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The Schwartz Kernel Theorem and the Frequency-Domain Theory of Nonlinear Systems

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Abstract The frequency domain theory of nonlinear analytic input-output maps is studied, directly in the frequency domain. By expanding the 'Fourier transform' of the input-output map as a Taylor series in an appropriate function space and applying Schwartz' kernel theorem, we obtain a general theory of nonlinear systems in the frequency domain. This obviates the necessity of using 'association of variables' in the time domain and leads to a much more general theory.
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1. Introduction

The frequency-domain theory of nonlinear systems began with a study of the Volterra series of a bilinear system of the form

$$\dot{x} = Ax + uDx + bu \quad , \quad x(0) = x_0$$

A typical term in the Volterra series solution of this equation is of the form

$$\int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_k} K_k(t, \tau_1, \cdots, \tau_k) u(\tau_1) \cdots u(\tau_k) d\tau_1 \cdots d\tau_k \qquad (1.1)$$

where the kernel K_k has a number of possible representations; for example, in triangular or regular forms (see [6]). A number of subtle transformations of the parameters $\tau_1, \cdots, \tau_k$ are then made in order to make (1.1) look like a k-dimensional convolution. The theory of k-dimensional Laplace transforms can then be applied and a k-dimensional 'transfer function' can be defined. These transfer functions then form a basis for a generalization of the classical theory of linear systems in the frequency domain (see [1],[2],[3],[4], [5]). As is pointed out in ([6]) the transfer functions may be used to determine the response of bilinear systems to simple inputs, using residue theory, but general inputs pose much more of a problem.

In this paper we show that by using the input-output map directly in the frequency domain and applying Schwartz' kernel theorem ([7]), all of the mysteries of the time-domain approach disappear and the frequency domain kernels are easy to obtain, but are generally distributions. By using the convolution algebra, we can even consider general nonlinear systems and develop a truly universal frequency domain theory.

In section 3 we consider analytic input-output maps and develop a general frequency-domain theory of nonlinear systems, giving as examples linear and bilinear systems. In section 4 we apply the kernel theorem to derive directly a frequency-domain Volterra-like representation based on kernel distributions. In sections 5 and 6 we show how the theory applies to time-varying bilinear systems (which may be distributed) and general linear-analytic systems.

2. Notation

$\mathcal{F}$ = Fourier transform
$L^p[a, b; X]$ = space of integrable functions with values in a (Banach) space X
$L^p[a, b] = L^p[a, b; \mathbb{R}]$
$L^p_T[a, b; X]$ = elements of $L^p[a, b; X]$ which are zero after T
$\mathcal{L}(X, Y)$ = bounded operators from X into Y
$\mathcal{L}(X) = \mathcal{L}(X, X)$
$f * g$ = convolution of f and g
δ = Dirac delta distribution
$\mathcal{D}'(X)$ = space of distributions on X
$\mathcal{C}_c^\infty(X)$ = space of infinitely differentiable functions with compact support on X
$\langle \cdot, \cdot \rangle$ = duality in $\mathcal{C}_c^\infty(X), \mathcal{D}'(X)$
$\mathbf{i} = (i_1, \cdots, i_n)$ (n-tuple of integers)
$|\mathbf{i}| = i_1 + \cdots + i_n$

$\mathbf{1}_k = (0, \cdots, 0, \underbrace{1}_{k}, 0, \cdots, 0)$

If f is an analytic function,

$$f^{(\mathbf{i})}(0) = \frac{\partial^{|\mathbf{i}|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} f(0) \quad .$$

3. Frequency Domain Theory of Nonlinear Systems

In this section we shall consider a causal nonlinear system S defined on the Hilbert space $L_T^2[0, \infty)$ of all measurable, square-integrable real-valued functions on the interval $[0, \infty)$ which are zero for $t > T$, i.e.

$$u(t) = 0 \quad \text{if} \quad t > T,$$

for some $T > 0$.

By causality, the output $y = S(u)$ is also zero for $t > T$. We shall assume that S maps $L_T^2[0, \infty)$ into itself. (For input-output stable systems we can take $T = \infty$.) Let $\mathcal{F}$ denote the Fourier transform. As is well-known,

$$\mathcal{F} : L^2(-\infty, \infty) \longrightarrow L^2(-\infty, \infty)$$

is an isomorphism and so $\mathcal{F}$ maps $L_T^2[0, \infty)$ one-to-one and isometrically into a subspace $\tilde{L}_T^2[0, \infty)$ say of $L^2(-\infty, \infty)$. (Note that this notation does not imply that $v(\omega) = 0$ for $\omega > T$ for any $v \in \tilde{L}_T^2[0, \infty)$.) We define the **transformed system** $\tilde{S}$ by

$$\tilde{S}(v) = \mathcal{F} S \mathcal{F}^{-1}(v) \quad . \tag{3.1}$$

Thus, $\tilde{S}$ is defined by the commutative diagram

$$\begin{array}{ccc} L_T^2[0,\infty) & \xrightarrow{S} & L_T^2[0,\infty) \\ \mathcal{F} \downarrow & & \downarrow \mathcal{F} \\ \tilde{L}_T^2[0,\infty) & \xrightarrow{\tilde{S}} & \tilde{L}_T^2[0,\infty) \end{array}$$

