The Response Spectrum Map, a Frequency Domain Equivalent to the Bifurcation Diagram

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Abstract

The Response Spectrum Map (RSM) is introduced as a frequency domain equivalent to the Bifurcation Diagram. The RSM is a map of the energy distribution of a system in the frequency domain, where subharmonics, superharmonics and chaos generation can be revealed. The RSM is used in this paper to qualitatively analyse and detect various dynamical states exhibited by a nonlinear system.

1 Introduction

Explicit solutions of nonlinear differential equations, such as the Duffing equation or Chua's circuit, which describe systems in terms of elementary functions or Volterra series are not always possible. In spite of this, geometric interpretation of the differential equations is often undertaken and useful information of a qualitative character can often be obtained.

The geometrical approach of solving differential equations adopted during the last two decades provides a useful insight into the realm of nonlinear oscillations. A dynamical system can have a rich variety of solutions: periodic, quasi-periodic or chaotic. Chaos is common in many nonlinear mechanical and electrical oscillators [Duffing,1918], [Ueda,1980], [Chua,1992], and has been observed in geomagnetic activity, human physiology, economics and fluid turbulence. It is also common in nonlinear systems to have different coexisting steady-state solutions, or to have several periodic and chaotic motions for the same parameter values but with different initial conditions.

Such a range of behaviour can be reflected in the Poincaré Map or the Bifurcation Diagram. As an alternative to the Bifurcation Diagram, and to provide complementary information in the frequency domain, the Response Spectrum Map is introduced in this paper. The Response Spectrum Map is shown to be a powerful new tool for the analysis of nonlinear system behaviour. The potential of this new tool is demonstrated in the analysis of the dynamical regimes detected for the Duffing equation, Chua's driven circuit, the Logistic equation and the Henon map. These are systems that have been used by many authors as a bench test in the study of nonlinear dynamics.

The paper is organised as follows: the Response Spectrum Map is introduced in Section 2 where it is compared with the Bifurcation Diagram. In Section 3 simulation results for four different systems are presented. The Response Spectrum Maps are used to detect strong nonlinear behaviour where subharmonics, superharmonics or chaotic states are generated.

2 Bifurcation Diagrams and Response Spectrum Maps

Bifurcation Diagrams are intrinsically related to Poincaré Maps. In the case of a non-autonomous system, such as the Duffing equation or Chua's driven circuit which are considered in the next section, the Poincaré Map is equivalent to sampling the trajectory of the solution at a rate equal to the forcing frequency [Parker and Chua, 1989]. Fixed points and closed orbits indicate a periodic solution. A fixed point of the Poincaré Map corresponds to a period-one solution and a k-periodic closed orbit corresponds to a kth-order subharmonic.

The Bifurcation Diagram can be seen as a succession of compressed Poincaré Maps, derived for a certain varying parameter. The point r of a Bifurcation Diagram for a non-autonomous system driven by $Acos(\omega t)$ can be defined as [Aguirre and Billings, 1994]

$$r = \{(y, A) \in \mathbb{R} \times I \quad | \quad y = y(t_i);\; A = A_0;\; t_i = t_0 + K_{ss} \times 2\pi/\omega\} \qquad (1)$$

where I is the interval $I = [A_i; A_f] \subset \mathbb{R}$, $0 \leq t_0 \leq 2\pi/\omega$, K_{ss} a constant.

In the case of an autonomous system the Bifurcation Diagram is the collection of all steady state solutions obtained when a parameter is varied. A detailed description of the Bifurcation Diagram algorithm generation for both autonomous and non-autonomous cases was given by Parker and Chua [1989].

The bifurcation diagram can reveal bifurcation points, subharmonics, cascades to chaos and chaotic states of a system, occurring when a parameter is varied. Both the Poincare map and the Bifurcation Diagram are based in the time domain, but since most of the dynamic behaviours noted above can generate particular patterns of energy distribution in the frequency domain there is clearly a need for complementary analysis techniques which are frequency domain based. It is well known for example that a limit cycle, as a periodic signal will have a discrete spectrum, while a chaotic signal will be described by a continuous spectrum. Limit cycles can be further classified as subharmonics or superharmonics of a certain order, according to the ratio they make with the driving frequency. Harmonic generation in particular is important for local Volterra series modelling, where the significance of the order of these harmonics can determine the order of the approximating Volterra series. A map showing the energy distribution in the frequency domain, similar to the way time domain effects are represented in a bifurcation diagram, should therefore be a concise representation of complex dynamic behaviour and provide more insight into the operation of a nonlinear system in the frequency domain.

The Response Spectrum Map (RSM) will be defined as the ensemble of response power spectra corresponding to a nonlinear system response described and generated by varying a system parameter. In analogy with the definition of a Bifurcation Diagram stated above for a varying parameter A, the Response Spectrum Map can be defined as

$$S_y : [0; f_N] \times [A_i; A_f] \rightarrow \mathbb{R}; \quad S_y(f, A) = S_{y_A}(f) \tag{2}$$

where $S_{y_A}(f)$ is the power spectrum estimated for the system response y_A, when the varying parameter has the value $A \in [A_i; A_f] \subset \mathbb{R}$. The frequency f is varied in the interval $[0; f_N]$, where f_N is the Nyquist frequency. The definition is valid not only for non-autonomous systems but also for the autonomous case.

It is well known that frequency domain analysis based on the conventional power spectrum has been applied to nonlinear systems showing complex nonlinear behaviour since the formal interpretation of the cascade to chaos, almost two decades ago, Feigenbaum [1980], and Cvitanović and Jensen [1982]. However a response power spectrum mapping of the type introduced in this paper has never been used before to the best of our knowledge. The Response Spectrum Map can show the well known cascade to chaos, but more importantly, the map reveals the evolution of the frequency domain features of the system associated with the extra-dimension of the varying parameter and the continuity or discontinuity of these features.

In practice the Response Spectrum Map is very easy to generate. The steady-state system response y_A is required, for a range of values of the parameter A, as in the case of the Bifurcation Diagram. The power spectrum of the system response is then computed for each value of A, using for example the Welch averaged periodogram method. When this is repeated for each value of A a complete map is obtained for the set of values $A \in [A_i; A_f]$. The frequency response map generates a three dimensional plot of the response power spectrum versus frequency and versus A. A logarithmic scale emphasises even more the features we are interested in, and all the examples analysed in this paper use a logarithmic scale. A two dimensional representation can be obtained if only a plan view or contour plot are considered. The response spectrum is dependent on both the system characteristics and the input properties.

