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Continuous time system identification using shifted Chebyshev polynomials

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A new instrumental variables based identification procedure is introduced to estimate linear and nonlinear continuous time models using a shifted Chebyshev basis in the presence of noise.

Introduction: Orthogonal polynomials have been used by many authors in continuous time system analysis, identification, and control. By introducing an operational matrix continuous time differential equations can be transferred to algebraic forms, from which the parameters can be estimated using least squares. In the system identification problem, however, the application of most of these approaches in the literature have ignored noise or assumed that the noise level is unrealistically low.

In this letter shifted Chebyshev polynomials will be applied to the identification of continuous time linear and nonlinear systems. If the noise contained in the measured output were ignored the estimates would be biased. An Instrumental Variable method is introduced to overcome the bias problem.

Chebyshev polynomials: The shifted Chebyshev polynomials are defined in $0 \le t \le t_f$ as [1]

$$T_i(t) = \cos[i\cos^{-1}(1-2t/t_f)] \qquad \text{for } i=1,2\ldots \tag{1}$$

In general, a time function $f(t) \in L^2$ in the time interval $0 \le t \le t_f$ can be expanded

$$f(t) \cong \sum_{i=0}^{m-1} f_i T_i(t) = f' T(t) \tag{2}$$

where

$f = [f_1 f_2 \cdots f_{m-1}]'$ is the Chebyshev coefficient vector, and

$T(t) = [T_0(t)\ T_1(t) \ldots T_{m-1}(t)]'$ is the Chebyshev vector.

The integration of the Chebyshev vector can be approximated [2] as

$$\int_0^t T(t)dt \cong PT(t) \qquad (3)$$

where P is called the operational matrix of integration.

The product of vectors $T(t)$ and $T'(t)$ with the shifted Chebyshev coefficient vector f can be expressed as

$$T(t)T'(t)f = \tilde{F}T(t) \qquad (4)$$

where $\tilde{F}$ is called the product operational matrix.

Continuous time identification: Consider a linear time-invariant continuous system as

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}\frac{dx}{dt} + a_n x = b_1 \frac{d^{n-1}u}{dt^{n-1}} + \ldots + b_{n-1}\frac{du}{dt} + b_n u \qquad (5)$$

with $\frac{d^i x}{dt^i} = 0,\ \frac{d^i x}{dt^i} = 0,\ i = 0,1,\ldots,n-1$

where $u(t)$ and $x(t)$ are the input and the output respectively.

If it is assumed that the output measurement is corrupted with a zero-mean white noise signal $e(t)$ with variance σ_e^2 which is independent of $u(t)$, then

$$y(t) = x(t) + e(t) \qquad (6)$$

with

$$y(t) \cong \sum_{i=0}^{m-1} y_i T_i(t) = y'T(t)$$
$$u(t) \cong \sum_{i=0}^{m-1} u_i T_i(t) = u'T(t) \qquad (7)$$

Substituting (6) into (5), Integrating n times and applying property (3) yields

$$y'T(t) + a_1 y'PT(t) + \ldots + a_n y'P^n T(t) = b_1 u'PT(t) + \ldots + b_n u'P^n T(t) + v'T(t) \qquad (8)$$

Equating coefficients of $T(t)$ in eqn (8) yields

$$y'+a_1 y' P+\ldots+a_n y' P^n = b_1 u' P+\ldots+b_n u' P^n + v' \quad (9)$$

Denote

$$\theta_{2n\times 1} = [a_1\ a_2\ \ldots\ a_n\ b_1\ b_2\ \ldots\ b_n]'$$
$$\Psi_{n\times 2n} = [P\,y\ \ P^{n-1}\,y \ldots P^n\,y\,P\,u\,P^{n-1}\,u\,P^n\,u]'$$

the least square estimates (LS): $\theta_{LS} = (\Psi'\Psi)^{-1}\Psi' y$ will be biased because v' is not white. An instrumental variable matrix Z' can then be introduced to overcome this problem[3][4] if Z' satisfies the condition that $Z'\Psi$ is nonsingular and $E(Z' v) = 0$.

The instrumental variable estimates $\theta_{IV} = (Z'\Psi)^{-1} Z' y$ will be unbiased .

Select

$$Z = [P^k\,y_F\ \ P^{k+1}\,y_F \ldots P^{k+n}\,y_F\ P\,u\ P^{n-1}\,u\ P^n\,u]' \quad (10)$$

where y_F represents the shifted Chebyshev coefficients of $y_F(t)$ which is naturally formed from the filtered input signal [5]

$$y_F(t) = N(s)u(t) \quad (11)$$

using a Runge-Kutta approximation. Here $N(s)$ is a pre-defined filter and s is the Laplace operator.