Lemma 3.1 *If S is an analytic function on $L_T^2[0, \infty)$, then $\tilde{S}$ is analytic on $\tilde{L}_T^2[0, \infty)$.*

Proof Since $\mathcal{F}$ is invertible and linear, it is an analytic map; hence $\tilde{S} = \mathcal{F} S \mathcal{F}^{-1}$ is analytic as a composition of analytic maps.□

Since $\tilde{S}$ is analytic we can expand it in a Taylor series:

$$\tilde{S}(v) = \sum_{i=0}^{\infty} M_i(v) \tag{3.2}$$

where M_i is a form of order i defined on $\tilde{L}_T^2[0,\infty)$.

Example 3.1

(1). Linear Systems:

Consider the causal linear system with zero initial condition

$$S: \quad y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Then we have

$$Y(i\omega) = G(i\omega)U(i\omega)$$

so that $\tilde{S}$ is simply multiplication by $G(i\omega)$. Note that we can write $\tilde{S}$ in an integral form:

$$Y(i\omega) = \int_{-\infty}^{\infty} \delta(\omega-\omega')G(i\omega')U(i\omega')d\omega' \tag{3.3}$$

with kernel the point distribution $\int_{-\infty}^{\infty} \delta(\omega-\omega')G(i\omega')$.

(2). Bilinear Systems:

Consider the causal single-input distributed bilinear system

$$\dot{x} = Ax + uDx + bu \quad , \quad x(0) = x_0 \quad , \quad x \in L^2(\Omega).$$

The solution is given by the Volterra series

$$\begin{aligned} x(t) \;=\; & e^{At}x_0 + \int_0^t e^{A(t-s)}bu(s)ds + \sum_{i=1}^{\infty}\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{i-1}} e^{A(t-\tau_1)}De^{A(\tau_1-\tau_2)}D\cdots \\ & \cdots De^{A(\tau_{i-1}-\tau_i)}De^{A\tau_i}x_0u(\tau_1)u(\tau_2)\cdots u(\tau_i)d\tau_1\cdots d\tau_i \\ & +\sum_{i=1}^{\infty}\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{i-1}}\int_0^{\tau_i} e^{A(t-\tau_1)}De^{A(\tau_1-\tau_2)}D\cdots De^{A(\tau_{i-1}-\tau_i)}De^{A(\tau_i-\tau_{i+1})}b \\ & u(\tau_1)u(\tau_2)\cdots u(\tau_{i+1})d\tau_1\cdots d\tau_{i+1} \end{aligned}$$

where e^{At} is the semigroup generated by A. The association of a series of frequency domain kernels is then a complicated procedure of extending each integral to be a function over the rectangle $[0,t]\times[0,t]\times\cdots\times[0,t]$ (i or $i+1$ factors) and then replacing the t's by $t_1,\cdots,t_i$ (or t_{i+1}). The integrals then become i (or $i+1$)-dimensional convolutions which can be operated on by an i (or $i+1$)-dimensional

Laplace transform. Only in simple cases can the resulting expressions be related easily to the time domain (see [6]).

We can derive a frequency-domain theory for this bilinear system directly by taking a one-sided Fourier transform (the system is assumed causal):

$$i\omega X = AX + \frac{1}{2\pi} U * DX + bU + x(0)$$

since the Fourier transform takes products to convolutions. Hence, if $i\omega \notin \sigma(A)$,

$$\left(I - \frac{1}{2\pi}(i\omega - A)^{-1} U * D\right) X = (i\omega - A)^{-1}(bU(\omega) + x(0)) \quad .$$

Suppose that A is a closed linear sectorial operator (see [8],[9]) with dense domain so that

$$\|(i\omega - A)^{-1}\| \le \frac{M}{|i\omega - a|}$$

for some real a and for all $i\omega$ in the sector

$$S_{a,\phi} = \{\lambda : \phi \le |\arg(\lambda - a)| \le \pi, \lambda \ne a\}$$

where $\phi \in (0, \pi/2)$. Thus, if $U \in L^1[0, \infty)$, then the operator K defined by

