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Abstract

Subharmonic generation is a complex nonlinear phenomenon which can arise from nonlinear oscillations, bifurcation and chaos. It is well known that single input single output Volterra series cannot currently be applied to model systems which exhibit subharmonics. A new modelling alternative is introduced in this paper which overcomes these restrictions by using local multiple input single output Volterra models. The generalised frequency response functions can then be applied to interpret systems with subharmonics in the frequency domain.

1 Introduction

The steady-state output of a linear system driven by a sine wave will consist of a sine wave of the same frequency but with a different amplitude and phase. Nonlinear systems driven by a sine wave may produce new frequency components, including components at integer multiples of the input frequency called harmonics, or new frequency components at fractions of the input frequency, called subharmonics.

Subharmonics are an important practical problem in the dynamics of nonlinear systems and are particularly evident in mechanical systems. In many industrial fields an important issue is the possibility of controlling subharmonics. Examples of real situations in which subharmonic generation can be a critical problem are given for example in Lefschetz [1956], Nayfeh and Mook [1979], Ishida et al [1992] or Ishida [1994]. A theoretical interest in studying subharmonics emerged over the last few decades from the connection between chaos and subharmonics. It has been noticed [Feigenbaum, 1980] that subharmonic generation is the first step on the route to chaos. Therefore studying subharmonics may provide more insight into the phenomenon of chaos.

Motivated by the theoretical and practical importance of subharmonics a lot of research has been carried out during the past few years. Subharmonic oscillations have been studied not only in nonlinear vibrations theory, but also in the context of nonlinear systems, dynamical systems applications and control [Thompson and Stewart, 1991]. Very often the tools employed for analysis belong to the differential geometry and topological vector spaces, especially when the subharmonics are associated with bifurcations and chaos [Guckenheimer and Holmes, 1983]. The literature devoted to this subject is therefore extensive. However a systematic theory on subharmonic oscillations does not seem to exist.

In the present paper the main objective is to introduce a new way of modelling and interpreting systems which exhibit subharmonics. It is well known that single input single output (SISO) Volterra models cannot currently be used to model systems which can produce subharmonic oscillations. A new modelling approach is introduced in this paper which overcomes these restrictions by using local multiple input single output (MISO) Volterra models. The advantage of the new approach is that all the existing knowledge regarding the interpretation, analysis and properties of Volterra models can be applied to reveal new insights into subharmonic systems. In particular the Generalised Frequency Response Functions (GFRF's) which are the frequency domain equivalents of the Volterra kernels can be used to study subharmonic systems in the frequency domain.

After introducing the main definitions and properties for subharmonic oscillations in Section 2, an overview of the current methods used in the analysis and modelling of subharmonics is given in Section 3. In Section 4 a new modelling methodology is introduced for subharmonic systems, based on multiple input single output (MISO) Volterra series. This methodology is applied to the Duffing oscillator. The particular case of the Duffing oscillator has recently been analysed in Billings and Boaghe [1999], where the Response Spectrum Map was introduced, which can reveal the presence of subharmonic oscillations.

The simulation examples described in this paper show that combining subharmonic modelling with the nonlinear GFRF's or transfer function generation may indeed reveal new features of a system exhibiting subharmonics. The new modelling principle is expected to improve the understanding of the complex problem of subharmonic and chaos generation.

2 Definitions and terminology

Nonlinear systems in the presence of forced oscillations and under suitable conditions admit subharmonics as a steady-state solution. Subharmonics are components where the frequency is an integral submultiple of the driving frequency.

Subharmonic oscillations have been encountered in many types of systems, including systems which are parametrically excited, stationary or non-stationary. Subharmonics have been detected in both nonlinear discrete systems with finite degrees of freedom and in continuous systems (beams, strings, plates, membranes) with infinite degrees of freedom.

Many terms have been associated with subharmonic phenomenon, including frequency demultiplication ([Stoker, 1957], [Nayfeh and Mook, 1979]]), subharmonic resonance ([Nayfeh and Mook, 1979], [Thompson and Stewart, 1991]), and subharmonic oscillation ([Stoker, 1957], [Cunningham, 1958], [Thompson and Stewart, 1991], [Rao, 1995]).

The term subharmonic resonance is also used in the literature in conjunction with Volterra series response functions. Schetzen [1980, p.152] remarked that nonlinear Volterra systems can have subharmonic resonances, where a subharmonic resonance refers to the local maximum of the output spectrum, appearing when the nonlinear system is excited with a fraction of the fundamental frequency of oscillation. The usage of the term subharmonic resonance in this context is therefore confusing, especially because subharmonic resonances meaning subharmonic oscillations, can not be generated by Volterra series, while subharmonic resonances described by Schetzen exist only for Volterra systems. An example of a subharmonic resonance is given in Billings and Boaghe [1999].

3 Subharmonic analysis and modelling

Subharmonic analysis belongs to the more general and vast realm of nonlinear oscillations. For nonlinear oscillations explicit solutions of the differential equations describing the system in terms of elementary functions are not always possible. In spite of this, geometric interpretation of the differential equations is often undertaken and useful information of a qualitative character can be obtained. However the geometric interpretation can not be applied for very complex systems where various analytic approximation methods of a more quantitative character are employed.

3.1 Quantitative analysis of subharmonics

The problem of determining the amplitude and frequency of weakly nonlinear oscillators dates back at least to Duffing [1918]. Since then several methods have been developed, such as the describing function method, the harmonic balance method, the method of multiple scales, and the averaging or perturbation method.

The describing function and harmonic balance methods are known to incorrectly predict oscillations in systems where there are none [Nayfeh and Mook, 1979]. The multiple scales and averaging methods allow the calculation of the amplitude and frequency of the response oscillations to any degree of accuracy, but extremely complex calculations are involved for orders of nonlinearity greater than two [Nayfeh and Mook, 1979]. Lindstedt's perturbation method gives only the periodic solution [Rao, 1995].

Lindstedt's perturbation method has been used by many authors to predict the possible appearance of subharmonics in the solution a nonlinear differential equation. Cunningham [1958] applied this method to the Duffing equation without dissipation and found that in certain cases subharmonics can be generated if the frequency of the input signal is at an integral multiple of the fundamental frequency of oscillation.

The iteration method can also be used to derive the conditions of existence for subharmonics. Stoker [1957] has found that in the case of a system described by the Duffing equation with dissipation

$$\omega^2 \ddot{y} + \omega c \dot{y} + \alpha y + \beta y^3 = A cos(\omega t) \tag{1}$$

if the external force amplitude A is prescribed and the parameters c, α, β are given, a third order subharmonic exists for

$$\omega \lessgtr 3\sqrt{\alpha + \frac{9}{16}\beta\frac{A^2}{64\alpha^2}} \quad \text{for} \quad \beta \lessgtr 0 \tag{2}$$

The subharmonic vibration results from a bifurcation from the harmonic solution at $\omega = 3\sqrt{\alpha + \frac{3}{2}\beta^2\frac{A^2}{64\alpha^2}}$.