One option for $N(s)$ is

$$N(s) = \frac{\hat{b}_1 s^{n-1} + \hat{b}_{n-1} s + \ldots + \hat{b}_n}{s^n + \hat{a}_1 s^{n-1} + \ldots + \hat{a}_{n-1} s + \hat{a}_n} \quad (12)$$

where $\hat{\theta} = [\hat{a}_1 \ldots \hat{a}_{n-1}\ \hat{a}_n\ \ \hat{b}_1\ \hat{b}_{n-1} \ldots \hat{b}_n]'$ represents the estimates at the previous step. This will be refered to as IV method one or IV1.

Another option for $N(s)$ is

$$N(s) = 1 \quad (13)$$

This will be refered to as IV method two or IV2.

The extension to the identification of continuous time nonlinear systems using shifted Chebyshev polynomials is not straightforward. Only bilinear and Hammerstein nonlinear systems can easily be handled by orthogonal polynomials. The latter case is illustrated by a simulation example in this letter using the shifted Chebyshev polynomials based IV method. The procedure is very similar to that of linear systems.

Simulation examples: Consider a linear time invariant second-order differential equation

$$\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_2x=b_2u \qquad (14)$$

The input-output data were obtained by simulating the above eqn with $a_1 = 1.8$, $a_2 = 1$ and $b_2 = 1$ at a sampling frequency of 50Hz over the interval $0 \le t \le 12$. Two kinds of input signal were chosen, i.e., $u(t) = 1$ and $u(t) = t^{0.8}$. The output signal was corrupted by an additive white noise sequence $e(t)$ with the variance σ_e^2 ($\sigma_e^2 = 0.0138$ when $u(t) = 1$ and $\sigma_e^2 = 0.0739$ when $u(t) = t^{0.8}$). The length of the shifted Chebyshev coefficients was chosen to be m=40.

Table 1 shows the results using the instrumental variable strategy based on equation (12) IV1 and (13) IV2 respectively.

Table 1. IV estimation results for linear system model (14)

Consider adding a second order nonlinear term $u^2(t)$ to eqn (14)

$$\frac{d^2x}{dt^2}+a_1\frac{dx}{dt}+a_2x=b_2u+c_2u^2, \quad c_2=8 \qquad (15)$$

All the simulation parameters remain as in the linear example except that $u(t) = e^{-0.2t}$. The length of the shifted Chebyshev polynomials was chosen as m=55. The results are listed in Table2.

Table 2. IV estimation results for nonlinear system model (15)

Conclusions: The property of the operational matrix makes it easy to convert an ordinary differential equation into an algebraic form which is suitable for digital computation.

For the identification of continuous time linear systems, it can be shown that when the noise is not negligible, the IV based approach introduced in this letter can be applied to obtain unbiased estimates.

For the identification of continuous time nonlinear systems only a restricted range of continuous time nonlinear terms can be treated by orthogonal polynomials. This means a greater computational overhead and the choice of the input signal becomes much more critical.

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References:

1 Shih, D.H and Kung, F.C.: Analysis and parameter estimation of non-linear systems via shifted Tschebyscheff expansions, Int. J. Systems Sci., 1986, **17**, (2), pp. 231-240.

2 Chou, J.H., and Horng, I.R.: Shifted Chebyshev series analysis of linear optimal control systems incorporating observers, Int. J. Control, 1985,**41**, (1), pp. 129-134.

3 Söderström, T., and Stoica, P.: Instrumental Variable Methods for System Identification, Springer-Verlag, 1983.

4 Young, P.C.: The instrumental variable method: A practical approach to identification and system parameter estimation, 7th IFAC Symposium on Identification and System Parameter Estimation, York, UK, 1985, pp. 1-15.

5 Ljung, L.: System identification--theory for the users, London: Prentice-Hall, 1987.

		$u(t) = 1$			$u(t) = t^{0.8}$		
	True	LS	IV1	IV2	LS	IV1	IV2
a_1	1.8	1.671	1.814	1.813	1.521	1.781	1.780
a_2	1.0	0.927	0.992	0.991	0.879	0.985	0.985
b_2	1.0	0.930	0.998	0.998	0.874	0.989	0.989

Table 1

		$u(t) = e^{-0.2t}$, σ_e^2=0.6811		
	True	LS	IV1	IV2
a_1	1.8	1.7220	1.7700	1.7697
a_2	1.0	0.9454	0.9498	0.9498
b_2	1.0	1.0815	0.9183	0.9190
c_2	8.0	7.4159	7.7543	7.7525

Table 2