$$KX = \frac{1}{2\pi}(i\omega - A)^{-1} \circ (U * (DX))$$

satisfies

$$\begin{aligned}
\|KX\|_{L^2(0,\infty,L^2(\Omega))} &\le \frac{1}{2\pi} \left\| \|(i\omega - A)^{-1}\|_{\mathcal{L}(L^2(\Omega))} \|U * (DX)\|_{L^2(\Omega)} \right\|_{L^2(0,\infty)} \\
&\le \frac{1}{2\pi} \left\| \|(i\omega - A)^{-1}\|_{\mathcal{L}(L^2(\Omega))} |U| * \|D\|_{\mathcal{L}(L^2(\Omega))} \|X\|_{L^2(\Omega)} \right\|_{L^2(0,\infty)} \\
&\le \frac{1}{2\pi} \left\| \frac{M}{|i\omega - a|} (|U| * \|D\|_{\mathcal{L}(L^2(\Omega))} \|X\|_{L^2(\Omega)}) \right\|_{L^2(0,\infty)} \\
&\le \frac{M}{2\pi} \left\| \frac{1}{|i\omega - a|} \right\|_{L^2(0,\infty)} \|U\|_{L^1(0,\infty)} \|D\|_{\mathcal{L}(L^2(\Omega))} \|X\|_{L^2(0,\infty,L^2(\Omega))} \\
&\le \frac{M}{4|a|} \|U\|_{L^1(0,\infty)} \|D\|_{\mathcal{L}(L^2(\Omega))} \|X\|_{L^2(0,\infty,L^2(\Omega))}
\end{aligned}$$

(by Young's inequality), i.e.

$$\|K\| \leq \frac{M}{4|a|}\|U\|_{L^1(0,\infty)}\|D\|_{\mathcal{L}(L^2(\Omega))} \quad .$$

Hence, if

$$\|U\|_{L^1(0,\infty)} < \frac{4|a|}{M\,\|D\|} \tag{3.4}$$

then

$$\|K\| \leq 1 \quad .$$

Now, under condition (3.4) we have

$$X = (I + K + K^2 + K^3 + \cdots)(i(\cdot) - A)^{-1}(bU(\cdot) + x(0))$$

by the Neumann series (see [9]). Let us evaluate explicitly the expression $\xi_p \doteq K^p(i\omega - A)^{-1}bU$. We have

$$\begin{aligned}
\xi_p &= \frac{1}{2\pi}K^{p-1}(i\omega - A)^{-1}\int_{-\infty}^{\infty} U(\omega-\omega_1)D(i\omega - A)^{-1}bU(\omega_1)d\omega_1 \qquad (3.5)\\
&= \frac{1}{(2\pi)^2}K^{p-2}(i\omega - A)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(\omega-\omega_2)U(\omega_2-\omega_1)U(\omega_1)D \times \\
&\quad (i\omega_2 - A)^{-1}D(i\omega_1 - A)^{-1}bd\omega_1 d\omega_2 \\
&\quad \cdots \\
&= \frac{1}{(2\pi)^p}(i\omega - A)^{-1}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} U(\omega-\omega_p)U(\omega_p-\omega_{p-1})\cdots U(\omega_2-\omega_1)U(\omega_1) \times \\
&\quad D(i\omega_p - A)^{-1}D\cdots D(i\omega_1 - A)^{-1}bd\omega_1\cdots d\omega_p \\
&= \frac{1}{(2\pi)^p}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \delta(\omega-\omega_{p+1})U(\omega_{p+1}-\omega_p)U(\omega_p-\omega_{p-1})\cdots U(\omega_2-\omega_1)U(\omega_1) \times \\
&\quad (i\omega_{p+1} - A)^{-1}D(i\omega_p - A)^{-1}D\cdots D(i\omega_1 - A)^{-1}bd\omega_1\cdots d\omega_{p+1}
\end{aligned}$$

This gives (almost) the standard kernel used in the well-known theory of frequency domain representations of bilinear systems (see [6]) multiplied by $\frac{1}{(2\pi)^p}\delta(\omega-\omega_{p+1})$, i.e.

$$K(\omega_1,\cdots,\omega_{p+1}) = \frac{1}{(2\pi)^p}\delta(\omega-\omega_{p+1})(i\omega_{p+1}-A)^{-1}D(i\omega_p-A)^{-1}D\cdots D(i\omega_1-A)^{-1}b \; .$$

It therefore provides a simple explanation for the terms in the Volterra kernels in the frequency domain and the response to any input $U(\omega)$ can easily be obtained.

4. General Input-Output Maps and the Kernel Theorem

In this section we show that the expression (3.5) is, in fact, a special case of a much more general result; namely, that any analytic input-output map on $L^2_T[0,\infty)$ can be written in terms of an infinite sequence of kernels. To do this we use Schwartz' kernel theorem (see [7]) which we can state in the form

Theorem 4.1 *The space of distributions $\mathcal{D}'(X\times Y)$ on the product space $X\times Y\subseteq \mathbb{R}^m\times\mathbb{R}^n$ is isomorphic to the set of continuous linear maps $\{K : C_c^\infty(Y)\to \mathcal{D}'(X)\}$.*$\square$

Under this isomorphism, $K(x,y)\in\mathcal{D}'(X\times Y)$ corresponds to $K : C_c^\infty(Y)\to \mathcal{D}'(X)$ where Kv is the distribution on X given by

$$C_c^\infty(X)\ni u\to\langle K(x,y),u(x)v(y)\rangle\ .$$

This is usually written in the form

$$(Kv)(x)=\int K(x,y)v(y)dy\ .$$

For example, $\delta(x-y)\in\mathcal{D}'(X\times Y)$ and

$$\int\delta(x-y)v(y)dy=v(x)$$

and so $\delta(x-y)$ is associated with the natural injection $C_c^\infty(Y)\hookrightarrow\mathcal{D}'(X)$.