The Response Spectrum Map can be seen as a projection of the information in the Bifurcation Diagram into the frequency domain. Notice that the Bifurcation Diagram and the Response Spectrum Map can be generated at the same time. While the former provides information about the intersection point in the time domain of the flow with a certain plane when a parameter is varied, the latter gives information about the response power spectrum in the frequency domain.

The Response Spectrum Map can be used to identify the various states of a system. States showing complex, strongly nonlinear behaviour, such as subharmonics and chaos are revealed. Mildly nonlinear behaviour, which corresponds to the case where Volterra series can be applied, can also be analysed using the Response Spectrum Map. Such an application is presented in Billings and Boaghe [1999]. The main advantage therefore is that the Response Spectrum Map, like the Bifurcation Diagram, can be applied to all nonlinear systems, both autonomous or non-autonomous, continuous or discrete. The insight that this new plot provides will be revealed with examples in the next section.

3 Simulation results

The examples which will be analysed in detail in this section are very popular in the literature devoted to bifurcation phenomena and chaos. They are given as nonlinear differential or difference equations which often cannot be solved analytically. A geometrical approach is often more appropriate, and this consists of identifying equilibrium points and attractors and the system dynamics around these.

Very often the Jacobian matrix is derived and the eigenvalues of this matrix are evaluated at the equilibrium points. If the real parts of the eigenvalues are different from zero, the equilibrium points are called hyperbolic, and the behaviour of the system is qualitatively the same as for the linearised system in a neighbourhood of the equilibrium, according to the Grobman - Hartman theorem.

A zero eigenvalue is considered a special case in ordinary differential equation textbooks, and the equilibrium point is called degenerate or non-hyperbolic. The dynamics near a non-hyperbolic equilibrium point are structurally unstable [Guckenheimer, Holmes, 1983]. The stability of a degenerate or non-hyperbolic point cannot be determined from the eigenvalues (for the autonomous case) or characteristic multipliers (for a period solution) alone. One possible way of analysing non-hyperbolic equilibrium points is by studying local bifurcations in parameter regions.

As a consequence, a system steady state solution is often calculated numerically, based on Poincaré Maps or Bifurcation Diagrams. In the examples below various nonlinear differential equations will be analysed for different parameter values, which will produce different dynamics, these will then be analysed using Bifurcation Diagrams and Response Spectrum Maps.

3.1 Example 1: The Duffing-Holmes Equation

A very well known example of a dynamical system is given by the Duffing-Holmes equation. Introduced by Holmes [1979], the equation has been used to model mechanical oscillations arising in two-well potential problems [Moon, 1987]. The non-autonomous case is considered in this example.

$$\ddot{y} + \alpha\dot{y} + \beta y + y^3 = Acos(\omega t) \tag{3}$$

The Bifurcation Diagram is analysed for the particular case $\alpha = 0.15$, $\beta = -1$ and $\omega = 1$ rad/sec, or driving frequency $f = 0.16$ Hz, which has been analysed previously in the literature in Aguirre and Billings [1995]

$$\ddot{y} + 0.15\dot{y} - y + y^3 = Acos(t) \tag{4}$$

This equation was simulated using a fourth-order Runge-Kutta algorithm with an integration interval of $\pi/15$, for an amplitude A varying in the range $0.22 \leq A \leq 0.35$.

The Bifurcation Diagram is given in Figure 1 (a) for a varying amplitude A. The Bifurcation Diagram shows a series of flip bifurcations, also referred to as period doubling or subharmonic bifurcations [Guckenheimer, Holmes, 1983], identified as parallel branches in the Bifurcation Diagram. For amplitude values between $0.22 \leq A \leq 0.2460$ the Bifurcation Diagram has two branches. It is not clear from the diagram if this corresponds to a second order subharmonic or to an interleaved map of a harmonic response. Furthermore the diagram shows a third order flip bifurcation occurring at $A = 0.246$. A second flip bifurcation visible in the Bifurcation Diagram occurs at $A = 0.2545$, followed by a period doubling cascade towards chaos. For amplitude values in $[0.2585; 0.3165]$ the system response is chaotic, after which the diagram has a single branch, corresponding to a periodic response with the same frequency as the driving input.

The Response Spectrum Map given in Figure 1 (b,c) gives a better insight into the dynamical regimes generated with the varying amplitude A. A detailed description of the dynamical regimes revealed in the Response Spectrum Map is given in Table 1. For the initial amplitude variation in $[0.220; 0.246]$ the Response Spectrum Map is not as ambiguous as the Bifurcation Diagram and shows that when the Bifurcation Diagram has two branches of variation, these correspond to interleaved response values. The RSM shows that the response has in fact a harmonic content of integer, even and odd, multiplies of the driving frequency.

In the window of variation $[0.246; 0.2585]$ preceding the chaotic regime, the Response Spectrum Map shows a succession of subharmonic and superharmonic responses concluded with a period doubling cascade towards chaos. Again the Response Spectrum Map gives a very clear representation for this particular interval of variation, while the Bifurcation Diagram shows only three response branches, corresponding to the third order subharmonic.

Finally, for amplitude values A in the interval $[0.3165; 0.3205]$ the map shows various dynamical regimes, given in Table 1, including second and fourth order subharmonics, period doubling cascade, chaos and even superharmonics, which are not visible in the Bifurcation Diagram. For the last interval of variation $[0.3205; 0.3500]$ both Bifurcation Diagram and Response Spectrum Map show a periodic response with the same period as the driving input. Moreover, the Response Spectrum Map shows that the response contains only odd multiples of the driving frequency, possibly as an effect of the third order nonlinearity in the Duffing Holmes equation (4).

