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MODE ANALYSIS OF A TUBULAR STRUCTURE OF COUPLED FIFTH-POWER NON-LINEAR OSCILLATORS FOR LARGE-INTESTINAL MODELLING

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ABSTRACT

A tubular structure of coupled non-linear oscillators provides a natural extension of hypothesized models for the electrical slow-wave activity of the mammalian intestines. In this paper fifth-power van der Pol dynamics are used for the oscillator units providing a zero stable state, which may be important in terms of the human large-intestine where periods of electrical silence may occur. The matrix Krylov-Bogolioubov linearisation method is used to provide mode analysis of this structure. Although more extended than for the conventional third-power case the theoretical analysis reveals a rich mode behaviour. After development of the mxn general cases, two particular examples of 3x4 and 4x4 structures are given. The theoretical results for the 3x4 case compare favourably with an experimental investigation using electronic implementation of van der Pol type oscillators.

1. Introduction

Coupled non-linear oscillators are being investigated increasingly both from a theoretical stand-point and for applicative modelling purposes. In the analysis presented in this paper the motivation is the electrical slow-wave activity of the mammalian gastro-intestinal tract. Since the initial conception of a mutually coupled non-linear oscillator model for slow-wave rhythms suggested by Nelsen and Becker in 1968 a number of modelling studies have been performed. Thus, a 1-dimensional chain of coupled van der Pol oscillators has been investigated for the small-intestine (Sarna et al, 1971), and a 2-dimensional array proposed for a gastric model (Sarna et al, 1972). For the human large-intestine, periods of electrical silence have prompted the hypothesis of an oscillator unit dynamic based on a fifth-power van der Pol equation (Linkens et al, 1976). In a recent paper a tubular structure for small-intestinal modelling, was analysed (Alian and Linkens, 1982), while the equivalent fifth-power tubular structure is studied in this paper.

On the theoretical side, the mode analysis of mutually coupled oscillators has also been steadily advancing. Of particular interest are computer-aided harmonic balancing methods and a matrix Krylov-Bovolioubov linearisation technique. The former method has been used on third-power chains and arrays. The latter method has been used with fifth-power dynamics for the two-oscillator case (Datardina and Linkens, 1978) and for a 1-dimensional chain (Endo and Ohta, .). It is this matrix Krylov-Bogolioubov method which is used in this paper to extend mode analysis to a tubular structure of fifth-power oscillators with reference to human large-intestinal electrical behaviour.

In Section 2 the fundamental mode equations are developed using the matrix decoupling and linearisation approach. In Section 3 the method of averaging is used to calculate the stationary amplitude values. Mode stability is then investigated in Section 4 for three cases viz., zero state, single modes, double nonresonant modes. Two particular examples are considered in Section 5, one of which is experimentally investigated via electronic implementaion in Section 6.

2. Derivation of the Fundamental Mode Equation

The proposed structure is a cylindrical structure shown in Fig. 1 where the unit oscillator has a fifth power nonlinear characteristic which is described by

$$I_{ij}\ (V_{ij}) = g_1 V_{ij} - g_3 V_{ij}^3 + g_5 V_{ij}^5$$

$$(g_1,\ g_3 \text{ and } g_5 > 0) \tag{1}$$

where i represents the location of the oscillator in each ring

j represents the location of the ring in the structure

$i = 1,2, \ldots, m; \quad j = 1,2, \ldots, n$

i.e. m is the number of oscillators in each ring and n is the number of rings in the whole structure.

Following the same procedures as in a previous paper (Alian and Linkens, 1982), the system equation can be written as

$$g_1 \frac{dV_{ij}}{dt} - 3g_3 V_{ij}^2 \frac{dV_{ij}}{dt} + 5g_5 V_{ij} \frac{dV_{ij}^4}{dt} + C \frac{d^2V_{ij}}{dt^2} + \frac{1}{L} V_{ij}$$

$$= \frac{1}{L_c} (V_{(i-1)j} - V_{ij}) + \frac{1}{L_c} (V_{i(j+1)} - V_{ij}) - \frac{1}{L_c} (V_{ij} - V_{(i+1)j})$$

$$- \frac{1}{L_c} (V_{ij} - V_{i(j-1)})$$

which can be arranged to be

$$C \frac{d^2V_{ij}}{dt^2} + (g_1 - 3g_3V_{ij}^2 + 5g_5V_{ij}^4) \frac{dV_{ij}}{dt} + (\frac{1}{L} + \frac{4}{L_c}) V_{ij}$$

$$- \frac{1}{L_c} V_{(i-1)j} - \frac{1}{L_c} V_{i(j+1)} - \frac{1}{L_c} V_{(i+1)j} - \frac{1}{L_c} V_{i(j-1)} = 0$$

Dividing the above equation by the value of the capacitor C and rearranging gives

$$\frac{d^2 V_{ij}}{dt^2} + \frac{g_1}{C} \left(1 - \frac{3g_3}{g_1} V_{ij}^2 + \frac{5g_5}{g_1} V_{ij}^4\right) \frac{dV_{ij}}{dt} + \left(\frac{1}{CL} + \frac{4}{CL_c}\right) V_{ij}$$

$$- \frac{1}{CL_c} V_{(i-1)j} - \frac{1}{CL_c} V_{(i+1)j} - \frac{1}{CL_c} V_{i(j-1)} - \frac{1}{CL_c} V_{i(j+1)} = 0 \qquad (2)$$

Substituting

$$V_{ij} = \sqrt[4]{\frac{g_1}{5g_5}}\; x_{ij} \qquad (3)$$

where x_{ij} is the normalised voltage in the structure, and its boundary values which represent the tube structure are given by

$$x_{oj} = x_{mj} \quad ; \quad x_{ij} = x_{(m+1)j} \qquad (4)$$

$$x_{io} = x_{i1} \quad ; \quad x_{in} = x_{i(n+1)} \qquad (5)$$

So, equation (2) becomes

$$\frac{d^2 x_{ij}}{dt^2} + \frac{g_1}{C} (1 - \beta x_{ij}^2 + x_{ij}^4) \frac{dx_{ij}}{dt} - \frac{1}{CL_c} x_{(i-1)j}$$

$$+ \left(\frac{1}{2CL} + \frac{2}{CL_c}\right) x_{ij} - \frac{1}{CL_c} x_{(i+1)j} - \frac{1}{CL_c} x_{i(j-1)} + \left(\frac{1}{2CL} + \frac{2}{CL_c}\right)$$

$$x_{ij} - \frac{1}{CL_c} x_{i(j+1)} = 0 \qquad (6)$$

where β is a function given by the parameters of the nonlinear characteristic of each oscillator as

$$\beta = \frac{3g}{\sqrt{5g_1 g_5}} \tag{7a}$$

Then, substituting

$$\tau = \sqrt{\frac{1}{2CL} + \frac{1}{CL_c}}\; t \tag{7b}$$

into (6) gives

$$\frac{d^2 x_{ij}}{d\tau^2} + \frac{g_1/C}{\sqrt{\frac{1}{2CL} + \frac{1}{CL_c}}} (1-\beta x_{ij}^2 + x_{ij}^4) \frac{dx_{ij}}{d\tau} - \frac{1}{1+\frac{L_c}{2L}} x_{(i-1)j}$$

$$+ \frac{\frac{1}{2CL} + \frac{2}{CL_c}}{\frac{1}{2CL} \quad \frac{1}{CL_c}} x_{ij} - \frac{1}{1+\frac{L_c}{2L}} x_{(i+1)j} - \frac{1}{1+\frac{L_c}{2L}} x_{i(j-1)}$$

$$+ \frac{\frac{1}{2CL} + \frac{2}{CL_c}}{\frac{1}{2CL} \quad \frac{1}{CL_c}} x_{ij} - \frac{1}{1+\frac{L_c}{2L}} x_{i(j+1)} = 0 \tag{8}$$

which can be written as

$$\frac{d^2 x_{ij}}{d\tau^2} + \xi (1-\beta\, x_{ij}^2 + x_{ij}^4) \frac{dx_{ij}}{d\tau} - \alpha x_{(i-1)j} + (1+\alpha)x_{ij} - \alpha\, x_{(i+1)j}$$

$$- \alpha\, x_{i(j+1)} + (1+\alpha)x_{ij} - \alpha\, x_{i(j+1)} = 0 \tag{9}$$

where $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

$$\alpha = \frac{2L}{2L+L_c} \qquad \xi = \frac{g_1}{\sqrt{\frac{C}{2L} + \frac{C}{L_c}}}$$

The above equation (9) can be expressed in matrix form by defining three matrices X, X_c and X_f

$$X = [x_{ij}] \quad (10a)$$

$$X_c = [x_{ij}^3] \quad (10b)$$

$$X_f = [x_{ij}^5] \quad (10c)$$

where each matrix is of order mxn.

Thus, (9) can be written in the matrix differential equation

$$X^{\cdot\cdot} + BX + XD = -(\xi X^{\cdot} - \frac{1}{3}\,\xi\beta X_c^{\cdot} + \frac{1}{5}\,\xi\, X_f^{\cdot}) \quad (11)$$

where B and D are two matrices given by

$$B = \begin{bmatrix} 1+\alpha & -\alpha & & & -\alpha \\ -\alpha & 1+\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 1+\alpha & -\alpha \\ -\alpha & & & -\alpha & 1+\alpha \end{bmatrix} \quad D = \begin{bmatrix} 1 & -\alpha & & & 0 \\ -\alpha & 1+\alpha & -\alpha & & \\ & \ddots & \ddots & \ddots & \\ & & -\alpha & 1+\alpha & -\alpha \\ 0 & & & -\alpha & 1 \end{bmatrix}$$

Applying the orthogonal transformation

$$X = P\,Y\,Q^T$$

into the matrix differential equation (11), it becomes

$$PY^{\cdot\cdot}Q^T + BPYQ^T + PYQ^TD = -\,(\xi\,PY^{\cdot}Q^T - \frac{1}{3}\,\beta\xi\,X_c^{\cdot} + \frac{1}{5}\xi\,\,X_f^{\cdot}) \quad (12)$$

and then multiplying by P^T from the left hand side and by Q from the right hand, we have

$$Y^{\cdot\cdot}+(P^TBP)Y+Y(Q^TDQ) = -(\xi Y^{\cdot}-\beta\xi P^TX_c^{\cdot}Q+\frac{1}{5}\xi\,P^TX_f^{\cdot}Q) \quad (13)$$

which is the fundamental equation. To solve it, we have first to solve the unperturbed differential equation in the same way as described in a previous paper (Alian & Linkens, 1982). Therefore, the elements of the two orthogonal matrices P and Q are given by (19) and (23) in Alian & Linkens (1982).

The mode y_{ij} is then given by

$$y_{ij} = A_{ij}\sin\,(\omega_{ij}\tau + \phi_{ij})\;;\quad \phi_{ij}\text{ is arbitrary} \quad (14)$$

where the frequency ω_{ij} is given by

$$\omega_{ij} = \sqrt{2}\{1+\alpha\left[1-\cos\frac{2(i-1)\pi}{m} - \cos\frac{(j-1)\pi}{n}\right]\} \tag{15}$$

To linearise the nonlinear terms of (13) using the equivalent linearization technique developed by Kryloff and Bogoliuboff, the whole equation should firstly be transferred into the Y-space. So, the elements of the nonlinear terms $P^T X_c Q$ and $P^T X_f Q$ must be written in the form of linear combinations of y_{ij} by linearising the higher nonlinear terms x_{ab}^3 and x_{ab}^5.

