Balanced models in Geophysical Fluid Dynamics: Hamiltonian formulation, constraints and formal stability

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1 Introduction

Most fluid systems, such as the three-dimensional compressible Euler equations, are too complicated to yield general analytical solutions, and approximation methods are needed to make progress in understanding aspects of particular flows. This chapter reviews derivations of approximate or reduced geophysical fluid equations which result from combining perturbation methods with preservation of the variational or Hamiltonian structure. Preservation of this structure ensures that analogues of conservation laws in the original "parent" equations of motion are preserved. Although formal accuracy in terms of a small parameter may be achieved with conservative asymptotic perturbation methods, asymptotic solutions are expected to diverge on longer time scales. Nevertheless, perturbation methods combined with preservation of the variational or Hamiltonian structure are hypothesized to be useful in a climatological sense because conservation laws associated with this structure remain to constrain the reduced fluid dynamics.

Variational and Hamiltonian formulations of fluid flows are of interest when effects of forcing and dissipation are of secondary importance, which is often the case on scales shorter than characteristic damping times or when nonlinearities remain dominant on longer time scales. Variational or Hamiltonian methods form a unifying framework to analyze various fluid phenomena. Applications of these methods include the systematic derivation and use of wave-activity conservation laws, classical linear and nonlinear stability theorems, saturation bounds on the growth of instabilities, statistical mechanics of geophysical fluid dynamics and conservative numerical integration (e.g. Fjørtoft 1950; Holloway 1986; Holm et al. 1985; McLachlan 1995; Morrison 1998; Salmon 1988a, 1998; Shepherd 1990, 1994; Vladimirov 1987, 1989).

In geophysical fluid dynamics the above-mentioned approximate or reduced models are generally called *balanced* models because certain types of waves have been eliminated relative to ones present in the original "parent" dynamics; e.g. an incompressible fluid is balanced relative to a compressible one because sound waves have been eliminated through the constraint of incompressibility. Elimination of certain types of waves can often be formalized through scaling, yielding relevant small parameters, and perturbation analysis. A well known example is the elimination of acoustic waves in the reduction from compressible to incompressible dynamics in which the Mach number, the ratio between a characteristic velocity scale and the speed of sound, is the relevant small parameter. Balanced equations thus result from singular perturbation methods, or equivalent approaches, which simplify the equations with essential singular terms and reduce the order (for example in time) of the system of differential equations. Although a perturbative approach appears to be most rigorous, one always has to realize that small parameters are a result of a scaling of the equations. This scaling tends to be a non-rigorous process, because although there may be a dominant characteristic time or spatial scale in the flow other scales can be excited and remain present due to nonlinear interactions. As an alternative to a formal perturbative approach, certain types of waves in the flow may be eliminated by imposing constraints based on observed characteristics or special insights in the fluid dynamical behaviour, which in light of the non-rigorous aspects of scaling often results into reduced systems of similar accuracy as the ones obtained via formal scaling and perturbation methods. This alternative, apparently less accurate, approach for finding constraints goes along with the observation that the notion of "balance" and the accuracy of solutions of balanced systems (analytical or numerical) hold often surprisingly well outside the realm of asymptotic perturbation theory. Examples of the numerical accuracy of solutions of geophysical balanced models are found in the context of coastal dynamics in Allen and Newberger (1993), in atmospheric dynamics in McIntyre and Norton (1998) and perhaps even in surf-zone dynamics where breaking waves on beaches lead to low Froude number balanced along-shore currents (e.g. Özkan-Haller 1997).

The history of numerical weather prediction also nicely illustrates the use of balanced models (e.g. Daley 1991). The first numerical weather prediction model was the barotropic quasi-geostrophic equation (see section 3.5 for a f-plane version), which crudely describes the motion of vortical structures and Rossby waves (e.g. Gill 1982) in a one-layer fluid. In this model, gravity waves and acoustic waves have been eliminated or filtered, and the Rossby number (the ratio of the local Earth's rotation time scale to the advective time scale) and aspect ratio (between vertical and horizontal spatial scales and velocity fields, respectively) are the relevant small parameters used in the approximation. In the 1960's the hydrostatic primitive equations (see section 3.2 for a planar version) replaced the (barotropic and baroclinic) quasi-geostrophic numerical weather prediction models. In these hydrostatic equations only acoustic waves have been filtered (except for the boundary-trapped Lamb mode, e.g. Gill (1982)); it was nevertheless still necessary to initialize or balance the data such as to eliminate spurious high-amplitude gravity waves. The concept of balance remains crucial in the initialization and interpretation phase of numerical weather prediction.

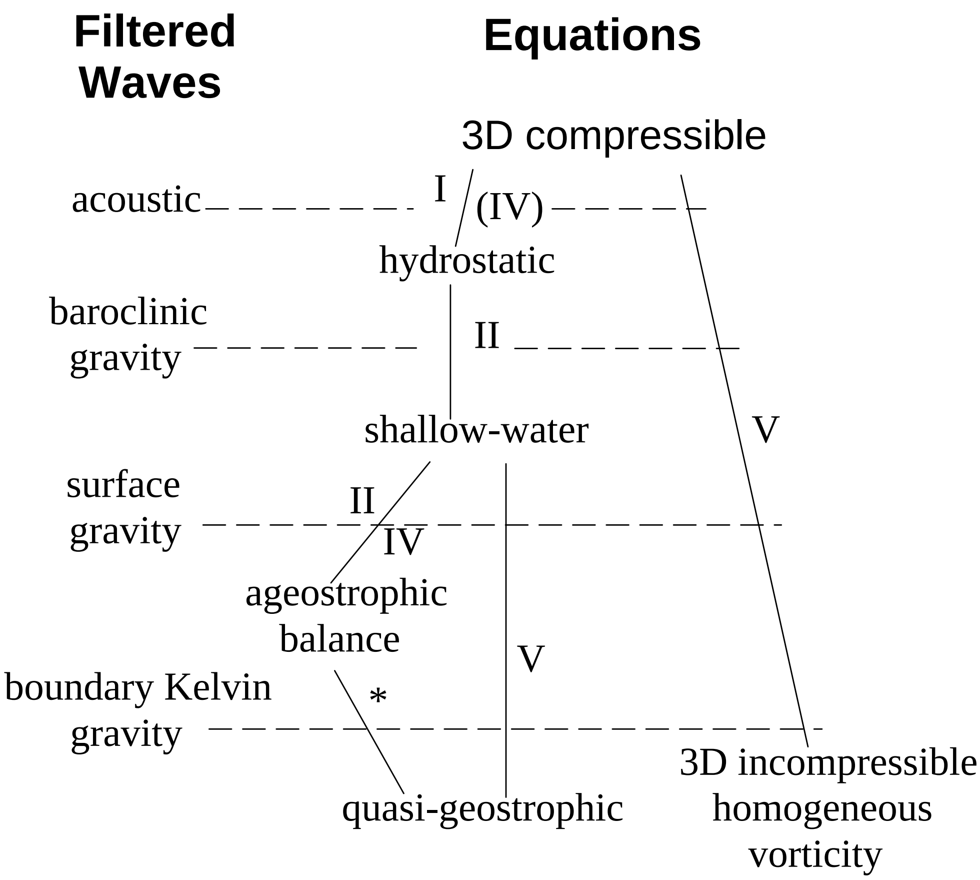

Figure 1 Sketch of the fluid systems considered in section 3. Connecting solid lines, going down, indicate the approximation route and Roman numerals the followed approximation approach. I-V denote various singular approximation methods defined in section 2 and 3 while "*" is a regular leading-order Rossby-number expansion. The left column under the heading "Filtered Waves" indicates the wave types filtered in the approximation *between* two fluid systems (dashed horizontal lines).

Theoretical analysis and numerical process studies of balanced models have greatly advanced our understanding in meteorology and oceanography and (nearly) inviscid fluid models are often the first ones to be studied (e.g. consider the analysis of quasi-geostrophic systems in Pedlosky (1987); and the analysis of cyclogenesis in various balanced systems in Snyder et al. (1991) and Maraki et al. 1999). A systematic derivation of reduced models with conservation laws has been and is important to understand geophysical flows. This chapter gives an account of some of the recent progress in deriving these conservative, geophysical balanced models.

Variational and Hamiltonian formulations, perturbative approaches based on slaving, and several constrained variational or Hamiltonian approximation approaches are introduced, and denoted by numerals I to V (Fig. 1), at first in section 2 for finite-dimensional systems because they facilitate a more succinct exposition of the essentials. (The more technical mathematical aspects of infinite-dimensional Hamiltonian systems are not considered here, see e.g. Marsden and Ratiu 1994.) Section 2 also contains several examples of finite-dimensional conservative fluid models. It additionally introduces the powerful energy-Casimir method which can be used to derive stability criteria for steady states of (non-canonical) Hamiltonian systems. In section 3 the Hamiltonian approximation approaches I-V are applied to various fluid models (Fig. 1) starting from the compressible Euler equations and finishing with the barotropic quasi-geostrophic and higher-order geostrophically balanced equations. The presentation of fluid examples runs in parallel with the general finite-dimensional treatment in section 2 which facilitates comparisons. In addition, I quote or derive stability criteria for all fluid examples. These criteria are summarized in Table 1 in the summary and discussion.

2 Finite-dimensional systems

Two variational principles, Hamilton's principle and its related action principle, are introduced in section 2.1. This action principle follows from Hamilton's principle via a Legendre transformation and yields Hamilton's equations of motion. Hamilton's equations open the route to the definition of the more general Poisson systems in section 2.2. Systematic approximations are introduced in section 2.3 using slaving principles and singular perturbations. These approximations yield constraints which will be imposed in various but related ways on variational and Hamiltonian formulations in section 2.4. A unified abstract treatment combining the derivation of constraints and balanced Hamiltonian dynamics is presented in section 2.5 together with a discussion of its limitations, which appear so severe that only the leading-order theory presented in section 2.6 seems to be applicable in practice. Finally, a review of the energy-Casimir method concerning stability criteria for steady states of Hamiltonian systems can be found in section 2.7.

2.1 Variational principles

2.1.1 Hamilton's principle

The equations of motion for a classical-mechanical system with generalized coordinates $q^i(t)$ and velocities $\dot{q}^i(t) \equiv dq^i(t)/dt$ as functions of time t follow from Hamilton's principle (e.g. Lanczos 1970, Arnold 1989, Marsden and Ratiu 1994)

$$\delta A[q^i] = \lim_{\epsilon \to 0} \frac{A[q^i + \epsilon\, \delta q^i] - A[q^i]}{\epsilon} = 0 \tag{2.1.1}$$

with the action $A[q^i]$ defined by

$$A[q^i] = \int_{t_0}^{t_1} \mathrm{d}t\, L(q^i, \dot{q}^i, t) \tag{2.1.2}$$

and its endpoint conditions by $\delta q^i(t_0) = \delta q^i(t_1) = 0$, where L is the Lagrangian and $i = 1, \ldots, K$. The familiar Euler-Lagrange equations appear when variations in Hamilton's principle (2.1.1) are performed and when the endpoint conditions are used to eliminate terms arising after integration by parts in time. They have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}. \tag{2.1.3}$$

A variety of dynamical systems can be derived from Hamilton's principle. For example, mono-atomic fluids consisting of N classical point particles, each with unit mass $\mathsf{m} = 1$, constitute a dynamical system with (generalized) positions q^i and velocities $\dot{q}^i$ (for $i = 1, \ldots, K = d\,N$ and with dimension d). Its dynamics is given by (2.1.3) for a Lagrangian $L = T - V$ being the kinetic energy T minus the potential energy V of the atoms. Alternatively, the dynamics (2.1.3) may be considered as the discretization of a continuous description of a fluid in terms of fluid parcels with unit mass $\mathsf{m} = 1$, (generalized) positions q^i and velocities $\dot{q}^i$. (Salmon (1983) uses such a discrete description of fluid parcels, along with an approximate representation of the potential energy, to perform numerical integrations of a blob of shallow water. Brenier (1996) provides another intriguing geometrical model of fluid parcel motion). More concretely, let us consider the following two finite-dimensional examples.

Example 1: Dynamics of a particle of unit mass in three spatial dimensions with position $\mathbf{q} = (q^1, q^2, q^3)^T = (x, y, z)^T$ (now $K = 3$) and potential energy $V(x, y, z)$ follows from Hamilton's principle as

$$\ddot{x} = -\frac{\partial V}{\partial x}, \quad \ddot{y} = -\frac{\partial V}{\partial y}, \quad \ddot{z} = -\frac{\partial V}{\partial z}, \tag{2.1.4}$$

which we recognize as Newton's equations of motion with a conservative force.

