision, or reconsidering it according to forecasted level of service.

In a simple extension of DP (3.11), user learning and forecasting behaviour is modelled through an exponential smoothing filter giving the *cost updating recursive equation* already described by the recursive equation (3.10) repeated below:

$$\mathbf{x}^t = \beta\, \mathbf{c}(\mathbf{f}^{t-1}; \boldsymbol{\mu}) + (1-\beta)\, \mathbf{x}^{t-1} \in \mathbf{c}(S_f) \qquad \text{with } \mathbf{x}^0 = \mathbf{c}(\mathbf{f}^0; \boldsymbol{\mu}),\ \mathbf{f}^0 \in S_f \tag{4.1a}$$

In addition, users may also review yesterday choice with a fixed probability, and their choice behaviour after reviewing can be simulated through a probabilistic path choice function. Let

$\alpha \in]0,1]$ be the probability of reconsidering yesterday choices; thus each day t, $(1-\alpha)\, d_{[i]}$ users simply repeat yesterday choice, assumed time invariant and common to all users.

$\mathbf{f}(\mathbf{x}^t; \alpha\, \mathbf{d}, \theta) = \alpha\, \mathbf{f}(\mathbf{x}^t; \mathbf{d}, \theta)$ be the vector of arc flows at day t due to the $\alpha\, \mathbf{d}$ users who have reconsidered yesterday choices, and behave according to forecasted costs $\mathbf{x}^t$.

Thus, an exponential smoothing filter gives the *flow updating recursive equation*:

$$\mathbf{f}^t = \alpha\, \mathbf{f}(\mathbf{x}^t; \mathbf{d}, \theta) + (1-\alpha)\, \mathbf{f}^{t-1} \in S_f \qquad \text{with } \mathbf{f}^0 \in S_f \tag{4.1b}$$

Comments made above for cost updating parameter β about multi-user assignment and calibration, as well as on numerical interpretation, apply to the flow updating parameter α too; in this case values in the range [0.4, 0.6] seem likely.

The recursive equations (4.1a) and (4.1b) define a (discrete-time) deterministic process model (DP), for demand assignment to a transportation network. The state at day t is defined by the vectors of arc forecasted costs and arc flows, $(\mathbf{x}^t, \mathbf{f}^t)$. On one hand the proposed DP (4.1) is a rather simple model of complex user behaviours; on the other hand, it allows for theoretical analysis of fixed-point stability described in next sub-sections. DP (4.1) is suitable for large scale application (through brute force approach), since the computer resources needed to run DP (4.1) are comparable to those for most solution algorithms for equilibrium model (3.8).

Some system states of DP (4.1) are worth of in-depth analysis, in particular fixed-points, the evolution over time of the system stops: $(\mathbf{x}^t, \mathbf{f}^t) = (\mathbf{x}^{t-1}, \mathbf{f}^{t-1}) = (\mathbf{x}^*, \mathbf{f}^*)$. This condition combined with equations (4.1) yields:

$$\mathbf{x}^* = \mathbf{c}(\mathbf{f}^*; \boldsymbol{\mu}) \in \mathbf{c}(S_f) \tag{4.2a}$$
$$\mathbf{f}^* = \mathbf{f}(\mathbf{x}^*; \mathbf{d}, \theta) \in S_f \tag{4.2b}$$

A fixed-point state described by above equations (4.2) is equivalent to the user equilibrium, as defined by equations (3.8); hence, the already discussed equilibrium existence and uniqueness conditions still apply.

Definition, existence and uniqueness of fixed-point states of DP (4.1) depend on cost function parameters $\boldsymbol{\mu}$ (including arc capacities), dispersion parameter θ (and possibly other parameters of the choice function), and demand flows **d**, but are not affected by the values of updating parameters α and β. On the other hand, provided that exactly one fixed-point exists the system evolves towards it only if it is stable, and stability of a fixed-point state is greatly affected by the values of updating parameters α and β, as discussed below.