The element h_{ij} of the matrix $(H=P^T X_c Q)$ is expressed as a linear combination of the element y_{ij} such that

$$h_{ij} = \sum_{a=1}^{m} \sum_{b=1}^{n} p_{ai}\, q_{bj}\, x_{ab}^3$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} \eta_{ij}(k,l)\, y_{kl} \tag{16}$$

The same procedures are used to linearise the fifth power nonlinear element which is

$$Z = P^T X_f Q = \left[z_{ij}\right]_{mn} \tag{17}$$

Therefore, the element z_{ij} can be expressed from (10c) and (17) as

$$z_{ij} = \sum_{a=1}^{m} \sum_{b=1}^{n} (p_{ai})\, (q_{bj})\, x_{ab}^5 \tag{18}$$

The element x_{ab}^5 can be linearised as in Appendix I.

Therefore, the element z_{ij} from the matrix Z can be written as

$$z_{ij} = \sum_{k=1}^{m} \sum_{l=1}^{n} \zeta_{ij}(k,l)\, y_{kl} \tag{19}$$

where $\zeta_{ij}(k,l)$ is given by the elements of the orthogonal matrices P and Q as

$$\zeta_{ij}(k,l) = \sum_{a=1}^{m} \sum_{b=1}^{n} p_{ai}\, q_{bj}\, L_{ij}(k,l) \tag{20}$$

where $L_{ij}(k,l)$ is calculated also from the elements of the orthogonal matrices P and Q and by using the values of the stationary amplitudes of modes, i.e. substituting (I-5) into (19), the value of the parameter $\zeta_{ij}(k,l)$ is given.

Now, substituting the element h_{ij} of the matrix $(H=P^T X_c Q)$ and the element z_{ij} of the matrix $(Z=P^T X_f Q)$ into the fundamental equation (13), then the equivalent linearised equation can be obtained as

$$\ddot{y}_{ij} + \omega_{ij}\, y_{ij} = -\{\xi \dot{y}_{ij} - \frac{1}{3}\xi\beta \sum_{k=1}^{m} \sum_{l=1}^{n} \dot{h}_{ij}(k,l)$$

$$- \frac{1}{5}\xi \sum_{k=1}^{m} \sum_{l=1}^{n} \dot{z}_{ij}(k,l)\} \qquad (21)$$

substituting (14) and (I-5) into (20) gives

$$\ddot{y}_{ij} + \omega_{ij}\, y_{ij} = -\{\xi\, \dot{y}_{ij} - \frac{1}{3}\xi\beta \sum_{k=1}^{m} \sum_{l=1}^{n} \eta_{ij}(k,l)\, \dot{y}_{kl}$$

$$+ \frac{1}{5}\xi \sum_{k=1}^{m} \sum_{l=1}^{n} \xi_{ij}(k,l)\, \dot{y}_{kl}\} \qquad (22)$$

Supposing that the left hand side of the above equation has a resonance centred around ω_{ij}, so that the mode frequencies which are not equal to ω_{ij} have little effect upon the solution provided that each mode frequency is separated enough, or that the Q-value of the resonance is fairly high, we can ignore all the y_{kl} terms of the right hand side except y_{ij}.

Therefore, the terms $\dot{h}_{ij}(k,l)$ and $\dot{z}_{ij}(k,l)$ in (22) can be replaced by $\dot{h}_{ij}(i,j)$ and $\dot{z}_{ij}(i,j)$ respectively, i.e. (22) becomes

$$\ddot{y}_{ij} + \omega_{ij}^2\, y_{ij} = -[\xi \dot{y}_{ij} - \frac{1}{3}\xi\beta\, \dot{h}_{ij}(i,j) + \frac{1}{5}\xi \dot{z}_{ij}(i,j)] \qquad (23)$$

where $h_{ij}(i,j)$ and $z_{ij}(i,j)$ are given by (16) and (24) respectively.
The elements $h_{ij}(i,j)$ and $z_{ij}(i,j)$ can be replaced for simplicity by h_{ij} and z_{ij} respectively, and $\eta_{ij}(i,j)$ and $\zeta_{ij}(i,j)$ can be replaced by η_{ij} and ζ_{ij}.

Thus the equivalent linearised equation (23) of the mode y_{ij} can be written as

$$\ddot{y}_{ij} + \omega^2_{ij}\, y_{ij} = -\left[\xi\, \dot{y}_{ij} - \frac{1}{3}\,\xi\beta\eta_{ij}\,\dot{y}_{ij} + \frac{1}{5}\,\zeta_{ij}\quad \dot{y}_{ij}\right] \tag{24}$$

where η_{ij} is given by

$$\eta_{ij} = \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi_{mn}(i,j,k,l)\; A^2_{kl} - \frac{3}{4}\psi_{mn}(i,j,i,j)\; A^2_{ij} \tag{25}$$

and ζ_{ij} is given by substituting the value L_{ij} in (I-5) into (20) to be

$$\zeta_{ij} = \frac{5}{8}\,\psi^*_{mn}(i,j,i,j)\; U^2_{ij} + \frac{15}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi^*_{mn}(i,j,k,l)\; U^2_{kl}$$

$$+ \frac{15}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi^*_{mn}(k,l,i,j) U_{ij}\; U_{kl}$$

$$+ \frac{15}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} \psi^{**}_{mn}(i,j,k,l,r,s) U_{kl}\; U_{rs}$$

where $(k,l) \neq (r,s) \neq (i,j)$ (26)

The value $\psi_{mn}\,(i,j,k,l)$ is exactly the same as that obtained in a previous paper (Alian & Linkens, 1982) while the value $\psi^*_{mn}(i,j,k,l)$ is given by

$$\psi^*_{mn}(i,j,k,l) = \sum_{a=1}^{m} \sum_{b=1}^{n} (p^2_{ai}) q^2_{bj}) (p^4_{ak}) (q^4_{bl})$$

$$= \sum_{a=1}^{m} (p^2_{ai}) (p^4_{ak}) . \sum_{b=1}^{n} (q^2_{bj}) (q^4_{bl}) \tag{27}$$

The value $\psi^{**}_{mn}(i,j,k,l,r,s)$ is given by

$$\psi^{**}_{mn}(i,j,k,l,r,s) = \sum_{a=1}^{m} \sum_{b=1}^{n} (p^2_{ai}) (q^2_{bj}) (p^2_{ak}) (q^2_{bl}) (p^2_{ar}) (q^2_{bs})$$

$$= \sum_{a=1}^{m} (p^2_{ai}) (p^2_{ak}) (p^2_{ar}) . \sum_{b=1}^{n} (q^2_{bj}) (q^2_{bl}) (q^2_{bs}) \tag{28}$$

To facilitate the calculation of the above two parameters ψ^{*}_{mn} and ψ^{**}_{mn}, it is better to follow the same procedures as mentioned for the calculation of the parameter ψ_{mn}, i.e. (27) can be rewritten in the form

$$\psi^{*}_{mn}(i,j,k,l) = \psi^{*}_{mn}(i,k).\psi^{*}_{n}(j,l) \tag{29}$$

where

$$\psi^{*}_{m}(i,k) = \sum_{a=1}^{m} p^{2}_{ai}\, p^{4}_{ak} \quad ; \quad i,k = 1,2, \ldots, m \tag{30a}$$

which leads to the fact that $\psi^{*}_{m}(i,k)$ is given by the elements of the columns i and k in the matrix P

and

$$\psi^{*}_{n}(j.l) = \sum_{b=1}^{n} q^{2}_{bj}\, q^{4}_{bl} \quad ; \quad j,l = 1,2, \ldots, n \tag{30b}$$

which is given by the elements of the columns j and l in the matrix Q.

Similarly, ψ^{**}_{mn} can be written as

$$\psi^{**}_{mn}(i,j,k,l,r,s) = \psi^{**}_{m}(i,k,r).\ \psi^{**}_{n}(j,l,s) \tag{31}$$

where

$$\psi^{**}_{m}(i,k,r) = \sum_{a=1}^{m} p^{2}_{ai}\cdot p^{2}_{ak}\cdot p^{2}_{ar} \tag{32a}$$

which is given by the elements of the columns i,k and r in the matrix P, and

$$\psi^{**}_{n}(j,l,s) = \sum_{b=1}^{n} q^{2}_{bj}\cdot\ q^{2}_{bl}\cdot\ q^{2}_{bs} \tag{32b}$$

which is the summation of the products of the squares of the elements in the matrix Q which are in the columns j, l and s.

3. Evaluation of the Stationary Amplitude Values by Averaging the Equivalent Linearised Equation

In order to investigate the stability of modes in such tubular structure with fifth power nonlinearities, it is necessary first to determine the stationary values of amplitude A_{ij} . Following the same procedures for the tube structure with third power nonlinearities, then the averaged equations can be obtained as

$$\dot{U}_{ij} = - \xi U_{ij} \{ 1 - \frac{1}{3} \beta \eta_{ij} + \frac{1}{5} \zeta_{ij} \} \qquad (33)$$

under the assumption that the amplitudes and phases are a slowly varying functions of time in quasiharmonic approximation.

The stationary values of amplitudes can be taken by putting all the first order time derivatives in the averaged equations (33) to zero, i.e.

$$\dot{U}_{ij} = 0$$

for $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

Consequently, the values of the parameters η_{ij} and ζ_{ij} should be determined in order that the stationary values of amplitudes can be evaluated.

From equation (25) η_{ij} can be determined using the values of the parameter ψ_{mn} . The parameters ζ_{ij} can be determined by (26) using the values of the parameters ψ^{*}_{mn} and ψ^{**}_{mn} .

For the case that the number of oscillators in each ring of the structure m = 3, the values $\psi^{*}_{3}(i,k)$ are given by the equation (30a) as

$$\psi^{*}_{m}(i,k) = \psi^{*}_{3}(i,k) = \sum_{a=1}^{3} p^{2}_{ai} p^{4}_{ak}$$

where $i,k = 1,2,3$; p_{ai} is the i^{th} column in the orthogonal matrix P and p_{ak} is the k^{th} column of the same matrix.

Using the above equation (35) the values ψ^{*}_{3} (i,k) can be represented for simplicity in square matrix form of order 3 as

$$\psi_3^*(i,k) = \begin{bmatrix} \frac{1}{9} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{9} & \frac{11}{36} & \frac{1}{6} \\ \frac{1}{9} & \frac{1}{36} & \frac{1}{4} \end{bmatrix} \tag{35}$$

If the number of oscillators in each ring is m=4, then $\psi_4^*(i,k)$ is given as

$$\psi_4^*(i,k) = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{16} & \frac{1}{8} & \frac{1}{16} & \frac{1}{8} \\ \frac{1}{16} & 0 & \frac{1}{16} & \frac{1}{4} \end{bmatrix} \tag{36}$$

Also, when the number of rings is n=4, then $\psi_4^*(j,l)$ is calculated as

$$\psi_4^*(j,l) = \begin{bmatrix} \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{3}{32} \\ \frac{1}{16} & \frac{5}{32} & \frac{1}{16} & \frac{1}{32} \\ \frac{1}{16} & \frac{3}{32} & \frac{1}{16} & \frac{3}{32} \\ \frac{1}{16} & \frac{1}{32} & \frac{1}{16} & \frac{5}{32} \end{bmatrix} \tag{37}$$

If we consider a tube oscillator system which consists of four rings, each ring comprises three oscillators, i.e. m=3 and n=4, then the corresponding values of $\psi_{34}(i,j,k,l)$ are given by Table 1, while $\psi^*_{34}(i,j,k,l)$ are given from (29) as

$$\psi^*_{mn}(i,j,k,l) = \psi^*_{34}(i,j,k,l) = \psi^*_3(i,k).\ \psi^*_4(j,l)$$

For simplicity, the values ψ^*_{mn} are represented in a table similar to that representing the values ψ_{mn}. Table 2 gives the values of $\psi^*_{34}(i,j,k,l)$,

$$i,k = 1,2,3 \quad ; \quad j,l = 1,2,3,4$$

Similarly, Table 3 gives the values of $\psi_{44}(i,j,k,l)$, while Table 4 gives the values of $\psi^*_{44}(i,j,k,l)$ which corresponds to the case in which the number of oscillators in each ring m=4 and the number of rings in the structure n=4.