Example 2: Euler-Lagrange equations for Lorenz's (1986) two-degree-of-freedom weather model with two coordinates $q \equiv q^1$ and $Q \equiv q^2$ (i.e. $K = 2$)

$$\ddot{q} - b\,\ddot{Q} + C\,\sin 2\,q \;=\; 0, \tag{2.1.5}$$

$$(1 + b^2)\,\ddot{Q} - b\,\ddot{q} + \frac{Q}{\epsilon^2} \;=\; 0, \tag{2.1.6}$$

readily follow from (2.1.1) and its endpoint conditions with Lagrangian

$$\begin{aligned} L(q, Q, \dot{q}, \dot{Q}) \;=\; & \frac{1}{2}\,\dot{q}^2 - b\,\dot{q}\,\dot{Q} + \frac{1}{2}\,(1 + b^2)\,\dot{Q}^2 - \\ & [-\frac{1}{2}\,C\,\cos 2\,q + \frac{1}{2}\,\frac{Q^2}{\epsilon^2}] \end{aligned} \tag{2.1.7}$$

which is the kinetic minus potential (terms in square brackets) energy. The coupling parameter between the pendulum (2.1.5) and the harmonic oscillator (2.1.6) is b, ϵ a small parameter, and C is proportional to the square of the (linearized) frequency of a pendulum.

In section 3.1.1 three-dimensional equations of motion for a compressible fluid are shown to arise from a Hamilton's principle wherein the Lagrangian is a functional, i.e. an integral over space.

2.1.2 Action principle

The Lagrangian $L(q^i, \dot{q}^i, t)$ in (2.1.2) is non-singular if the determinant of the Jacobian of the transformation between the two coordinate pairs $\{q^i, \dot{q}^i\}$ and $\{q^i, p_i\}$ is nonzero ($i = 1, \ldots, K$), in which conjugate momentum p_i is defined as

$$p_i \equiv \frac{\partial L(q^i, \dot{q}^i, t)}{\partial \dot{q}^i}. \tag{2.1.8}$$

In other words L is convex in $\dot{q}$. Consequently a Legendre transform

$$H(q^i, p_i, t) = p_i\,\dot{q}^i - L(q^i, \dot{q}^i, t) \tag{2.1.9}$$

is well defined (see Lanczos 1970, Arnold 1989, and Marsden and Ratiu 1994; also for a geometrical interpretation), and the Hamiltonian H is a function of the q^i, p_i, and t only. $\dot{q}^i(p_i, q^i, t)$ is now defined by the extremal conditions $\partial H/\partial \dot{q}^i = 0$. Under this transformation Hamilton's principle changes into the action principle

$$\delta \int_{t_0}^{t_1} \mathrm{d}t\, L(q^i, \dot{q}^i, t) = \delta \int_{t_0}^{t_1} \mathrm{d}t\, \Big\{p_i\,\dot{q}^i - H(q^i, p_i, t)\Big\} = 0 \tag{2.1.10}$$

for variations δq^i and δp_i and endpoint conditions $\delta q^i(t_0) = \delta q^i(t_1) = 0$. Its variations yield $N \equiv 2K$ first-order equations, that is, Hamilton's equations

$$\frac{dq^i}{dt} \;=\; \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}. \tag{2.1.11}$$

Example 3: The action principle corresponding to Example 1 is

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ u\,\dot{x} + v\,\dot{y} + w\,\dot{z} - \left(\frac{1}{2}\,(u^2 + v^2 + w^2) + V(x, y, z) \right) \right\} = 0 \quad (2.1.12)$$

with $u = \partial L/\partial \dot{x}, v = \partial L/\partial \dot{y}$ and $w = \partial L/\partial \dot{z}$.

Example 4: We may verify that Hamilton's equations corresponding to the Euler-Lagrange equations for Lorenz's (1986) model of Example 2 follow from the action principle (2.1.10) with $N = 2$ and Hamiltonian

$$H = -\frac{1}{2}\,C\,\cos 2\,q + \frac{1}{2}\left(p^2 + \frac{Q^2}{\epsilon^2} + (P + b\,p)^2 \right) \quad (2.1.13)$$

in which we have derived momenta $p \equiv p_1 = \dot{q} - b\,\dot{Q}$ and $P \equiv p_2 = (1 + b^2)\,\dot{Q} - b\,\dot{q}$ following (2.1.8). Conversely, we may derive $\dot{q}, \dot{Q}$ from the extremal conditions $\partial L/\partial p_i = 0$ with $L = \dot{q}^i\,p_i - H(q^i, p_i)$.

Sometimes dynamical systems do not arise from Hamilton's principle or from a related action principle in terms of generalized coordinates and momenta, but rather from an action principle in terms of some variables z. Consider the action principle

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ a_m(z)\,\frac{dz^m}{dt} - H(z^m) \right\} \quad (2.1.14)$$

with endpoint variations $\delta z^m(t_0) = \delta z^m(t_1) = 0$, Hamiltonian H, functions $a_m(z)$ of z, $m = 1, \ldots, N$ and $N = 2K$. Variation (2.1.14) with respect to δz^n yields the equations

$$\tilde{K}_{nm}\frac{dz^m}{dt} = \frac{\partial H}{\partial z^n}, \quad (2.1.15)$$

where it is assumed that

$$\tilde{K}_{nm} \equiv \frac{\partial a_m}{\partial z^n} - \frac{\partial a_n}{\partial z^m} \quad (2.1.16)$$

is a non-singular tensor. If $z = \{q^i, p_i\}$ and

$$\tilde{\mathbf{K}} = \begin{pmatrix} 0 & -\mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} \quad (2.1.17)$$

then (2.1.14) equals (2.1.10); here $\mathbf{I}$ is the $K \times K$ unit matrix. Since $\tilde{\mathbf{K}}$ is invertible we may define a tensor $\mathbf{J} \equiv (\tilde{\mathbf{K}})^{-1}$ and rewrite (2.1.15) as generalized Hamiltonian equations

$$\frac{dz^i}{dt} = J^{ij}\,\frac{\partial H}{\partial z^j}, \quad (2.1.18)$$

which include the canonical Hamilton's equations (2.1.11). Since $\mathbf{J}$ is non-singular, transformations $\{z^m\} \to \{q^i, p_i\}$ may be defined, at least locally, by virtue of Darboux's theorem (see e.g. Arnold 1989) such that $\mathbf{J}$ takes the canonical form

$$\mathbf{J}^c = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}. \tag{2.1.19}$$

If global canonical, so-called Darboux, coordinates exist, then (2.1.14) may be rewritten in the form (2.1.10). Action princple (2.1.2) often provides a more convenient description than Hamilton's principle or the canonical action principle (2.1.10) when (non-canonical) variables z are more meaningful or when global canonical coordinates are difficult to define.

Example 5: An action principle (2.1.14) for Lorenz's (1986) model is

$$\delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ x_3 \frac{d\phi}{dt} - \epsilon(x_5 + b\, x_3)\frac{dx_4}{dt} - H \right\} = 0 \tag{2.1.20}$$

with respect to variations $\delta z = \delta\phi, \delta x_3, \delta x_4$, and δx_5, respectively, subject to endpoint conditions $\delta\phi(t_{0,1}) = \delta x_4(t_{0,1}) = 0$, and with Hamiltonian

$$H = -\frac{1}{2} C \cos 2\phi + \frac{1}{2}(x_3^2 + x_4^2 + x_5^2) \tag{2.1.21}$$

(Bokhove and Shepherd 1996). The action principle (2.1.20) yields Lorenz's (1986) model in a reduced format

$$\begin{aligned} \frac{d\phi}{dt} &= x_3 - b\, x_5, \quad \frac{dx_3}{dt} = -C \sin 2\phi, \\ \frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \quad \frac{dx_5}{dt} = \frac{x_4}{\epsilon} + b\, C \sin 2\phi. \end{aligned} \tag{2.1.22}$$

Variational principle (2.1.20) is identical to the variational principle (2.1.10) in Example 4 when we make the identification $q = \phi$, $p = x_3$, $Q = \epsilon\, x_4$ and $P = -(x_5 + b\, x_3)$.

A Lagrangian action principle for three-dimensional compressible flows is derived in section 3.1.2 via a Legendre transform of a relevant Hamilton's principle.

2.2 Hamiltonian formulation

The mathematical structure of equations (2.1.18) gives rise to Poisson systems. Such systems have the form

$$\frac{dF}{dt} = [F, H], \tag{2.2.1}$$

where H is the Hamiltonian and F is an arbitrary function of the variables z. The Poisson bracket $[\cdot, \cdot]$ is defined by

$$[F, G] = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j}, \tag{2.2.2}$$

where G is another arbitrary function of z and $\mathbf{J}$ is a tensor (here $i = 1, \ldots, N$ for arbitrary N). The system (2.2.1), (2.2.2) is Hamiltonian if and only if bracket (2.2.2) satisfies the following *conditions* for arbitrary functions F, G, K:

(i) skew-symmetry $[F, G] = -[G, F]$,

(ii) Jacobi's identity $[F, [G, K]] + [K, [F, G]] + [G, [K, F]] = 0$, and

(iii) Leibniz's rule

$$[F\, G, K] = F\, [G, K] + G\, [F, K]. \tag{2.2.3}$$

By using (2.2.2) to evaluate (2.2.3) these conditions imply the following conditions, which define a cosymplectic tensor $\mathbf{J}$:

(i) skew-symmetry $J^{ij} = -J^{ji}$,

(ii) Jacobi's identity

$$J^{im} \frac{\partial J^{jk}}{\partial z^m} + J^{km} \frac{\partial J^{ij}}{\partial z^m} + J^{jm} \frac{\partial J^{ki}}{\partial z^m} = 0, \tag{2.2.4}$$

(iii) Condition (2.2.3)(iii) is automatically guaranteed by the form (2.2.2), because derivatives obey Leibniz's rule (regarding functionals, see Olver [1986]).

Jacobi's identity is often difficult to prove; it is a quadratic identity which means that in perturbation approaches the various orders get mixed. Substitution of $F = z^i$ into (2.2.1) yields the Hamiltonian equations (2.1.18). Note that a cosymplectic tensor satisfying conditions (2.2.4) (i)–(ii) does not need to be invertible. Poisson systems therefore generalize the Hamiltonian systems with invertible $\mathbf{J}$ which were introduced at the end of section 2.1.2. Historically, the theory of Hamiltonian dynamics originated in the realm of classical mechanics, where the following *canonical* Poisson bracket

$$[F, G] = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i}, \tag{2.2.5}$$

for $z = \{q^i, p_i\}$ and with $N = 2K$ even, arises from the canonical equations of motion (2.1.11) and the corresponding cosymplectic tensor is (2.1.19). The bracket (2.2.5) satisfies conditions (2.2.3) (i)–(iii). The significance of these conditions led to a generalized definition of Hamiltonian systems of the form (2.2.1), (2.2.2) for more general, non-canonical, Poisson brackets.

This generalization, however, has important consequences. In contrast to the Poisson bracket (2.2.5), the bracket (2.2.2) is neither necessarily canonical nor even-dimensional, and this permits the existence of nontrivial Casimir invariants C, which are solutions of $[C, G] = 0$ for arbitrary G. The invariance of the Casimirs readily follows from this definition since

$$\frac{dC}{dt} = [C, H] = 0. \tag{2.2.6}$$

Casimir invariants span the kernel of the cosymplectic tensor $\mathbf{J}$ (Littlejohn 1982) since condition $[C, G] = 0$ implies that

$$J^{ij} \frac{\partial C}{\partial z^j} = 0, \tag{2.2.7}$$

and vectors with components $\partial C / \partial z^j$ thus span the null space of $\mathbf{J}$.

Other invariants of (continuous) Hamiltonian systems are related to symmetries of the Hamiltonian through Noether's theorem (e.g. Lanczos 1970, Olver 1986, Arnold 1989). When a Hamiltonian is invariant under time translation conservation of energy ensues, $dH/dt = [H, H] = 0$, and when a Hamiltonian is invariant under spatial translations conservation of momentum ensues.