Now, returning to the input-output map $S : L^2_T[0,\infty)\longrightarrow L^2_T[0,\infty)$ we have seen that if S is analytic it induces a map

$$\tilde{S}:\tilde{L}^2_T[0,\infty)\longrightarrow\tilde{L}^2_T[0,\infty)\ ,$$

which is also analytic. Writing

$$\tilde{S}(v)=\sum_{i=0}^{\infty}M_i(v)$$

in the form of a Taylor series we have

$$M_i(v)=\frac{F^i\tilde{S}(0)}{i!}v^{(i)}$$

where F is the Fréchet derivative and $v^{(i)}=\underbrace{(v,\cdots,v)}_{i}$. Thus, M_i determines a symmetric multilinear form

$$\bar{M}_i(v_1,\cdots,v_i)\in\mathcal{L}(\Lambda,\mathcal{L}(\Lambda,\cdots,\mathcal{L}(\Lambda,\Lambda)\cdots))$$

where $\Lambda = \tilde{L}^2_T[0,\infty)$.
Theorem 4.2 *$\bar{M}_i$ can be represented by a kernel distribution*

$$K_i \in \mathcal{D}'([0,\infty) \times [0,\infty) \times \cdots \times [0,\infty))$$

Proof This follows from theorem 4.1 by induction. For example, consider

$$\bar{M}_2 \in \mathcal{L}(\Lambda, \mathcal{L}(\Lambda, \Lambda)) \; .$$

Restricting $\bar{M}_2(v_1, \cdot)$ to $\mathcal{C}^\infty_c([0,\infty))$ we can write

$$\bar{M}_2(v_1, v_2)(\omega) = \int) v_2(\omega_2) d\omega_2$$

where $K(\omega, \omega_2; v_1) \in \mathcal{D}'([0,\infty) \times [0,\infty))$ for each v_1. Again, restricting $\bar{M}_2(\cdot, v_2)$ to $\mathcal{C}^\infty_c([0,\infty))$ we have

$$\bar{M}_2(v_1, v_2)(\omega) = \int K(\omega, \omega_1, \omega_2) v_1(\omega_1) v_2(\omega_2) d\omega_1 d\omega_2$$

for some $K_2 \in \mathcal{D}'([0,\infty) \times [0,\infty) \times [0,\infty))$, since $K(\omega, \omega_2; v_1)$ induces a map $K' : \mathcal{C}^\infty_c([0,\infty)) \to \mathcal{D}'([0,\infty) \times [0,\infty))$ given by

$$K'(v)(\omega_1, \omega_2) = K(\omega_1, \omega_2; v) \; \square$$

Example 4.1 We have seen explicit examples of the kernels in example 3.1; for a linear system we have

$$K_1(\omega_1, \omega_2) = \delta(\omega_1 - \omega_2) G(i\omega_1)$$

and for a bilinear system

$$K_p(\omega_1, \cdots, \omega_{p+1}) = \delta(\omega_1 - \omega_2)(i\omega_2 - A)^{-1} D (i\omega_3 - A)^{-1} D \cdots D (i\omega_{p+1} - A)^{-1} b. \; \square$$

5. Time-Varying Bilinear Systems

In this section we shall consider the time-varying bilinear system

$$\dot{x} = A(t)x + uD(t)x + b(t)u \; , \; x(0) = x_0 \in L^2(\Omega) \tag{5.1}$$

where $A(t)$ is a sectorial operator for each $t \geq 0$. Since the convolution algebra is associative we can write the Fourier transform of this equation in the form

$$i\omega X = \tilde{A} * X + \frac{1}{(2\pi)^2} U * \tilde{D} * X + \frac{1}{2\pi} \tilde{b} * U + x(0) \tag{5.2}$$

where $\tilde{A} = \mathcal{F}(A(t)), \tilde{D} = \mathcal{F}(D(t))$ and $\tilde{b} = \mathcal{F}(b(t))$, assuming they exist. Thus, for example, we assume that $\tilde{A}$ exists in the strong sense:

$$\int_0^\infty A(t) v e^{-i\omega t} dt$$

exists in $L^2(\Omega)$ for all $v \in \cap_{t \geq 0} \mathcal{D}(A(t))$.

Consider the equation

$$(\Gamma X)(\omega) \doteq i\omega X(\omega) - (\tilde{A} * X)(\omega) = S(\omega) \tag{5.3}$$

for X in terms of S. In the time domain we have

$$\dot{x} - A(t)x = s(t) \ , \ x(0) = 0$$

so that

$$x(t) = \int_0^t \Phi(t, \tau) s(\tau) d\tau$$

where Φ is the evolution operator generated by $A(t)$. Thus,

$$\|x(t)\|_{L^2(\Omega)} \leq \int_0^t \|\Phi(t,\tau)\|_{\mathcal{L}(L^2(\Omega))} \|s(\tau)\| \, d\tau \ .$$