Substituting η_{ij} and ζ_{ij} from (25) and (26) respectively into (33) $\dot{U}_{ij}$ can be written as

$$\begin{aligned}\dot{U}_{ij} = -\zeta U_{ij}\Big[1-\beta\{\tfrac{1}{2}\sum_{k=1}^{m}\sum_{l=1}^{n}\psi_{mn}(i,j,k,l)U_{kl} - \tfrac{1}{4}\psi_{mn}(i,j,i,j)U_{ij}\}\\ +\{\tfrac{1}{8}\psi^*_{mn}(i,j,i,j)U^2_{ij} + \tfrac{3}{8}\sum_{k=1}^{m}\sum_{l=1}^{n}\psi^*_{mn}(i,j,k,l)U^2_{kl}\\ +\tfrac{3}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\psi^*_{mn}(k,l,i,j)U_{ij}U_{kl}\\ +\tfrac{3}{4}\sum_{k=1}^{m}\sum_{l=1}^{n}\sum_{r=1}^{m}\sum_{s=1}^{n}\psi^{**}_{mn}(i,j,k,l,r,s)U_{kl}U_{rs}\}\Big] \quad (38)\end{aligned}$$

which is called the averaged equation of the system.

4. Investigation of the Stability Problem

The stability of every mode of oscillation can be considered by using the averaged equation (33). The stationary states of those oscillatory modes are determined by reducing the first-order time derivatives in the averaged equations to zero. The stability of such stationary states is then determined by linearizing the average equations around the stationary values to obtain the Jacobian matrix $\left[J_{ij}(t,u) \right]$ of the structure as

$$J_{ij}(t,u) = \frac{d(U^{\cdot}_{ij})}{dU_{kl}} \tag{39}$$

which can be calculated by differentiating (34), therefore for (i,j) = (t,u)

$$J_{ij}(i,j) = -\xi\Big[1-\frac{1}{2}\beta \sum_{k=1}^{m} \sum_{l=1}^{n} \psi_{mn}(i,j,k,l)U_{kl} + \frac{3}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi^{*}_{mn}(i,j,k,l)U^{2}_{kl}$$

$$(k,l) \neq (i,j)$$

$$+ \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi^{*}_{mn}(k,l,i,j)U_{kl}U_{ij}$$

$$(k,l) \neq (i,j)$$

$$+ \frac{3}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} \psi^{*}_{mn}(i,j,k,l,r,s)U_{kl}\cdot U_{rs} \Big]$$

$$(k,l) \neq (r,s) \neq (i,j) \tag{40a}$$

but for (i,j) ≠ (t,u)

$$J_{ij}(t,u) = -\xi\Big[-\frac{1}{2}\beta\psi_{mn}(i,j,t,u)U_{ij} + \frac{3}{4}\psi^{*}_{mn}(i,j,t,u)U_{ij}U_{tu}$$

$$+ \frac{3}{4}\psi^{*}_{mn}(t,u,i,j)U^{2}_{ij} + \frac{3}{2} \sum_{k=1}^{m} \sum_{l=1}^{n} \psi^{**}_{mn}(i,j,t,u,k,l)U_{kl}U_{ij}\Big]$$

$$(k,l) \neq (i,j) \neq (t,u) \tag{40b}$$

Then, the stability of a mode is determined by examining the eigenvalues of the corresponding Jacobian matrix which are the roots of the characteristic equation

$$\left| \{ J_{ij} (t,u)\} - sI \right| = 0 \qquad (41)$$

where I is a unit matrix of order mnxmn.

4.1 Zero State

For the case of nonoscillation, i.e. the amplitude assumption is

$U_{ij} = 0$ for all $i = 1,2, \ldots, m$; $j = 1,2, \ldots, n$

$$\text{therefore } J_{ij}(t,u) = 0 \qquad \text{for } (i,j) \neq (t,u) \qquad (42a)$$

$$\text{and } J_{ij}(i,j) = -\xi \qquad \text{for } (i,j) = (t,u) \qquad (42b)$$

Substituting (42) into (41), then the characteristic equation can be reduced to

$$\prod_{i=1}^{m} \prod_{j=1}^{n} \left[J_{ij}(i,j) - sI \right] = 0 \qquad (43)$$

since the nondiagonal elements of the Jacobian matrix are all zeros, (43) becomes

$$(s + \xi)^{m+n} = 0 \qquad (44)$$

Thus, all the real parts of the eigen values of the characteristic equation are negative, which confirms that the nonoscillation state is stable in the tube structure with fifth power nonlinearities.

4.2 Single Ordinary Mode

To investigate the stability of single ordinary mode, the restriction of amplitudes can be

$$U_{iojo} \neq 0 \; ; \; U_{ij} = 0 \qquad (45)$$

for all i and j except $(i,j) = (i_o,j_o)$

i.e. the mode (i_o, j_o) is the only one which is supposed to be excited.

The stationary amplitude of the mode (i_o, j_o) can be obtained by putting $\dot{U}_{i_o j_o}$ in (33) to zero, which gives

$$1 - \frac{1}{3}\beta\,\eta_{i_o j_o} + \frac{1}{5}\zeta_{i_o j_o} = 0 \qquad (46)$$

Substituting the amplitude assumption (45) into (46) using (34), the stationary amplitude $U_{i_o j_o}$ can be derived from the condition

$$1 - \frac{1}{4}\beta\psi_{mn}(i_o, j_o, i_o, j_o)U_{i\,j} + \frac{1}{8}\psi^*_{mn}(i_o, j_o, i_o, j_o)U^2_{i_o j_o} = 0 \qquad (47a)$$

Dividing the whole equation by $(\psi^*_{mn}(i_o, j_o, i_o, j_o)/8)$ gives

$$U^2_{i_o j_o} - 2\beta\frac{\psi_{mn}(i_o, j_o, i_o, j_o)}{\psi^*_{mn}(i_o, j_o, i_o, j_o)}U_{i_o j_o} + 8/\psi^*_{mn}(i_o, j_o, i_o, j_o) = 0 \qquad (47b)$$

which is the ordinary second-order equation and therefore has the general roots

$$U_{i_o j_o} = \beta\frac{\psi_{mn}(i_o, j_o, i_o, j_o)}{\psi^*_{mn}(1_o, j_o, i_o, j_o)} \pm \sqrt{\beta^2\frac{\psi^2_{mn}(i_o, j_o, i_o, j_o)}{\psi^{*2}_{mn}(i_o, j_o, i_o, j_o)} - \frac{32}{\psi^*_{mn}(i_o, j_o, i_o, j_o)}}$$

Neglecting the solution with (-ve) sign, because it corresponds to an unstable limit cylce, the stationary value $U_{i_o j_o}$ can be given as

$$U_{i_o j_o} = \frac{\psi_{mn}(i_o, j_o, i_o, j_o)}{\psi^*_{mn}(i_o, j_o, i_o, j_o)}\left[\beta + \sqrt{\beta^2 - 8\frac{\psi^*_{mn}(i_o, j_o, i_o, j_o)}{\psi^2_{mn}(i_o, j_o, i_o, j_o)}}\right] \qquad (48)$$

Applying the orthogonal transformation into the solution (14), the structure of a single mode (i_o, j_o) becomes

$$x_{ij} = p_{ii_o} q_{jj_o} \sqrt{U_{i_oj_o}} \sin(\omega_{i_oj_o}\tau + \phi_{i_oj_o}) \quad (49)$$

Substituting the restrictions of amplitudes (45) into the Jacobian matrix gives its elements as

$$J_{ij}(t,u) = 0 \quad \text{for } (i,j) \neq (t,u) \quad (50a)$$

$$J_{ij}(i,j) = -\xi\Big[1 - \frac{1}{2}\beta\psi_{mn}(i,j,i_oj_o)U_{i_oj_o} + \frac{3}{8}\psi^*_{mn}(i,j,i_o,j_o)U^2_{i_oj_o}\Big]$$
$$\text{for } (i,j) = (t,u) \quad (50b)$$

Therefore the characteristic equation of the Jacobian (41) can be reduced to

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\Big[J_{ij}(i,j) - s\Big] = 0 \quad (51)$$

since the nondiagonal elements $J_{ij}(t,u)$ are all zeros as in (50a)

In order for a single mode (i,j) to be stable

$$J_{ij}(i,j) < 0 \quad (52)$$

for all i=1,2, ...,m ; j=1,2, ...,n

From (50b) and the condition (52), the stability criterion (SC) for a single mode (i_o,j_o) can be obtained by the condition

$$(SC)_{ij} = 1 - \frac{1}{2}\beta\psi_{mn}(i,j,i_o,j_o)U_{i_oj_o} + \frac{3}{8}\psi^*_{mn}(i,j,i_o,j_o)U^2_{i_oj_o} > 0 \quad (53)$$

Substituting $U_{i_oj_o}$ from (48) into (53) (SC) is given by

$$(SC)_{ij} = 1 - \frac{1}{2}\beta\psi_{mn}(i,j,i_o,j_o)\,\frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\Big[\beta + \sqrt{\beta^2 - R_{i_oj_o}}\Big]$$

$$+\frac{3}{8}\psi^*_{mn}(i,j,i_o,j_o)\ \frac{\psi^2_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\left[\beta^2+2\beta\sqrt{\beta^2-R_{i_oj_o}}+\beta^2-R_{i_oj_o}\right] \quad (54)$$

where

$$R_{i_oj_o} = 8\,\frac{\psi^*_{mn}(i_o,j_o,i_o,j_o)}{\psi^2_{mn}(i_o,j_o,i_o,j_o)} \quad (55)$$

which is a constant value for any specific mode (i_o,j_o) and given the two corresponding fixed values $\psi_{mn}(i_o,j_o,i_o,j_o)$ and $\psi^*_{mn}(i_o,j_o,i_o,j_o)$

More simply the stability criterion (SC) can be rearranged to be

$$(SC)_{ij} = 1-3\,\frac{\psi_{mn}(i,j,i_o,j_o)}{\psi_{mn}(i_o,j_o,i_o,j_o)} + \frac{1}{4}\,\frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}$$

$$\left[3\psi^*_{mn}(i,j,i_o,j_o)\ \frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)} - 2\psi_{mn}(i,j,i_o,j_o)\right]\beta^2$$

$$+2\,\frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\left[3\psi^*_{mn}(i,j,i_o,j_o)\ \frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\right.$$

$$\left.-2\psi_{mn}(i,j,i_o,j_o)\right]\beta\sqrt{\beta^2-R_{i_oj_o}} \quad (56)$$

Defining

$$F_{ij}(i_o,j_o) = 1-3\,\frac{\psi_{mn}(i,j,i_o,j_o)}{\psi_{mn}(i_o,j_o,i_o,j_o)} \quad (57a)$$

$$H_{ij}(i_o,j_o) = \frac{1}{4}\,\frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\left[3\psi^*_{mn}(i,j,i_o,j_o)\ \frac{\psi_{mn}(i_o,j_o,i_o,j_o)}{\psi^*_{mn}(i_o,j_o,i_o,j_o)}\right.$$

$$\left.-2\psi_{mn}(i,j,i_o,j_o)\right] \quad (57b)$$

Then, $(SC)_{ij}$ can be written in a simple form as

$$(SC)_{ij} = F_{ij}(i_o,j_o) + H_{ij}(i_o,j_o)\beta^2 + H_{ij}(i_o,j_o)\beta \sqrt{\beta^2 - R_{i_o,j_o}}$$

and should be greater than zero for all i=1,2,...,; j=1,2,...,n (58)

While, the value $R_{i_oj_o}$ is fixed and given by (55), the values of F_{ij} are given by (57a) and determined by the values of the parameter $\psi_{mn}(i,j,i_o,j_o)$ and the value $\psi_{mn}(i_o,j_o,i_o,j_o)$. $\psi(i_o,j_o,i_o.j_o)$ is the value which lies in the diagonal of the table representing ψ_{mn}. The values $\psi_{mn}(i,j,i_o,j_o)$ are all the other values in the $(i_o,j_o)^{th}$ column. For m=3 and n=4, the corresponding F_{ij} is calculated by Table 2 and the same for the other structures.