When the cosymplectic tensor $\mathbf{J}$ is invertible no nontrivial Casimirs exist and the conditions (2.2.3) on $\mathbf{J}$ can then be translated into *linear* conditions on the symplectic tensor $\tilde{\mathbf{K}}$

(i) skew-symmetry $\tilde{K}_{ij} = -\tilde{K}_{ji}$,

(ii) Jacobi's identity

$$\frac{\partial \tilde{K}_{ij}}{\partial z^k} + \frac{\partial \tilde{K}_{jk}}{\partial z^i} + \frac{\partial \tilde{K}_{ki}}{\partial z^j} = 0. \tag{2.2.8}$$

Example 6: The original model derived by Lorenz (1986), which we encountered in various disguises in Examples 2, 4, and 5, reads

$$\begin{aligned} \frac{dx_1}{dt} &= -x_2\, x_3 + b\, x_2\, x_5, \quad \frac{dx_2}{dt} = x_1\, x_3 - b\, x_1\, x_5, \quad \frac{dx_3}{dt} = -x_1\, x_2, \\ \frac{dx_4}{dt} &= -\frac{x_5}{\epsilon}, \quad \frac{dx_5}{dt} = \frac{x_4}{\epsilon} + b\, x_1\, x_2. \end{aligned} \tag{2.2.9}$$

Its Hamiltonian formulation is

$$\frac{dF}{dt} = [F, H'] \tag{2.2.10}$$

with Poisson bracket (satisfying conditions (2.2.3)(i)–(iii))

$$
\begin{aligned}
[F,G] \;=\; & \frac{\partial F}{\partial x_1} x_2 \left(b\,\frac{\partial G}{\partial x_5} - \frac{\partial G}{\partial x_3} \right) + \frac{\partial F}{\partial x_2} x_1 \left(\frac{\partial G}{\partial x_3} - b\,\frac{\partial G}{\partial x_5} \right) + \\
& \frac{\partial F}{\partial x_3}\left(x_2\,\frac{\partial G}{\partial x_1} - x_1\,\frac{\partial G}{\partial x_2} \right) - \frac{1}{\epsilon}\frac{\partial F}{\partial x_4}\frac{\partial G}{\partial x_5} + \frac{\partial F}{\partial x_5}\left(-b\,x_2\,\frac{\partial G}{\partial x_1} + b\,x_1\frac{\partial G}{\partial x_2} + \frac{1}{\epsilon}\frac{\partial G}{\partial x_4} \right)
\end{aligned} \tag{2.2.11}
$$

and Hamiltonian

$$
H' = H + \frac{3}{2}\,C = \frac{1}{2}\,(x_1^2 + 2\,x_2^2 + x_3^2 + x_4^2 + x_5^2). \tag{2.2.12}
$$

The Casimir invariant $C = \frac{1}{2}\,(x_1^2 + x_2^2)$ shows why the parameter C has been taken constant in previous appearances of Lorenz's model in Examples 2, 4, and 5. Note that variable ϕ in Example 5 follows from the polar transformation $x_1 = \sqrt{2\,C}\,\cos\,\phi, x_2 = \sqrt{2\,C}\,\sin\,\phi$.

A Hamiltonian formulation of the four-component version (2.1.22) in Example 5, augmented with $dC/dt = 0$, may be derived from (2.2.10)–(2.2.12) by a transformation of variables $x_1, \ldots, x_5$ to ϕ, x_3, x_4, x_5 and C. Derivatives of functions F are then calculated from the chain rule

$$
\begin{aligned}
\delta F \;&=\; \sum_{i=1}^{5} \frac{\partial F}{\partial x_i}\,\delta x_i \\
&=\; \frac{\partial F}{\partial \phi}\,\delta\phi + \frac{\partial F}{\partial x_3}\,\delta x_3 + \frac{\partial F}{\partial x_4}\,\delta x_4 + \frac{\partial F}{\partial x_5}\,\delta x_5 + \frac{\partial F}{\partial C}\,\delta C.
\end{aligned} \tag{2.2.13}
$$

The new bracket in terms of ϕ, x_3, x_4, x_5 and C follows by substituting these results into (2.2.11), and with (2.2.12) and (2.2.1) system (2.1.22) may be derived.

Example 7: A very simple non-dimensionalized, three-component model describing shallow-water flows with radial and time coordinate dependence only, is the following non-canonical system. Use of (2.2.1) with $F = u(t), v(t)$ or $h(t)$ alternately, with Hamiltonian

$$
H = \frac{1}{2}\Big(h\,(u^2 + v^2) + h^2 \Big) - h, \tag{2.2.14}
$$

and Poisson bracket

$$
[F,G] = q\left(\frac{\partial F}{\partial u}\,\frac{\partial G}{\partial v} - \frac{\partial G}{\partial u}\,\frac{\partial F}{\partial v} \right) + \frac{\partial G}{\partial h}\,\frac{\partial F}{\partial u} - \frac{\partial F}{\partial h}\,\frac{\partial G}{\partial u}, \tag{2.2.15}
$$

yields the equations of motion

$$
\frac{du}{dt} = \mu\,v + \frac{1}{2}\,(u^2 + 3\,v^2) + h - 1,\ \ \frac{dv}{dt} = -(\mu + v)\,u,\ \ \frac{dh}{dt} = -h\,u \tag{2.2.16}
$$

in which $q = (\mu + v)/h$ is the "potential vorticity" and μ a dimensionless Coriolis parameter. The reader may compare the Hamiltonian (2.2.14) and Poisson bracket (2.2.15) with the Hamiltonian (3.4.9) and bracket (3.4.14) of the continuous shallow-water equations. In this simple fluid model the radial velocity is defined by $u(t)\,J_1(r)$, the azimuthal velocity by $v(t)\,J_1(r)$, and the depth of the fluid by $h(t)\,J_0(r)$, in which J_0, J_1 are Bessel functions of the first kind of zeroth and first order. $h = 1$ is the constant non-dimensional rest depth. The model may be derived systematically from the (axisymmetric) Hamiltonian formulation of the shallow-water equations, which we will encounter in section 3.4, and is a Galerkin projection of the nonlinear terms onto the linear modes (e.g. Tribbia (1981) and Leith (1996)). The Casimir invariant of this model is $C(q)$ since it obeys $[C, G] = 0$. The model is integrable because with the two constants of motion H and q the one remaining equation may always be solved by quadratures.

Hamiltonian formulations for a three-dimensional compressible fluid are given in sections 3.1.2 and 3.1.3 in Lagrangian and Eulerian coordinates, respectively.

2.3 Slaving principles and singular perturbations

Constraints often arise from perturbation approaches or physical insights and can then be imposed on the relevant "parent" variational principles or Hamiltonian dynamics in order to derive constrained balanced dynamics. Slaving principles are a systematic approach to derive constraints formally correct to the desired order of accuracy in an appropriate small parameter ϵ, which has arisen after suitable scaling of the relevant dynamical equations.

Let us consider a prototypical system of dimensionless ordinary or partial differential equations, singular in a small parameter ϵ, of the form

$$\frac{\partial s}{\partial t} = S(s, f; \epsilon), \tag{2.3.1}$$

$$\frac{\partial f}{\partial t} + \frac{\Gamma f}{\epsilon} = F(s, f; \epsilon). \tag{2.3.2}$$

In system (2.3.1), (2.3.2) f denotes $O(1)$ fast variables and s denotes $O(1)$ slow variables, $S(s, f; \epsilon)$ and $F(s, f; \epsilon)$ are $O(1)$ nonlinear vector operators, and Γ is an $O(1)$ invertible linear operator independent of ϵ and time. For notational simplicity s, f, S, F, and Γ are not written as vectors. Normal-mode solutions of the form $f \sim \exp(i\omega_f \tau)$ with $|\omega_f| \geq 1$, fast time scale $\tau = \epsilon^{-1} t$ and ω_f the spectrum of operator Γ can often be found for the linearized version of system (2.3.1), (2.3.2) (Kreiss 1980). On the fast time scale τ the slow variables s are unequivocally slow, i.e. $\partial s/\partial \tau = 0$, and s then projects onto the exact invariant slow manifold of the *linearized* equations.

In contrast the fast variables oscillate with $O(1)$ frequency on the fast time scale. Whether a well-defined time scale separation in the linearized dynamics remains valid for nonlinear dynamics is questionable. The least one can hope for is that perturbation methods like a power series of s and f in ϵ yield asymptotic solutions which define a quasi-invariant slow manifold. On this manifold, which has the dimensionality of the phase space of s, the motion is hoped to be devoid of fast oscillations for a finite time when ϵ is sufficiently small.

In order to write a particular (geophysical fluid) model in the form (2.3.1), (2.3.2) it is in general necessary to redefine the origin around fixed points of the system (or to introduce an appropriate basic state for partial differential equations). In that case s and f are considered to be perturbation variables and, accordingly, any linearization would take place about $s = 0$ and $f = 0$.

It turns out that expansion of both s and f into a power series in ϵ, $s = s^{(0)} + \epsilon s^{(1)} + \ldots$, leads to non-uniform convergence when the leading order dynamics in $s^{(0)}$ has positive Liapunov exponents (Warn et al. 1995), an example being two-dimensional incompressible or quasi-geostrophic turbulence as leading-order $s^{(0)}$-dynamics. The dynamics for $s^{(1)}$ is linear and grows without bound when the leading-order dynamics has positive Liapunov exponents, because the homogeneous system for $s^{(1)}$ is equivalent to the linearized version of the decoupled leading-order system for $s^{(0)}$ (Warn et al. 1995). Multiple time-scale analysis seems to offer no alternative when the power series gets disordered due to the presence of positive Liapunov exponents.

Rather than expanding both variables s and f we will therefore follow Warn et al. (1995) and assume that the dynamics lies on a nearly invariant slow manifold given by the slaving principle (e.g. Van Kampen 1985)

$$f = U(s) \qquad \Longleftrightarrow \qquad \tilde{f} \equiv f - U(s) = 0, \tag{2.3.3}$$

which may be exactly or only asymptotically valid. A partial differential or functional equation, the so-called superbalance equation (cf. Lorenz 1980), for $U(s)$ results after substitution of the *slaving principle* into the fast equation (2.3.2) while using the slow equation (2.3.1)

$$-\frac{\Gamma\, U(s)}{\epsilon} = \frac{dU(s)}{dt} - F(s, U(s); \epsilon) = \frac{\partial U(s)}{\partial s}\, S(s, U(s); \epsilon) - F(s, U(s); \epsilon). \tag{2.3.4}$$

Approximate (asymptotic) solutions of this superbalance equation may be found by substitution of the power series $U(s) = U^{(0)}(s) + \epsilon\, U^{(1)}(s) + \ldots$ in ϵ into (2.3.4) and evaluation at every order in ϵ: this amounts to a *modified* expansion of f into a power series of ϵ only, as a function or functional of s. Substitution of

$$s = s, \qquad f = f^{(0)} + \epsilon\, f^{(1)} + \ldots \tag{2.3.5}$$

into (2.3.1), (2.3.2) now yields a hierarchy of balance models. Just like in (2.3.4), time derivatives of the fast variables need to be evaluated in a special way:

$$\frac{\partial f}{\partial t} = \Big[\mathcal{T}^{(0)} + \epsilon\,\mathcal{T}^{(1)} + \ldots\Big]\,\frac{\partial s}{\partial t} = \Big[\mathcal{T}^{(0)} + \epsilon\,\mathcal{T}^{(1)} + \ldots\Big]\,S(s, f; \epsilon), \qquad (2.3.6)$$

where $\mathcal{T}^{(n)}$ denotes the derivative of $f^{(n)} = f^{(n)}(s)$ with respect to s, before equating powers of ϵ. The hierarchy of models at different orders is:

$$\begin{aligned}
O(1/\epsilon): \qquad f &= 0, && (2.3.7)\\
O(1): \qquad f &= \epsilon\,\Gamma^{-1} F(s, 0; 0), \qquad \frac{\partial s}{\partial t} = S(s, 0; 0), && (2.3.8)\\
O(\epsilon): \qquad f &= \epsilon\,\Gamma^{-1}\Big\{F(s, f; \epsilon)\Big|_{O(\epsilon)} - \epsilon\,\Gamma^{-1}\frac{\partial F(s, 0; 0)}{\partial s} S(s, 0; 0)\Big\}, &&\\
\frac{\partial s}{\partial t} &= S(s, f; \epsilon)\Big|_{O(\epsilon)}. && (2.3.9)\\
\vdots \quad & \quad \vdots &&\\
O(\epsilon^n): \qquad f &= \epsilon\,\Gamma^{-1}\Big\{F(s, f; \epsilon) - \mathcal{T} S(s, f; \epsilon)\Big\}\Big|_{O(\epsilon^n)}, && (2.3.10)\\
\frac{\partial s}{\partial t} &= S(s, f; \epsilon)\Big|_{O(\epsilon^n)}, && (2.3.11)
\end{aligned}$$

where $(\cdot)|_{O(\epsilon^n)}$ denotes inclusion of all powers in ϵ up to and including terms of order ϵ^n.

Alternatively, we may also use the ϵ-ordering in the superbalance equation (2.3.4) to define linear or nonlinear iterative approximations $U_n(s)$ of $U(s)$, e.g. by starting with $U_0(s) = 0$.

Example 8: The slaving approach is readily illustrated for the Lorenz equations (2.1.22) or (2.2.9), which may be seen to fit the prototypical format (2.3.1)–(2.3.2). The slaving ansatz implies that

$$U_4 = U_4(\phi, x_3) \quad \text{and } U_5 = U_5(\phi, x_3). \qquad (2.3.12)$$

Differentiation of these relations with respect to time and use of the equation of motion (2.1.22) gives the corresponding superbalance equations

$$\begin{aligned}
-\frac{1}{\epsilon}\,U_5 &= \frac{dU_4}{dt} = \frac{\partial U_4}{\partial \phi}\,(x_3 - b\,x_5) - \frac{\partial U_4}{\partial x_3}\,C\,\sin 2\phi,\\
\frac{1}{\epsilon}\,U_4 + b\,C\,\sin 2\phi &= \frac{dU_5}{dt} = \frac{\partial U_5}{\partial \phi}\,(x_3 - b\,x_5) - \frac{\partial U_5}{\partial x_3}\,C\,\sin 2\phi. \qquad (2.3.13)
\end{aligned}$$

Substitution of $U_4 = U_4^{(0)} + \epsilon\,U_4^{(1)} + \ldots,\ U_5 = U_5^{(0)} + \epsilon\,U_5^{(1)} + \ldots$ into (2.3.13), following the outlined expansion procedure, gives the leading and first order results

$$U_4^{(0)} = U_5^{(0)} = 0, \quad U_4^{(1)} = -b\,C\,\sin 2\phi, \quad U_5^{(1)} = 0, \qquad (2.3.14)$$

which readily extend to higher order via algebraic manipulations. It is also straightforward to show that substitution of an expansion of x_4 and x_5, in a power series of ϵ, into the fast equations gives the same result.