A different approach is considered in Nayfeh and Mook [1979] in which the method of multiple scales is applied. Third order subharmonics are derived also for the system described by the Duffing equation with dissipation

$$\ddot{y} + 2\epsilon\mu\dot{y} + \omega_0^2 y + \epsilon\alpha y^3 = A cos(\omega t) \tag{3}$$

Subharmonic approximate solutions are obtained if the following condition is fulfilled

$$\frac{\sigma}{\mu} - \sqrt{\frac{\sigma^2}{\mu^2} - 63} \leq \frac{63\alpha\Lambda^2}{4\omega_0\mu} \leq \frac{\sigma}{\mu} + \sqrt{\frac{\sigma^2}{\mu^2} - 63} \tag{4}$$

where $\omega = 3\omega_0 + \epsilon\sigma$, $\Lambda = \frac{A}{2}\frac{1}{\omega_0^2 - \omega^2}$ and ϵ is a small parameter, $\epsilon \ll 1$. It is also found that α and σ should have the same sign and $\Lambda^2 < \frac{4\omega_0\sigma}{27\alpha}$.

More recently some other methods were developed for the analysis of subharmonic oscillations, based on classical methods of analysis. In a review paper Ishida [1994] summarises some of the various numerical and analytical methods to study steady-state responses of subharmonic oscillations. Ishida describes the subharmonic oscillations observed in gas-turbine engines and in the high-pressure turbo-pump of the space shuttle main engine.

A frequency domain method of analysis was used in Ishida et al [1992] to analyse non-stationary subharmonic oscillations in a rotating shaft. An improved-averaging scheme to calculate approximate solutions of the main resonance and subharmonics, using Floquet theory, was proposed in Tsuda et al [1992], where the difficulty of obtaining analytical approximate periodical steady-state solutions with high accuracy for oscillating systems was emphasised. A class of relaxation algorithms has recently been proposed in Frey and Norman [1992] and Frey [1998] for the efficient analysis of both non-autonomous and autonomous oscillating systems.

It is also interesting to mention an application of Volterra series to nonlinear oscillations. Chua and Tang [1982] derived a method to determine the frequency and amplitude of the fundamental harmonic, for a nonlinear autonomous oscillating system, using Volterra series. The method is of interest in this context because it represents the first rigorous application of the Volterra series to nonlinear oscillations.

3.2 Qualitative analysis of subharmonics

The geometrical approach adopted during the last two decades provides useful insight into the realm of nonlinear oscillations, of a qualitative nature, based on the geometrical and topological properties of differential equation solutions and iterated maps. Methods from dynamical systems and bifurcation theories have been applied in the analysis of nonlinear oscillations. Issues such as existence and uniqueness of the differential equations, Poincaré Maps and structural stability, also became related to nonlinear oscillations.

A dynamical system can have a rich variety of solutions: periodic, quasi-periodic or chaotic. It is also common in nonlinear systems to have different co-existing steady-state solutions, or to have several periodic and chaotic motions for the same parameter values but with different initial conditions. As pointed out in Soliman and Thompson [1992], where there is more than one solution coexisting at the local bifurcation, a simple jump to resonance may become indeterminate in the sense that it is not possible to predict onto which solution the system will settle. Prediction of such behaviour can be made in a qualitative manner using the Poincaré Map, the Bifurcation Diagram or the Response Spectrum Map [Billings and Boaghe, 1999].

In the case of a non-autonomous system, such as the Duffing equation considered in the next section, the Poincaré Map is equivalent to sampling the trajectory of the solution at a rate equal to the forcing frequency [Parker and Chua, 1989]. Fixed points and closed orbits indicate a periodic solution. A fixed point of the Poincaré Map corresponds to a period-one solution and a k-periodic closed orbit corresponds to a kth-order subharmonic.

The Bifurcation Diagram can be seen as a succession of compressed Poincaré Maps, providing a graphical representation of the bifurcation phenomenon, when a certain parameter is varied. The term bifurcation was originally used by Poincaré to describe the point where equilibrium solutions split into a family of differential equations. The bifurcation concept is therefore related to structural stability. The stability of a fixed point in a discrete system is determined by the eigenvalues of the first derivative of the map evaluated at that point. Bifurcations occur when the linearised map is degenerate, in other words when at least one eigenvalue of the discrete map has unit modulus [Guckenheimer and Holmes, 1983]. When the system parameters are varied the eigenvalue may pass through the unit modulus value, at which point a bifurcation occurs.

The case in which the eigenvalue of a fixed point has the value -1 is a special case. The eigenvalue -1 is associated with flip bifurcations, also referred to as period doubling or subharmonic bifurcations. If the subharmonic is unstable, subharmonic bifurcations are continuously generated, this phenomenon is called a period doubling cascade, or period doubling route to chaos. Sometimes the motion found at the end of the period doubling cascade is no longer periodical, but chaotic. In some other cases a reverted period doubling cascade is produced. Therefore period doubling may be a first indication that the system is becoming chaotic.

The Poincaré Map and the Bifurcation Diagram are qualitative descriptors of the system dynamics in the time domain. In the frequency domain a similar means of analysis is the Response Spectrum Map [Billings and Boaghe, 1999]. The Response Spectrum Map can be seen as a projection of the information in the Bifurcation Diagram into the frequency domain. While the former provides information about the intersection point in the time domain of the flow with a certain plane when a parameter is varied, the latter gives information about the response power spectrum in the frequency domain. The Response Spectrum Map can therefore be used to identify the various states of a system such as subharmonics and chaos.

In conclusion the identification and analysis of steady-states with subharmonics is a difficult task. Studies of these effects often require a good mastery of dynamical systems techniques and of topological and bifurcational procedures. In a few cases analytic approximate methods can be applied with success.

3.3 Modelling subharmonics

A central issue in system theory is system modelling. In general system models can be classified as being either implicit or explicit. Differential or difference equation models are an example of implicit models in which the system response is expressed as an implicit operation on the system input, as opposed to explicit models where the operation on the system input is explicit. An example of an explicit model for nonlinear systems is the Volterra series representation.

System modelling is also an important problem for nonlinear systems with subharmonics. Modelling subharmonics with explicit and implicit types of models has been reported in the literature. In particular, modelling with the explicit Volterra series was considered as an exciting alternative, given all the advantages of the Volterra series approach. The periodic steady stated theorem formulated in [Boyd et al. 1984] provides a time domain equivalent of the fact that Volterra series cannot represent systems with subharmonics, by proving that Volterra nonlinear systems respond in the time domain to a periodic excitation with a periodic output with exactly the same period:

Theorem 3.1 *[Periodic steady state theorem] If the input of a nonlinear system described by a Volterra series operator N is periodic with period T for $t \geq 0$, then the output Nu approaches a steady state, which is also periodic with period T.*

Moreover Volterra series represent or approximate time-invariant nonlinear systems with fading memory. Fading memory is related to the concept of a unique steady state and independence from the initial conditions [Boyd and Chua, 1985]. For a subharmonic oscillation to exist in a physical system initial conditions are crucially important. In other words systems with subharmonics have infinite memory and therefore they do not belong to the Volterra class of systems.