The values $H_{ij}(i_o,j_o)$ are given by (57b), therefore they are calculated by using the values $\psi_{mn}(i,j,i_o,j_o)$ and $\psi^*_{mn}(i,j,i_o,j_o)$. The values $\psi_{mn}(i,j,i_o,j_o)$ are given as mentioned above. Similarly, the value $\psi^*_{mn}(i_o,j_o,i_o,j_o)$ is that one which lies in the diagonal of the table ψ^*_{mn} and the values $\psi^*_{mn}(i,j,i_o,j_o)$ are the other values in the same $(i_o,j_o)^{th}$ column of the table.

Thus, for a cylindrical structure consisting of m oscillators in each ring and n rings, there are (mxn) values of the parameter F_{ij} corresponding to any mode (i_o,j_o) and another (mxn) values for the parameter H_{ij}. Consequently, there are (mxn) values for the Stability Criterion (SC) concerning any specific mode (i_o,j_o) and represented by $(SC)_{ij}$, where i=1,2,...,m and j=1,2,...,n.

From equation (58), it is now clear that the mode (i_o,j_o) is stable if the values of the Stability Criterion (SC) are all positive.

Thus, the investigation of the stability of a single mode (i_o,j_o) in the proposed tube structure is performed by employing the following procedures.

a. Determination of the orthogonal matrices $[p_{ij}]$ and $[q_{ij}]$ for $i=1,2,\ldots,m$; $j=1,2,\ldots,n$

b. Calculation of the parameters ψ_{mn}

c. Calculation of the parameters $\psi^{*}_{mn}(i,j,k,l)$ through the two parameters $\psi^{*}_{m}(i,k)$ and $\psi^{*}_{n}(j,l)$

d. Calculation of the value $R_{i_o j_o}$,

e. Calculation of the values F_{ij} and H_{ij} given by (57) for all $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$ using the values in the $(i_o,j_o)^{th}$ column of the two tables corresponding to ψ_{mn} and ψ^{*}_{mn}

f. Substution of the values of R_{iojo} , F_{ij} and H_{ij} into the Stability Criterion (SC) to investigate the stability of the mode (i_o,j_o). In the case that all the calues $(SC)_{ij}$ are positive, then the mode (i_o,j_o) is then stable.

4.3 Non resonant double modes

Now we consider the stability of nonresonant double-mode oscillations. The restrictions of amplitudes for a nonresonant double-mode consisting of $U_{i_o j_o}$ and $U_{r_o s_o}$ becomes

$$U_{i_o j_o} \neq 0 \;;\; U_{r_o s_o} \neq 0 \;;\; U_{ij} = 0$$

$$\text{for all } i=1,2,\ldots m \;;\; j=1,2,\ldots,n \tag{59}$$

except $(i,j) = (i_o,j_o), (r_o,s_o)$, also $(i_o,j_o) \neq (r_o,s_o)$

The structure of the solution of the mode x_{kl}, corresponding to that nonresonant double-mode, is exactly the same as (55).

Following the same procedures used for investigating the stability of nonresonant double-modes in the tube structure of third-power non-linearities (Alian and Linkens, 1982), the characteristic equation of the Jacobian takes the form

$$\begin{bmatrix} J_{i_1j_1}(i_o,j_o)-s & J_{i_oj_o}(r_o,s_o) \\ J_{r_os_o}(i_o,j_o) & J_{r_os_o}(r_o,s_o)-s \end{bmatrix} \prod_{i=1}^{m}\prod_{j=1}^{n}\left[J_{ij}(i,j) \quad -s \right] = 0$$

where $i=1,2,\ldots,m;\ j=1,2,\ldots,n;\ (i,j) \neq (i_o,j_o),\ (r_o,s_o)$

$$(i_o,j_o) \neq (r_o,s_o) \tag{60}$$

where $J_{ij}(i,j)$ is given from (40a) using the amplitude conditions (59) as

$$J_{ij}(i,j) = -\xi\Big[1-\tfrac{1}{2}\beta\psi_{mn}(i,j,i_o,j_o)U_{i_oj_o} - \tfrac{1}{2}\beta\psi_{mn}(i,j,r_o,s_o)U_{r_os_o}$$
$$+\tfrac{3}{8}\psi^{*}_{mn}(i,j,i_o,j_o)U^2_{i_oj_o} + \tfrac{3}{8}\psi^{*}_{mn}(i,j,r_o,s_o)U^2_{r_os_o}$$
$$+\frac{3}{2}\psi^{*}_{mn}(i,j,i_o,j_o,r_o,s_o)U_{i_oj_o}U_{r_os_o}\Big] \tag{61a}$$

The element $J_{i_oj_o}(r_o,s_o)$ can be given from 40b using the amplitude conditions (59) as

$$J_{i_oj_o}(r_o,s_o) = -\xi\Big[-\tfrac{1}{2}\beta\psi_{mn}(i_o,j_o,r_o,s_o)U_{i_oj_o} + \tfrac{3}{4}\psi^{*}_{mn}(i_o,j_o,r_o,s_o)U_{i_oj_o}U_{r_os_o}$$
$$+\tfrac{3}{4}\psi^{*}_{mn}(r_o,s_o,i_o,j_o)U^2_{i_oj_o}\Big] \tag{61b}$$

Similarly, $J_{r_os_o}(i_o,j_o)$ can be written as

$$J_{r_os_o}(i_o,j_o) = -\xi\Big[-\tfrac{1}{2}\beta\psi_{mn}(r_o,s_o,i_o,j_o)U_{r_os_o} + \tfrac{3}{4}\psi^{*}_{mn}(r_o,s_o,i_o,j_o)U_{i_oj_o}U_{r_os_o}$$
$$+\tfrac{3}{4}\psi^{*}_{mn}(i_o,j_o,r_o,s_o)U^2_{r_os_o}\Big] \tag{61c}$$

From (60) and (61), the stability criterion for the nonresonant double-mode can therefore be written as

a. $J_{i_oj_o}(i_o,j_o) + J_{r_os_o}(r_o,s_o) < 0$ (62a)

b. $J_{i_oj_o}(i_o,j_o)\, J_{r_os_o}(r_o,s_o) - J_{i_oj_o}(r_o,s_o)\, J_{r_os_o}(i_o,j_o) > 0$ (62b)

c. $J_{ij}(i,j) < 0$ (62c)

The structure of the nonresonant double mode (i_o, j_o) and (r_o, s_o) is given by

$$x_{ij} = p_{ii_o} q_{jj_o} \sqrt{U_{i_o j_o}} \sin(\omega_{i_o j_o} \tau + \phi_{i_o j_o}) + p_{iro} q_{jso} \sqrt{U_{r_o s_o}} \sin(\omega_{r_o s_o} \tau + \phi_{r_o s_o}) \quad (63)$$

Thus, the stability criterion of the nonresonant double mode produced by a cylindrical oscillator system with fifth-power nonlinear characteristic is given by (62) and its structural solution is given by (63).

5. Solved Examples of Cylindrical structures with Fifth Power Nonlinear Characteristic

5.1 3 x 4 structure

Consider the number of oscillators in each ring m = 3, and the number of rings in the structure n = 4.

Using the condition (44), it is clear that the non-oscillation state is stable.

To investigate the stability of modes, the Stability Criterion $(SC)_{ij}$ of (19) should be positive for all values i = 1,2,3 and j = 1,2,3,4.

Mode (1,1) is stable for $\beta^2 > 8$

Mode (1,2) is stable for $8.8889 < \beta^2 < 16$

Mode (1,3) is stable for $\beta^2 > 8$

Mode (1,4) is stable for $8.8889 < \beta^2 < 16$

Mode (2,1) is unstable for any value of β

Mode (2,2) is unstable for any value of β

Mode (2,3) is unstable for any value of β

Mode (2,4) is unstable for any value of β

Mode (3.1) is stable for $\beta^2 > 8$

Mode (3,2) is stable for $8.8889 < \beta^2 < 16$

Mode (3.3) is stable for $\beta^2 > 8$

Mode (3.4) is stable for $8.8889 < \beta^2 < 16$

Thus, the tube structure with three oscillators in each ring and four rings reproduces four stable modes for

$8 < \beta^2 < 8.8889$

But for the range of β such that

$8.8889 < \beta^2 < 16$

the structure can reproduce twice as many stable modes, i.e. we can obtain eight stable modes, and it again reproduces four stable modes for

$\beta^2 > 16$

For the stability of the nonresonant double modes, the condition (62) must be satisfied. In such a case the stationary amplitudes cannot be calculated analytically, and a computer is used to solve the equations produced by equating the corresponding equations (29) to zero.

The stationary amplitudes of the stable modes can be calculated using equation (48), i.e.

$$U_{11} = U_{13} = 12\ \{\ \beta + \surd(\beta^2 - 8)\ \}$$

$$U_{12} = U_{14} = 7.2\ \{\beta + \surd(\beta^2 - 8.8889)\ \}$$

$$U_{31} = U_{33} = 8\ \{\ \beta + \surd(\beta^2 - 8)\ \}$$

$$U_{32} = U_{34} = 4.8\ \{\ \beta + \surd(\beta^2 - 8.8889)\ \}$$

Substituting the elements of the corresponding matrices P and Q, the stationary amplitudes U_{ij}, and the angular frequencies ω_{ij} into equation (43), the structure of the stable mode (i.1) is given as

$$[x_{ij}] = \begin{bmatrix} \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \end{bmatrix} \sqrt{12(\beta+\sqrt{\beta^2-8})}\sin(2\pi.2300t+\phi_{11})$$

$$= \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \sqrt{\left[\beta + \sqrt{\beta^2-8}\right]} \sin (2\pi . 2300t + \phi_{11})$$

where ϕ_{11} is an arbitrary phase.

The structure of other modes can be obtained in a similar way as for mode (1,1).

The amplitudes of the stable modes against the parameter β can be seen in Fig. 2 and Fig. 3. Also, for the different values of β, Fig. 4 shows the number of stable modes excited by the above oscillator system.

5.2 <u>4x4 Structure</u>

If the number of oscillators in each ring of the previous structure is increased by one oscillator of the same type, i.e. the tube consists of four rings and each ring contains four oscillators.

It can be shown that:

The nonoscillation state is stable for any value of the nonlinearity factor ξ.

Using the stable Stability Criterion (58), the values of ψ_{44} in Table 3, and the values of ψ^*_{44} in Table 4, the stability of various modes (which is dependent on the parameter β) are obtained as follow:

Mode (1,1) is stable for $\beta^2>8$

Mode (1,2) is stable for $8.8889<\beta^2<16$

Mode (1,3) is stable for $\beta^2>8$

Mode (1,4) is stable for $8.8889<\beta^2<16$

Mode (2,1) is stable for $\beta^2>8$

Mode (2,2) is stable for $8.8889<\beta^2<16$

Mode (2,3) is stable for $\beta^2>8$

Mode (2,4) is stable for $8.8889<\beta^2<16$

Mode (3,1) is stable for $\beta^2>8$

Mode (3,2) is stable for $8.8889<\beta^2<16$

Mode (3,3) is stable for $\beta^2>8$

Mode (3,4) is stable for $8.8889<\beta^2<16$

Mode (4,1) is stable for $\beta^2>8$

Mode (4,2) is stable for $8.8889<\beta^2<16$

Mode (4,3) is stable for $\beta^2>8$

Mode (4,4) is stable for $8.8889<\beta^2<16$

Thus, it is clear that the oscillator system can produce eight stable modes for any value of β such that

$$8<\beta^2<8.8889$$

and sixteen stable modes for

$$8.8889<\beta^2<16$$

and the system is again capable of reproducing only eight stable modes for

$$16<\beta^2$$

It is valuable to notice that when the number of oscillators in each ring (m) has been increased by one oscillator to be m = 4 (which is an even number), the number of stable modes has been increased by a factor of two for the same value of β.