Fluid dynamical examples of these expansion or iteration slaving procedures for the rapidly rotating shallow-water and stratified Boussinesq equations can be found in Allen (1993), Warn et al. (1995) and in Bokhove (1997). Slaving principles for compressible barotropic flows and the shallow-water equations with Mach and Rossby numbers as small parameter, respectively, are considered in sections 3.3 and 3.5.

2.4 Constrained Hamiltonian formulation

Given a "parent" variational principle we may want to impose constraints on the parent dynamics. In principle this can be done either by *direct substitution* of the constraints into the variational principle, although this may not always be possible for implicit constraints, or by augmentation of constraints on the variational principle with Lagrange multipliers (e.g. Lanczos 1970). In either case one has to ensure consistency and check whether the Lagrange multipliers can be solved under the newly constructed constrained dynamics. If there is an inconsistency more, so-called, secondary constraints may arise which need to be imposed as well, and so forth, but ultimately we arrive either at a case where constrained dynamics and constraints form a closed system or one that reveals an inconsistency which suggests that the imposed set of constraints is ill-posed.

In the previous section we argued that approximation methods based on slaving principles yield asymptotic constraints in a systematic manner and this forms hopefully some safeguard against ill-posed constraints, at least for small values of ϵ. Hereafter we assume that constraints obey their consistency requirements although this must always be checked.

An example of direct substitution of constraints is presented in section 3.6.3 where ageostrophic barotropic equations are derived from their Eulerian shallow-water parent action principle.

2.4.1 Lagrange multipliers

Assume we have (asymptotic) constraints of the form $\tilde{f}(z) = f - U(s) = 0$ (cf. (2.3.3)). Defining $z = \{s, f\}$ a constrained form of action principle (2.1.14) is

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ a_m(z) \frac{dz^m}{dt} - H + \lambda_\alpha(t) \tilde{f}^\alpha \right\} \tag{2.4.1}$$

for independent variations δz^m and $\delta\lambda_\alpha(t)$, where $\lambda_\alpha(t)$ are Lagrange multipliers. The Lagrange multipliers enforce the constraints at every time t and are therefore a function of time. Variation of (2.4.1) with respect to $\lambda_\alpha(t)$ yields the constraints $\tilde{f^\alpha} = 0$. The invertibility of $\tilde{\mathbf{K}}$ guarantees that a constrained Hamiltonian formulation exists with

$$\begin{aligned} \frac{dF}{dt} &= [F, H]^* = [F, H] - [F, \lambda_\alpha \tilde{f^\alpha}] \\ &= \frac{\partial F}{\partial z^i} J^{ij} \Big(\frac{\partial H}{\partial z^j} - \lambda_\alpha \frac{\partial \tilde{f^\alpha}}{\partial z^j} \Big) \end{aligned} \tag{2.4.2}$$

or, alternatively,

$$\frac{dF}{dt} = [F, H^*] \tag{2.4.3}$$

with parent bracket (2.2.2) and Hamiltonian

$$H^* = H - \lambda_\alpha \tilde{f^\alpha}. \tag{2.4.4}$$

2.4.2 Dirac's theory

Dirac's theory of constrained Hamiltonian dynamics deals with (time-independent) Lagrangians $L(q^i, \dot{q}^i)$ that are singular (Dirac 1950, 1958, 1964; Sudarshan and Mukunda 1974; for a geometric account see Gotay, Nester and Hinds 1978 and Marsden and Ratiu 1994). Singular Lagrangians may, for example, arise after scaling and truncation of a non-singular parent Lagrangian. Take $i = 1, \ldots K$ such that $N = 2K$. Normally, the momenta p_i defined by (2.1.8) are independent, but for a singular Lagrangian a Legendre transformation is less trivial because the Jacobian

$$\frac{\partial p_i}{\partial \dot{q}^j} = \frac{\partial^2 L}{\partial \dot{q}^i \, \partial \dot{q}^j} \tag{2.4.5}$$

is singular. This implies that the transformation between the coordinate pairs $\{q, \dot{q}\}$ and $\{q, p\}$ is singular and the corank of the Jacobian (2.4.5) is, say, N_c. Consequently, N_c of the K equations (2.1.8) may be expressed in terms of constraints involving the p's and q's, e.g. $N_c = 2\,M$ constraints of the form $\tilde{f^\alpha}(p, q) \simeq 0$ (cf. [2.3.3]). (Isolated singularities do not necessarily inhibit a global calculation of the Legendre transform, see Sewell (1999).)

The variations of δp_i and δq^i are no longer independent because of the constraints and the Legendre transformation is non-unique. Independent variations of δp_i and δq^i can nevertheless still be enforced by introducing Lagrange multipliers $\lambda_\alpha(t)$ (e.g. Lanczos 1970). The constrained action principle is

$$\begin{aligned} 0 \;=\; & \int_{t_0}^{t_1} \mathrm{d}t \Big\{ \Big(-\frac{\partial H}{\partial q^i} - \frac{dp_i}{dt} \Big) \delta q^i + \Big(-\frac{\partial H}{\partial p_i} + \frac{dq^i}{dt} \Big) \delta p_i \Big\} + \\ & \int_{t_0}^{t_1} \mathrm{d}t \Big(\lambda_\alpha(t)\, \delta \tilde{f^\alpha}(q^i, p_i) + \tilde{f^\alpha}(q^i, p_i)\, \delta\lambda_\alpha(t) \Big), \end{aligned} \tag{2.4.6}$$

where the variations of $\lambda_\alpha(t)$ yield the constraints, and where $H(q^i, p_i) = p_i\,\dot q^i - L(q^i, \dot q^i)$. The equations of motion follow from (2.4.6)

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} - \lambda_\alpha \frac{\partial \tilde{f}^\alpha}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} + \lambda_\alpha \frac{\partial \tilde{f}^\alpha}{\partial q^i}, \tag{2.4.7}$$

and correspond to dynamics (2.4.3) with Hamiltonian (2.4.4) when the bracket $[\cdot,\cdot]$ is identified with the canonical bracket (2.2.5).

Example 9: Suppose Hamilton's principle for Example 1 is scaled such that the term $\dot z^2$ (or w^2) is $O(\delta^2)$ relative to the horizontal kinetic energy terms. Hamilton's principle truncated to $O(1)$ then becomes

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ \frac{1}{2}\,(\dot x^2 + \dot y^2) - V(x,y,z) \right\}, \tag{2.4.8}$$

and this Lagrangian is singular since the vertical conjugate momentum $f_1 = w \equiv \partial L/\partial \dot z = 0$. Variation of (2.4.8) with respect to δz now yields a secondary constraint $\partial V/\partial z = 0$. Variation of the constrained action principle

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ u\,\frac{dx}{dt} + v\,\frac{dy}{dt} + w\,\frac{dz}{dt} - H^{(0)}(x,y,z,u,v) + \lambda\,w \right\}, \tag{2.4.9}$$

with the $O(1)$ Hamiltonian

$$H^{(0)}(x,y,z,u,v) = \frac{1}{2}\,(u^2+v^2) + V(x,y,z), \tag{2.4.10}$$

with respect to $\lambda, u, v, w, x, y, z$ yields the constraint and constrained dynamics

$$\begin{aligned} f_1 &\equiv w = 0, \quad \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = -\lambda, \quad \frac{du}{dt} = -\frac{\partial V}{\partial x}, \\ \frac{dv}{dt} &= -\frac{\partial V}{\partial y}, \quad \frac{dw}{dt} = [w, H^{(0)}] - \lambda\,[w,w] = -\frac{\partial V}{\partial z}, \end{aligned} \tag{2.4.11}$$

respectively, with

$$[F,G] = \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial u_i} - \frac{\partial G}{\partial x_i}\frac{\partial F}{\partial u_i}$$

and $x_1 = x$, $u_1 = u$, etc. Notice that $-\lambda$ has taken the rôle of vertical velocity. Consistency requires now that $dw/dt \simeq 0$, yielding the secondary constraint $f_2 \equiv \partial V/\partial z = 0$. Consistency of $f_2 = 0$ under constrained dynamics (2.4.11) requires that

$$\frac{df_2}{dt} \simeq [f_2, H^{(0)}] - \lambda\,[f_2, w] = \frac{\partial^2 V}{\partial x\,\partial z}\,u + \frac{\partial^2 V}{\partial y\,\partial z}\,v - \lambda\,\frac{\partial^2 V}{\partial z^2} = 0 \tag{2.4.12}$$

which determines λ and closes the system for a suitable potential $V(x,y,z)$.

A fluid example analogous to Example 9 is given in section 3.2, where a singular variational principle for a compressible fluid in Lagrangian variables emerges after elimination of the vertical velocity in the kinetic energy on the basis of scaling arguments involving the aspect ratio of vertical to horizontal velocities. The resulting constrained dynamics are the familiar hydrostatically balanced fluid equations used in meteorology.

2.4.3 Postulation of constrained dynamics

So-far we have imposed constraints on variational principles and a constrained bracket will arise in all (consistent) cases. Alternatively, we can postulate dynamics (2.4.2) with constrained dynamics

$$\frac{dF}{dt} = [F,H]^* \simeq [F,H] - \lambda_\alpha [F,\tilde{f}^\alpha] = [F,H^*] \tag{2.4.13}$$

given a set of constraints (2.3.3) and (non-canonical) Hamiltonian parent dynamics (2.2.1) with bracket $[F,G]$. The symbol "$\simeq$" in (2.4.13) denotes that the brackets must be calculated before the constraints (2.3.3) are applied, following Dirac (1950, 1958, 1964) and Sudarshan and Mukunda (1974). Invariance of the constraints under the constrained dynamics, i.e. consistency,

$$\frac{d\tilde{f}^\beta}{dt} \simeq 0 = [\tilde{f}^\beta, H] - \lambda_\gamma\, [\tilde{f}^\beta, \tilde{f}^\gamma] \tag{2.4.14}$$

in principle ensures that the Lagrange multipliers can be solved. Although we can usually trace all Hamiltonian dynamics back to a variational principle it may not always be convenient or physically meaningful to do so, in which case postulating non-canonical constrained dynamics may be useful. Note that Casimirs C of the parent bracket $[F,G]$ remain Casimirs for dynamics (2.4.13).

In section 3.6.2 the ageostrophic barotropic equations are derived from a postulated constrained bracket with the shallow-water Hamiltonian formulation as parent dynamics. In this case the Lagrange multipliers will be proportional to the velocity which advects fluid parcels.

Subsequently, it is assumed that for all $\tilde{f}^\beta$ the tensor $\mathbf{\Delta}$, defined by

$$\Delta^{\beta\gamma} \equiv [\tilde{f}^\beta, \tilde{f}^\gamma], \tag{2.4.15}$$

is non-singular, which implies that it must have an even number of components. The condition

$$\det|\mathbf{\Delta}| \neq 0 \tag{2.4.16}$$

is a restriction suitable for the elimination of pairs of fast variables. In physical terms, fast modes refer for example to pairs of acoustic- and gravity-wave modes, which may be eliminated by applying constraints. Consequently, one may introduce the inverse **C** of the transpose of **Δ**. In components **C** is written as

$$C_{\beta\alpha}\,[\tilde{f^\beta}, \tilde{f^\gamma}] = \delta^\gamma_\alpha, \tag{2.4.17}$$

where δ^γ_α is the Kronecker-delta symbol, and it follows that **C** is skew-symmetric. Multipliers λ_α can now be determined from consistency requirements (2.4.14) to be

$$\lambda_\alpha = C_{\beta\alpha}\,[\tilde{f^\beta}, H]. \tag{2.4.18}$$

The constrained dynamics (2.4.13) thus takes the form

$$\begin{aligned} \frac{dF}{dt} &\simeq [F,H] - C_{\beta\alpha}\,[\tilde{f^\beta}, H]\,[F, \tilde{f^\alpha}] \\ &= [F,H] + [F, \tilde{f^\alpha}]\,C_{\alpha\beta}\,[\tilde{f^\beta}, H]. \end{aligned} \tag{2.4.19}$$

This gives rise to the so-called *Dirac bracket*

$$[F,G]^* \simeq [F,G] + [F, \tilde{f^\alpha}]\,C_{\alpha\beta}\,[\tilde{f^\beta}, G], \tag{2.4.20}$$

which may be shown to satisfy the conditions (2.2.3) (i)–(iii) when the bracket $[\cdot,\cdot]$ itself satisfies these conditions (Dirac (1950)). Any function $K(\tilde{f^\gamma})$ of the constraints (2.3.3) satisfies $[F,K]^* = 0$, because

$$\begin{aligned} [F,K]^* &\simeq [F,K] + [F, \tilde{f^\alpha}]\,C_{\alpha\beta}\,[\tilde{f^\beta}, \tilde{f^\gamma}]\,\frac{\partial K}{\partial \tilde{f^\gamma}} \\ &= [F,K] - [F, \tilde{f^\alpha}]\,\delta^\gamma_\alpha\,\frac{\partial K}{\partial \tilde{f^\gamma}} = 0. \end{aligned} \tag{2.4.21}$$

Thus, the result is that for a transformation from $\{s^i, f^\alpha\}$ to $\{\tilde{s^j} = s^j, \tilde{f^\alpha} = f^\alpha - U^\alpha(s) \simeq 0\}$ an explicit reduction of the dynamics to the constrained slaving manifold may be established. After this transformation, one needs only to refer to functions of $\tilde{s}$ since (2.4.21) renders any reference to the dependence on $\tilde{f}$ superfluous. Consequently, the variations of H with respect to $\tilde{f^\alpha}$ can be neglected, and one may use $H(\tilde{s}, \tilde{f} = 0)$. Again the original Casimirs C are also Casimirs of the Dirac bracket; moreover, the constraints can be considered as additional Casimirs.