The implicit differential or difference equations have proven to be more successful in modelling nonlinear systems with subharmonics. In fact it has been argued [Pearson, 1994] that the key feature required in nonlinear models capable of generating subharmonics is recursion: the present output must depend on previous outputs and not just on the past history of the inputs as in the Volterra model. However, the implicit model provides no insight into the internal system mechanism in the way Volterra models do.

In the next section a new methodology is proposed in an attempt to overcome several restrictions associated with existing methods for modelling systems with subharmonics. The new approach is based on using MISO Volterra models and will be illustrated using the Duffing oscillator. The new models are then further analysed in both the time and the frequency domain to illustrate the simplicity and advantages of the new approach.

4 Modelling subharmonics with MISO Volterra series

As concluded above, it is impossible to model subharmonics with single input single output (SISO) Volterra series, because Volterra series respond to a periodic excitation with a periodic signal with the same period. But if the given input could be modified somehow to create an input with a period equal to the subharmonic oscillation, Volterra series modelling could be applied. In this section the possibility of modelling subharmonics with multiple input single output (MISO) Volterra series is investigated for the first time.

It is well known that initial conditions are crucially important for a subharmonic to exist in a nonlinear system. Consequently subharmonics cannot be produced in a Volterra system, where the dependence on the initial conditions gradually fades out. However, local Volterra series can be derived for a particular steady-state of the system. If a local Volterra series representation exists for a state of the system which features subharmonics, then the local Volterra series will not be expected to be equal to another local Volterra series, corresponding to a different state of the system.

The modified input which could be used for Volterra series modelling of a subharmonic should have the same period as the subharmonic. If the subharmonic considered for modelling has the period n times the original input period T, a modified input u_{mod} should have the period n times the real input period. The modified input can then be considered to be n-dimensional, $u_{mod} = [u_1; \ldots; u_i; \ldots; u_n]$, with the ith component u_i taken as

$$u_i(t) = \begin{cases} 0 & t \in [0; T) \\ \ldots & \\ u(t) & t \in [(i-1)T; iT) \\ 0 & t \in [iT; (i+1)T) \\ \ldots & \\ 0 & t \in [(n-1)T; nT) \end{cases} \tag{5}$$

Consequently, for the individual components $u_i(t)$ it can be verified that

$$u_1(t) + \ldots + u_i(t) + \ldots + u_n(t) = u(t) \tag{6}$$

and

$$u_i(t) = u_1(t - iT) \tag{7}$$

To allow for MISO Volterra modelling, the system which generates subharmonics in the steady state y when excited with the input u should have the internal structure given in Figure 1. The information required for modelling is the order n of the subharmonic and the period T of the input signal u. This information can be readily obtained by analysing the system, in the form of a given model or from a system identification study (eg. a NARMAX model) using a Bifurcation Diagram and a Response Spectrum Map. This was illustrated in the examples given in Billings and Boaghe [1999] and will be used in the examples below.

The system given in Figure 1 (a) is composed of a time multiplexor M and a MISO system S. The time multiplexor generates the n-dimensional input $u_{mod} = [u_1; \ldots; u_n]$, based on the original input $u(t)$, the subharmonic order n and input period T. The modified input u_{mod} is applied to the system S, for which the output is the subharmonic $y(t)$. For the system S both input u_{mod} and output y have the period nT, therefore a local MISO Volterra model can be identified directly from the data generated from the simulation.

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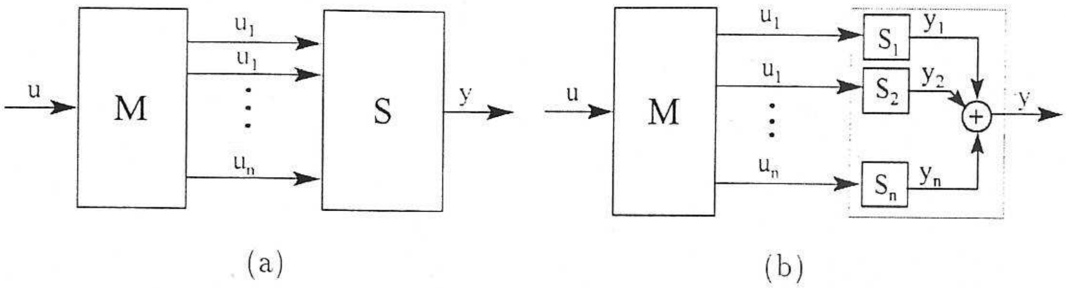

Figure 1: Internal structure of a nonlinear system generating subharmonics for modelling with (a) MISO Volterra series (b) MISO Volterra series without cross-product $u_i u_j$ terms

To simplify the internal system mechanism the Volterra series to be derived can be prevented from containing cross-product terms of the type $u_i u_j$, $i \neq j$, so that individual contributions from every input u_i can be separated in the Volterra model. If individual contributions at the input are separated, the system S in Figure 1 (b) can be decomposed into n independent subsystems $S_1, \ldots, S_n$, one for every individual input u_i. In this case the MISO Volterra model will not contain cross-product terms between inputs and the generalised frequency response functions can then easily be derived for the individual subsystems S_i.

To summarise, the modelling procedure is given below:

i. Detect a subharmonic steady state using the Bifurcation Diagram for a simulated implicit difference or differential equation, or any other model form.

ii. Identify the order n and the period of oscillation nT of the subharmonic $y(t)$, by visual inspection or using the Response Spectrum Map.

iii. Generate an n-dimensional modified input u_{mod} based on the original input $u(t)$, using a time multiplexor.

iv. Identify a MISO discrete-time Volterra model for the system with output $y(t)$ and the n-dimensional modified input u_{mod}.

The methodology given above for modelling with MISO Volterra series will be applied in the next section to the Duffing equation, where subharmonics can be generated for example by the Duffing-Ueda or Duffing-Holmes cases. For both cases the input considered is a periodic sine wave, however the modelling procedure given above is not limited to sine waves. The Volterra series are obtained in the time-domain as MISO models based on just the input or exogenous variables, and the Volterra kernels are further analysed in the frequency domain.

To illustrate the ideas in the simplest way all the examples below start from an assumed known differential equation model of the system. But the method is based on simulations of the system model for the input u_{mod} and can therefore be just as easily applied to an identified NARMAX, neural network or any other model that represents the system.

4.1 Example 1: Duffing-Ueda equation

In this example the Duffing equation is considered in the form also known as the Duffing-Ueda model

$$\ddot{y} + k\dot{y} + y^3 = u(t) \tag{8}$$

For the present analysis the Duffing-Ueda equation (8) was simulated for $k = 0.1$ and $u(t) = Acos(t)$, where $0 \leq A \leq 12$. A fourth-order Runge-Kutta algorithm with an integration interval of $\pi/3000$ was used to simulate the response of the system to the sinusoidal input. The input and output signals were further sampled at periods of $T_s = \pi/60$ sec.