The analysis carried out in sub-section 3.4 may be quite straightforwardly be applied to DP (4.1) clearly distinguishing the role of updating parameters α and β, which only affects the size of the stability region, and that of all the other parameters, which only affects the eigenvalues ω_a^*. This approach cannot be followed for more general models.

Conditions for fixed-point (local) stability can be used to check whether the fixed-point is an attractor, without running the underlying deterministic process model. These conditions are based on a spectral analysis of the Jacobian matrix of DP (4.1). (For more details see Cantarella and Cascetta, 1995; Cantarella and Velonà, 2003; Cantarella, 2013).

It is worth noting that DP (4.1) is, *dissipative*, that is it converges anyway to some kind of attractor, over the whole state space, that is from any starting state, and any combination of parameters. Since DP (4.1) is dissipative only three types of bifurcations may occur:

- a Pitchfork bifurcation, in this case several fixed-points exist, only some of them being stable, and the system evolves towards a stable fixed-point that depends on the starting state, unless the starting state is exactly a non-stable fixed-point;
- a Flip bifurcation, and the system evolves towards a periodic attractor, then possibly to an a-periodic one.
- a Neumark bifurcation, and the system evolves towards a quasi-periodic attractor.

Assuming that the arc flow function has a symmetric negative semi-definite Jacobian matrix, $\mathbf{J_f}(\mathbf{x})$, as it occurs for invariant choice functions, the following results hold.

- If the Jacobian matrix, $\mathbf{J_c}(\mathbf{f})$, of arc cost function is symmetric, Pitchfork or Flip bifurcations only may be observed. Hence quasi-periodic attractors may only occur with arc cost function with asymmetric Jacobian.
- If the Jacobian matrix, $\mathbf{J_c}(\mathbf{f})$, of arc cost function is symmetric positive semi-definite, Flip bifurcations only may be observed. Hence, multiple fixed-points may not occur.

A relevant research perspective regards multi-user class assignment to model distribution of updating parameters, as well as of dispersion and possibly other parameters among users. In this case, the extension of explicit stability analysis is by no means straightforward. This way e.g. systematic vs. non-systematic users or ATIS-equipped vs. non-equipped users might be differentiated, this extension is rather straightforward, but does not allows explicit stability analysis.

Most other approaches to user learning and forecasting are based on moving average filters (eg Iida et al., 1992), possibly with respect to differences between actual and forecasted costs in previous days. Other approaches to modelling user inertia to change and how much users are prone to review their habit are briefly described below; they are not suitable for the explicit stability analysis carried out in sub-section (3.4) distinguishing the role of each parameter, unless otherwise stated.

In aggregate approaches based on extra utility models for conditional path choice model where the path chosen the previous day is given an extra utility, expressing the so-called transition cost to a different alternative (an example of such an approach is in Cascetta and Cantarella, 1991).

In aggregate approaches to modelling the effect of reliability of (possibly different) information sources the flow updating parameter α depends on the aggregate reliability, thus may change over time. Examples of this kind models, often called *bounded-rationality models*, are mostly based on probabilistic (or deterministic) threshold filters with respect to differences between actual and forecasted costs; in this case the flow updating parameter α is the results of a switching choice model. Modelling effects of an ATIS reliability is addressed by Bifulco et al. (2009) through a modelling approach consistent with this paper; the analysis is further developed in Bifulco et al. (2013) requiring deterministic process models where today state depends on today state too apart the yesterday one (the inverse function method described in section 2 has been applied in this case, as already noted.). Still, embedding this approach in a (complete) multi-user framework allowing for the explicitly stability analysis is still an open issue.

In disaggregate approaches (examples are reported by Cascetta & Cantarella, 1993; Chang & Mahamassani, 1988, 2004), a flow updating parameter α is defined for each path separately depending on the difference between experienced and forecasted arc costs. The use of probabilistic thresholds leads to path choice switching models. This approach is rather effective when only two paths are available between each O-D pair, since there is no need of any path choice function. Indeed, when more than two paths are available, a conditional path choice function should be applied to model path choice behaviour of users who decide to reconsider their yesterday choice. This approach seems better suited for disaggregate assignment through stochastic process models, which are out of the scope of this paper.