Now assume that

$$\|\Phi(t,\tau)\|_{\mathcal{L}(L^2(\Omega))} \leq C e^{\theta(t-\tau)}.$$

Then, if $\varepsilon > \theta$, we have

$$e^{-\varepsilon t} \|x(t)\|_{L^2(\Omega)} \leq C \int_0^t e^{(-\varepsilon+\theta)(t-\tau)} e^{-\varepsilon \tau} \|s(\tau)\| \, d\tau \ .$$

Let $w = e^{-\varepsilon t}$ and let $H_w \doteq L^2_w([0,\infty); L^2(\Omega))$ denote the space of all measurable functions $x(t)$ such that

$$\|x\|_{H_w} = \|x(t)\|_{L^2_w([0,\infty);L^2(\Omega))} \doteq \left(\int_0^\infty e^{-2\varepsilon t} \|x(t)\|^2_{L^2(\Omega)} \, dt \right)^{\frac{1}{2}} < \infty \ .$$

Then $\mathcal{F}$ is an isomorphism of H_w to $\tilde{H}_w$ and by Parseval's theorem

$$\|x\|_{H_w} = \left(\int_{-\infty}^{\infty} X_\varepsilon^2(\omega)d\omega\right)^{\frac{1}{2}} \doteq \|X_\varepsilon\|_{\tilde{H}_w}$$

where

$$X_\varepsilon(\omega) \doteq \int_0^\infty e^{-\varepsilon t}\, \|x(t)\|_{L^2(\Omega)}\, e^{-i\omega t} d\omega \ .$$

Hence we have proved

Lemma 5.1 The operator Γ defined in (5.3) maps $\tilde{H}_w$ into itself, is invertible and

$$\left\|\Gamma^{-1}\right\|_{\mathcal{L}(\tilde{H}_w)} \leq C\int_0^\infty e^{(-\varepsilon+\theta)t} dt \ . \ \square$$

From (5.2) we have

$$\left(I - \frac{1}{(2\pi)^2}\Gamma^{-1}U * \tilde{D}*)\right) X = \frac{1}{2\pi}\Gamma^{-1}\tilde{b} * U + \Gamma^{-1}x(0) \ .$$

If $X \in \tilde{H}_w$ we have, for $K \doteq \frac{1}{(2\pi)^2}\Gamma^{-1}U * \tilde{D}*$,

$$\|KX\|_{\tilde{H}_w} \leq \frac{1}{(2\pi)^2}\left\|\Gamma^{-1}\right\|_{\mathcal{L}(\tilde{H}_w)} \|U\|_{L^1(-\infty,\infty)} \left\|\tilde{D}\right\|_{L^1(-\infty,\infty;\mathcal{L}(L^2(\Omega)))} \|X\|_{\tilde{H}_w}$$

and so, if

$$\frac{1}{(2\pi)^2}\frac{C}{\varepsilon-\theta}\|U\|_{L^1(-\infty,\infty)} \left\|\tilde{D}\right\|_{L^1(-\infty,\infty;\mathcal{L}(L^2(\Omega)))} < 1$$

we have

$$\|K\|_{\mathcal{L}(\tilde{H}_w)} < 1 \ .$$

Hence, as in section 3, we have

$$X = (I + K + K^2 + K^3 + \cdots)\left(\frac{1}{2\pi}\Gamma^{-1}\tilde{b} * U + \Gamma^{-1}x(0)\right) \ .$$

Consider the term

$$\xi_p \doteq \frac{1}{2\pi}K^p(\Gamma^{-1}\tilde{b} * U) \ .$$

In order to express this in integral form, note that we have the maps

$$\begin{aligned} C_c^\infty([0,\infty)\otimes L^2(\Omega)) &\hookrightarrow L_w^2([0,\infty);L^2(\Omega)) \xrightarrow{\Gamma^{-1}} L_w^2([0,\infty);L^2(\Omega)) \\ &\hookrightarrow \mathcal{D}'([0,\infty)\otimes L^2(\Omega)) \end{aligned}$$

which are continuous and so, by the kernel theorem, their composition is given by a kernel. Since the first and last maps are injections, this induces a kernel representation on Γ^{-1}. We write

$$(\Gamma^{-1}X)(\omega) = \int_{-\infty}^{\infty} \gamma(\omega,\omega_1)X(\omega_1)d\omega_1 \; ,$$

where $\gamma \in \mathcal{D}(([0,\infty)\times[0,\infty))\otimes L^2(\Omega))$. We have

$$\begin{aligned}
\xi_p &= \frac{1}{2\pi}K^{p-1}\Gamma^{-1}(U*\tilde{D}*)\Gamma^{-1}\tilde{b}*U \\
&= \frac{1}{2\pi}K^{p-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\gamma(\omega,\omega_5)\tilde{D}(\omega_5-\omega_4-\omega_3)\gamma(\omega_3,\omega_2) \\
&\quad \times\tilde{b}(\omega_2-\omega_1)U(\omega_1)U(\omega_5)d\omega_1 d\omega_2 d\omega_3 d\omega_4 d\omega_5 \\
&= \cdots \\
&= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}}_{3p+2}\gamma(\omega,\omega_{3p+2})\tilde{D}(\omega_{3p+2}-\omega_{3p+1}-\omega_{3p})\gamma(\omega_{3p},\omega_{3p-1}) \\
&\quad \times\tilde{D}(\omega_{3p-1}-\omega_{3p-2}-\omega_{3p-3})\cdots\gamma(\omega_6,\omega_5)\tilde{D}(\omega_5-\omega_4-\omega_3)\gamma(\omega_3,\omega_2) \\
&\quad \times\tilde{b}(\omega_2-\omega_1)U(\omega_{3p+2})U(\omega_{3p-1})\cdots U(\omega_5)U(\omega_1)d\omega_1\cdots d\omega_{3p+2} \, .
\end{aligned}$$