For this equation various dynamic regimes were noticed, for different amplitudes of the input signal. The Bifurcation Diagram and the Response Spectrum Map, for a varying amplitude A of the input signal $u(t) = Acos(t)$, are given in Figure 2. The Response Spectrum Map shows that the frequency of the sinusoidal input $f = \frac{1}{2\pi} = 0.159Hz$ is present for all input values A and that subharmonic generation is evident for certain sets of amplitude values. In this example the subharmonic of order $\frac{1}{n} = \frac{1}{3}$ at $0.053Hz$ obtained for amplitude $A = 6$ in the model (8) will be considered.

In order to derive a Volterra MISO model a modified input is produced. The modified input is generated from the original input signal, knowing the order of the subharmonic n and the time period T. Both n and T parameters can be readily obtained from the Response Spectrum Map given in Figure 2. For the amplitude value $A = 6$, the order is $n = 3$ and the time period is $T = \pi/60 secs$. Figure 3 shows the original input u and the modified 3-dimensional input, with the components u_1, u_2 and u_3. By inspecting the waveforms given in Figure 3, the relations given in (6) and (7) can be easily verified.

For a system with the input $[u_1; u_2; u_3]$ a discrete-time MISO Volterra model

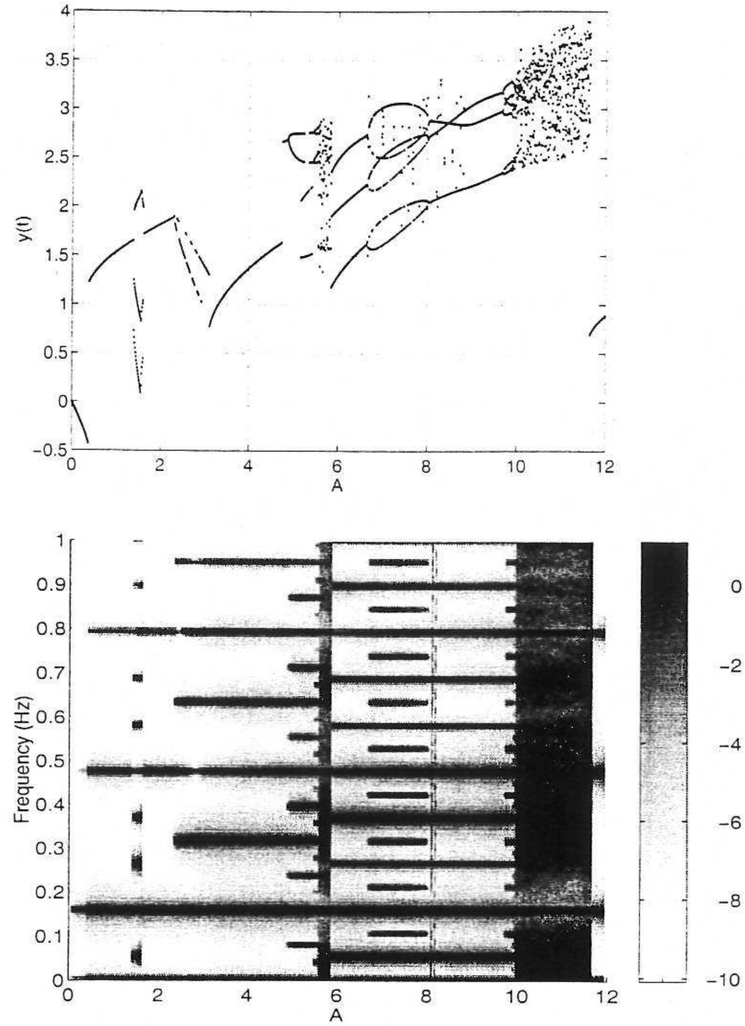

Figure 2: Bifurcation Diagram and Response Spectrum Map (plan view) for the Duffing-Ueda equation (8)

was identified. In the present study the MIMO Orthogonal Least Squares (OLS) method [Billings et al. 1989] was applied but there are several alternatives. The main advantage of the OLS method is that it can automatically select the terms in the model. Notice that because the MISO consists of a set of Volterra models which only involve the inputs, bias which can be induced by noise should not be a problem and the estimation is therefore relatively straightforward.

Volterra models with 3, 5 and 7 orders of nonlinearity were estimated, providing correspondingly increased degrees of accuracy. For a 3rd order of nonlinearity the model predicted output was already very good, therefore the model selected for

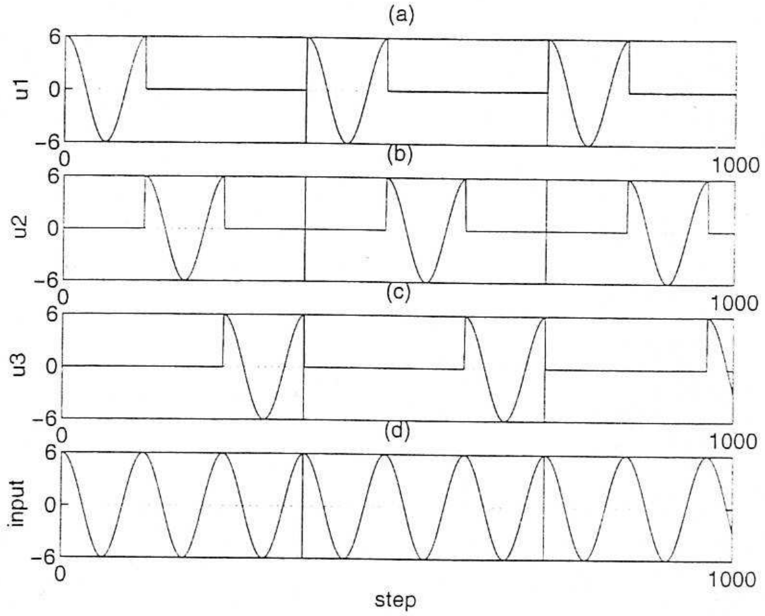

Figure 3: Modified input signal $[u_1; u_2; u_3]$ with the components (a) u_1; (b) u_2; (c) u_3 and (d) original input