More general models are given in Cantarella & Cascetta (1995), for instance a matrix is used for the convex combination underlining the exponential filter. But, in more general models, not based on exponential smoothing filters, fixed-point states may not be equivalent to the user equilibrium; in this case ad hoc existence and uniqueness conditions should be developed.

The deterministic process model (4.1) is suitable for large applications, as already stated. On the other hand, the application of stability conditions as such to a large scale network seems quite hard, since it requires the computation of the eigenvalues of large matrices; in this approximation through matrix norms may be applied. This expression of stability conditions may be included as a constraint in optimization models for Transportation Supply Design with equilibrium assignment. This approach has been applied by Cantarella et al. (2012) for signal setting with equilibrium at a large scale.

5. Continuous-time models based on deterministic choice

Thus far, our analysis has focused on models in which choice decisions, conditional on the past, are represented through probabilistic choice models based on random utility theory. It is well known that as the variance in the random terms tends to zero, then so we approach a case of "deterministic choice". This is a potentially confusing term since all of the models we consider in the present paper (including those with probabilistic choice) are deterministic processes, but the distinction is that in the case of "deterministic choice" all users make decisions without the presence of unexplained/random variation. Such dynamical models are thus potentially a way of exploring stability with respect to "deterministic" (Wardrop) user equilibrium.

In such cases, we may potentially explore systems in either discrete-time or continuous-time; however, all existing literature on this topic, to the authors' knowledge, has focused on continuous-time systems, primarily since in such case it is possible to derive some rather general results for certain classes of adjustment process. Although corresponding discrete-time systems have not been explored to date, the comments made in sections 3.3 and 3.5 could be trivially extended to the present case; thus, we can think of stability in continuous-time as establishing the existence of a sufficiently slow rate of adjustment for stability in discrete-time, but without providing insights into the stronger property of whether particular adjustment rates will or will not be stable. We therefore shall not show the explicit extension of the results in section 3.3 and 3.5 to the limit case of deterministic choice since, although some care is needed (due to non-uniqueness issues, as discussed below), the essential elements of the arguments made in that section transfer in a straightforward way.

Thus our focus will be on continuous time DP models based on deterministic choice. One of the advantages of analysing dynamical systems in continuous rather than discrete time is that, generally speaking, it is more straightforward to find general theoretical results than for discrete-time systems. Aside from this it seems appealing to represent time as continuous, since it surely is in reality. However, this brings with it a significant difficulty, namely how to represent in a single model the very different kinds of phenomena that happen on a shorter 'within-day' time-scale (e.g. the congestion interactions at vehicles traverse a network on a particular day) with those on a longer 'between-day' time-scale (e.g. drivers reviewing their choice of route on a subsequent trip in response to trip experiences in the recent past, but not in response to 'instantaneous' conditions). While continuous-time models as proposed to date therefore seem rather more distant from the real-life system than their discrete-time counterparts, the study of the former kind of system can still be informative; for example, as we discussed in section 3.5, they provide insights into properties of discrete time models.

Informally we can understand continuous systems as describing a kind of general trend, without getting into the detail of what exactly that would mean. Perhaps due to the difficulty in synchronizing the within- and between-day scales, it seems that virtually all analyses of continuous-time traffic assignment models to date have adopted an 'instantaneous' network loading relationship, and it is therefore one that we shall also adopt here. That is, at any time instant t, a path flow instantaneously propagates to each of its component arcs at that time t. In turn these are used to generate instantaneous arc travel costs, through steady-state arc performance functions, which are then components of the instantaneous path travel cost at time instant *t*. We refer to this below as 'the instantaneous loading assumption'. (Although effectively the same occurs in a discrete time process but just over a discrete time *t*, such a model offers the possibility of a different interpretation due to the fact that we may think of each 'epoch' of the discrete time process as having a duration of its own, allowing us to refer separately to 'within-day' and 'between-day' time.)