The stationary amplitudes of the stable modes can be calculated in a similar way as illustrated in the previous example as

$$U_{11} = U_{13} = U_{31} = U_{33} = 16 \{\beta + \surd(\beta^2 - 8)\}$$

$$U_{12} = U_{14} = U_{32} = U_{34} = 9.6 \{\beta + \surd(\beta^2 - 8.8889)\}$$

$$U_{21} = U_{23} = U_{41} = U_{43} = 8 \{\beta \pm \surd(\beta^2 - 8)\}$$

$$U_{22} = U_{24} = U_{42} = U_{44} = 4.8 \{\beta + \surd(\beta^2 - 8.8889)\}$$

Substituting the elements of the corresponding matrices P and Q, the stationary amplitudes U_{ij}, and the angular frequencies ω_{ij} into equation

(45), the structure of the stable modes can then be obtained in a similar way as shown in the previous example.

The number of stable modes in the above tube oscillator system against the important parameter β is shown in 5.

6. Experimental Investigation for Tube Oscillator System with Fifth-power Nonlinear Characteristic

In this section we report some preliminary observations on a 12-cell tube oscillator which comprises four rings with three oscillators in each ring.

Experimental results for the same structure have been discussed in (Alian & Linkens, 1982) when each unit oscillator has cubic nonlinear characteristic. In the following, the two diodes in the electronic circuit of each oscillator are connected in the circuit diagram as shown in Fig. 6. The fifth-power nonlinear characteristic which is produced by any unit oscillator in the structure is similar to that of Fig. 7.

The twelve oscillators are connected in a cylindrical oscillator structure. The oscillators in the structure are mutually coupled by inductances. This inductive coupling is obtained by switching on all the switches of coupling inductors in the two directions. One direction represents the mutual coupling between oscillators in the same ring, the other represents the coupling between oscillators in different rings.

The nonoscillation state has been observed on an oscilloscope. The nonoscillation state has never been observed in the case of tube oscillator system with the third power nonlinear characteristic. But the other types of modes and multimodes have been observed in both oscillator systems.

The mode frequencies are nearly the same as for the tube system with third power nonlinearity. This agrees with the theoretical analysis which has already been discussed in this paper. The number of stable modes produced by the oscillator system is varied by changing the parameter β, which characterises the nonlinearity of oscillators.

For the amplitudes of the excited modes, their values agree well with the theoretical ones which have been calculated in the solved example and given by Fig. 2 and Fig. 3.

The theoretical amplitude of the mode (1,1) is a function of the parameter β. Fig. 2 shows the different values of the amplitude A_{11} against the parameter and gives an indication that the real amplitude value must be greater than $(A_{11})_{min}$ which is given as

$$(A_{r_{11}})_{min} = P_{k_1} q_{1_1} (A_{11})_{min} .k_v$$

$$= 5.8259/2 \sqrt{3} .k_v$$

k_v is the voltage scale factor (it has been measured to be 4.5). Thus $(A_{r_{11}})_{min}$ is calculated as

$$(A_{r_{11}})_{min} = 7.568 \quad V$$

The above amplitude value corresponds to the value $\beta = 2.828$, at which the mode (1,1) is just capable of being excited as a stable one. When increasing the value of the parameter β, the resultant amplitude also increases, e.g., if $\beta = 4$, the corresponding value of A_{11} is given from Fig. 3 as

$$A_{11} = 9.052$$

therefore, the real value of amplitude $A_{r_{11}}$ is given as

$$A_{r_{11}} = 1.759 \quad V$$

For the rest of the modes which are reproduced by the electronic model under investigation , it has been shown that the experimental results of the mode frequencies and amplitudes agree well with the theoretical results.

7. Conclusions

It has been shown that the complex model of a tubular structure of intercoupled fifth-power non-linear oscillators is amenable to the theoretical mode anlaysis. The manipulations are more extensive than for the third-power case, but reveal a rich modal behaviour. As expected, the zero state is proved to be a stable condition for such a model. The stability criteria for the modes turn out to be more complex than for the third-power case, and since they are not in directly calculable form, some computer assistance is required in the investigation of particular cases. The non-linear parameter β plays an important part in determining the number of stable modes, which a particular case can support. Thus for the 3 x 4 case considered theoretically and experimentally 0,4 or 8 modes are stable depending on the particular value of β .

The paper has concentrated on the single mode behaviour and pointed to the approach for double mode investigation. The stability of degenerate and non-degenerate modes requires further study and extensive manipulation, but could form the basis of further work on this structure. The investigation of particular cases based on the general development for single mode stability would benefit from a computer-aided approach, which is also an area of extension for this work.

From the analysis presented it is clear that the introduction of a fifth-power term into the unit oscillator dynamic increases the mode repertoire. In terms of large-intestinal modelling this may be of considerable importance since confusion exists regarding the basic behaviour patterns which can be observed in the mammalian large-intestine. Thus, considerable debate continues as to the presence of periods of electrical silence, and the range of frequencies which can be recorded from the colon. If a fifth-power tubular structure is a necessary model structure then such confusion is very likely to continue, since the number of possible stable modes would almost inevitably make the recorded situation illustrate a very complicated pattern of switching between modes for such a structure. Such mode switching complexity would almost certainly occur in the presence of noise which is endemic in biological systems. Finally, it should be noted that although the algebraic manipulation involved in this theoretical analysis is extensive, an equivalent investigation of mode stability via simulation would be almost impossible.

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APPENDIX I

Linearisation of the term x^5_{ab}

$$x_{ab} = \sum_{k=1}^{m} \sum_{1=1}^{n} p_{ak}\, q_{b1}\, y_{k1} \qquad \text{(I-1)}$$

So, x^5_{ab} becomes

$$x^5_{ab} = \sum_{k=1}^{m} \sum_{1=1}^{n} (p_{ak})(q_{b1}) y^5_{k1}$$

$$+ 5 \sum_{k=1}^{m} \sum_{1=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} (p^4_{ak})(q^4_{b1})(p_{ar})(q_{bs}) y^4_{k1}\, y_{rs}$$

$$+ 10 \sum_{k=1}^{m} \sum_{1=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} (p^3_{ak})(q^3_{b1})(p^3_{ar})(p^2_{ar})(q^2_{bs})\, y^3_{k1}\, y^2_{rs}$$

$$+ 10 \sum_{k=1}^{m} \sum_{1=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{m} \sum_{u=1}^{n} p_{ak} q_{b1} p_{ar} q^3_{bs} y_{k1} y_{rs} y^3_{tu}\, q^3_{bu}$$

$$+ 15 \sum_{k=1}^{m} \sum_{1=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{m} \sum_{u=1}^{n} p_{ak} q_{b1} p_{ar} q_{bs} p^2_{at} q^2_{bu} y^2_{k1} y_{rs} y^2_{tu}$$

$$+ 10 \sum_{k=1}^{m} \sum_{1=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{m} \sum_{u=1}^{n} \sum_{v=1}^{m} \sum_{w=1}^{n} p_{ak} q_{b1} p_{ar} q_{bs} p_{at} q_{bu}$$

$$p_{at} q_{bu} p^2_{av} p^2_{bw} y_{k1} y_{rs} y_{tu} y^2_{vw}$$

$$+ (y_{k1}\ y_{rs}\ y_{tu}\ y_{vw}\ y_{gh} \text{ terms})$$

$$(k,p) \neq (r,s) \neq (t,u) \neq (v,w) \neq (g,h) \qquad \text{(I-2)}$$

Since it is assumed that there is no resonant interaction between the modes, the $u_{k1}\ y_{rs}\ y_{tu}\ y_{vw}\ y_{gh}$ terms can be ignored, but all the other terms cannot be ignored.

Expanding these terms and neglecting the higher harmonics in quasiharmonic analysis:

$$y_{kl}^{5} = \tfrac{5}{8} A_{kl}^{4} y_{kl}$$

$$y_{kl}^{4} y_{rs} = \tfrac{3}{8} A_{kl}^{4} y_{rs}$$

$$y_{kl}^{3} y_{rs}^{2} = \tfrac{3}{8} A_{kl}^{2} A_{rs}^{2} y_{kl}$$

$$y_{kl} y_{rs} y_{tu}^{3} = 0$$

$$y_{kl}^{2} y_{rs} y_{tu}^{2} = \tfrac{1}{4} A_{kl}^{2} A_{tu}^{2} y_{rs}$$

$$y_{kl} y_{rs} y_{tu} y_{vw} = 0$$

Substituting the above terms into (18) x_{ab}^{5} becomes

$$x_{ab}^{5} = \tfrac{5}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} (p_{ak}^{5}) (q_{bl}^{5}) A_{kl}^{4} y_{kl}$$

$$+ \frac{15}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} (p_{ak}^{4})(q_{bl}^{4})(p_{ar})(q_{bs}) A_{kl}^{4} y_{rs}$$

$$+ \frac{15}{8} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} p_{ak}^{3} q_{bl}^{3} p_{ar}^{2} q_{bs}^{2} A_{kl}^{2} A_{rs}^{2} y_{kl}$$

$$+ \frac{15}{4} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{m} \sum_{u=1}^{n} p_{ak}^{2} q_{bl}^{2} p_{ar} q_{bs} p_{at}^{2} q_{bu}^{2} A_{kl}^{2} A_{tu}^{2} y_{rs}$$

where $(k,l) \neq (r,s) \neq (t,u)$ (I-3)

For simplicity x_{ab}^{5} can be written in the form of a linear combination as

$$x_{ab}^{5} = \sum_{k=1}^{m} \sum_{l=1}^{n} L_{ij} (k,l) \, y_{kl} \qquad \text{(I-4)}$$

where

$$L_{ij}(k,l) = \tfrac{5}{8} p_{ak}^{5} q_{bl}^{5} A_{kl}^{4}$$

$$+ \frac{15}{8} \sum_{r=1}^{m} \sum_{s=1}^{n} p_{ar}^{4} q_{bs}^{4} p_{ak} q_{bl} A_{rs}^{4}$$

$$+ \frac{15}{4} \sum_{r=1}^{m} \sum_{s=1}^{n} p_{ak}^{3} q_{bl}^{3} p_{ar}^{2} q_{bs}^{2} A_{kl}^{2} A_{rs}^{2}$$

$$+ \frac{15}{4} \sum_{r=1}^{m} \sum_{s=1}^{n} \sum_{t=1}^{m} \sum_{u=1}^{n} p_{at}^{2} q_{bu}^{2} p_{ar}^{2} q_{bs}^{2} p_{ak} q_{bl} A_{tu}^{2} A_{rs}^{2}$$