further analysis was a 3rd order MISO Volterra model

$$\begin{aligned}
y_1(k) &= -0.6685 + 1.6766u_1(k-3) + 0.8832u_1^2(k-3) - 1.4688u_1(k-1) + \\
&+ 0.0213u_1^3(k-1) - 0.0239u_1^3(k-3) + 0.0048u_1(k-1)u_1^2(k-2) + \\
&+ 0.3323u_1^2(k-1) - 1.1966u_1(k-1)u_1(k-3) \qquad (9) \\
y_2(k) &= 0.0195u_2^3(k-3) - 1.2882u_2^2(k-3) - 1.0715u_2(k-3) - 1.0424u_2^2(k-1) + \\
&+ 2.3687u_2(k-1)u_2(k-3) - 0.0202u_2^3(k-1) + 1.4514u_2(k-1) \qquad (10) \\
y_3(k) &= -0.9846u_3^3(k-3) + 0.6732u_3(k-1)u_3^2(k-3) - 0.1331u_3(k-3) + \\
&+ 0.7935u_3(k-1) + 1.0905u_3(k-1)u_3(k-2)u_3(k-3) + 5.7359u_3^2(k-3) + \\
&+ 6.0159u_3^2(k-1) - 11.754u_3(k-1)u_3(k-3) - 0.7903u_3^3(k-1) \qquad (11)
\end{aligned}$$

where

$$y(k) = y_1(k) + y_2(k) + y_3(k) \qquad (12)$$

It should be noticed that according to the general internal structure given in Figure 1 (b), the model has no cross product input terms (no terms of the type u_1u_2), therefore the identification procedure is easier, and consists of fitting 3 SISO Volterra models which are further combined to give the system output.

The model predicted output is plotted in Figure 4, where it can be compared with the original output. They match almost exactly and some improvement is

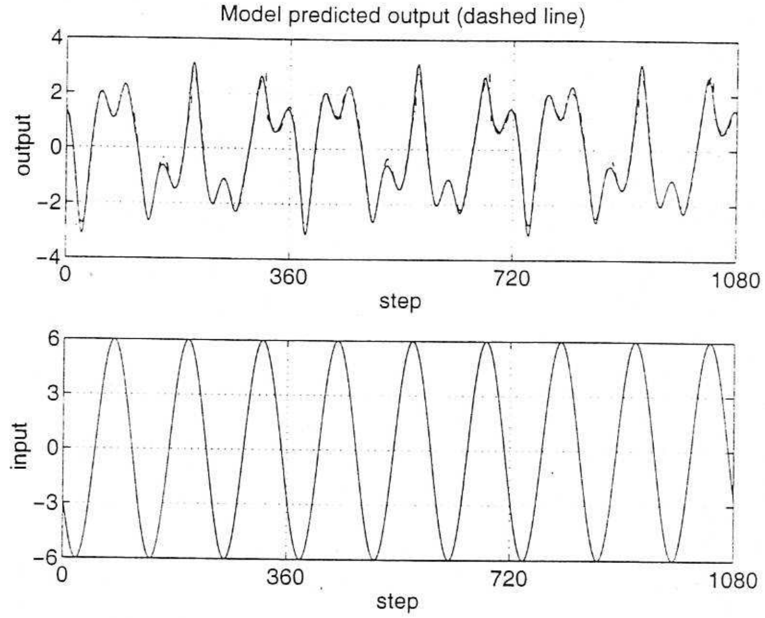

Figure 4: Model (12) predicted output (dashed line), original output (solid line), and input signal

obtained for an increased order of nonlinearity. In Figure 4 the original input signal is also shown, with a period 3 times smaller than the resulting subharmonic.

It is important to emphasise that one single Volterra model cannot represent this system and any attempt to find such a model results in very poor model predictions. But by using the Response Spectrum Map to determine an appropriate input and then by estimating in this case three single input single output models and combining these gives the final MISO Volterra representation. The result is that now the MISO model should be an excellent representation of the system with the advantage that the properties, analysis and interpretation of this model can now be studied using all the methods developed for Volterra systems. But now these can be applied to severely nonlinear systems and to study subharmonic phenomena.

The Volterra kernels of the MISO model (9)-(11) can now be analysed in the frequency domain. There are no cross-product terms between individual input components, and the model (12) can be considered to be composed of 3 independent submodels, one for each input component. By applying the probing method to the submodels (9)-(11), following the methodology given in Peyton Jones and Billings [1989] the Generalised Frequency Response Functions are derived next, one for each submodel. There will be 3 different linear functions $H_1(j\omega)$, 3 different functions $H_2(j\omega_1, j\omega_2)$, etc. The linear functions $H_1(j\omega)$ are given in Figure 5.

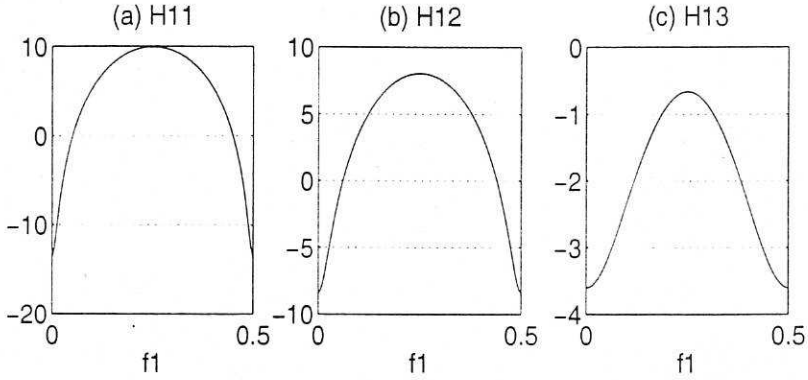

Figure 5: Generalised Frequency Response Function $H_1(j\omega)$ for (a) u_1, (b) u_2, (c) u_3 in equation (12)

For all input components the functions $H_1(\omega)$ have the same parabolic shape for the magnitude, with different maximum values at the normalised frequency $f_1 =$ 0.25 (corresponding to $9.54Hz$). The functions $H_2(j\omega_1, j\omega_2)$ are given in Figure 6, together with the plan images. These also have a similar shape, with a maximum at $f_1 = f_2 = 0.25$.

The functions $H_3(j\omega_1, j\omega_2, j\omega_3)$ are represented in Figure 7, for the section $f_1 = f_3$. Again the first two of these have similar features, with high magnitudes for $f_1 + f_2 + f_3 = 0.25$ the normalised frequency (corresponding to $9.54Hz$), and a minimum magnitude for $f_1 + f_2 + f_3 = 0$. Only for the third subsystem are the lines somewhat distorted, showing high magnitude for $f_1 + f_2 + f_3 = 0$ and a minimum magnitude on the frequency lines $f_1 + f_2 + f_3 = 0.25$, which is the direct opposite to the first two third order frequency response functions.