Here, the kernel is to be interpreted in terms of the tensor product of distributions.

6. General Nonlinear Systems

We now come to consider the gerenal finite-dimensional nonlinear system

$$\begin{aligned}
\dot{x} &= f(x) + ug(x) \\
y &= h(x)
\end{aligned} \tag{6.1}$$

where f, g are analytic vector fields defined on $\mathbb{R}^n$, and h is an analytic function defined on $\mathbb{R}^n$. For simplicity of exposition we shall assume that u and y are scalars; the general case is treated in the same way. Also we suppose that $f(0) = 0$.

We shall use the Carleman linearization method (see [1]) to write (6.1) as an infinite-dimensional bilinear system. Thus, we define

$$\phi_{i_1\cdots i_n} = x_1^{i_1}\cdots x_n^{i_n} \; . \tag{6.2}$$

By Taylor's theorem,

$$f_k(x) = \sum_{\substack{\mathbf{i}=\mathbf{0}\\ \mathbf{i}\neq\mathbf{0}}}^{\infty} \frac{x^{\mathbf{i}}}{\mathbf{i}!} f_k^{(\mathbf{i})}(0)\ ,\ g_k(x) = \sum_{\substack{\mathbf{i}=\mathbf{0}\\ \mathbf{i}\neq\mathbf{0}}}^{\infty} \frac{x^{\mathbf{i}}}{\mathbf{i}!} g_k^{(\mathbf{i})}(0)\ ,\ h(x) = \sum_{\substack{\mathbf{i}=\mathbf{0}\\ \mathbf{i}\neq\mathbf{0}}}^{\infty} \frac{x^{\mathbf{i}}}{\mathbf{i}!} h^{(\mathbf{i})}(0)$$

where $\mathbf{i} = (i_1, \cdots, i_n)$, $\mathbf{i}! = i_1! \cdots i_n!$, $x^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$, $f = (f_1, \cdots, f_n)^T$ and $g = (g_1, \cdots, g_n)^T$. Hence

$$\begin{aligned}
\dot{\phi}_{\mathbf{i}} &= \dot{\phi}_{i_1 \cdots i_n} \\
&= \sum_{k=1}^{n} i_k x_1^{i_1} \cdots x_k^{i_k - 1} \cdots x_n^{i_n} \dot{x}_k \\
&= \sum_{k=1}^{n} i_k x_1^{i_1} \cdots x_k^{i_k - 1} \cdots x_n^{i_n} \left(\sum_{\substack{\mathbf{j}=\mathbf{0}\\ \mathbf{j}\neq\mathbf{0}}}^{\infty} \frac{x^{\mathbf{j}}}{\mathbf{j}!} f_k^{(\mathbf{j})}(0) + u \sum_{\substack{\mathbf{j}=\mathbf{0}\\ \mathbf{j}\neq\mathbf{0}}}^{\infty} \frac{x^{\mathbf{j}}}{\mathbf{j}!} g_k^{(\mathbf{j})}(0) \right) \\
&= \sum_{k=1}^{n} \sum_{\substack{\mathbf{j}=\mathbf{0}\\ \mathbf{j}\neq\mathbf{0}}}^{\infty} \frac{i_k}{\mathbf{j}!} x^{\mathbf{i}+\mathbf{j}-\mathbf{1}_k} f_k^{(\mathbf{j})}(0) + u \sum_{k=1}^{n} \sum_{\substack{\mathbf{j}=\mathbf{0}\\ \mathbf{j}\neq\mathbf{0}}}^{\infty} \frac{i_k}{\mathbf{j}!} x^{\mathbf{i}+\mathbf{j}-\mathbf{1}_k} g_k^{(\mathbf{j})}(0) \\
&= \sum_{\mathbf{j}=\mathbf{i}}^{\infty} \sum_{k=1}^{n} \frac{i_k}{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)!} f_k^{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)}(0) \phi_{\mathbf{j}} + \sum_{\mathbf{j}=\mathbf{i}}^{\infty} \sum_{k=1}^{n} \frac{i_k}{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)!} g_k^{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)}(0) \phi_{\mathbf{j}}
\end{aligned}$$