Table 1: Dynamic regimes for the Duffing Holmes equation (4) extracted from the Response Spectrum Map in Figure 1 (b) and (c)

Parameter A	Dynamic regime
$[0.2200; 0.2460]$	superharmonics: $n = k$, $k = 1, 2, 3$
$[0.2460; 0.2470]$	subharmonics: $n = \frac{k}{3}$, $k = 1, 3, 5, 7, 9$
$[0.2470; 0.2480]$	superharmonics: $n = k$, $k = 1, 2, 3$
$[0.2480; 0.2510]$	subharmonics: $n = \frac{k}{3}$, $k = 1, 3, 5, 7, 9$
$[0.2510; 0.2515]$	superharmonics: $n = k$, $k = 1, 2, 3$
$[0.2515; 0.2540]$	subharmonics: $n = \frac{k}{3}$, $k = 1, 3, 5, 7, 9$
$[0.2540; 0.2545]$	superharmonics: $n = k$, $k = 1, 2, 3$
$[0.2545; 0.2570]$	subharmonics: $n = \frac{k}{3}$, $k = 1, 2, 3, 4, 5, 6, 7, 8, 9$
$[0.2570; 0.2585]$	period doubling cascade
$[0.2585; 0.3165]$	chaos
$[0.3165; 0.3175]$	subharmonics: $n = \frac{k}{2}$, $k = 1, 2, 3, 4, 5, 6, 7$
$[0.3175; 0.3180]$	subharmonics: $n = \frac{k}{4}$, $k = 1, 2, 3, 4, 5, 6, 7, 8$
$[0.3180; 0.3185]$	period doubling cascade
$[0.3185; 0.3195]$	superharmonics: $n = k$, $k = 1, 3$
$[0.3195; 0.3205]$	chaos
$[0.3205; 0.3500]$	superharmonics: $n = k$, $k = 1, 3$

3.2 Example 2: Chua's Driven Circuit

Chua's circuit is well known as a classical example of a system showing a rich variety of complex dynamical behaviour ranging from subharmonic oscillations, period doubling to chaos generation. The circuit description is given for example in Chua [1992]. The driven version is considered in this example, similar to the case analysed by Murali and Lakshmanan [1993], and described by the following system of nonlinear differential equations

$$\begin{aligned} \dot{x} &= \alpha(y - h(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y + u(t) \end{aligned} \tag{5}$$

where

$$h(x) = \begin{cases} m_1 x + (m_0 - m_1), & x \geq 1 \\ m_0 x, & |x| \leq 1 \\ m_1 x - (m_0 - m_1), & x \leq -1 \end{cases} \tag{6}$$

and $\alpha = 7$, $\beta = 100/7$, $m_0 = -1/7$, $m_1 = 2/7$ and $u(t) = Asin(3 \times t)$. This system of nonlinear differential equations was simulated using a fourth-order Runge-Kutta algorithm with an integration interval of $\pi/24$, for an amplitude A varying in the range $0.70 \leq A \leq 2.50$. The frequency of the driving input in this example was 0.47 Hz. The Bifurcation Diagram and Response Spectrum Maps were generated and are displayed in Figure 2 for the variable x in equation (5).

Both maps show much more complex dynamics taking place for this system compared to the previous example. There are about 11 windows of chaotic behaviour for the input amplitude varying in the interval $[0.70; 2.50]$. A detailed description of the chaotic regimes detected in the Response Spectrum Map is given in Table 2. As in the previous example the Bifurcation Diagram is less explanatory and accurate, compared to the Response Spectrum Map.

The initial succession of dynamic states is visible in both maps. For a values of amplitude A in the interval $[0.70; 0.78]$ the system response has a spectrum composed of integer multiples of the input 0.47 Hz driving frequency. This interval is followed by a succession of period doubling towards a chaotic regime starting at $A = 0.95$. The Response Spectrum Map shows a reversed period doubling taking place after the chaotic interval $[0.950; 1.075]$, this is not visible in the Bifurcation Diagram. Again visible in both maps is the 9th-order subharmonic generated in the interval $[1.205; 1.281]$, given in the Bifurcation Diagram as 9 distinct branches. Visible in the Bifurcation Diagram is the 5th-order subharmonic generated in the interval $[1.4; 1.5]$. However the Response Spectrum Map shows that in this interval the dynamical states contain also a period doubling cascade and chaos. The interval $[2.170; 2.270]$ clearly shows in both diagrams the presence of the 7th-order subharmonic, the Bifurcation Diagram showing in this case 7 distinct branches. For the final interval of variation $[2.465; 2.500]$ the Response Spectrum Map again shows the presence of superharmonics, similar to the first interval of variation.

Response Spectrum Maps and Bifurcation Diagrams were also generated for the variable y and z in equation (5), and it was found that even if the Bifurcation Diagrams were not identical in absolute values, the Response Spectrum Maps as expected showed the same succession of dynamical regimes as for the variable x in equation (5).

Table 2: Dynamic regimes for the Chua's driven circuit (5) extracted from the Response Spectrum Map in Figure 2 (b) and (c)

Parameter A	Dynamic regime
$[0.700; 0.780]$	superharmonics: $n = k,\ k = 1, 2, \ldots, 7$
$[0.780; 0.900]$	subharmonics: $n = \frac{k}{2},\ k = 1, 2, \ldots, 15$
$[0.900; 0.950]$	subharmonics: $n = \frac{k}{4},\ k = 1, 2, \ldots, 31$ followed by period doubling cascade
$[0.950; 1.075]$	chaos
$[1.075; 1.080]$	subharmonics: $n = \frac{k}{4},\ k = 1, 2, \ldots, 31$
$[1.080; 1.090]$	subharmonics: $n = \frac{k}{2},\ k = 1, 2, \ldots, 15$
$[1.090; 1.205]$	chaos
$[1.205; 1.281]$	subharmonics: $n = \frac{k}{9},\ k = 1, 3, 5, \ldots$
$[1.281; 1.285]$	subharmonics: $n = \frac{k}{18},\ k = 1, 3, 5, \ldots$ followed by period doubling cascade
$[1.285; 1.341]$	chaos
$[1.341; 1.344]$	subharmonics: $n = \frac{k}{7},\ k = 1, 2, 3, \ldots$
$[1.344; 1.400]$	chaos
$[1.400; 1.440]$	subharmonics: $n = \frac{k}{5},\ k = 1, 3, 5, \ldots$
$[1.440; 1.445]$	subharmonics: $n = \frac{k}{10},\ k = 1, 3, 5, \ldots$
$[1.445; 1.450]$	subharmonics: $n = \frac{k}{20},\ k = 1, 3, 5, \ldots$ followed by period doubling cascade
$[1.450; 1.475]$	chaos followed by reversed period doubling
$[1.475; 1.485]$	subharmonics: $n = \frac{k}{10},\ k = 1, 3, 5, \ldots$
$[1.485; 1.505]$	subharmonics: $n = \frac{k}{5},\ k = 1, 3, 5, \ldots$
$[1.505; 1.545]$	chaos
$[1.545; 1.550]$	superharmonics: $n = \frac{k}{13},\ k = 1, 3, 5, \ldots$
$[1.545; 1.550]$	subharmonics: $n = \frac{k}{13},\ k = 1, 3, 5, \ldots$
$[1.550; 2.105]$	chaos
$[2.105; 2.130]$	subharmonics: $n = \frac{k}{3},\ k = 1, 2, 3, \ldots$
$[2.130; 2.170]$	chaos
$[2.170; 2.270]$	subharmonics: $n = \frac{k}{7},\ k = 1, 3, 5, \ldots$
$[2.270; 2.410]$	chaos
$[2.410; 2.440]$	reversed period doubling towards subharmonics: $n = \frac{k}{2},\ k = 1, 2, \ldots$
$[2.440; 2.450]$	chaos
$[2.450; 2.460]$	subharmonics: $n = \frac{k}{2},\ k = 1, 2, \ldots$
$[2.460; 2.465]$	chaos
$[2.465; 2.500]$	superharmonics: $n = k,\ k = 1, 2, \ldots$