Example 10: Direct substitution of the slaving ansatz or constraints (2.3.12) into the action principle (2.1.20) of Example 5 is equivalent to using (2.4.13) with Lagrange multipliers λ_α, which in Lorenz's model may be eliminated explicitly. The balanced Dirac bracket becomes

$$[F,G]^* = \left(\frac{\partial F}{\partial \phi}\frac{\partial G}{\partial x_3} - \frac{\partial F}{\partial x_3}\frac{\partial G}{\partial \phi}\right)\Big(1 + \epsilon g_L(\phi, x_3)\Big)^{-1} \tag{2.4.22}$$

with $g_L(\phi, x_3)$ defined by

$$g_L(\phi, x_3) = -\left(b + \frac{\partial U_5}{\partial x_3}\right)\frac{\partial U_4}{\partial \phi} + \frac{\partial U_4}{\partial x_3}\frac{\partial U_5}{\partial \phi}, \tag{2.4.23}$$

together with the Hamiltonian

$$H^* = -\frac{1}{2}C\cos 2\phi + \frac{1}{2}\left(x_3^2 + U_4^2(\phi, x_3) + U_5^2(\phi, x_3)\right). \tag{2.4.24}$$

The bracket (2.4.22) obeys all mathematical requirements by construction for any $U_4(\phi, x_3)$ or $U_5(\phi, x_3)$, assuming the denominator in (2.4.22) is nonzero.

The Hamiltonian balanced equations derived from $dF/dt = [F, H^*]^*$ and (2.4.22)–(2.4.24) are

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{1}{1+\epsilon\, g_L(\phi, x_3)}\left(x_3 + U_4\frac{\partial U_4}{\partial x_3} + U_5\frac{\partial U_5}{\partial x_3}\right), \\ \frac{dx_3}{dt} &= -\frac{1}{1+\epsilon\, g_L(\phi, x_3)}\left(C\sin 2\phi + U_4\frac{\partial U_4}{\partial \phi} + U_5\frac{\partial U_5}{\partial \phi}\right). \end{aligned} \tag{2.4.25}$$

2.5 Slaved Hamiltonian dynamics [1]

Rather than first deriving constraints and subsequently imposing these on a variational or Hamiltonian formulation, it will be shown that a balanced Hamiltonian formulation arises in one singular Hamiltonian perturbation approach based on a slaving principle. To use the ideas of slaving principles and singular perturbations, it will be assumed that the prototypical singular system (2.3.1), (2.3.2) is Hamiltonian. The linearized Hamiltonian is then necessarily quadratic in the fast variables f.

A valid prototypical Hamiltonian version of (2.3.1), (2.3.2), but perhaps not the most general, can be hypothesized to have the form

$$\frac{dF}{dt} = [F, H] \tag{2.5.1}$$

with Poisson bracket

$$\begin{aligned} [F, G] = &\frac{\partial F}{\partial s^i} J_{ss}^{ij} \frac{\partial G}{\partial s^j} + \frac{\partial F}{\partial s^i} J_{sf}^{i\alpha} \frac{\partial G}{\partial f^\alpha} + \frac{\partial F}{\partial f^\beta} J_{fs}^{\beta j} \frac{\partial G}{\partial s^j} - \\ &\frac{\partial F}{\partial f^\beta} \frac{\Gamma_\gamma^\beta (A^{-1})^{\gamma\alpha}}{\epsilon} \frac{\partial G}{\partial f^\alpha} + \frac{\partial F}{\partial f^\beta} J_{ff}^{\beta\alpha} \frac{\partial G}{\partial f^\alpha} \end{aligned} \tag{2.5.2}$$

[1] Joint work with T.G. Shepherd.

satisfying skew-symmetry and Jacobi's identity, and a Hamiltonian H whose small-amplitude form is quadratic in f^α at leading order in ϵ:

$$\frac{\partial H}{\partial f^\alpha} = A_{\alpha\nu}\, f^\nu + \epsilon\, r_\alpha(s, f; \epsilon) \tag{2.5.3}$$

for an invertible constant matrix $\mathbf{A}$ (the Greek lowercase symbols range from $1, \ldots, 2M$ such that there are an even number of fast variables) and with $r_\alpha = O(1)$. Furthermore, it is assumed that J_{ss}^{ij}, $J_{ff}^{\beta\alpha}$, and $J_{sf}^{i\alpha} = -J_{fs}^{\alpha i}$ are $O(1)$. All these definitions and assumptions are in agreement with the $O(1)$-magnitude of S and F in (2.3.1)–(2.3.2). The cosymplectic tensor corresponding to (2.5.2) has the form

$$\mathbf{J} = \begin{pmatrix} J_{ss} & J_{sf} \\ J_{fs} & -\frac{\Gamma A^{-1}}{\epsilon} + J_{ff} \end{pmatrix}, \tag{2.5.4}$$

and the cosymplectic form of the equations of motion is

$$\frac{ds^i}{dt} = J_{ss}^{ij}\,\frac{\partial H}{\partial s^j} + J_{sf}^{i\alpha}\,\frac{\partial H}{\partial f^\alpha}, \tag{2.5.5}$$

$$\frac{df^\beta}{dt} = -\frac{\Gamma_\gamma^\beta\,(A^{-1})^{\gamma\alpha}}{\epsilon}\,\frac{\partial H}{\partial f^\alpha} + J_{fs}^{\beta j}\,\frac{\partial H}{\partial s^j} + J_{ff}^{\beta\alpha}\,\frac{\partial H}{\partial f^\alpha}. \tag{2.5.6}$$

In section 2.3, it was pointed out that the system (2.3.1), (2.3.2) has fixed points or a basic state corresponding to $s = 0, f = 0$. Consequently, the Hamiltonian prototypical system is assumed to have a quadratic Hamiltonian in f for $\epsilon \to 0$. This is not true in general, especially not for non-canonical Hamiltonian systems encountered in fluid dynamics. However, H can be replaced by the pseudo energy P, which will be defined in section 2.7, without changing the dynamics such that the small-amplitude form of P is quadratic in f.

In order to construct Hamiltonian balanced models, the slaving principle or approximations thereof will be considered as the following set of constraints

$$\tilde{f^\alpha} = f^\alpha - U^\alpha(s) \simeq 0 \tag{2.5.7}$$

which may be exact invariants of the full flow. The tensor $\mathbf{\Delta}$ defined by

$$\begin{aligned}\Delta^{\alpha\beta} \equiv [\tilde{f^\alpha}, \tilde{f^\beta}] = {} & -\frac{\Gamma_\gamma^\alpha\,(A^{-1})^{\gamma\beta}}{\epsilon} + J_{ff}^{\alpha\beta} + \frac{\partial U^\alpha}{\partial s^i}\,J_{ss}^{ij}\,\frac{\partial U^\beta}{\partial s^j} - \\ & \frac{\partial U^\alpha}{\partial s^i}\,J_{sf}^{i\beta} - J_{fs}^{\alpha j}\,\frac{\partial U^\beta}{\partial s^j}\end{aligned} \tag{2.5.8}$$

will be nonzero for sufficiently small ϵ, and hence invertible, because both $\mathbf{\Gamma}$ and $\mathbf{A}$ are assumed to be invertible. Note that $\mathbf{\Delta}$ is then necessarily even-dimensional. The tensor $\mathbf{C}$ defined by the inverse of $\mathbf{\Delta}^T$ is $C_{\beta\alpha}\,[\tilde{f^\beta}, \tilde{f^\gamma}] = \delta_\alpha^\gamma$.

Consider a coordinate transformation $\tilde{s} = s, \tilde{f} = f - U(s)$, transforming the bracket (2.5.2) into

$$\begin{aligned}[F,G] \quad = \quad & \frac{\partial F}{\partial \tilde{s^i}} J^{ij}_{ss} \frac{\partial G}{\partial \tilde{s^j}} + \frac{\partial F}{\partial \tilde{s^i}} \left(J^{i\alpha}_{sf} - J^{ij}_{ss} \frac{\partial U^\alpha}{\partial \tilde{s^j}} \right) \frac{\partial G}{\partial \tilde{f^\alpha}} + \\ & \frac{\partial F}{\partial \tilde{f^\beta}} \left(J^{\beta j}_{fs} - J^{ij}_{ss} \frac{\partial U^\beta}{\partial \tilde{s^i}} \right) \frac{\partial G}{\partial \tilde{s^j}} + \frac{\partial F}{\partial \tilde{f^\beta}} \left(- \frac{\Gamma^\beta_\gamma (A^{-1})^{\gamma\alpha}}{\epsilon} + J^{\beta\alpha}_{ff} + \right. \\ & \left. \frac{\partial U^\beta}{\partial \tilde{s^i}} J^{ij}_{ss} \frac{\partial U^\alpha}{\partial \tilde{s^j}} - \frac{\partial U^\beta}{\partial \tilde{s^i}} J^{i\alpha}_{sf} - J^{\beta j}_{fs} \frac{\partial U^\alpha}{\partial \tilde{s^j}} \right) \frac{\partial G}{\partial \tilde{f^\alpha}}, \end{aligned} \tag{2.5.9}$$

and together with (2.5.1) and the Hamiltonian H, the following equations of motion appear by definition

$$\frac{d\tilde{s^i}}{dt} = [\tilde{s^i}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}} + [\tilde{s^i}, \tilde{f^\alpha}] \frac{\partial H}{\partial \tilde{f^\alpha}}, \tag{2.5.10}$$

$$\frac{d\tilde{f^\beta}}{dt} = [\tilde{f^\beta}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}} + [\tilde{f^\beta}, \tilde{f^\alpha}] \frac{\partial H}{\partial \tilde{f^\alpha}}. \tag{2.5.11}$$

If the constraints (2.5.7) are (approximately) conserved by the primitive equations (2.5.5), (2.5.6), then it follows from (2.5.11) that (approximately)

$$\frac{d\tilde{f^\beta}}{dt} \simeq [\tilde{f^\beta}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}} + [\tilde{f^\beta}, \tilde{f^\alpha}] \frac{\partial H}{\partial \tilde{f^\alpha}} = 0, \tag{2.5.12}$$

where the notation $\simeq$ indicates that the brackets have to be calculated before the constraints are applied. Using (2.4.17), the expression (2.5.12) may be rewritten as

$$\frac{\partial H}{\partial \tilde{f^\alpha}} \simeq C_{\alpha\beta} [\tilde{f^\beta}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}}, \tag{2.5.13}$$

and after substituting (2.5.13) into (2.5.10) one obtains the balanced dynamics

$$\frac{d\tilde{s^i}}{dt} \simeq [\tilde{s^i}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}} + [\tilde{s^i}, \tilde{f^\alpha}] C_{\alpha\beta} [\tilde{f^\beta}, \tilde{s^j}] \frac{\partial H}{\partial \tilde{s^j}}, \tag{2.5.14}$$

$$\tilde{f^\alpha} \simeq 0. \tag{2.5.15}$$

Note that, after the brackets are calculated in (2.5.14), the Hamiltonian is subject to derivatives with respect to $\tilde{s}$ only. The Hamiltonian formulation which follows from the dynamics (2.5.14), (2.5.15) is

$$\frac{dF}{dt} = [F, H]^* \tag{2.5.16}$$

and the cosymplectic tensor corresponding to (2.5.14) gives the Dirac bracket $[\cdot, \cdot]^*$ (Dirac 1950, 1958, 1964; and Sudarshan and Mukunda 1974) we encountered before

$$[F, G]^* \simeq [F, G] + [F, \tilde{f^\alpha}] C_{\alpha\beta} [\tilde{f^\beta}, G]. \tag{2.5.17}$$

The conclusion is that the slaving approach for a prototypical singular Hamiltonian system (2.5.1), (2.5.2) systematically yields the (approximate) Hamiltonian balanced dynamics (2.5.16), (2.5.17). Although in the derivation of (2.5.17) F, G, and H are initially restricted to be functions of $\tilde{s}$ only, it may be verified directly or via Dirac's theory that this restriction may be lifted.