(I-5)

where $(i,j) \neq (k,l) \neq (r,s)$

(i,j) \ (k,l)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)
(1,1)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,2)	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24
(1,3)	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12	1/12
(1,4)	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8	1/12	1/24	1/12	1/8
(2,1)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,2)	1/12	1/8	1/12	1/24	1/8	3/16	1/8	1/16	1/24	1/16	1/24	1/48
(2,3)	1/12	1/12	1/12	1/12	1/8	1/8	1/8	1/8	1/24	1/24	1/24	1/24
(2,4)	1/12	1/24	1/12	1/8	1/8	1/16	1/8	3/16	1/24	1/48	1/24	1/16
(3,1)	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,2)	1/12	1/8	1/12	1/24	1/24	1/16	1/24	1/48	1/8	1/16	1/8	1/16
(3,3)	1/12	1/12	1/12	1/12	1/24	1/24	1/24	1/24	1/8	1/8	1/8	1/8
(3,4)	1/12	1/24	1/12	1/8	1/24	1/48	1/24	1/16	1/8	1/16	1/8	3/16

Table 1. The values $\psi_{mn}(i,j,k,l)$ when $m=3$, $n=4$

(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	
1/9	1/6	1/9	1/6	1/6	1/4	1/6	1/4	1/6	1/4	1/6	1/4	
1/9	5/18	1/9	1/18	1/6	5/12	1/6	1/12	1/6	5/12	1/6	1/12	
1/9	1/6	1/9	1/6	1/6	1/4	1/6	1/4	1/6	1/4	1/6	1/4	
1/9	1/18	1/9	5/18	1/6	1/12	1/6	5/12	1/6	1/12	1/6	5/12	
1/9	1/6	1/9	1/6	11/36	11/24	11/36	11/34	1/6	1/4	1/6	1/4	
1/9	5/18	1/9	1/18	11/36	55/72	11/36	11/72	1/6	5/12	1/6	1/12	$X\frac{1}{16}$
1/9	1/6	1/9	1/6	11/36	11/24	11/36	11/24	1/6	1/4	1/6	1/4	
1/9	1/18	1/9	5/18	11/36	11/72	11/36	55/72	1/6	1/12	1/6	5/12	
1/9	1/6	1/9	1/6	1/36	1/24	1/36	1/24	1/4	3/8	1/4	3/8	
1/9	5/18	1/9	1/18	1/36	5/72	1/36	1/72	1/4	5/8	1/4	1/8	
1/9	1/6	1/9	1/6	1/36	1/24	1/36	1/24	1/4	3/8	1/4	3/8	
1/9	1/18	1/9	5/18	1/36	1/72	1/36	5/72	1/4	1/8	1/4	5/8	

Table 2. The values of $\psi^{*}_{mn}(i,j,k,l)$ for m=3 and n=4

(i,j) \ (k,l)	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)
(1,1)	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
(1,2)	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32
(1,3)	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
(1,4)	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32
(2,1)	1/16	1/16	1/16	1/16	1/8	1/8	1/8	1/8	1/16	1/16	1/16	1/16	0	0	0	0
(2,2)	1/16	3/32	1/16	1/32	1/8	3/16	1/8	1/16	1/16	3/32	1/16	1/32	0	0	0	0
(2,3)	1/16	1/16	1/16	1/16	1/8	1/8	1/8	1/8	1/16	1/16	1/16	1/16	0	0	0	0
(2,4)	1/16	1/32	1/16	3/32	1/8	1/16	1/8	3/16	1/16	1/32	1/16	3/32	0	0	0	0
(3,1)	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
(3,2)	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32
(3,3)	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16	1/16
(3,4)	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32	1/16	1/32	1/16	3/32
(4,1)	1/16	1/16	1/16	1/16	0	0	0	0	1/16	1/16	1/16	1/16	1/8	1/8	1/8	1/8
(4,2)	1/16	3/32	1/16	1/32	0	0	0	0	1/16	3/32	1/16	1/32	1/8	3/16	1/8	1/16
(4,3)	1/16	1/16	1/16	1/16	0	0	0	0	1/16	1/16	1/16	1/16	1/8	1/8	1/8	1/8
(4,4)	1/16	1/32	1/16	3/32	0	0	0	0	1/16	1/32	1/16	3/32	1/8	1/16	1/8	3/16

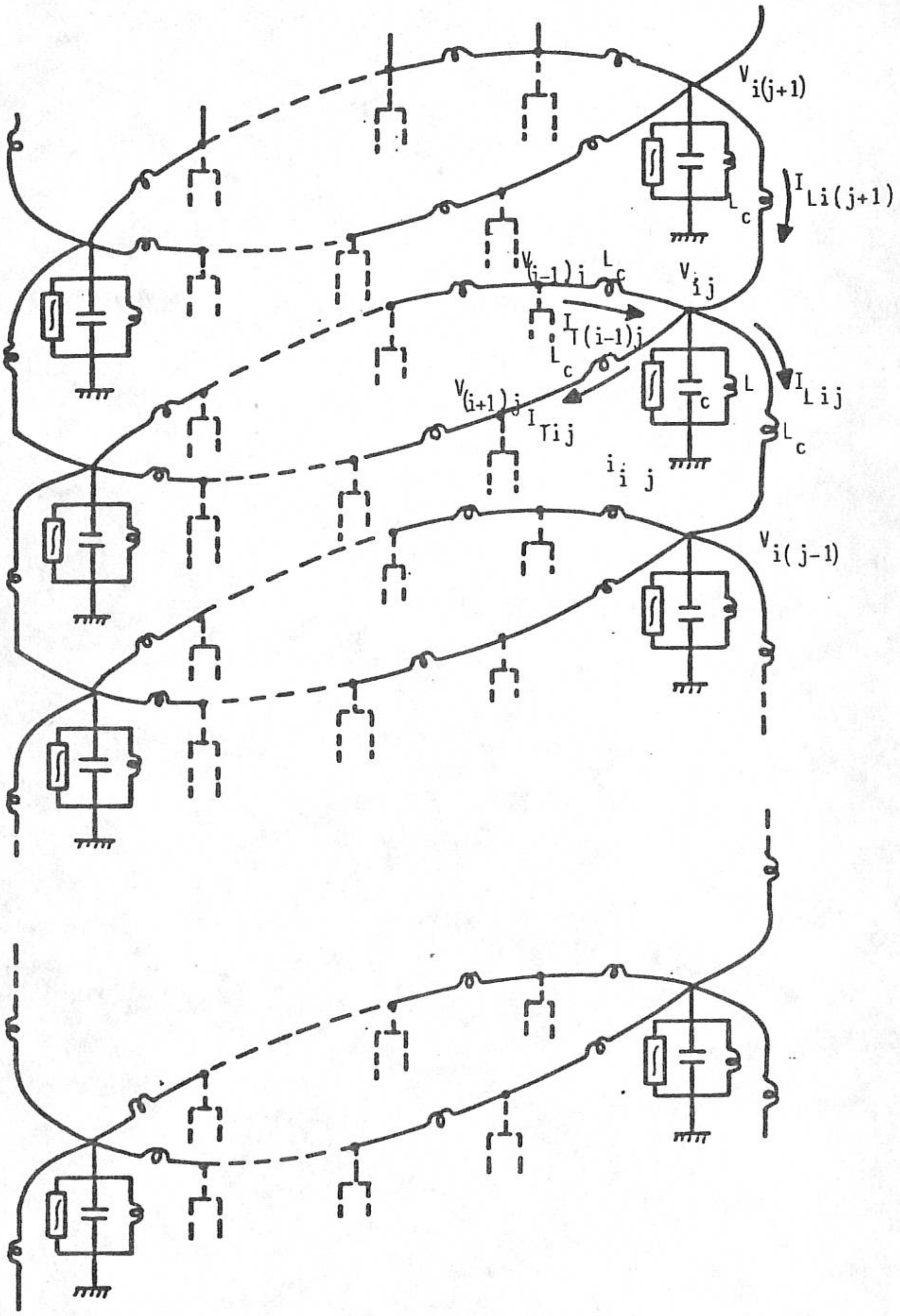
Table 3. The values of $\psi_{mn}(i,j,k,l)$ when m=4, n=4

(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)	
1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	
1/4	5/8	1/4	1/8	1/2	5/4	1/2	1/4	1/4	5/8	1/4	1/8	1/2	5/4	1/2	1/4	
1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	
1/4	1/8	1/4	5/8	1/2	1/4	1/2	5/4	1/4	1/8	1/4	5/8	1/2	1/4	1/2	5/4	
1/4	3/8	1/4	3/8	1	3/2	1	3/2	1/4	3/8	1/4	3/8	0	0	0	0	
1/4	5/8	1/4	1/8	1	5/2	1	1/2	1/4	5/8	1/4	1/8	0	0	0	0	
1/4	3/8	1/4	3/8	1	3/2	1	3/2	1/4	3/8	1/4	3/8	0	0	0	0	
1/4	1/8	1/4	5/8	1	1/2	1	5/2	1/4	1/8	1/4	5/8	0	0	0	0	$\times\frac{1}{64}$
1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	
1/4	5/8	1/4	1/8	1/2	5/4	1/2	1/4	1/4	5/8	1/4	1/8	1/2	5/4	1/2	1/4	
1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	1/4	3/8	1/4	3/8	1/2	3/4	1/2	3/4	
1/4	1/8	1/4	5/8	1/2	1/4	1/2	5/4	1/4	1/8	1/4	5/8	1/2	1/4	1/2	5/11	
1/4	3/8	1/4	3/8	0	0	0	0	1/4	3/8	1/4	3/8	1	3/2	1	3/2	
1/4	5/8	1/4	1/8	0	0	0	0	1/4	5/8	1/4	1/8	1	5/2	1	1/2	
1/4	3/8	1/4	3/8	0	0	0	0	1/4	3/8	1/4	3/8	1	3/2	1	3/2	
1/4	1/8	1/4	5/8	0	0	0	0	1/4	1/8	1/4	5/8	1	1/2	1	5/2	

Table 4. The values of $\psi^*_{mn}(i,j,k,l)$ for m=4 and n=4

FIGURE CAPTIONS

1. Tube structure for large-intestinal model, using fifth-power non-linear characteristic.
2. Mode amplitudes in a 3x4 system versus non-linear parameter .
3. Mode aplitudes in a 3x4 system versus non-linear parameter .
4. Number of stably excited modes in a 3x4 system versus non-linear parameter .
5. Number of stably excited modes in a 4x4 system versus non-linear paramter .
6. Oscillator circuit with fifth power non-linear characteristic.
7. Fifth-power type characteristic of non-linear conductance for an electronic oscillator.



V i(j+1)
I Li(j+1)
L c
V (i-1) j
L c
V i j
I T(i-1)j
L c
I L i j
L
c
V (i+1) j
I Tij
L c
i i j
V i(j-1)

Fig. 1.