4.2 Example 2: Duffing-Holmes equation

In this section the Duffing equation is considered in the form also known as the Duffing-Holmes model

$$\ddot{y} + 1.5\dot{y} - y + y^3 = Acos(t) \tag{13}$$

This equation was simulated using a fourth-order Runge-Kutta algorithm with an integration interval of $\pi/15$, for an amplitude A varying in the range $1 \leq A \leq 1.6$. The Bifurcation Diagram and the Response Spectrum Map given in Figure 8 show cascades of period doubling, for different amplitudes of the input signal varying in

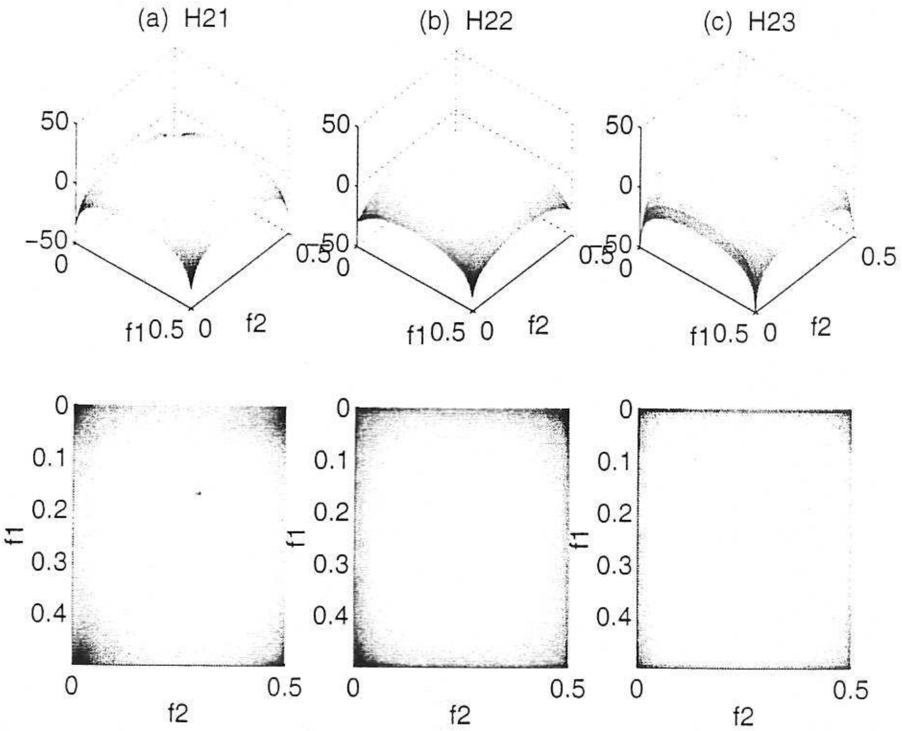

Figure 6: Generalised Frequency Response Function $H_2(j\omega_1, j\omega_2)$ for (a) u_1, (b) u_2, (c) u_3 in equation (12)

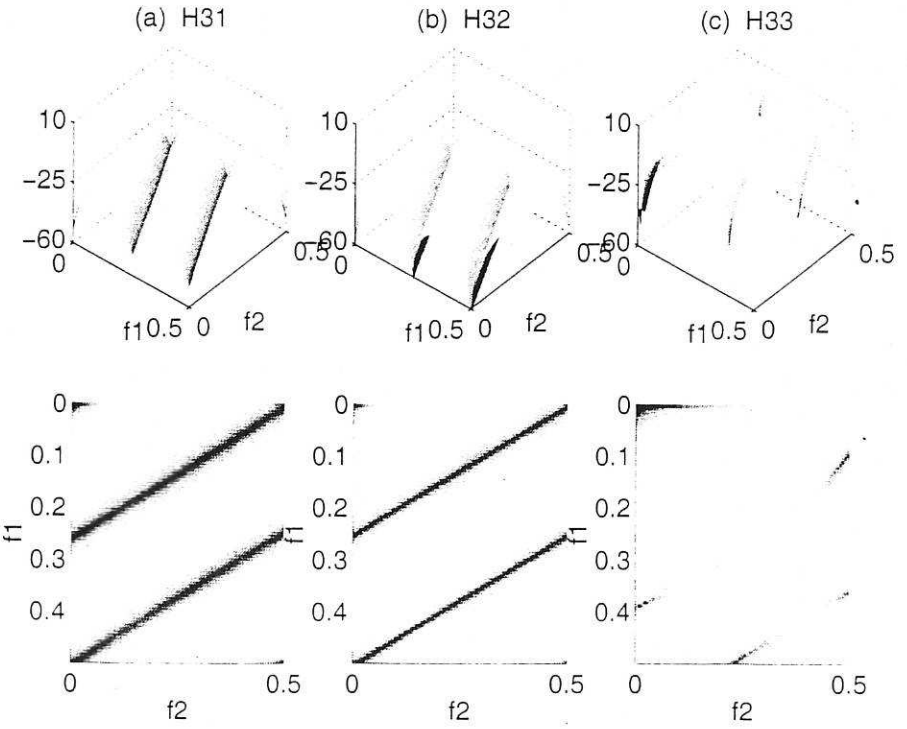

Figure 7: Generalised Frequency Response Function $H_3(j\omega_1, j\omega_2, j\omega_1)$ for (a) u_1, (b) u_2, (c) u_3 in equation (12)

the range $1 \leq A \leq 1.6$. In this example the subharmonic of order $\frac{1}{n} = \frac{1}{2}$ at $0.089Hz$ obtained for amplitude $A = 1.2$ in the model (13) will be considered.

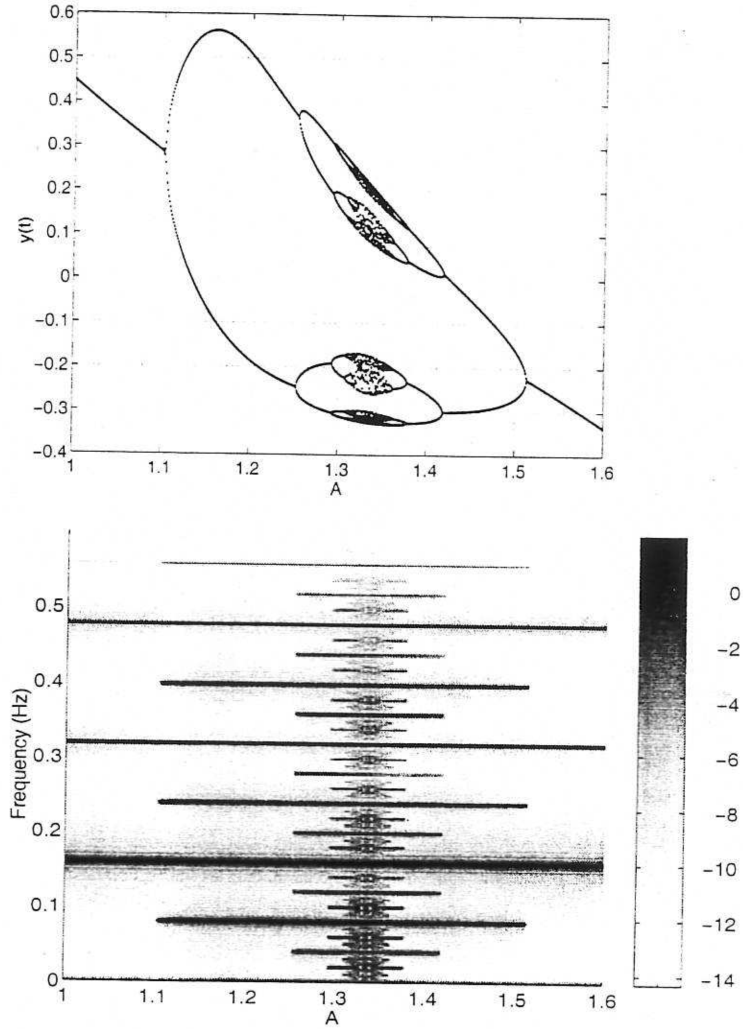

Figure 8: Bifurcation Diagram and Response Spectrum Map (plan view) for the Duffing-Holmes equation (13)

As in the previous example, a modified input is produced first. The modified input is generated from the original input signal, knowing the order of the subharmonic n and the time period T. By examining the Response Spectrum Map given in Figure 8, the parameters n and T can be obtained. For the amplitude value $A = 1.2$, the order is $n = 2$ and the time period is $T = \pi/60secs$. The original and the modified 2-dimensional input signals are given in Figure 9.