A few remarks are useful. First, when the constraints (2.3.3) are exactly conserved by the full dynamics, then it should be no surprise to us that the balanced dynamics are Hamiltonian, since the primitive equations are Hamiltonian. It is then unnecessary to modify the original, unconstrained dynamics, as is done in Dirac's approach, because the unconstrained dynamics do conserve the constraints already. The balanced dynamics (2.5.16), (2.5.17) are than nothing else than a reduced description of the evolution on the constrained manifold defined by (2.5.7). Second, the situation is more complex when the constraints are approximate. In that case, the constraints are not exactly conserved by the unconstrained dynamics given by the primitive equations (2.5.5), (2.5.6). When the dynamics is modified by adding additional forces that constrain the dynamics to the constrained manifold $\tilde{f}^\alpha \simeq 0$, the Dirac bracket is interpreted as a modification of the forces in the full dynamics such that the constraints are conserved exactly by the modified dynamics. It is the classical way in which Lagrange multipliers are interpreted as the forces of reaction exerted on account of given constraints (Lanczos 1970). This modified (Hamiltonian) dynamics thus deviates relative to the original, unconstrained dynamics. Even when $\tilde{f} = 0$ initially $d\tilde{f}/dt$, as determined by the unconstrained dynamics, is not; the difference of the unconstrained and the modified dynamics will then generally increase in time. Although trajectories of the parent and constrained dynamics with the same initial conditions are expected to diverge, the constrained Hamiltonian dynamics may preserve long-term stability and statistical properties better than non-conservative approximations. Third, further simplifications may in principle be realized by expansion of the bracket (2.5.17) in a Taylor series in ϵ, after expressing the constraints in a power series in ϵ. It is then tempting to try to truncate (2.5.17) and the Hamiltonian separately, in such a way that skew-symmetry and most notably Jacobi's identity are preserved. In the next section, it is shown that a leading-order truncation always preserves the Hamiltonian structure.

2.6 Leading-order Hamiltonian perturbations [2]

For many, and most notably, continuous systems the construction of a Dirac bracket is too complicated. Rather than finding the inverse of $[\tilde{f}^\beta, \tilde{f}^\alpha]$ or an explicit solution of Lagrange multipliers λ_α, these quantities may be solved for in a perturbative way such that skew-symmetry and Jacobi's identity are preserved. In other words, for constraints $\tilde{f}^\alpha$ formally accurate to $O(\epsilon^{n+1})$ the

[2]Joint work with T.G. Shepherd.

challenge is to find a cosymplectic truncation to $O(\epsilon^n)$ of the Dirac bracket (2.5.17), since there are no mathematical restrictions on truncations of the Hamiltonian. If the Hamiltonian is also truncated to $O(\epsilon^n)$, then the resulting balanced dynamics can be shown to contain all terms up to $O(\epsilon^n)$ (plus some higher order terms).

The cosymplectic tensor corresponding to (2.5.17) is

$$\breve{J}^{ij}_{\tilde{s}\tilde{s}} \;=\; [\tilde{s^i}, \tilde{s^j}] + [\tilde{s^i}, \tilde{f^\beta}]C_{\beta\alpha}[\tilde{f^\alpha}, \tilde{s^j}] = J^{ij}_{\tilde{s}\tilde{s}} + J^{i\beta}_{\tilde{s}\tilde{f}}C_{\beta\alpha}J^{\alpha j}_{\tilde{f}\tilde{s}}. \tag{2.6.1}$$

It may be seen from (2.4.17), (2.5.8) and (2.5.9) that $[\tilde{s}, \tilde{f}] = O(1)$ and $\mathbf{C} = O(\epsilon)$. The leading-order truncation of (2.6.1) is thus

$$\breve{J}^{ij}_{\tilde{s}\tilde{s}}\Big|_{O(1)} \equiv J^{ij}_{\tilde{s}\tilde{s}}\Big|_{O(1)}, \tag{2.6.2}$$

which is shown to obey Jacobi's identity by the following argument.

Consider Jacobi's identity $[F,[G,K]] + [K,[F,G]] + [G,[K,F]] = 0$ for the bracket (2.5.9) with arbitrary functions F, G, K. If $F = F(\tilde{s}), G = G(\tilde{s})$ and $K = K(\tilde{s})$ then Jacobi's identity is

$$J^{im}_{\tilde{s}\tilde{s}}\frac{\partial J^{jk}_{\tilde{s}\tilde{s}}}{\partial \tilde{s}^m} + J^{i\mu}_{\tilde{s}\tilde{f}}\frac{\partial J^{jk}_{\tilde{s}\tilde{s}}}{\partial \tilde{f}^\mu} + \text{cyclic permutations} = 0, \tag{2.6.3}$$

and if $F = F(\tilde{s}), G = G(\tilde{s})$ and $K = K(\tilde{f})$ then Jacobi's identity is

$$\begin{aligned} J^{im}_{\tilde{s}\tilde{s}}\frac{\partial J^{j\kappa}_{\tilde{s}\tilde{f}}}{\partial \tilde{s}^m} + J^{i\mu}_{\tilde{s}\tilde{f}}\frac{\partial J^{j\kappa}_{\tilde{s}\tilde{f}}}{\partial \tilde{f}^\mu} + J^{jm}_{\tilde{s}\tilde{s}}\frac{\partial J^{\kappa i}_{\tilde{f}\tilde{s}}}{\partial \tilde{s}^m} + J^{j\mu}_{\tilde{s}\tilde{f}}\frac{\partial J^{\kappa i}_{\tilde{f}\tilde{s}}}{\partial \tilde{f}^\mu} \quad &+ J^{\kappa m}_{\tilde{f}\tilde{s}}\frac{\partial J^{ij}_{\tilde{s}\tilde{s}}}{\partial \tilde{s}^m} + \\ \left(\frac{\Gamma^\kappa_\gamma (A^{-1})^{\gamma\mu}}{\epsilon} + J^{\kappa\mu}_{\tilde{f}\tilde{f}}\right)\frac{\partial J^{ij}_{\tilde{s}\tilde{s}}}{\partial \tilde{f}^\mu} \quad &= \quad 0. \end{aligned} \tag{2.6.4}$$

Since it is assumed that $J_{\tilde{s}\tilde{s}}$ and $J_{\tilde{s}\tilde{f}}$ are $O(1)$ (2.6.4) gives at leading order $(\partial J^{ij}_{\tilde{s}\tilde{s}}/\partial \tilde{f}^\mu)|_{O(1)} = 0$. Substitution of this result into (2.6.3) then implies that at leading order

$$\left\{J^{im}_{\tilde{s}\tilde{s}}\frac{\partial J^{jk}_{\tilde{s}\tilde{s}}}{\partial \tilde{s}^m}\right\}\Big|_{O(1)} + \text{cyclic permutations} = 0, \tag{2.6.5}$$

which proves the claim that the truncated leading-order version of the Dirac bracket is cosymplectic.

At leading order, the modified expansion gives $f = 0$ (cf. 2.3.7), and one finds $\tilde{s} = s, \tilde{f} = f$. The balanced dynamics at leading order (cf. (2.3.8)) is thus cosymplectic when the abstract prototypical parent dynamics is Hamiltonian. In summary, the leading-order singular Hamiltonian method reduces the original bracket (2.5.2) with Hamiltonian $H(s, f)$ to the constrained bracket

$$[F', G']_0 = \frac{\partial F'}{\partial s^i} J^{ij}_{ss} \frac{\partial G'}{\partial s^j} \tag{2.6.6}$$

with Hamiltonian $H^{(0)} = H(s,0)$ (or pseudo energy $P^{(0)}$) and $F' \equiv F(s)$, etc. Higher-order cosymplectic truncations or other simplifications of the Dirac bracket have not been obtained. $O(\epsilon)$ truncations of the Dirac bracket generally do not preserve Jacobi's identity automatically since it is a quadratic identity. We expect therefore that some higher-order terms need to be included in order to satisfy Jacobi's identity, but further research is necessary to identify these terms.

In section 3 we present two examples of the Hamiltonian leading-order perturbation approach: a modified Mach-number expansion of barotropic three-dimensional compressible fluid equations around a resting basic state of constant density which provides a systematic Hamiltonian derivation of the incompressible Euler equations in section 3.3, and a modified Rossby-number expansion of rotating shallow-water equations which provides a systematic Hamiltonian derivation of the barotropic quasi-geostrophic equations in section 3.5.

2.7 Formal stability

Steady states of the Hamiltonian equations (2.1.18) are solutions $z^i = Z^i$ of

$$0 = J^{ij}(z)\,\frac{\partial H}{\partial z^j}\bigg|_{z=Z}. \tag{2.7.1}$$

Let us consider the stability of these steady states Z. Decomposing $z = Z + z'$ into steady state and perturbation component z' we find

$$\frac{d(z')^i}{dt} = \left(J^{ij}(z)\,\frac{\partial^2 H}{\partial z^j\,\partial z^l} + \frac{\partial J^{ij}}{\partial z^l}\,\frac{\partial H}{\partial z^j}\right)\bigg|_Z (z')^l + O(z'^2). \tag{2.7.2}$$

When the cosymplectic tensor J^{ij} is non-singular, $(\partial H/\partial z^j)|_Z = 0$. Liapunov or norm stability (Arnol'd 1989) is now achieved when the Hessian matrix H_{zz} is sign definite. All eigenvalues of the linearized system are purely imaginary under these conditions. For Hamiltonians of the form $H = \frac{1}{2}\sum p_i^2 + V(q)$ stability is ensured when the second variations $\delta^2 H$, leading to a Hessian $H_{qq} \equiv V_{qq}$, are positive definite.

For non-canonical systems with nontrivial Casimirs the stability of steady states may be established with the energy-Casimir method. The method is useful when steady states are constrained extremals of the energy, i.e. Casimir(s) C_k are imposed as constraints on the Hamiltonian by Lagrange multipliers λ_k such that the pseudo energy $P = H(z) + \lambda_k\,C_k(z) - H(0) - \lambda_k\,C_k(0)$ is extremal at the steady state

$$\frac{\partial P}{\partial z}\bigg|_Z = 0. \tag{2.7.3}$$

The pseudo energy P may replace H as the Hamiltonian, since Casimirs are by definition invariants of the flow

$$\frac{dC_k}{dt} = [C_k, H] = 0, \tag{2.7.4}$$

and is an invariant that is quadratic to leading order in the disturbances z'. For singular J^{ij} (2.7.2) becomes

$$\begin{aligned} \frac{d(z')^i}{dt} &= \left\{ J^{ij}(z) \frac{\partial^2 P}{\partial z^j \partial z^l} + \frac{\partial J^{ij}}{\partial z^l} \frac{\partial P}{\partial z^j} \right\}\Bigg|_Z {z'}_l + O(z'^2) \\ &= \frac{\partial^2 P}{\partial z^j \partial z^l}\Bigg|_Z (z')^l + O(z'^2). \end{aligned} \tag{2.7.5}$$

Recall that J is in general not invertible. When the second variation $\delta^2 P$ is sign definite, i.e. Hessian P_{zz} is sign definite, then it follows that the steady state $z = Z$ is a Liapunov or norm stable solution of (2.1.18) (Arnol'd 1989). All eigenvalues are either imaginary or zero, the latter corresponding to the null eigenvectors of $J^{ij}(Z)$, when the Hessian of P is sign definite. Complications may now arise when J^{ij} changes rank in which case the number of null eigenvectors $\partial C/\partial z^j$ changes (e.g. Morrison 1998).

For infinite-dimensional systems the energy-Casimir method proceeds along similar lines but formal stability, i.e. sign definiteness of the second variation of functional pseudo energy $\mathcal{A} = \mathcal{H} + \mathcal{C}$, no longer implies Liapunov or norm stability but only linear stability. Only formal stability criteria will be considered in the following fluid examples. More details on Liapunov or norm stability in fluid systems can be found in Arnol'd (1966, 1989), Holm et al. (1985), McIntyre and Shepherd (1987), Shepherd (1990, 1994), Marsden and Ratiu (1994) and references therein.

The invariant P is known in the literature as pseudo energy, free energy or energy-Casimir invariant, and is by construction a finite-amplitude invariant which is quadratic to leading order in the disturbances to a given steady state. The continuum analogue of P is often denoted as wave activity $\mathcal{A}$. In fluid systems, for example, basic state and disturbances may be associated with zonal mean and eddy components.