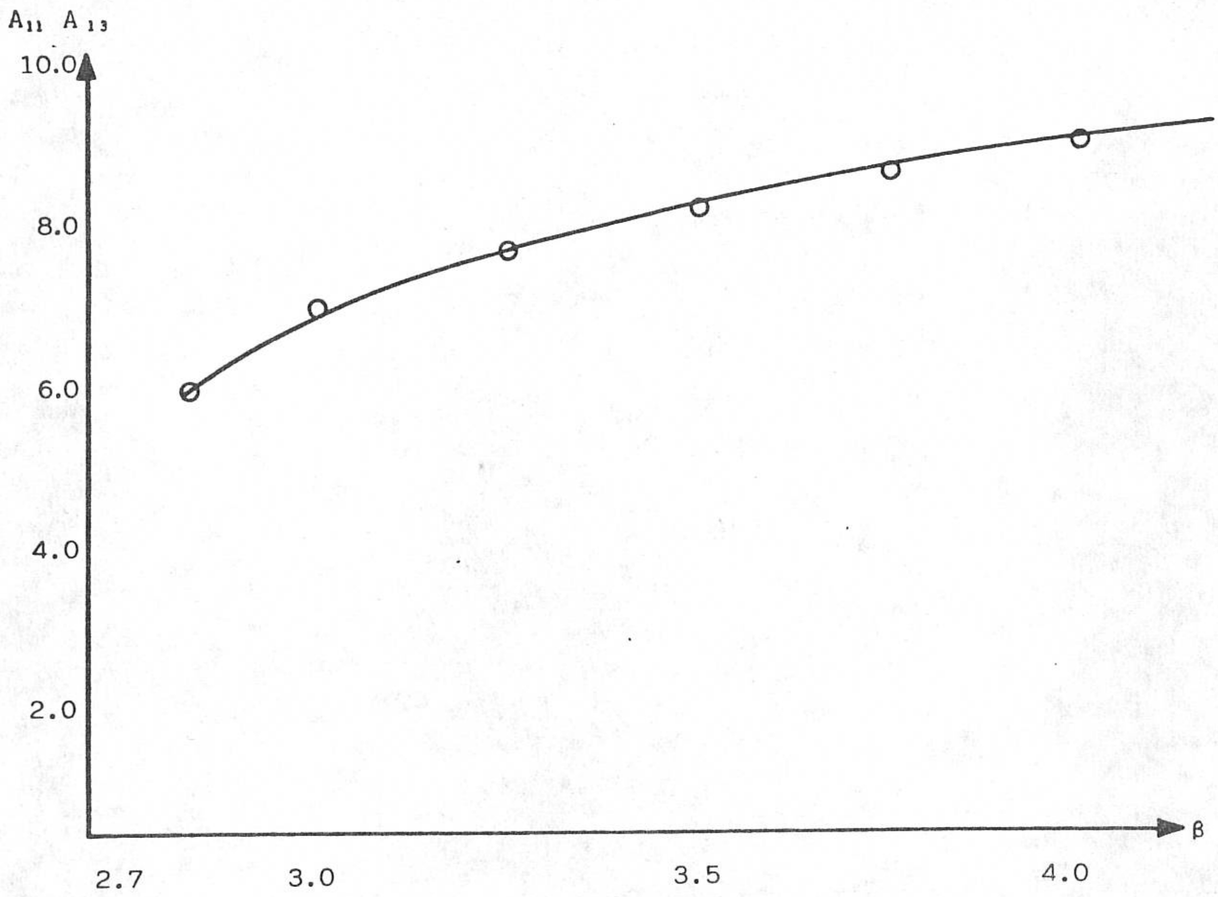

A11 A13
10.0
8.0
6.0
4.0
2.0
β
2.7
3.0
3.5
4.0

Fig. 2a

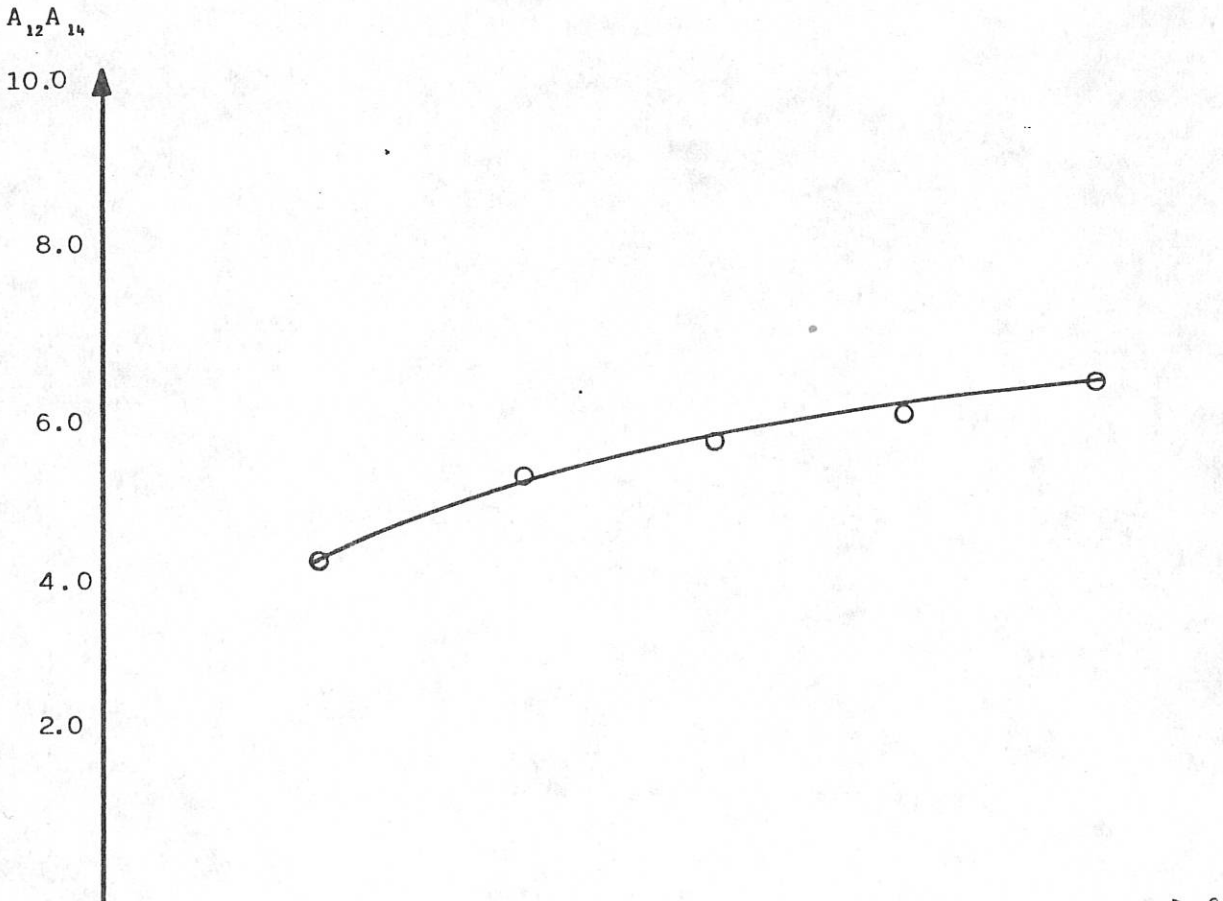

A12A14
10.0
8.0
6.0
4.0
2.0
β
2.7
3.0
3.5
4.0

Fig. 2b

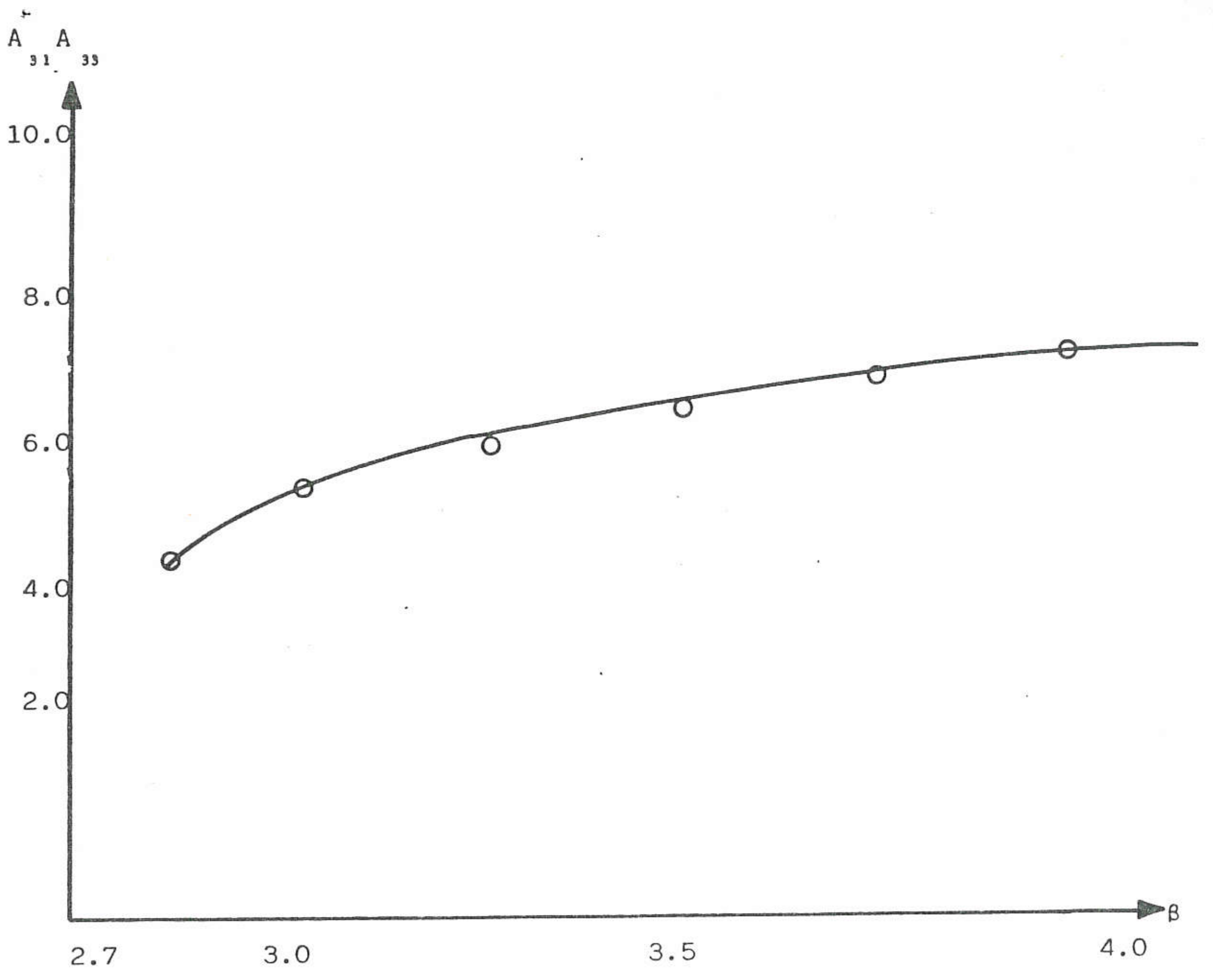


Fig. 3a.