The Bifurcation Diagram was obtained in this example by recording the steady state response once at every input cycle, for a maximum value of the input $sin(3t)$. However it is well known that for a different strobing point the Bifurcation Diagram would have been different. However this is not the case for the Response Spectrum Map, which depends on the total variation of the system response, and not only on points collected at every other cycle. The Response Spectrum Map is therefore an invariant descriptor of the system dynamics.

Finally it should be emphasised that the Response Spectrum Map gives a very accurate and clear representation of the dynamical regimes taking place in Chua's driven equation. The map shows in a simple pictorial way a whole variety of complex behaviour. Subharmonics of order 2, 3, 4, 6, 7, 9, 10, 13, together with period doubling sequences are visible in the Response Spectrum Map for this example, in a short interval of amplitude variation. It is this simple interpretation, from the graphical RSM plots that allows easy identification of sub and superharmonic oscillations that provides valuable and complementary insights into the system behaviour compared to the Bifurcation Diagram.

3.3 Example 3: The Logistic Equation

Another classical example of a nonlinear system with complex nonlinear behaviour is the logistic equation

$$y(k) = A\Big(1 - y(k-1)\Big)y(k-1) \qquad (7)$$

where A is a parameter which when varied produces a range of complex dynamic behaviour. The logistic equation became popular when Feigenbaum discovered the period doubling mechanism in 1978 and since then it is widely used as a benchmark for the study of chaos and bifurcations.

The logistic equation is used in this example as an autonomous system for which the Bifurcation Diagram and Response Spectrum Map are generated and used for system analysis. Equation (7) was simulated and the response $y(k)$ recorded recursively. The Logistic equation is a discrete system for which the sampling step is one unit, or 1 Hz. The Nyquist frequency in the Response Spectrum Map therefore will be 0.5 Hz. The initial value was $y(1) = 0.65$ and the parameter A was varied in the interval $[2.8; 3.9]$.

The Bifurcation Diagram and Response Spectrum Map are shown in Figure 3. The Bifurcation Diagram has a series of period doubling cascades, which correspond to subharmonic generation visible in the Response Spectrum Map. Table 3 gives a detailed description of the dynamical regimes which take place when A is varied in the interval $[2.8; 3.9]$. The Bifurcation Diagram starts with a single branch representation in the interval $[2.8; 3.0]$. Because the system is autonomous, the meaning of the Bifurcation Diagram representation is slightly different than in the non-autonomous cases analysed in the previous two examples. The Bifurcation Diagram shows in this case the steady state response after the transients have died out. Therefore the single branch representation in the Bifurcation Diagram corresponds to a single valued response, with no cycles. This is confirmed by the Response Spectrum Map which shows no frequency components in the interval $[2.8; 3.0]$ except the spike at 0 Hz, corresponding to the steady state value.

When the amplitude is larger than 3, a period doubling cascade occurs. The Bifurcation Diagram is represented in the interval $[3.00; 3.45]$ by a two-branch plot, corresponding to a two-points limit cycle, or a period two limit cycle. As expected the Response Spectrum Map shows two frequency components, at 0 Hz and 0.5 Hz. A further period doubling occurs in the interval $[3.45; 3.54]$ represented in the Response Spectrum Map by the generation of another frequency component of 0.25 Hz. The period doubling succession is clearly visible in the Response Spectrum Map in the interval $[3.54; 3.60]$, more clearly than in the Bifurcation Diagram, which is even more opaque in the interval $[3.60; 3.83]$, where only a cloud of points can be seen. Over this interval the Response Spectrum Map shows various dynamical regimes, including 3rd, 5th and 6th order subharmonics, together with the corresponding period doubling cascades, as shown in Table 3. Just before the final window of chaos a 3 point limit cycle is generated, shown in the Response Spectrum Map by a frequency component at 0.33 Hz.

Again it can be concluded that the Response Spectrum Map as defined in Section 2 allows accurate system analysis, and provides additional insight compared to the Bifurcation Diagram. The Response Spectrum Map is therefore not only complementary but in cases such as the one analysed here could be used to replace the Bifurcation Diagram.