For a system with the input $[u_1 : u_2]$ represented in Figure 9 a discrete-time MISO Volterra model was identified, using the MIMO OLS method [Billings et al.

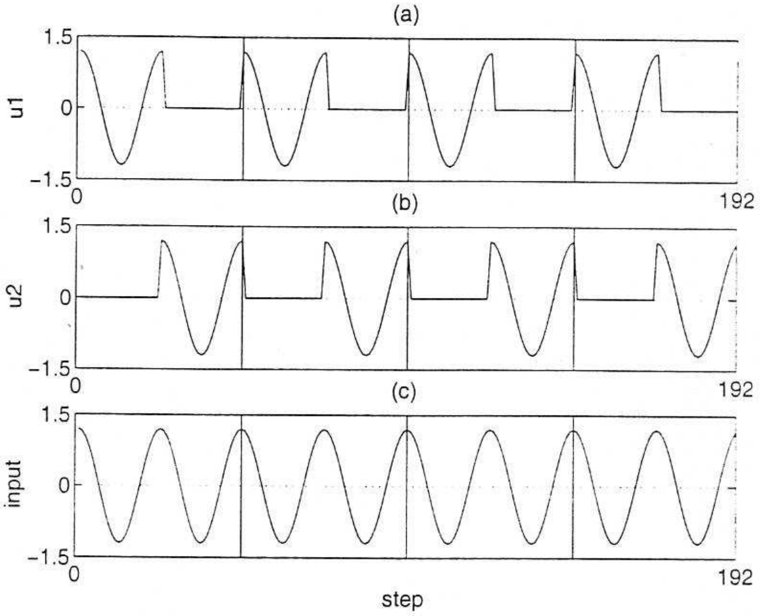

Figure 9: Modified input signal $[u_1; u_2]$ with the components (a) u_1: (b) u_2 and (c) original input

1989]. The Volterra model selected had a 3rd order of nonlinearity and no cross product input terms (no terms of the type $u_1 u_2$)

$$\begin{aligned} y_1(k) &= 5.6235u_1(k-3) - 0.0859u_1^3(k-3) + 2.9122u_1(k-1) - 0.0833u_1^3(k-1) + \\ &+ 5.7643u_1(k-2)u_1(k-3) - 12.0268u_1(k-1)u_1(k-2) - 7.9389u_1(k-2) + \\ &+ 5.7980u_1^2(k-1) - 3.0893u_1^2(k-3) + 3.6287u_1^2(k-2) \qquad (14) \\ y_2(k) &= 14.6176u_2(k-1)u_2(k-2) + 6.6054u_2^2(k-3) + 6.0183u_2(k-3) - \\ &- 10.1978u_2(k-2) + 0.4071u_2(k-2)u_2^2(k-1) - 7.8590u_2^2(k-1) + 0.4472 - \\ &- 13.5093u_2(k-2)u_2(k-3) - 0.3316u_2^3(k-1) + 4.0732u_2(k-1) \qquad (15) \end{aligned}$$

where

$$y(k) = y_1(k) + y_2(k) \qquad (16)$$

As in the preceding example, the model predicted output was estimated and used for model validation. Figure 10 shows a good agreement between the original and the estimated output. In Figure 10 the original input signal is also shown, with a period 2 times smaller than the resulting subharmonic.

The Volterra model (16) has no cross-product terms between individual input components, therefore the Volterra kernels can be further analysed in the frequency domain. By applying the probing method to the submodels (14)-(15), following the methodology given in Peyton Jones and Billings [1989], the Generalised Frequency

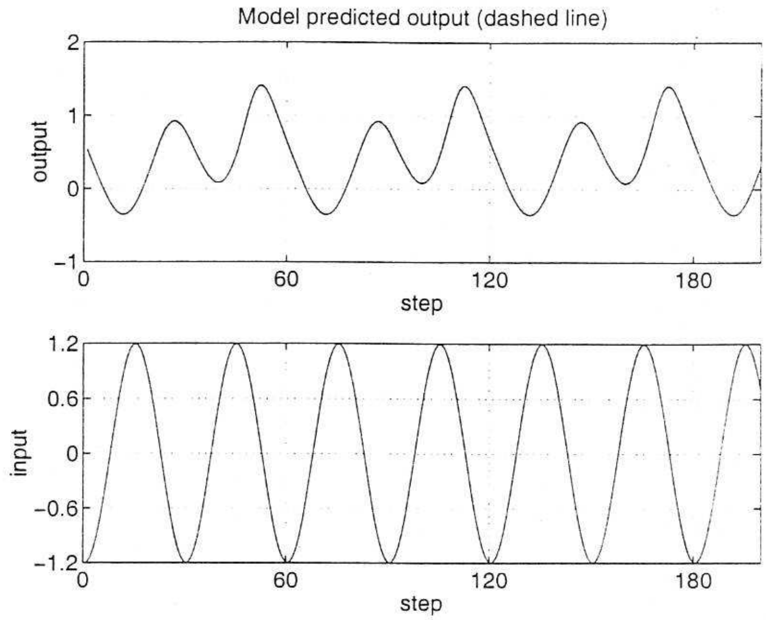

Figure 10: Model (16) predicted output (dashed line), original output (solid line), and input signal

Response Functions are derived next, one for each submodel. The first order GFRF's are given in Figure 11.

The functions $H_1(j\omega)$ have the same parabolic shape for both input components, with different maximum values, of $24dB$ and $26dB$ respectively, at the normalised frequency $f_1 = 0.5$ (corresponding to $2.38Hz$). They are also similar to the left hand side half of the functions $H_1(j\omega)$ derived in the previous example, given in Figure 5.

The functions $H_2(j\omega_1, j\omega_2)$ are given in Figure 12, together with the plan images. The shape of the functions $H_2(j\omega_1, j\omega_2)$ are again very similar. Both functions $H_2(j\omega_1, j\omega_2)$ have a maximum amplitude near $f_1 = f_2 = 0.5$ (corresponding to $2.38Hz$), of $27dB$ and $31dB$ respectively, and a minimum near the origin of the input frequencies.