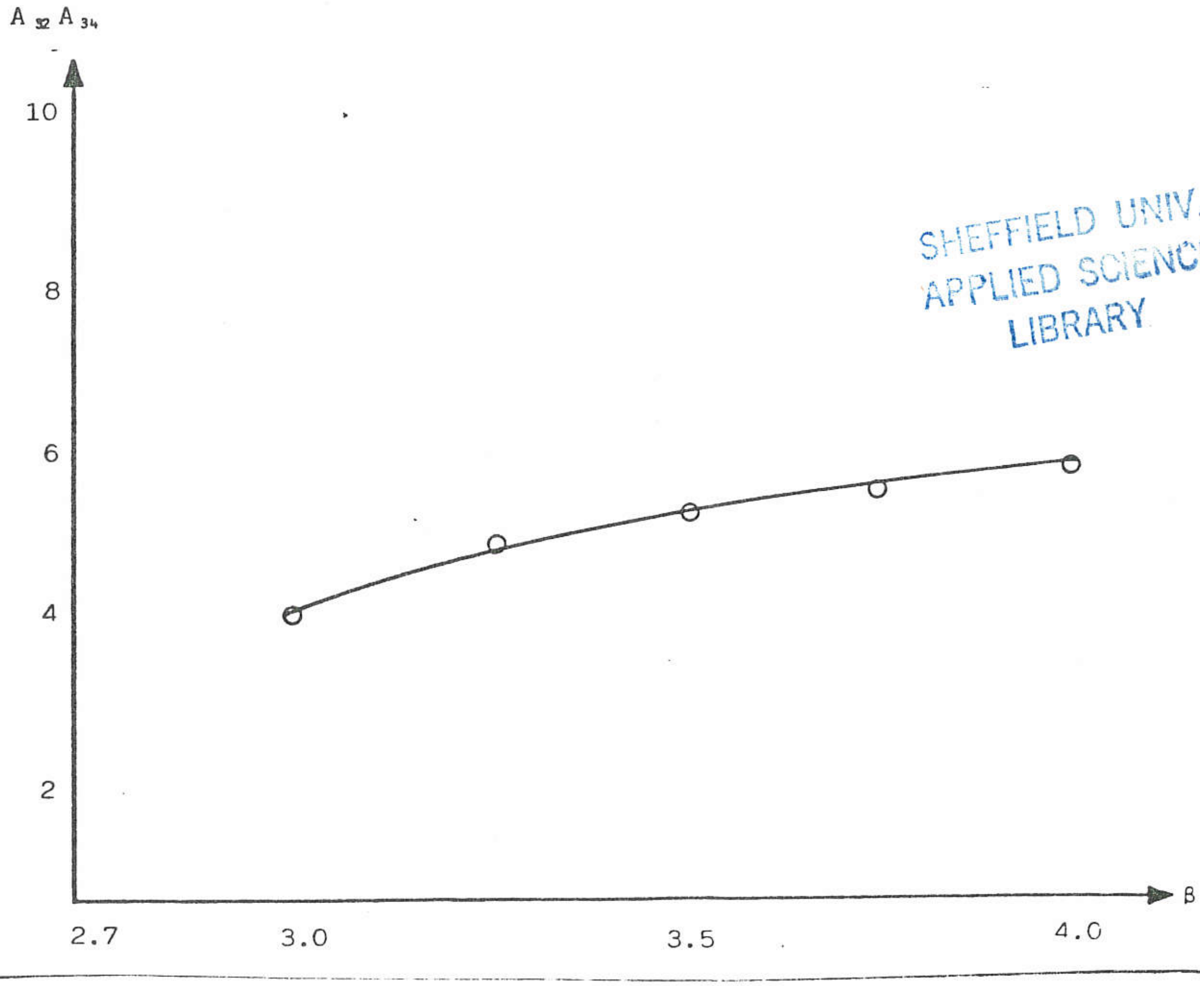
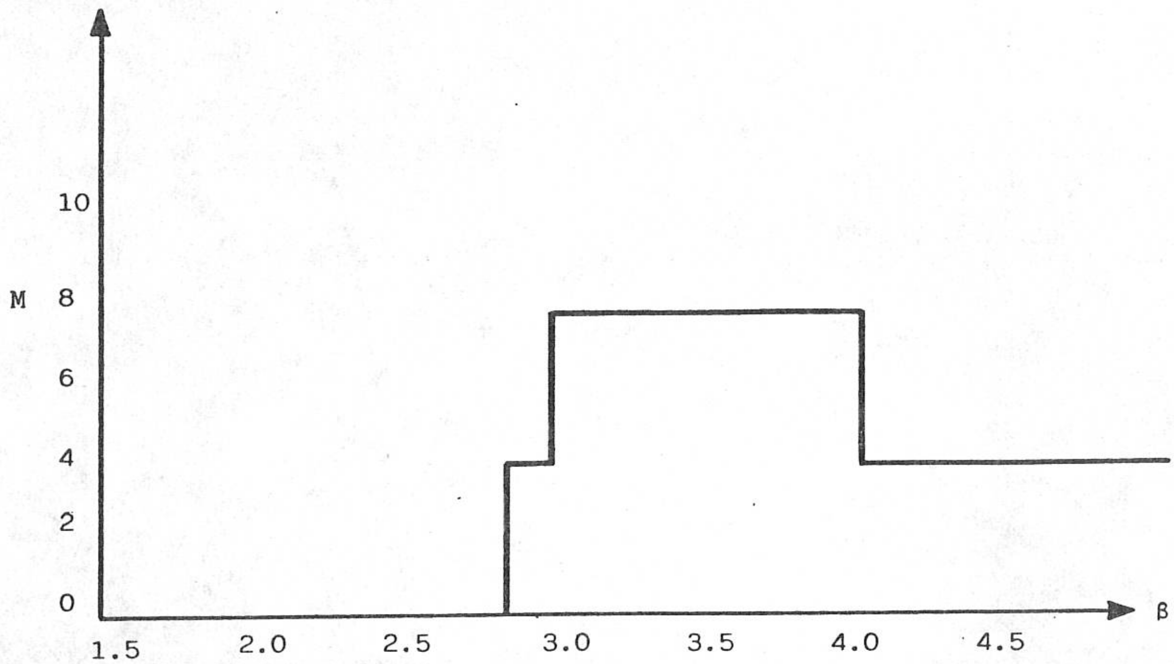
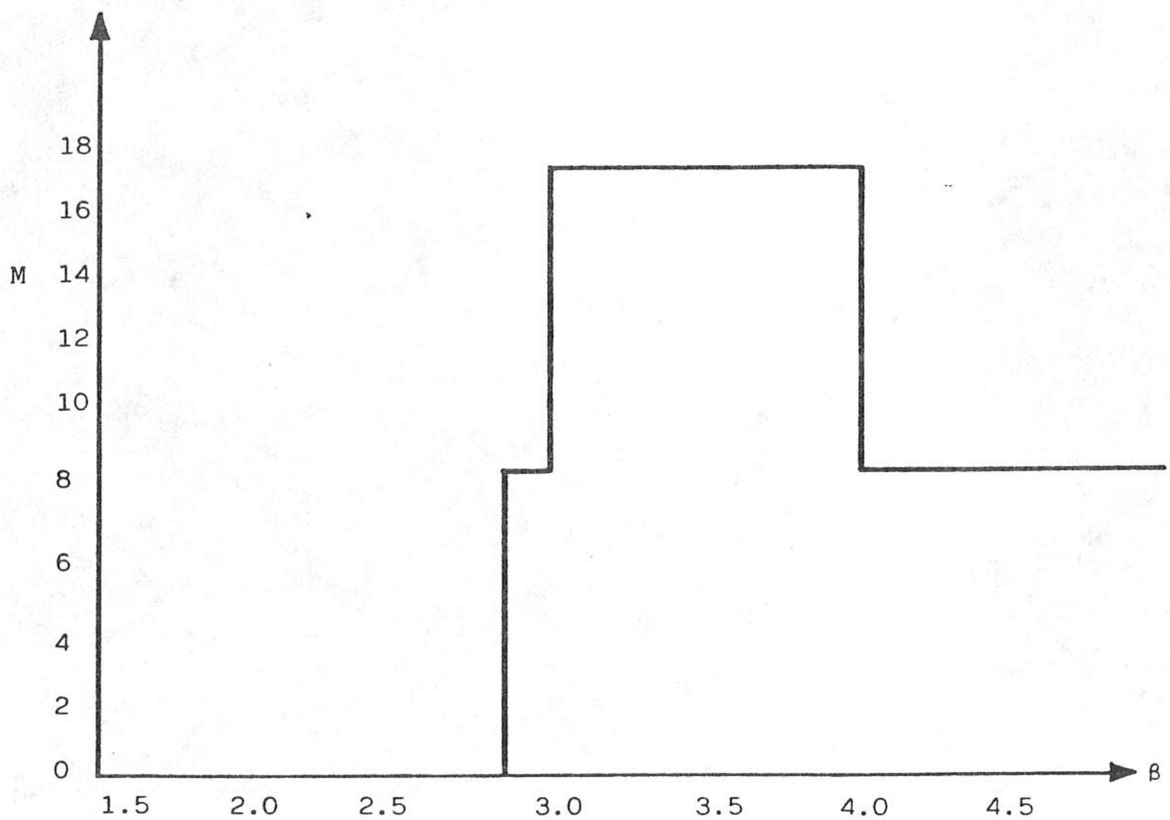


Fig. 3b



M
10
8
6
4
2
0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
β

Fig. 4



M
18
16
14
12
10
8
6
4
2
0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
β

Fig. 5

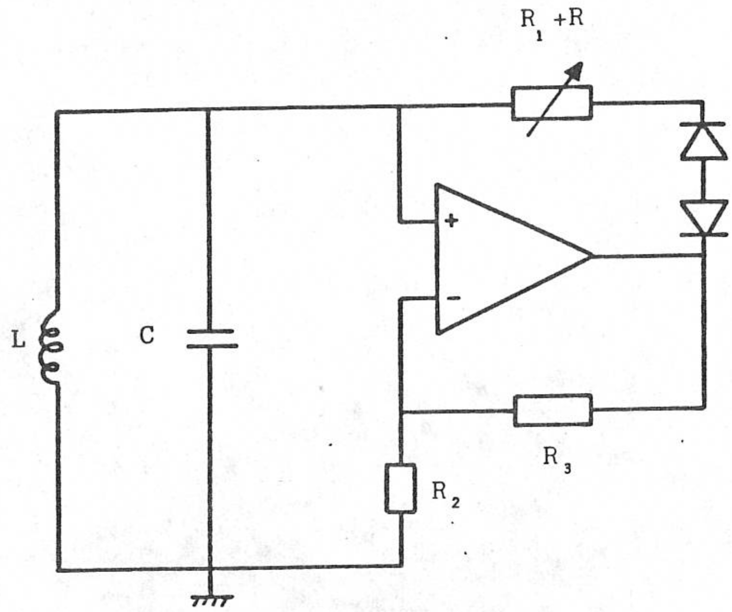


Fig. 6.

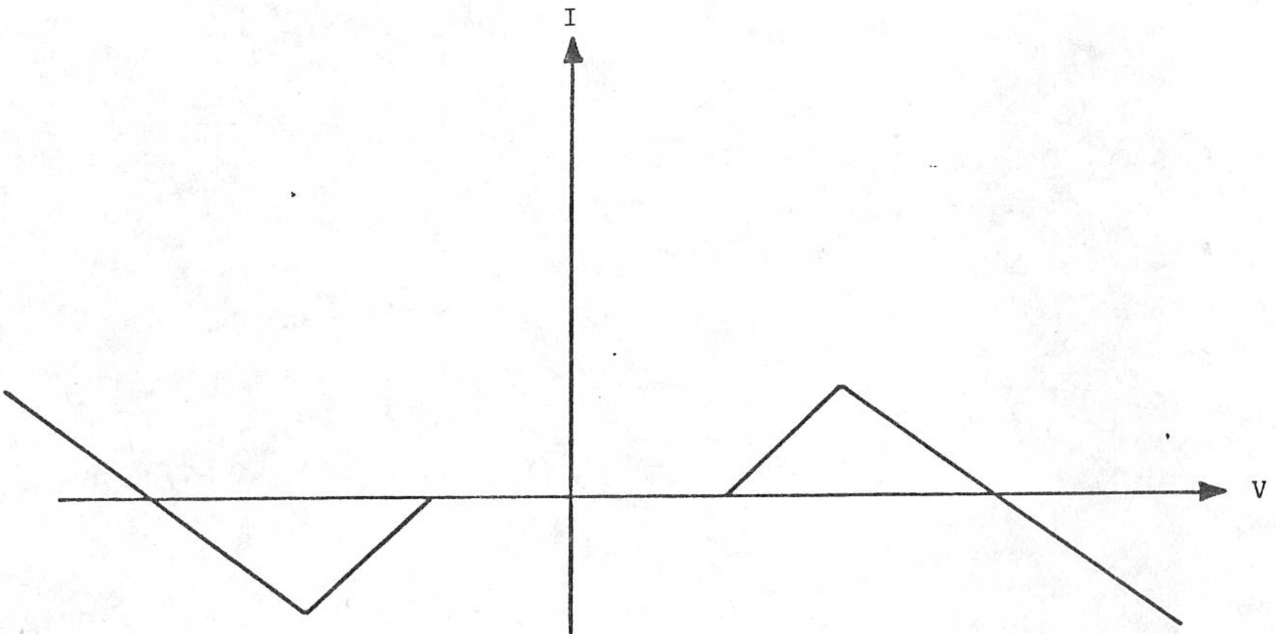


Fig. 7.