3.4 Example 4: The Henon Map

For the final example a system which is both autonomous and multi-variable was chosen. The Henon map, which is more often discussed in connection with chaos

Table 3: Dynamic regimes for Logistic equation (7) extracted from the Response Spectrum Map in Figure 3 (b) and (c)

Parameter A	Dynamic regime
$[2.800; 3.000]$	underdamped response
$[3.000; 3.450]$	2 points limit cycle: $f = \frac{1}{2}$ Hz
$[3.450; 3.540]$	4 points limit cycle: $f = \frac{k}{4}$ Hz, $k = 1, 2$
$[3.540; 3.565]$	8 points limit cycle: $f = \frac{k}{8}$ Hz, $k = 1, 2, 3, 4$
$[3.565; 3.600]$	16 points limit cycle: $f = \frac{k}{16}$ Hz, $k = 1, 2, 3, 4, 5, 6, 7, 8$ followed by period doubling cascade
$[3.600; 3.625]$	chaos
$[3.625; 3.635]$	6 points limit cycle: $f = \frac{k}{6}$ Hz, $k = 1, 2, 3$ followed by period doubling cascade
$[3.635; 3.662]$	chaos
$[3.662; 3.664]$	8 points limit cycle: $f = \frac{k}{8}$ Hz, $k = 1, 2, 3, 4$
$[3.664; 3.740]$	chaos
$[3.740; 3.750]$	5 points limit cycle: $f = \frac{k}{5}$ Hz, $k = 1, 2$ followed by period doubling cascade
$[3.750; 3.830]$	chaos
$[3.830; 3.840]$	3 points limit cycle: $f = \frac{k}{5}$ Hz, $k = 1$ followed by period doubling cascade
$[3.840; 3.900]$	chaos

and fractal dimension, is a system described by the following system of equations

$$\begin{cases} x(k) = 1 + bx(k-1)^2 + y(k-1) \\ y(k) = ax(k-1) \end{cases} \tag{8}$$

For the present analysis the parameter a was set to $a = 0.3$ and b took values in the interval $[-1.4; -1]$. As in the previous example the variables x and y are generated recursively, at every unit time step. The sampling frequency is therefore 1 Hz, as in the previous example, giving a maximum frequency in the Response Spectrum Map of 0.5 Hz, which is the Nyquist frequency, above which the spectrum is symmetrical. The Response Spectrum Map and Bifurcation Diagram were generated for both subsystems x and y and these are shown in Figure 4 and 5. Even if the Bifurcation Diagrams are different for the two subsystems, the dynamical regimes are identical, as seen in the Response Spectrum Maps. A map generated for a subsystem is therefore sufficient to identify and describe the dynamic behaviour generated when the parameter b is varied. The succession of the dynamic regimes extracted from the Response Spectrum Map is given in Table 4.

For the initial interval of variation $[-1.40; -1.31]$ the response is chaotic, as seen in both Response Spectrum Map and the Bifurcation Diagram. This interval is followed by a 14 points limit cycle represented in the Response Spectrum Map by frequency components at multiples of $0.07 = \frac{1}{14}$ Hz, up the 0.5 Hz Nyquist frequency. A reversed period doubling precedes the chaotic window at $[-1.30; -1.26]$. This is followed by a reversed period doubling clearly seen in the Response Spectrum Map, starting with a 28 points limit cycle for $b = -1.26$, and ending with a 7 points limit cycle at $b = -1.1771$. In the interval $[-1.1771; -1.0560]$ a chaotic regime can be distinguished in both maps, with some additional detail revealed by the Response Spectrum Map, in which 10, 18 and 14 point limit cycles are identified. In the final interval of variation $[-1.056; -1.000]$ a reversed period doubling takes place, ending with a 4 point limit cycle.

4 Conclusions

The Response Spectrum Map was introduced in this paper as a frequency domain counter part to the Bifurcation Diagram. Four examples of systems with complex nonlinear dynamics were used to illustrate strong nonlinear behaviour including limit cycles, subharmonics and chaos. The Response Spectrum Map was computed for these systems and was shown to provide a pictorial display of the system characteristics, in a manner even more accurate than the Bifurcation Diagram.

In particular bifurcation points are seen in a Response Spectrum Map as abrupt discontinuities and qualitative changes in the response spectrum. Multiple branches in a Bifurcation Diagram occur not only for subharmonics but also for interleaved responses, and in such cases the true type of dynamics can only be revealed in the frequency domain. Moreover, the Bifurcation Diagram is not invariant for a non-autonomous system, but it can change with the position of the strobing point. This does not happen for the Response Spectrum Map, which is independent of the strobing point. It has also been noticed that for autonomous systems single valued system responses which appear as branches in a Bifurcation Diagram, resembling period one limit cycles, reveal their true nature only in the frequency domain. Finally, the Response Spectrum Map can also be used to quantify the degree in

Table 4: Dynamic regimes for Henon equation extracted from the Response Spectrum Map in Figure 4 and 5 (b) and (c)

Parameter b	Dynamic regime
$[-1.4000; -1.3100]$	chaos
$[-1.3100; -1.3000]$	14 points limit cycle: $f = \frac{k}{14}$ Hz, $k = 1, 2, 3$ followed by reversed period doubling
$[-1.3000; -1.2600]$	chaos
$[-1.2600; -1.2300]$	28 points limit cycle: $f = \frac{k}{28}$ Hz, $k = 1, \ldots, 13$ followed by reversed period doubling
$[-1.2300; -1.1771]$	chaos
$[-1.1771; -1.1770]$	10 points limit cycle: $f = \frac{k}{10}$ Hz, $k = 1, 2, 3, 4, 5$
$[-1.1770; -1.1015]$	chaos
$[-1.1015; -1.1002]$	10 points limit cycle: $f = \frac{k}{10}$ Hz, $k = 1, 2, 3, 4, 5$
$[-1.1002; -1.0960]$	chaos
$[-1.0960]$	18 points limit cycle: $f = \frac{k}{18}$ Hz, $k = 1, \ldots, 9$
$[-1.0961; -1.0909]$	chaos
$[-1.0909; -1.0907]$	14 points limit cycle: $f = \frac{k}{14}$ Hz, $k = 1, \ldots, 7$
$[-1.0907; -1.0560]$	chaos
$[-1.0560; -1.0000]$	reversed period doubling towards 4 points limit cycle: $f = \frac{k}{4}$ Hz, $k = 1, 2$

which a system is chaotic, by displaying a spectrum with a corresponding degree of continuity.

The examples considered clearly show that a combined analysis and interpretation of both the Bifurcation Diagram and the Response Spectrum Map provides a very clear insight into the operation of even very complex nonlinear systems.