Example 11: Consider perturbations of the low-order shallow-water system (2.2.16) in Example 7 around a non-resting (i.e. depth $h \neq 0$ or $h \neq 1$) steady state $Z = \{\bar{u}, \bar{v}, \bar{h}\}$ defined by $\bar{u} = 0, \bar{h} = 1 - \mu\,\bar{v} - (3/2)\,\bar{v}^2$. The pseudo-energy invariant is defined as

$$\begin{aligned} P(u, v, h; \bar{u}, \bar{v}, \bar{h}) &= H(u, v, h) + C(u, v, h) - H(\bar{u}, \bar{v}, \bar{h}) - C(\bar{u}, \bar{v}, \bar{h}) \\ = \frac{1}{2}\left(h\,(u^2 + v^2) + \quad h^2\right) &- h + C(q) - \frac{1}{2}\left(\bar{h}\,\bar{v}^2 + \bar{h}^2\right) + \bar{h} - C(\bar{q}) \end{aligned} \tag{2.7.6}$$

with $\bar{q} \equiv (\mu + \bar{v})\,\bar{h}$. The first variation of P vanishes at this steady state when we choose

$$\left.\frac{\partial P}{\partial z}\right|_Z = 0 \quad \Leftrightarrow \quad C'(\gamma) \equiv dC(\gamma)/d\gamma = -\bar{h}^2(\gamma)\,\bar{v}(\gamma) \tag{2.7.7}$$

with $\gamma = \bar{q}$ at the steady state. If we define disturbance quantities in the usual way, $q = \bar{q} + q'$, $h = \bar{h} + h'$, $u = u'$, $v = \bar{v} + v'$; then P may be recast as

$$\begin{aligned} P \;=\; & \frac{1}{2\,(\bar{h}+h')}\Big((\bar{h}+h')\,v' + \bar{v}\,h'\Big)^2 + \frac{1}{2}\Big(1 - \frac{\bar{v}^2}{(\bar{h}+h')} - \frac{[C'(\bar{q})]^2}{\bar{h}^2\,C''(\bar{q})}\Big)\,h'^2 + \\ & \frac{1}{2}\,(\bar{h}+h')\,u'^2 + \frac{1}{2}\,C''(\bar{q})\,\Big(q' - \frac{C'(\bar{q})\,h'}{\bar{h}\,C''(\bar{q})}\Big)^2 + O(q'^3, v'\,h') \end{aligned} \tag{2.7.8}$$

which is positive definite (assuming $\bar{h} + h' > 0$) when

$$\frac{\bar{v}^2}{(\bar{h}+h')} < 1 + \frac{[C'(\bar{q})]^2}{\bar{h}^2\,C''(\bar{q})} \quad \text{and} \quad C''(\bar{q}) > 0. \tag{2.7.9}$$

Similar conditions appear later as formal stability conditions for the full shallow-water and its quasi-geostrophic and ageostrophic balanced counterparts in sections 3.4, 3.5 and 3.6, respectively. The principal difference between (2.7.9) and the shallow-water (balanced model) formal stability criteria is the second term on the right-hand-side of the first (subsonic) inequality (2.7.9), which is absent in the continuum case, and which may arise because mass does not appear as a separate Casimir invariant in the vortex model (2.2.16). (Concerning stability in this vortex model, it may be more convenient to write it in canonical form by treating q as a parameter for given initial conditions. The dynamics is then constrained explicitly onto the symplectic leaves labelled by $q(t) \equiv q(0)$ and is a one-degree-of-freedom Hamiltonian system with parameter q.)

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3 Applications to fluids

In section 2, variational or Hamiltonian formulations of the parent dynamical system have been changed by imposing constraints on these formulations. It was argued that these constraints should be chosen systematically based on perturbative or iterative methods in order to ensure formal accuracy. In addition, we may thus hope to avoid ill-posed dynamical systems. In particular, slaving methods and singular perturbations based on the presence of small parameters in the system were presented as a systematic way to find suitable constraints. Alternatively, scaling of the parent variational principle and subsequent truncation to the required order could lead to a singular variational principle. Constrained conservative dynamics were then shown to arise via: (I) *Dirac's theory* for these singular Lagrangians (section 2.4.2), (II) *direct substitution* of the constraints in the parent variational principle (introduction of section 2.4), (III) imposing the constraints onto this parent principle via *Lagrange multipliers* (section 2.4.1), (IV) *postulation* of Dirac's constrained Poisson bracket formulation for a given parent Hamiltonian system (section 2.4.3), or (V) slaved Hamiltonian dynamics and *leading-order Hamiltonian* perturbations (sections 2.5 and 2.6).

The Lagrangian or Hamiltonian formulation of three-dimensional compressible fluid equations will be our starting point, i.e. our parent formulation, in the forthcoming presentation of reduced or *balanced* Hamiltonian fluid systems. The diagram in figure 1 outlines the hierarchy of fluid systems considered here and the roman numerals along the connecting lines indicate which one of the above approaches is used. Going down in the hierarchy of fluid systems a reduction of the types of waves supported by each system is apparent as follows: compressible dynamics support acoustic, gravity and vorticity waves; hydrostatic and incompressible shallow-water equations support gravity and vorticity waves, while the ageostrophic equations only support vorticity and boundary Kelvin (gravity) waves and the quasi-geostrophic equations only vorticity waves. Each set of constraints thus filters specific types of waves deemed less important in the physical problem at hand.

Although the applications form a hierarchy, it is possible to start reading at different points. Study of sections 3.1 and 3.2 is required to understand a variational hydrostatic approximation. Study of sections 3.1 and 3.3 provides a Hamiltonian derivation of incompressible hydrodynamics. Alternatively, the shallow-water equations in section 3.4 can be regarded as parent dynamics for a Hamiltonian derivation of barotropic quasi-geostrophic dynamics in section 3.5 and ageostrophic barotropic equations in section 3.6.

Apart from a focus on the systematic fashion in which all these systems are derived, emphasis will also be put on a comparison of formal stability criteria. For each system these criteria are either stated with reference to the literature or are derived for novel systems.

3.1 Compressible Euler equations

3.1.1 Lagrangian Hamilton's principle

The compressible three-dimensional fluid equations will be derived from an infinite-dimensional extension of the finite-dimensional Hamilton's principle (2.1.1) (e.g. Salmon (1988a) and Morrison (1998)).

The analogues of the finite-dimensional generalized coordinates q and $\dot{q}$ are the positions $\boldsymbol{\xi}(\mathbf{a},\tau) = (\xi_1,\xi_2,\xi_3)^T$, and their time derivatives $\partial\boldsymbol{\xi}(\mathbf{a},\tau)/\partial\tau$ as functions of a continuum of fluid-parcel labels $\mathbf{a} = (a,b,c)^T$ and time τ. Hamilton's principle for this three-dimensional compressible fluid is

$$\delta S[\boldsymbol{\xi}] = \delta \int_{\tau_0}^{\tau_1} \mathrm{d}\tau\, \mathcal{L}[\boldsymbol{\xi}, \frac{\partial \boldsymbol{\xi}}{\partial \tau}] = 0 \qquad (3.1.1)$$

with endpoint conditions

$$\delta\xi_i(\mathbf{a},\tau_0) = \delta\xi_i(\mathbf{a},\tau_1) = 0 \tag{3.1.2}$$

in a three-dimensional domain D, with $i = 1, 2, 3$. The Lagrangian functional $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L}[\boldsymbol{\xi}, \frac{\partial \boldsymbol{\xi}}{\partial \tau}] &= \int_D \mathrm{d}\mathbf{a}\, \rho_0(\mathbf{a}) \left\{ \left[\frac{1}{2}\frac{\partial \xi_i}{\partial \tau} + R_i(\xi_1, \xi_2) \right] \frac{\partial \xi_i}{\partial \tau} - \right. \\ & \left. U\Big(s_0(\mathbf{a}), \rho(\boldsymbol{\xi}, \mathbf{a})\Big) - g\,\xi_3 \right\} \end{aligned} \tag{3.1.3}$$

in which $U(s, \rho)$ is the internal energy, $s_0(\mathbf{a})$ the conserved value of entropy s on a fluid parcel, and ρ the density defined by

$$\rho(\mathbf{a}, \tau) = \rho(\boldsymbol{\xi}(\mathbf{a}, \tau), \mathbf{a}) \equiv \frac{\rho_0(\mathbf{a})}{\mathcal{T}}. \tag{3.1.4}$$

Here, the Jacobian $\mathcal{T}$ between fluid parcel positions $\xi_i(\mathbf{a}, \tau)$ and fluid labels a_j is given by

$$\mathcal{T} \;\equiv\; \det \left| \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{a}} \right| = \nabla_a \xi_1 \cdot \nabla_a \xi_2 \times \nabla_a \xi_3, \tag{3.1.5}$$

with $\nabla_a = (\partial/\partial a, \partial/\partial b, \partial/\partial c)^T$. The Coriolis force is included via the term $R_i\, \partial\xi_i/\partial\tau$ in (3.1.3) with $R_3 = 0$, $\mathsf{f}(\xi_1, \xi_2) = \hat{\mathbf{z}} \cdot \nabla_3 \times \mathbf{R}$, $\hat{\mathbf{z}}$ the unit vector in the vertical and $\nabla_3 \equiv (\partial/\partial\xi_1, \partial/\partial\xi_2, \partial/\partial\xi_3)^T$. The reference density $\rho_0(\mathbf{a})$ may be chosen such that $\rho(\mathbf{a}, \tau = 0) \equiv \rho_0(\mathbf{a})$. An element of mass dm is thus defined by

$$\mathsf{dm} = \rho_0(\mathbf{a})\, \mathrm{d}\mathbf{a} = \rho(\boldsymbol{\xi}, t)\, \mathrm{d}\boldsymbol{\xi} \tag{3.1.6}$$

with time denoted by t in an Eulerian framework. The first step in a variation of internal energy U is obtained from the first law of thermodynamics

$$\mathrm{d}U = T\,\mathrm{d}s - p\,\mathrm{d}(1/\rho), \tag{3.1.7}$$

with T the temperature and p the pressure, as follows: $\delta\, U = (p/\rho^2)\, \delta\rho$ since in label coordinates one has $\delta s = 0$. Pressure p is via an equation of state

$$p = p(s, \rho) \tag{3.1.8}$$

related to entropy and density. Further manipulation of the functional

$$\int_D \mathrm{d}\mathbf{a}\, \rho_0(\mathbf{a})\, (p/\rho^2)\, \delta\rho$$

requires use of (3.1.4), integration by parts and valid boundary conditions such as no normal flow through solid boundaries.

The resulting equations of motion are

$$\rho_0(\mathbf{a})\frac{\partial^2\xi_p}{\partial\tau^2}+\rho_0(\mathbf{a})\left(\frac{\partial R_p}{\partial\xi_j}-\frac{\partial R_j}{\partial\xi_p}\right)\frac{\partial\xi_j}{\partial\tau}=-A_{pj}\frac{\partial p}{\partial a_j}-g\rho_0(\mathbf{a})\delta_{p3} \tag{3.1.9}$$

where A_{pj} is defined by

$$A_{pj}\equiv\frac{1}{2}\epsilon_{pqr}\epsilon_{jkl}\frac{\partial\xi_q}{\partial a_k}\frac{\partial\xi_r}{\partial a_l}, \tag{3.1.10}$$

with the permutation symbol ϵ_{pqr} and $p,q,r=1,2,3$; together with (3.1.7), (3.1.8), boundary conditions and initial conditions such as

$$\boldsymbol{\xi}(\mathbf{a},0)\;\equiv\;\mathbf{a}. \tag{3.1.11}$$

(A detailed calculation is found in Bokhove (1996) with particular emphasis on incorporating the boundary conditions.) The continuity equation emerges from (3.1.4) as

$$\frac{\partial\rho(\mathbf{a},\tau)}{\partial\tau}=-\frac{\rho(\mathbf{a},\tau)^2}{\rho_0(\mathbf{a})}\left[\frac{1}{2}\epsilon_{pqr}\,\epsilon_{jkl}\,\frac{\partial^2\xi_p}{\partial\tau\,\partial a_j}\,\frac{\partial\xi_q}{\partial a_k}\,\frac{\partial\xi_r}{\partial a_l}\right] \tag{3.1.12}$$

and the entropy equation is $\partial s_0(\mathbf{a})/\partial\tau=0$. Alternatively, (3.1.7) may be used to derive a temperature–pressure equation

$$T\,\frac{\partial s}{\partial\tau}=c_p\,\frac{\partial T}{\partial\tau}-\frac{\beta_e\,T}{\rho}\,\frac{\partial p}{\partial\tau}=0$$

in which p, T, specific heat $c_p=T\,(\partial s/\partial T)_p$ and coefficient of thermal expansion $\beta_e=-(1/\rho)\,(\partial\rho/\partial T)_p$ are directly measurable quantities.