The instantaneous loading assumption means that simple relationships then exist between arc flows/costs and path flows/costs. In particular, at any given time instant *t*, the $|K_{[i]}|$-dimensional path cost vector $\mathbf{w}_{[i]}(t)$ for user class *i* is then related to a corresponding arc cost vector $\mathbf{c}(t)$ through the arc path cost consistency relationship (3.4), i.e. through $\mathbf{w}_{[i]}(t) = \mathbf{B}_{[i]}^{\mathrm{T}}\ \mathbf{c}(t)$, for $i = 1,2,\ldots,m$. Similarly at any time instant *t*, the $|K_{[i]}|$-dimensional path flow vector $\mathbf{h}_{[i]}(t)$ for class *i* is related to a corresponding arc flow vector $\mathbf{f}(t)$ through (3.5) by $\mathbf{f}(t) = \sum_{[i]} \mathbf{B}_{[i]}\,\mathbf{h}_{[i]}(\mathrm{t})$.

If we assume there to be a time-independent congestion function $\mathbf{c}(.)$ given by (3.7) that maps (instantaneous) arc flows $\mathbf{f}(t)$ onto (instantaneous) arc costs $\mathbf{c}(t) = \mathbf{c}(\mathbf{f}(t))$, it follows that we may then also write down a time-independent functional relationship which maps the concatenated vector of all user class path flows $\mathbf{h}(t) = (\mathbf{h}_1(t), \mathbf{h}_2(t),\ldots, \mathbf{h}_m(t))^{\mathrm{T}}$ onto user class path costs $\mathbf{w}_1(t), \mathbf{w}_2(t),\ldots$:

$$\mathbf{w}_{[i]}(t) = \mathbf{w}_{[i]}(\mathbf{h}(t)) = \mathbf{B}_{[i]}^{\mathrm{T}}\mathbf{c}(\textstyle\sum_j \mathbf{B}_j\mathbf{h}_j(\mathrm{t})) \qquad i = 1,2,\ldots,m\,.$$

Below, it is sometimes more convenient to work with congestion relationships on the arc level and sometimes on the path level, but these are entirely equivalent through the relationship above. Finally, we remark that as stated in §3, for each user class *i* we assume there to be a time-independent, non-empty and finite set $K_{[i]}$ of elementary paths available, and that there is a time-independent demand flow of $d_{[i]} \geq 0$.

As noted in §2, for continuous-time systems the specification is typically in terms of a differential equation, with a dot over a variable denoting a time-derivative. It seems that the first person to explicitly write down such a system in a traffic network assignment context was Smith (1984a), which is most readily specified in the path domain. Denoting $\mathbf{h}(t) = (\mathbf{h}_1(t), \mathbf{h}_2(t),\ldots)^{\mathrm{T}}$, Smith considered the system:

$$\dot{\mathbf{h}} = \textstyle\sum_{[i]} \sum_{k,l\in K_{[i]}\ (k\neq l)} h_{[i]r}(t)\ \max(0, w_{[i]k}(\mathbf{h}(t)) - w_{[i]l}(\mathbf{h}(t)))\ \boldsymbol{\Delta}_{kl}$$

where for each user class *i* and for each $k, l \in K_{[i]}$ $(k \neq l)$ the path-swap indicator vector $\boldsymbol{\Delta}_{kl}$ is the vector of dimension $\sum_{[i]} |K_{[i]}|$ with –1 in the k^{th} element, +1 in the l^{th} element, and zeroes elsewhere[1]. That is to say, this model assumes drivers on a higher cost path than an alternative will switch to the alternative at a rate proportional to the product of the flow on the higher cost path and the difference in cost. Under the assumption of weakly monotone path cost functions, the Wardrop equilibria are in general non-unique, but form a convex set which coincides with the point equilibria of this system. Under this weak monotonicity condition, Smith established global convergence of his system to a point equilibrium in this convex set.