5 Acknowledgements

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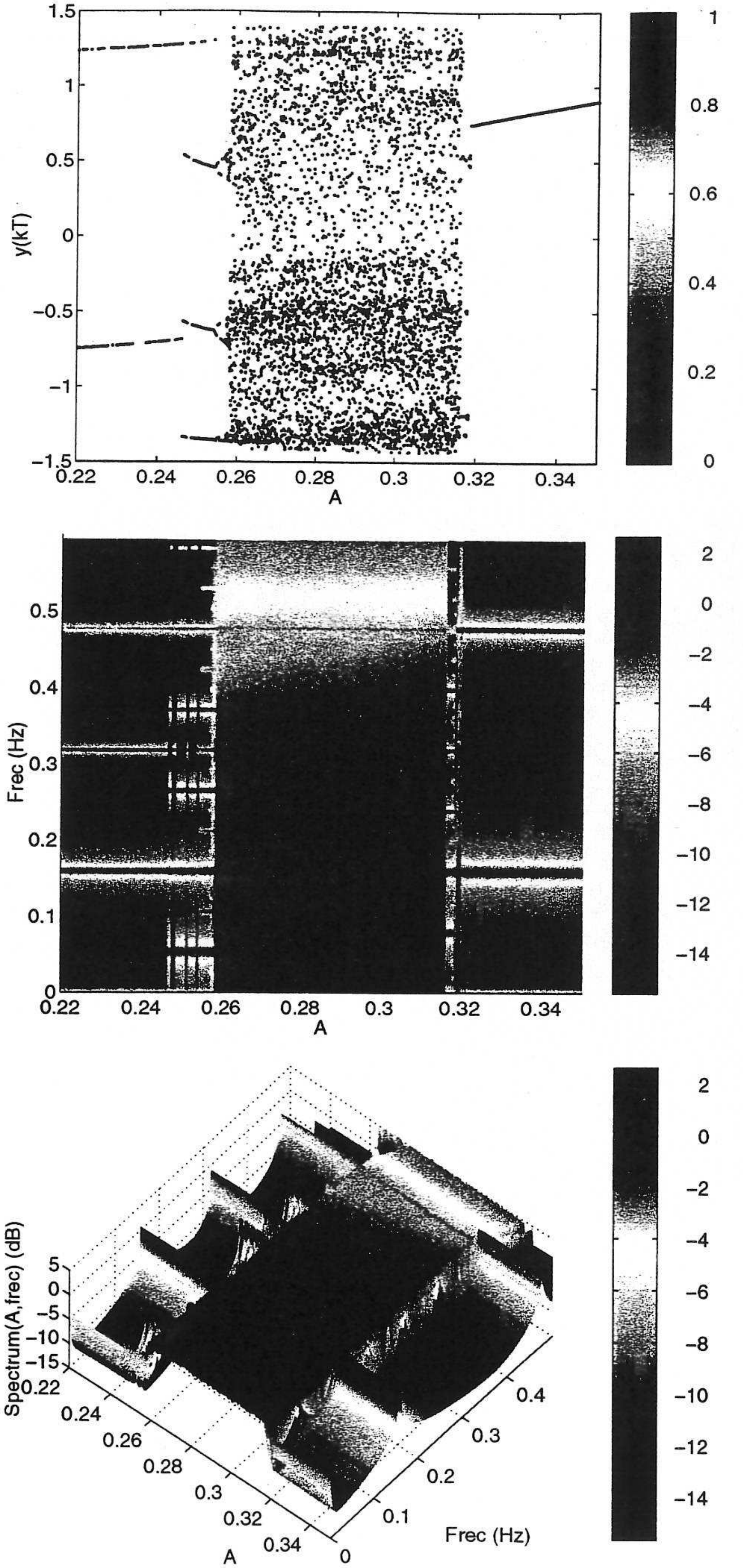

Figure 1: Bifurcation Diagram and Response Spectrum Map for the Duffing-Holmes equation: (a) bifurcation diagram (b) plan view (c) 3-dimensional view