where $\mathbf{1}_k = (0, \cdots, 1, \cdots, 0)$ with 1 in the k^{th} place. Let $\phi = (\phi_{i_1, \cdots, i_n})$ denote the infinite-dimensional rank-n tensor with components $\phi_{i_1, \cdots, i_n}$, $\mathbf{i} \geq \mathbf{0}$, and let A and B be the tensor operators defined by

$$\begin{aligned}
(A\phi)_{\mathbf{i}} &= \sum_{\mathbf{j}=\mathbf{i}}^{\infty} \sum_{k=1}^{n} \frac{i_k}{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)!} f_k^{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)}(0) \phi_{\mathbf{j}} \\
(B\phi)_{\mathbf{i}} &= \sum_{\mathbf{j}=\mathbf{i}}^{\infty} \sum_{k=1}^{n} \frac{i_k}{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)!} g_k^{(\mathbf{j}-\mathbf{i}+\mathbf{1}_k)}(0) \phi_{\mathbf{j}}
\end{aligned}$$

Then equation (6.1) can be written in the form

$$\begin{aligned}
\dot{\phi} &= A\phi + uB\phi \\
y &= c\phi
\end{aligned} \tag{6.3}$$

where

$$c\phi = \sum_{\mathbf{i}=\mathbf{0}}^{\infty} \frac{h^{(\mathbf{i})}(0)}{\mathbf{i}!} \phi_{\mathbf{i}}\ .$$

In order to define the 'size' of operators involving A and B we introduce
Definition 6.1 Let $\mathcal{T}$ be the set of all tensors

$$\mathcal{T} = \{(r_{i_1,\cdots,i_n}) : r_{i_1,\cdots,i_n} \in \mathbb{R} \ , \mathbf{i} \geq \mathbf{0}\}$$

such that the norm

$$\left\|(r_{i_1,\cdots,i_n})\right\| = \left(\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{|r_{i_1,\cdots,i_n}|^2}{(i_1! \cdots i_n!)} \right)^{\frac{1}{2}} < \infty \ . \ \square$$

Lemma 6.1 $\mathcal{T}$ is a Banach space. □
Since

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \frac{(x_1^{i_1} \cdots x_n^{i_n})^2}{(i_1! \cdots i_n!)} = \exp(x_1^2 + \cdots + x_n^2)$$

we see that tensors of the form (6.2) belong to a subset $\mathcal{S}$ (not, of course, a linear subspace) of $\mathcal{T}$.

Suppose that P is a map from $\mathcal{S}$ into $\mathcal{S}$. We define the **norm** of P by

$$||P|| = \sup_{\substack{\phi \in \mathcal{S} \\ \phi \neq 0}} \frac{||P\phi||}{||\phi||} \ .$$

Now consider the unforced equation

$$\dot{x} = f(x) \ , \ x(0) = x_0 \tag{6.4}$$

which has the linear representation

$$\dot{\phi} = A\phi \ .$$

Suppose that $f(0) = 0$ that $x = 0$ is an exponentially stable equilibrium point of (6.4) and that $\mathcal{B} \ni 0$ is a subset of the basin of uniform exponential stability of (6.4). It can be seen that the solution of (6.4), for $x_0 \in \mathcal{B}$ is given by

$$\phi(t) = e^{At}\phi_0 \ , \ \phi_0 = (x^{\mathbf{i}}(0))$$

where $\phi(t) = (x^{\mathbf{i}}(t))$ and $x(t)$ is the solution of (6.4). Since $x_0 \in \mathcal{B}$ we have

$$|x_i(t)| \leq L \, ||x_0|| \, e^{-at} \ , \ \mathit{for\ each}\ i \in \{1, \cdots, n\}$$

for some $L, a > 0$. Hence,

$$\begin{aligned}\left\|e^{At}\right\| &= \sup_{\phi_0 \neq 0} \frac{\left\|e^{At}\phi_0\right\|}{\|\phi_0\|} \\ &\leq e^{-at} \exp\left(\frac{Ln\,\|x_0\|^2}{2} - \frac{\|x_0\|^2}{2}\right) .\end{aligned}$$

We have therefore proved:

Theorem 6.1 *If the conditions above hold and $\mathcal{B}$ is bounded then*

$$\left\|e^{At}\right\|_{\mathcal{O}(\mathcal{S},\mathcal{S})} \leq Me^{-at}$$

for some M where $\mathcal{O}(\mathcal{S},\mathcal{S})$ is the space of (nonlinear) operators from $\mathcal{S}$ into $\mathcal{S}$. □

Remark If the system (6.4) is only asymptotically stable (but not exponentially stable) a similar inequality holds for some function $v(t)$ replacing e^{-at} with $v(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Corollary 6.1 *The Fourier transform*

$$\mathcal{F}(e^{At}) \doteq \int_0^\infty e^{At}e^{-i\omega t}dt$$

exists in $\mathcal{O}(\mathcal{S},\mathcal{S})$ with

$$\left\|\mathcal{F}(e^{At})\right\|_{\mathcal{O}(\mathcal{S},\mathcal{S})} \leq \frac{M}{a} . \ \square$$