Friesz et al. (1994) and Zhang and Nagurney (1996) both considered the stability of the *elastic demand* Wardrop equilibrium state. Friesz et al. assumed the OD demand

[1] This assumes that the paths are labelled 1 up to the total number of paths for *all* user classes, so we must start with some overall path label set *K* of which the { $K_1, K_2, \ldots, K_m$) are then subsets.

vector $\mathbf{d}(\mathbf{u})$ to be a function of the m-vector of OD travel costs $\mathbf{u}$. Using the vector pair $(\mathbf{u}(t), \mathbf{h}(t))$ as state variable, they considered the system:

$$\dot{u}_i = \kappa_{[i]} \{ \max(0, u_{[i]}(t) + \alpha\, (d_{[i]}(\mathbf{u}(t)) - \textstyle\sum_{k\in K_{[i]}} h_k(t))) - u_{[i]}(t)\} \qquad (i = 1,2,\ldots,m)$$
$$\dot{h}_k = \eta_k \{ \max(0, h_k(t) - \beta\, (w_{[i]k}(\mathbf{h}(t)) - u_{[i]}(t))) - h_k(t)\} \qquad (k\in K_{[i]}\,;\, i = 1,2,\ldots,m)$$

for given values of the parameters $\kappa_{[i]} > 0$ ($i = 1,2,\ldots,m$), $\eta_k > 0$ ($k\in K_{[i]}$; $i = 1,2,\ldots,m$), $\alpha > 0$ and $\beta > 0$. The justification for this model was as a representation of a traveller information system, which was able to inform drivers of the equilibrium path costs for the previous day. Friesz et al established that there exists a sufficiently large value of α and small value of β to ensure asymptotic stability of the set of path-based, elastic Wardrop equilibrium solutions with respect to this system, under the assumption that both the arc cost functions and negative demand functions are continuous and strictly monotone.

The approach of Zhang and Nagurney (1996) differed from Friesz et al, firstly, in the respect that they assumed the demand functions to be invertible (in the terminology of Friesz et al.); i.e. that there exist inverse demand functions $(\zeta_1(\mathbf{d}), \zeta_2(\mathbf{d}), \ldots, \zeta_m(\mathbf{d}))$ such that $\zeta_{[i]}(\mathbf{d}(\mathbf{u})) = u_{[i]}$ ($i = 1,2,\ldots,m$). But since we always have that the OD demands are the sum of path flows:

$$d_{[i]} = \textstyle\sum_{k\in K_{[i]}} h_k \qquad (i = 1,2,\ldots,m)$$

then we can re-write the inverse demand functions as dependent on $\mathbf{h}$, rather than $\mathbf{d}$; let us suppose that the inverse demand functions written in this way are $(\xi_1(\mathbf{h}), \xi_2(\mathbf{h}), \ldots, \xi_m(\mathbf{h}))$. We may then consider the following continuous time system in state variable $\mathbf{h}(t)$ only:

$$\begin{aligned} \dot{h}_k &= \zeta_{[i]}(\mathbf{h}(t)) - w_{[i]k}(\mathbf{h}(t)) && \text{if } h_k(t) > 0 \qquad (k\in K_{[i]}\,;\, i = 1,2,\ldots,m) \\ &= \max(0, \zeta_{[i]}(\mathbf{h}(t)) - w_{[i]k}(\mathbf{h}(t))) && \text{if } h_k(t) = 0 \end{aligned}$$

Zhang and Nagurney establish global asymptotic stability of the Wardrop elastic equilibrium with respect to this system, under the assumptions that the arc cost functions and negative inverse demand functions are continuous and strictly monotone. As an alternative, when monotonicity assumptions cannot be guaranteed, they show how local asymptotic stability may be tested by classical techniques from the dynamical systems literature, explicitly determining the eigenvalues (characteristic values) of a Jacobian matrix of dimension equal to the number of paths with non-zero equilibrium flow, with asymptotic stability guaranteed if these eigenvalues all have negative real parts.

More recently Yang & Zhang (2009) – building on the theory set out in Zhang et al (2001) – went on to show how the systems proposed by Smith (1984), Friesz et al. (1994) and Zhang and Nagurney (1996) [as reviewed above], along with several others models proposed for transportation problems, were examples of a general family characterised by being a *Rational Behaviour Adjustment Process* (RBAP):

> '*A day-to-day route choice adjustment process is called a RBAP with fixed travel demand if the aggregated travel cost of the entire network decreases based on the previous day's path travel costs when path flows change from day to day*.'