The functions $H_3(j\omega_1, j\omega_2, j\omega_3)$ are represented in Figure 13, for the section $f_1 = f_3$. For the first model the function $H_3(j\omega_1, j\omega_2, j\omega_1)$ shows high magnitude for $f_1 + f_2 + f_3 = 0.5$ normalised frequency, with a maximum value of $-15dB$, and a minimum magnitude is produced when $f_1 + f_2 + f_3 = 0.25$. For the second model the function $H_3(j\omega_1, j\omega_1)$ shows a maximum magnitude of $-2dB$ for $f_1 = f_2 = f_3 = 0.5$ normalised frequency and a minimum magnitude is produced when

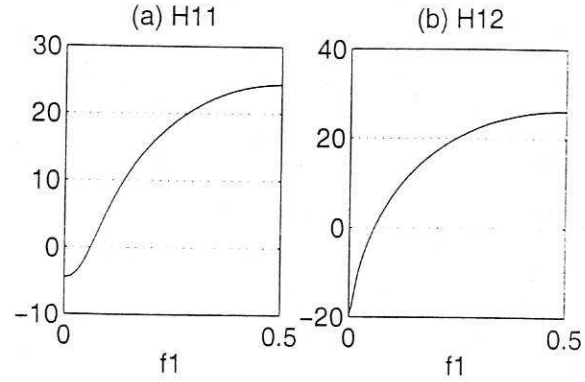

Figure 11: Generalised Frequency Response Functions $H_1(j\omega)$ for (a) u_1 (b) u_2 in equation (16)

$f_1 = f_2 = f_3 = 0$. By comparing the maximum values of the Generalised Frequency Response Functions it is apparent that the significance of the nonlinearities in the model decreases with the order.

The frequency response functions obtained in both examples have graphical representations with similar features. This may suggest that the H's previously derived correspond not only to the local Volterra kernels. but they can also be related to the original underlying system. which is the Duffing equation. and are therefore a global feature. This idea will be the starting point for further investigations into subharmonic analysis.

5 Conclusions

The objective of this paper was to investigate the modelling. analysis and interpretation of nonlinear systems with subharmonics. After an introduction to nonlinear subharmonic oscillations. various techniques for subharmonic analysis were reviewed and possible ways of modelling subharmonics were discussed. It was found that methods from system dynamics. topology. bifurcation theory and nonlinear oscillations can be applied to subharmonic analysis and modelling.

A new modelling approach was introduced based on a MISO Volterra series model representation. The advantage of this approach is that it is relatively simple to apply and once the model has been obtained all the well known methods of analysis for Volterra series. which have until now been restricted to mildly nonlinear

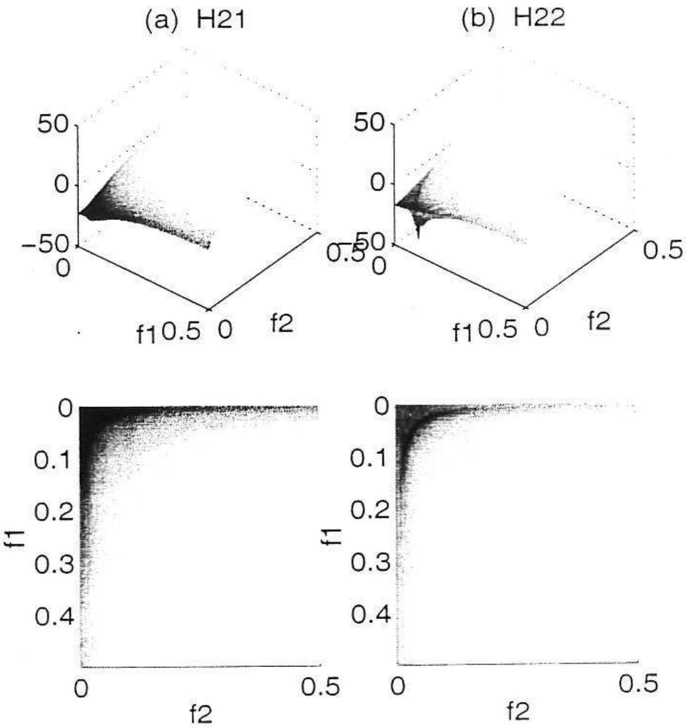

Figure 12: Generalised Frequency Response Functions $H_2(j\omega_1, j\omega_2)$ for input (a) u_1, (b) u_2 in equation (16)

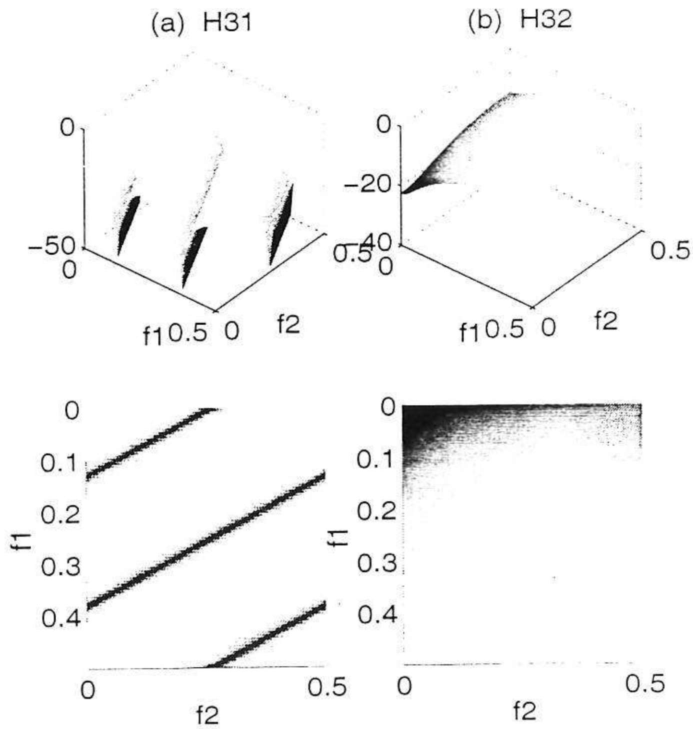

Figure 13: Generalised Frequency Response Functions $H_3(j\omega_1, j\omega_2, j\omega_1)$ for input (a) u_1, (b) u_2 in equation (16)

systems only, can now be applied to nonlinear systems which exhibit subharmonics.

The new approach was applied to modelling subharmonics associated with various steady-states of the Duffing oscillator, and the GFRF's of the MISO models were used to analyse the system properties in the frequency domain. It was interesting to find common characteristics in the GFRF representations for different local subharmonics of the Duffing oscillator, suggesting that the GFRF's derived describe some invariant features of the system and not just the local behaviour. This idea will form the starting point for further investigations into the analysis of systems with subharmonics.

6 Acknowledgements

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