From (6.3) we have

$$\phi(t) = e^{At}\phi(0) + \int_0^t u(s)e^{A(t-s)}B\phi(s)ds .$$

Hence,

$$\Phi(i\omega) = \mathcal{F}(e^{At})(i\omega)\phi(0) + \frac{1}{2\pi}\mathcal{F}(e^{At})(i\omega)(U(i\omega) * B\Phi(i\omega))$$

and so

$$\left(I - \frac{1}{2\pi}\mathcal{F}(e^{At})(i\omega)U(i\omega) * B\right)\Phi(i\omega) = \mathcal{F}(e^{At})(i\omega)\phi(0) . \tag{6.5}$$

Let

$$K = \frac{1}{2\pi}\mathcal{F}(e^{At})(i\omega)U(i\omega) * B .$$

Then

$$\|K\|_{L^2([0,\infty);\mathcal{S})} \leq \frac{M}{2\pi a} \|U\|_{L^1(0,\infty)} \|B\|_{\mathcal{S}} \ .$$

If

$$\|U\|_{L^1(0,\infty)} < \frac{2\pi a}{M \|B\|_{\mathcal{S}}}$$

then

$$\|K\|_{L^2([0,\infty);\mathcal{S})} < 1$$

and we have, from (6.5),

$$\Phi(i\omega) = (I + K + K^2 + \cdots)\mathcal{F}(e^{At})(i\omega)\phi(0) \ .$$

In a similar way in which (3.5) was obtained, we have

$$\begin{aligned} K^p\mathcal{F}(e^{At})(i\omega)\phi(0) \;=\; & \frac{1}{(2\pi)^p} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{p+1} \delta(\omega - \omega_{p+1}) R(i\omega_{p+1}; A) B \\ & R(i\omega_p; A)B \cdots BR(i\omega_1; A)\phi(0)U(\omega_{p+1} - \omega_p)U(\omega_p - \omega_{p-1}) \times \\ & \cdots U(\omega_2 - \omega_1) d\omega_1 \cdots d\omega_{p+1} \end{aligned} \tag{6.6}$$

where we have written

$$R(i\omega; A) = \mathcal{F}(e^{At})(i\omega) \ .$$

Example 6.1 In order to illustrate the theory in this section, we shall consider the scalar system

$$\dot{x} = -x^3 + xu$$

keeping it as simple as possible to make the computations clear. Thus,

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -2 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & -3 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and so

$$e^{At} = \begin{pmatrix} 1 & 0 & -t & 0 & 3\frac{t^2}{2!} & 0 & -\frac{3.5}{3!}t^3 & 0 & \frac{3.5.7}{4!}t^4 & 0 & \cdots \\ 0 & 1 & 0 & -2t & 0 & \frac{2.4}{2!}t^2 & 0 & -\frac{2.4.6}{3!}t^3 & 0 & \frac{2.4.6.8}{4!}t^4 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} .$$

Thus,

$$\mathcal{F}(e^{At}) = \begin{pmatrix} \frac{1}{i\omega} & 0 & -\frac{1}{(i\omega)^2} & 0 & 3\frac{1}{(i\omega)^3} & 0 & -5.3\frac{1}{(i\omega)^4} & 0 & \cdots \\ 0 & \frac{1}{i\omega} & 0 & -\frac{2}{(i\omega)^2} & 0 & 2 & 0 & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} . \qquad (6.7)$$

Note that the unforced system

$$\dot{x} = -x^3 \ , \ x(0) = x_0$$

has solution given by

$$x^2(t) = \frac{x_0^2}{1 + 2tx_0^2}$$

and it follows that $\left\| e^{At} \right\| \leq \frac{1}{(1+2tx_0^2)^{1/2}}$. A bound on $\left\| \mathcal{F}(e^{At}) \right\|$ can easily be found as

$$\begin{aligned} \left\| \mathcal{F}(e^{At}) \right\| &\leq 1 + \left| \int_0^\infty \frac{x_0^2 \cos \omega t}{(1+2tx_0^2)^{3/2}} dt \right| + \left| \int_0^\infty \frac{x_0^2 \sin \omega t}{(1+2tx_0^2)^{3/2}} dt \right| \\ &\doteq \alpha \end{aligned}$$

Hence if

$$\|U\|_{L^1(0,\infty)} < \frac{2\pi}{\alpha \|B\|_S} = \frac{2\pi}{\alpha |x_0|} \ , \ (\text{since } \|B\|_S = |x_0|)$$

the series in (6.6) exists with

$$B = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 3 & \\ 0 & & & \ddots \end{pmatrix}$$

and $R(i\omega; A)$ given by (6.7).□

7. Conclusions

In this paper we have considered the general frequency domain theory of nonlinear systems and have shown that by working directly in the frequency domain the need for complicated 'association of variables' is not necessary. We have seen that Schwartz' kernel theorem gives a sequence of distributional kernels which have a direct relation to the kernels obtained from the Volterra series approach in the time domain. By applying global linearization techniques we can even derive explicit expressions for these kernels. The application of these methods to resonances in general nonlinear systems is an important step in understanding the frequency domain behaviour of nonlinear equations. We shall examine this in a future paper.

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