They showed furthermore that if in any RBAP system a path flow becomes stationary over days, then it must be a user equilibrium path flow.

Most recently, there has been an interest in *arc-based* continuous-time systems, in contrast to the path-based methods described so far. This was motivated by the observations of He et al (2010), who noted two deficiencies with path-based methods in the context of continuous time systems that approach user equilibrium. These were namely that different path-flow equilibria would arise from different initial path-flows, and that the path-based methods neglect a kind of interdependence between paths. [These considerations only apply to continuous time systems based on deterministic choice behaviour.] In response they proposed the following system with the arc flow vector $\mathbf{f}(t)$ as state variable:

$$\dot{\mathbf{f}} = \delta\left(\operatorname{argmin}\left\{\mathbf{y} \in S_f : \lambda \mathbf{c}(\mathbf{f})^{\mathrm{T}}\mathbf{y} + (1-\lambda)D(\mathbf{f},\mathbf{y})\right\} - \mathbf{f}\right)$$

where $\delta > 0$ and $0 < \lambda < 1$ are given parameters, S_f is the feasible arc flow set (as defined in section 3), and $D(. , .)$ is a distance metric. Assuming the arc cost functions to be separable between arcs, by which we may write $\mathbf{c}(\mathbf{f}) = (c_1(f_1), c_2(f_2),\ldots, c_n(f_n))$, He et al propose the use of the metric:

$$D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} \int_{x_i}^{y_i} \left(c_i(w) - c_i(x_i)\right) dw \,.$$

Under the assumption of continuously differentiable and monotonically increasing arc cost functions, He et al showed that $\mathbf{f}$ is a fixed point of the system above if and only if $\mathbf{f}$ is a user equilibrium flow pattern.

He et al's results a have since been extended in two ways. Guo & Liu (2011) showed that the approach may be adapted to analyse boundedly rational behaviour, establishing that $\mathbf{f}$ was a fixed point of the resulting system if and only if $\mathbf{f}$ was a boundedly rational user equilibrium arc flow pattern. In parallel work, and returning to the user equilibrium framework, Han & Du (2012) recently extended He et al's result by considering a family of metrics satisfying the conditions:

- $D(. ,.)$ is a nonnegative function;
- $D(\mathbf{x}, \mathbf{y}) = 0 \Leftrightarrow \mathbf{x} = \mathbf{y}$;
- $D(\mathbf{x}, \mathbf{y})$ is differentiable and strictly convex in $\mathbf{y}$ for each $\mathbf{x}$.

For such a family, and under the same assumptions on the arc cost functions as He et al made, they showed that $\mathbf{f}$ is a fixed point of He et al's dynamical system (for any D in the family above) if and only if $\mathbf{f}$ is a user equilibrium. Moreover they establish that such a fixed point is globally asymptotically stable over S_f for this family of dynamical systems. Finally, they extended this result to asymmetric, non-separable arc cost functions, though for a much more restricted family of metrics. Under the assumption that $\mathbf{c}(\mathbf{f})$ is continuous and strictly monotone (in the vector sense), and for the family of distance metrics satisfying the gradient condition $D_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) = - D_{\mathbf{y}}(\mathbf{x}, \mathbf{y})$, they again establish that any fixed point of He et al's system is globally asymptotically stable and the unique user equilibrium for that problem. An example of a distance metric is given satisfying the required condition, namely:

$$D(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y})^{\mathrm{T}} \mathbf{A}\, (\mathbf{x} - \mathbf{y})$$

for some constant, positive definite matrix $\mathbf{A}$.

6. Conclusions

While the existence of travellers' adaptive travel choice behaviour has been recognised for many decades, and while it implicitly underpins the notion of network equilibrium, the last twenty years has seen a growing body of work in which dynamical models are recognised in their own right, not only as a means of justifying long-run equilibrium analysis. In the present paper we have sought to highlight, and present a unified treatment of, the theoretical foundation of such models, focusing specifically on those models that may be represented as a deterministic dynamical system. A particular contribution has been to bring together discrete and continuous time systems in one paper; this is more significant than might first be imagined, since the two bodies of work seem to have grown almost independently, apparently from somewhat different ideologies.