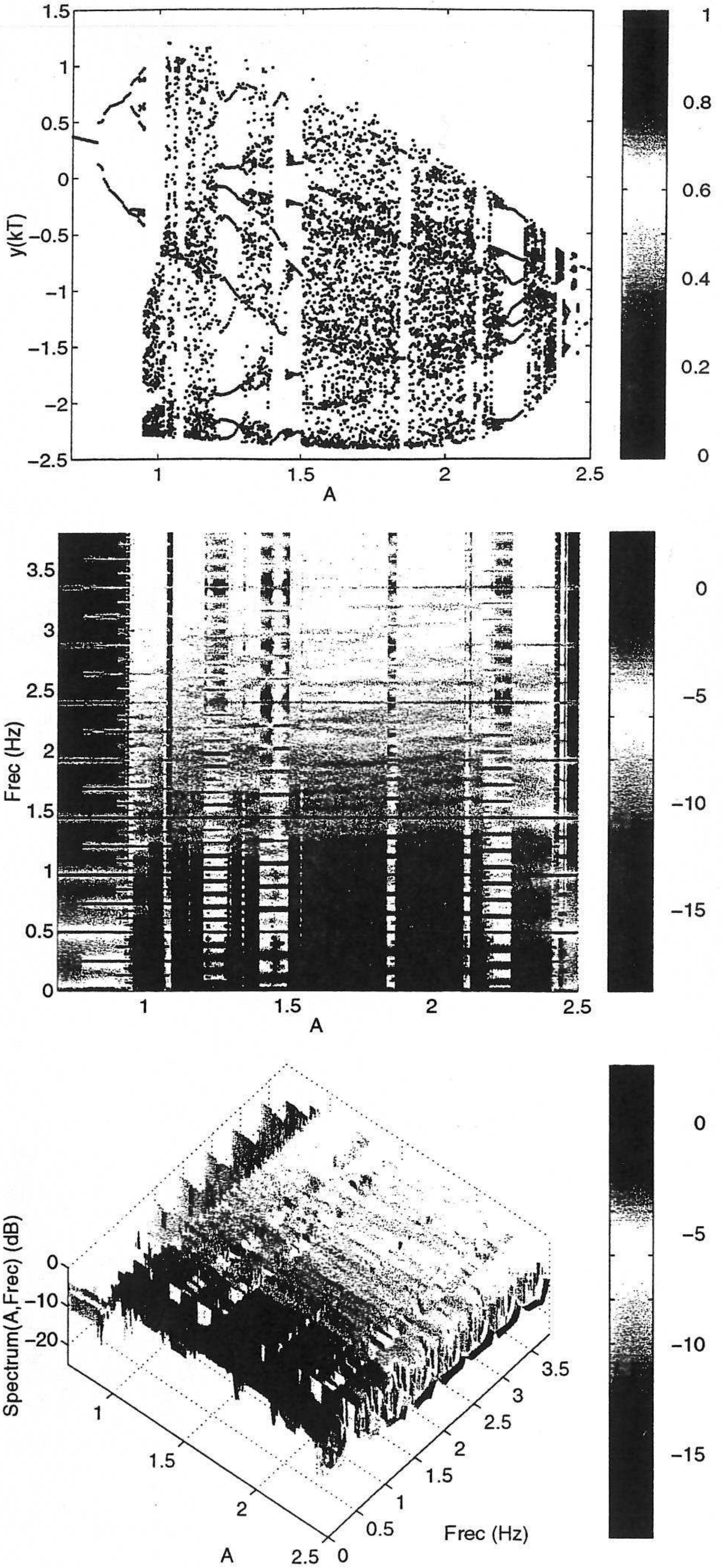

Figure 2: Bifurcation Diagram and Response Spectrum Map for the driven Chua's circuit: (a) bifurcation diagram (b) plan view (c) 3-dimensional view

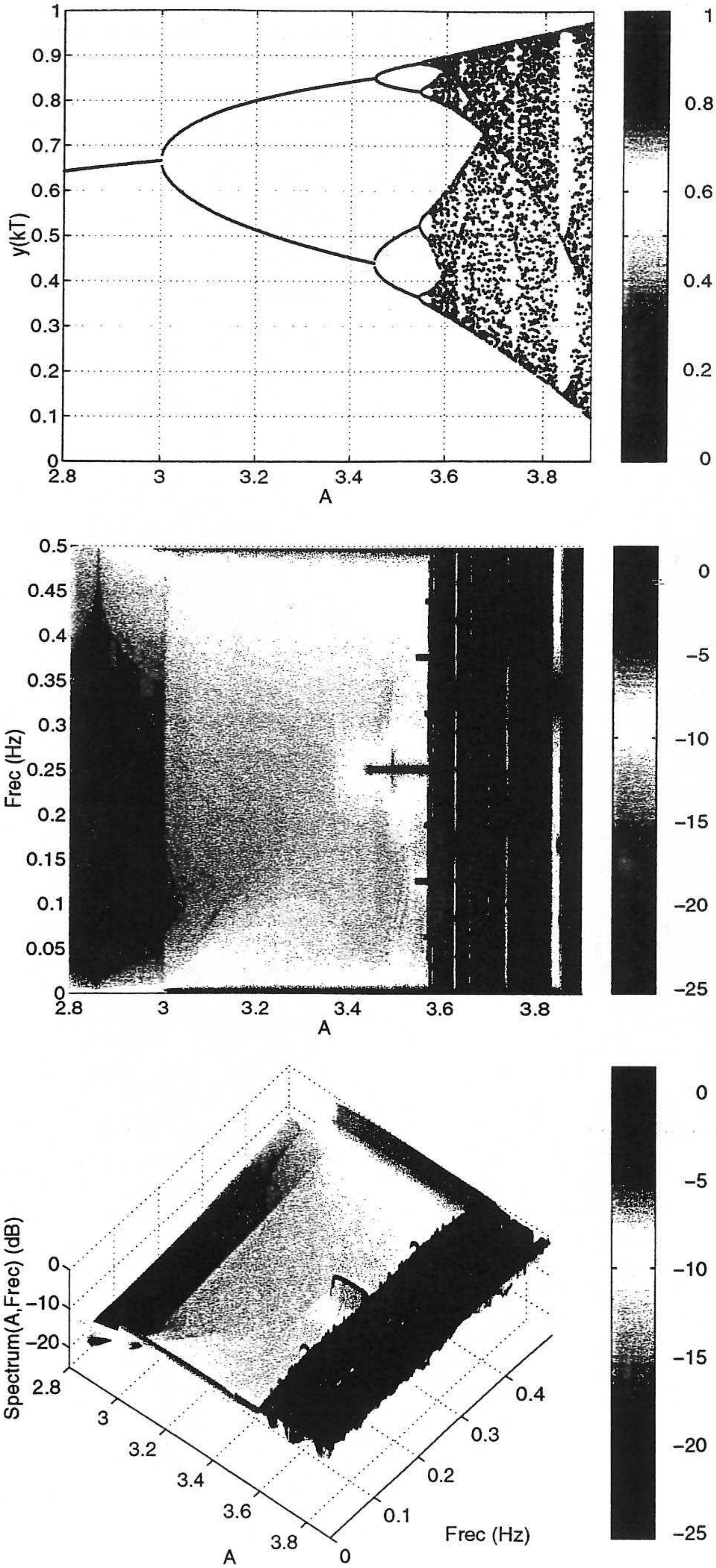

Figure 3: Bifurcation Diagram and Response Spectrum Map for the logistic equation: (a) bifurcation diagram (b) plan view (c) 3-dimensional view