As said in the introduction day-to-day dynamics concerns the evolution of a transportation system that occurs in similar periods over successive days, thus it is ontologically discrete, and naturally described by discrete-time dynamic process. These models try to explicitly mimic real evolution over time, and could be easily calibrated against real data. In addition, they can easily be compared with stochastic process models, usually formulated over discrete time. Moreover, for numerical solution or computer simulation discrete-time is more convenient, and so even if the model is specified originally in continuous-time, it will typically be discretised for computational purposes, as stressed below.

Adopting a continuous-time representation of the system, on the other hand, seems more appealing from the point of view that in the real world, time is typically considered to be continuous; surely, then, the discrete time system is only an approximation? This can be a somewhat misleading line of thought, however, due to the difficulty in dealing with the two different time-scales over which within-day traffic interactions occur, and between-day updating of travel choices occurs. A continuous-time model which could deal with both would indeed be attractive, but rather complex due to the lagged effect of daily experiences on subsequent decisions. Virtually all models considered in the literature are not so complex, and do not separate these scales, meaning that it is more difficult to understand which real-world phenomena these models are aiming to capture. It seems that a more plausible explanation for the continuous-time approaches we have reviewed here is that they are intended as smooth *approximations* to an underlying discrete day-to-day adjustment process. These approximations themselves may not have any direct real-world interpretation; rather their value is in the light they shed on the underlying discrete-time world. In particular, as discussed in section 3.5, it is possible to use stability analysis of a continuous-time model to infer stability properties of a related discrete-time model, in a consistent way with results obtained by directly analyzing the discrete-time model. On the other hand continuous-time models need discretization to be solved, thus this kind of models are less relevant from the solution point-of-view.

As far as choice modelling is concerning, probabilistic choice models (PCM's), derived from random utility theory, when compared with deterministic choice model (WCM), derived from Wardrop I principle, provide a more realistic description of user choice behaviour, since additional parameters model several source of uncertainty, regarding both the users, such as perception errors, dispersion of a user behaviour over days, dispersion among users, … , and the modeller, such as aggregation errors (due to area zoning), dispersion of supply characteristics (for instance due to weather conditions), missing attributes, attribute measurement errors, … , thus WCM may be considered a limit case when all sources of uncertainty may be neglected.

Discrete- or continuous-time dynamic assignment models based on PCM's, with respect to those based on WCM, show the following features, besides those reviewed in the subsection 3.1 and in the appendix with respect to user equilibrium assignment:

- they naturally include the equilibrium pattern (given by SUE modelled by a fixed-point model) as a fixed-point state, also allowing for weaker uniqueness conditions;
- they provide consistent and equivalent results with respect to path or arc variables (this distinction is only meaningful for dynamic models based on WCM);
- they can easily be compared with stochastic process models (see below).

In a companion paper, we have provided a similar treatment of stochastic process models, which emerge from a quite different theoretical backdrop (Watling & Cantarella, 2013a). Together, these papers extend and update the treatment of these two kinds of modelling approach as described in the unifying framework of Cantarella & Cascetta (1995). Our objective has been to highlight the theoretical pedigree of this modelling approach and the tools available for its analysis; we have not had the space to also consider the many computational methods proposed for implementing such models, but a study of such approaches would be a natural next step for the interested reader.

In the *discrete*-time models considered in sections 3 and 4, we made the compromise of assuming that steady-state link performance functions could represent the within-day scale as a first approximation, with the possibility that these can be replaced by more sophisticated dynamic network loading methods as required (see, as some examples: Cascetta & Cantarella, 1991; Hu & Mahmassani, 1997; Balijepalli & Watling, 2005; Liu et al, 2006; Friesz et al, 2011). Adopting a continuous day-to-day scale, on the other hand, makes it rather more difficult to distinguish the processes occurring over the two time-scales. Zhang et al ('Assumption 5', 2001) notably propose such a distinction with two separate continuous time-scales, yet these do not appear 'synchronized' to a common overall time-scale (it is as if two separate clocks are running) and so the model is rather abstract. Smith and Wisten (1995) and Friesz et al (1996) have made some advances in this direction.

Our review suggests that the field is now sufficiently mature that the range of alternative methods, theoretical results and tools are now ready to be put to good practical use in analysing real-life systems. Certainly, we are not suggesting that the models will not evolve, but rather it seems now that it is time for the models to be used more widely in empirical studies, so that we might better understand their strengths and the areas in which they need to be improved. We would suggest that a particularly good, open area for publication would be studies which tried to match the theoretical tools with actual real-life phenomena. This may include diverse aspects such as clarifying what we might *mean* by a 'day', considering how we might deal with unusual events or seasonal effects, considering the spatial and temporal transferability of the behavioural specifications, or considering more deeply how travellers actually 'learn' from personal experience, experience of others and any information sources.

Our approach in the present paper has not been to present day-to-day dynamics as a replacement for equilibrium analysis, but rather as an additional and extended option: equilibrium sits within the framework of day-to-day dynamic analysis. For relatively stable systems that quickly re-stabilise following a systematic change or perturbation, and for policies and measures which are readily captured by their assumptions, equilibrium analyses remain a sensible choice. However, there seem to be many cases in which it seems to be more difficult to justify the premises of equilibrium analysis, such as in the cases of incidents, information systems, responsive control and networks subject to high levels of variability. Although these models are only just starting to be considered in a more practical context, early experience suggests that even in cases that might seem more amenable to equilibrium analysis (such as a bridge closure), day-to-day models are able to capture 'irreversible' effects that would never arise from a traditional Wardrop or so-called stochastic user equilibrium analysis (He & Liu, 2012). While it seems premature to suggest that we have reached the 'end of equilibrium' (a phrase coined by Goodwin, 1998), at least we may now ask whether the decision processes that we model stabilise sufficiently quickly that equilibrium is a reasonable approximation. To be clear, we could always have *asked* this question; the difference is that now we have an alternative approach to consider.

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A. Appendix

Traditionally user equilibrium assignment has been addressed assuming no uncertainty due to modeller and/or user errors, thus leading to (deterministic) User Equilibrium (UE) where route choice is based on the I Wardrop principle. It is modelled through nonlinear optimization and variational inequalities, for which very efficient solution algorithms are available, at least for cost functions with symmetric Jacobian. 25 years afterwards the user equilibrium with probabilistic path choice models, usually named Stochastic User equilibrium (SUE), was introduced, since

- SUE includes a more realistic description of user route choice behaviour, after all it has at least one more parameter, and UE can be considered as a limit case of SUE. Additional parameters model several source of uncertainty, regarding both the users and the modeller, which can hardly be neglected, such as:
  - perception errors, dispersion of a user behaviour over days, dispersion among users, …
  - aggregation errors, missing attributes, attribute measurement errors, …

Some useful mathematical features of fixed-point models for SUE with respect to any kind of models available for UE are given below.

- Uniqueness of flows per user class, and of route flows.
- Flows depend on cost through a continuous, c. differentiable function with symmetric Jacobian, under very mild assumptions met by all models used in current practice.
- These models for SUE allow for weak uniqueness and convergence conditions, including non necessarily increasing cost functions, which cannot be extended to UE, however modelled.
- These models can be solved through simple and feasible algorithms proved converging under very mild assumptions even for cost functions with asymmetric Jacobian.

  The SUE arc flow pattern is less sensible to input data such as demand flow (less than proportional) with respect to UE pattern (more than proportional), thus there is no need of a high convergence threshold, 10^{-3} being enough in most cases, to be compared with 10^{-6}, or even less, often required for UE solution.

Moreover,

- Fixed-point models for SUE, as well as all the related analysis, can easily be extended to deal with SUE with variable demand including any kind of demand models, whilst models for DUE require that the inverse demand function, not available in the general case, and anyway hard to define and compute, apart from other limiting assumptions. Thus SUE approached through fixed-point models is the only option for equilibrium assignment with variable demand.