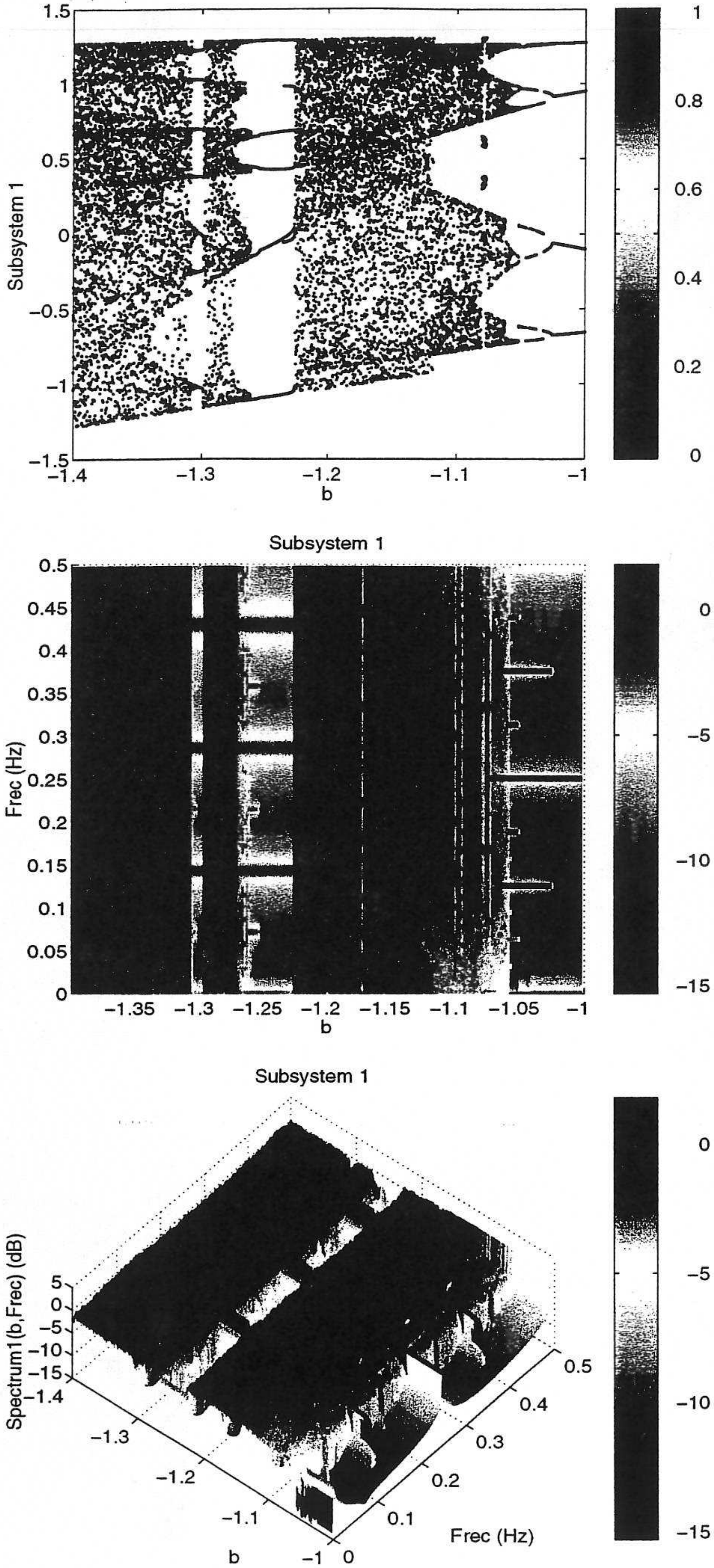

Figure 4: Bifurcation Diagram and Response Spectrum Map for the Henon equation, subystem 1: (a) bifurcation diagram (b) plan view (c) 3-dimensional view

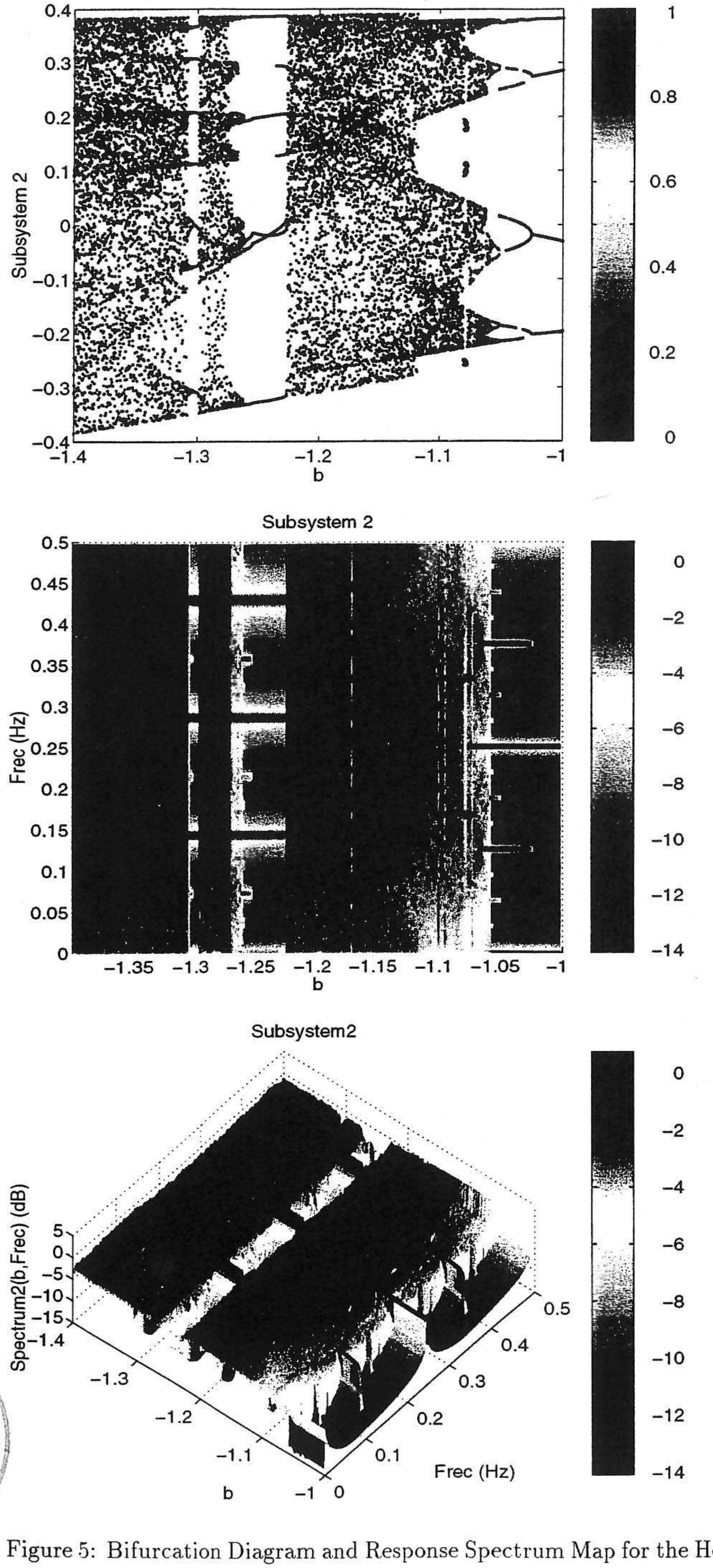

Figure 5: Bifurcation Diagram and Response Spectrum Map for the Henon equation, subystem 2: (a) bifurcation diagram (b) plan view (c) 3-dimensional view