3.1.2 Lagrangian action principle and Hamiltonian formulation

In extension to the finite-dimensional case (cf. (2.1.9)), the Legendre transform of the Lagrangian $\mathcal{L}$ in (3.1.3) of the infinite-dimensional compressible fluid is

$$\begin{aligned}\mathcal{H}[\xi_i,\pi_i^*]&=\int_D \mathrm{d}\mathbf{a}\,\pi_i^*\,\frac{\partial\xi_i}{\partial\tau}-\mathcal{L}[\xi_i,\frac{\partial\xi_i}{\partial\tau}],\\ &=\int_D \mathrm{d}\mathbf{a}\,\rho_0(\mathbf{a})\left\{\frac{1}{2}\left[\frac{\pi_i^*}{\rho_0(\mathbf{a})}-R_i\right]^2+U(s_0,\rho)+g\,\xi_3\right\}\end{aligned} \tag{3.1.13}$$

or

$$\mathcal{H}[\xi_i,\pi_i]=\int_D \mathrm{d}\mathbf{a}\,\rho_0(\mathbf{a})\left\{\frac{1}{2}\left[\frac{\pi_i}{\rho_0(\mathbf{a})}\right]^2+U(s_0,\rho)+g\,\xi_3\right\}. \tag{3.1.14}$$

The generalized momenta $\pi_i^*(\mathbf{a},\tau)$ and $\pi_i(\mathbf{a},\tau)$ are defined by (cf. (2.1.8))

$$\pi_i^*(\mathbf{a},\tau)\equiv\frac{\delta\mathcal{L}}{\delta\left[\dfrac{\partial\xi_i(\mathbf{a},\tau)}{\partial\tau}\right]}=\rho_0(\mathbf{a})\left[\frac{\partial\xi_i(\mathbf{a},\tau)}{\partial\tau}+R_i\right]=\pi_i(\mathbf{a},\tau)+\rho_0(\mathbf{a})\,R_i. \tag{3.1.15}$$

The action principle (cf. (2.1.10))

$$0 = \delta \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \left\{ \int_D \mathrm{d}\mathbf{a}\, \pi_i^* \frac{\partial \xi_i}{\partial \tau} - \mathcal{H}[\xi_i^*, \pi_i] \right\} \tag{3.1.16}$$

yields Hamilton's equations (cf. 2.1.11) for the fields $\boldsymbol{\xi}(\mathbf{a}, \tau)$ and $\boldsymbol{\pi}^*(\mathbf{a}, \tau)$. Alternatively, one can use

$$0 = \delta \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \left\{ \int_D \mathrm{d}\mathbf{a} \left(\pi_i + \rho_0(\mathbf{a})\, R_i \right) \frac{\partial \xi_i}{\partial \tau} - \mathcal{H}[\xi_i, \pi_i] \right\}. \tag{3.1.17}$$

A corresponding Hamiltonian formulation is

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F}, \mathcal{H}] \tag{3.1.18}$$

with bracket

$$[\mathcal{F}, \mathcal{G}] \;=\; \int_D \mathrm{d}\mathbf{a} \left\{ \frac{\delta \mathcal{F}}{\delta \xi_i} \frac{\delta \mathcal{G}}{\delta \pi_i} - \frac{\delta \mathcal{F}}{\delta \pi_i} \frac{\delta \mathcal{G}}{\delta \xi_i} + \rho_0(\mathbf{a})\, \mathsf{f}(\xi_1, \xi_2)\, \epsilon_{ho} \frac{\delta \mathcal{F}}{\delta \pi_h} \frac{\delta \mathcal{G}}{\delta \pi_o} \right\} \tag{3.1.19}$$

(while in terms of the fields $\boldsymbol{\xi}(\mathbf{a}, \tau)$ and $\boldsymbol{\pi}^*(\mathbf{a}, \tau)$ the infinite-dimensional analogue of the Poisson bracket (2.2.5) appears), where $i = 1, 2, 3$ and $\epsilon_{ho} \equiv \epsilon_{ho3}$ is the two-dimensional permutation symbol with $h, o = 1, 2$; and with Hamiltonian (3.1.14). In addition, thermodynamic equations (3.1.7), (3.1.8), initial conditions (3.1.11), and suitable boundary conditions are required. If

$$\mathcal{F} = \boldsymbol{\pi}(\mathbf{a}_0, \tau) = \int_D \mathrm{d}\,\mathbf{a}\; \boldsymbol{\delta}(\mathbf{a} - \mathbf{a}_0)\, \boldsymbol{\pi}(\mathbf{a}, \tau)$$

then $\delta \mathcal{F} / \delta \boldsymbol{\pi}(\mathbf{a}, \tau) = \boldsymbol{\delta}(\mathbf{a} - \mathbf{a}_0)$. Substitution of this result into (3.1.19) together with the required variations of the Hamiltonian, whose evaluation proceeds similarly to variation of Hamilton's principle, yields the momentum equations in label coordinates. The class of admissible functionals $\mathcal{F}$ and $\mathcal{G}$ is restricted to ones for which the continuum bracket (3.1.19) does not involve products of delta functions.

The infinite-dimensional analogues of (2.2.3) which define a proper Poisson bracket are skew-symmetry

$$[\mathcal{F}, \mathcal{G}] = -[\mathcal{G}, \mathcal{F}], \tag{3.1.20}$$

and Jacobi's identity

$$[\mathcal{F}, [\mathcal{G}, \mathcal{K}]] + [\mathcal{G}, [\mathcal{K}, \mathcal{F}]] + [\mathcal{K}, [\mathcal{F}, \mathcal{G}]] = 0, \tag{3.1.21}$$

and both conditions are satisfied by (3.1.19).

3.1.3 Hamiltonian formulation in Eulerian coordinates

After a transformation from Lagrangian label coordinates **a** to horizontal and vertical Eulerian coordinates, $\mathbf{x} = (x, y)^T = \boldsymbol{\xi}$ and z, respectively, the Hamiltonian (3.1.14) becomes

$$\mathcal{H} = \int_D \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\rho\left\{\frac{1}{2}|\mathbf{u}|^2 + U(s,\rho) + g\,z\right\}, \tag{3.1.22}$$

where $s, \rho, \mathbf{u} \equiv (u, v, w)^T$ are now all functions of $\mathbf{x}, z$ and t. Variation of (3.1.22), using (3.1.7), yields

$$\delta\mathcal{H} = \int_D \mathrm{d}\mathbf{x}\,\mathrm{d}z\left\{\rho\,\mathbf{u}\cdot\delta\mathbf{u} + \rho\,T\,\delta s + \left(\frac{1}{2}|\mathbf{u}|^2 + g\,z + U + \frac{p}{\rho}\right)\delta\rho\right\}. \tag{3.1.23}$$

Variations of the Hamiltonian lead to five independent variations in terms of $s, \rho, \mathbf{u}$ in Eulerian coordinates as opposed to the six variations in terms of Lagrangian variables $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ in Lagrangian label coordinates. This reduction is due to the particle-relabelling symmetry in the Lagrangian Hamiltonian which leaves the Hamiltonian invariant under those rearrangements of particle labels **a** that leave the density (3.1.4), i.e. the Jacobian between label and coordinate space that appears in the internal energy, invariant. More details on this symmetry and the associated potential vorticity conservation laws are found in Salmon (1983, 1988a), and Padheye and Morrison (1996).

Reduction of the Hamiltonian system to this reduced set of new variables s, ρ and $\mathbf{u}$ is possible because the transformation of the Poisson bracket from the larger set of original variables $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ is closed, i.e. the new bracket is entirely specified in terms of the reduced set of new variables. The theory of reduction of a Hamiltonian system based on a symmetry in the Hamiltonian and the closure property of the Poisson bracket under the transformation to a variable set of reduced dimension is well known, see for example Sudarshan and Mukunda (1974), Holm et al. (1985), Olver (1986) and Morrison (1998).

After the reduction procedure we arrive at a Hamiltonian formulation (3.1.18) of a compressible fluid in Eulerian coordinates (originally recorded by Morrison and Green (1982)) with Hamiltonian (3.1.22), Poisson bracket

$$\begin{aligned}[\mathcal{F},\mathcal{G}] \;=\; &\int_D \mathrm{d}\mathbf{x}\,\mathrm{d}z\left\{\frac{(\boldsymbol{\omega} + \hat{\mathbf{z}}\,f)}{\rho}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{u}}\times\frac{\delta\mathcal{G}}{\delta\mathbf{u}} - \frac{\delta\mathcal{F}}{\delta\rho}\nabla_3\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{u}} + \frac{\delta\mathcal{G}}{\delta\rho}\nabla_3\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{u}} + \right.\\ &\left.\frac{1}{\rho}\nabla_3 s\cdot\left(\frac{\delta\mathcal{F}}{\delta\mathbf{u}}\frac{\delta\mathcal{G}}{\delta s} - \frac{\delta\mathcal{G}}{\delta\mathbf{u}}\frac{\delta\mathcal{F}}{\delta s}\right)\right\},\end{aligned} \tag{3.1.24}$$

and thermodynamic relations, where we have defined vorticity $\boldsymbol{\omega} \equiv \nabla_3\times\mathbf{u}$ and gradient $\nabla_3 = (\partial_x, \partial_y, \partial_z)^T$. Notice that (3.1.24) is a non-canonical bracket which satisfies (3.1.20) and (3.1.21) because these identities are preserved under the reduction procedure (a direct verification of Jacobi's identity is more tedious). The resulting equations of motion are the familiar compressible Euler equations

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_3)\,\mathbf{u} + \mathrm{f}\,\hat{\mathbf{z}}\times\mathbf{v} \;=\; -\frac{1}{\rho}\nabla_3\,p - \hat{\mathbf{z}}\,g, \tag{3.1.25}$$

$$\rho_t + \nabla_3\cdot(\rho\,\mathbf{u}) \;=\; 0, \tag{3.1.26}$$

$$s_t + (\mathbf{u}\cdot\nabla_3)\,s \;=\; 0 \tag{3.1.27}$$

with, e.g., ρ_t denoting the partial time derivative of ρ.

3.1.4 Formal stability

Formal stability criteria of non-resting fully three-dimensional basic states have, to my knowledge, not been proven for compressible flows. The static stability case, however, was considered by Vladimirov (1987, 1989), see also references therein. For a basic state $\mathbf{u} = 0, \rho = \rho_0(z)$ and $s = S_0(z)$ these formal stability criteria include static stability with the Brunt-Väisälä frequency

$$\begin{aligned} N^2 &\equiv \frac{g}{c_p}\frac{\partial S_0}{\partial z} > 0, \quad \text{and} \\ c_0^2 &= \left.\frac{\partial p(\rho, s)}{\partial \rho}\right|_{\rho_0,S_0} > 0, \quad \left.\frac{\partial^2 U}{\partial\rho\,\partial s}\right|_{\rho_0,S_0} > 0, \end{aligned} \tag{3.1.28}$$

where c_0 is the speed of sound evaluated at the resting basic state.

3.2 Hydrostatic equations

3.2.1 Aspect ratio truncation and Dirac's theory

Hydrostatically balanced flows arise directly by scaling the Lagrangian $\mathcal{L}$ in Hamilton's principle. The term $(\partial\xi_3/\partial\tau)^2$ in (3.1.3) scales as the square of the aspect ratio $\delta \equiv W/U = D/L$ relative to other kinetic energy terms for a vertical velocity and length scale W and D, and horizontal velocity and length scale U and L. Such a scaling is useful in meteorology where the atmosphere is effectively a thin layer of the order of 10 km around the earth relative to large horizontal flow scales of the order of 100 to 1000 km, so that $\delta \ll 1$. After truncation to $O(1)$ a modified Hamilton's principle results

$$\begin{aligned} 0 = \delta S_c[\xi_h] \;=\; & \delta\int_{\tau_0}^{\tau_1}\mathrm{d}\tau\int_D \mathrm{d}\mathbf{a}\,\rho_0(\mathbf{a})\left\{\left[\frac{1}{2}\frac{\partial\xi_h}{\partial\tau} + R_h(\xi_1,\xi_2)\right]\frac{\partial\xi_h}{\partial\tau} - \right. \\ & \left. U(s_0,\rho) - g\,\xi_3\right\} \end{aligned} \tag{3.2.1}$$

with $h = 1, 2$, in which the vertical velocity $\partial\xi_3(\mathbf{a},\tau)/\partial\tau$ is absent. Variation of (3.2.1) with respect to $\delta\xi_h$ with endpoint conditions (3.1.2), thermodynamic relations (3.1.7), (3.1.8), initial conditions (3.1.11), and suitable boundary conditions yields the hydrostatic equations of motion in the form

$$\rho_0(\mathbf{a})\left[\frac{\partial^2\xi_h}{\partial\tau^2}+\left(\frac{\partial R_h}{\partial\xi_j}-\frac{\partial R_j}{\partial\xi_h}\right)\frac{\partial\xi_j}{\partial\tau}\right] \;=\; -A_{hj}\,\frac{\partial p}{\partial a_j}, \tag{3.2.2}$$

$$0=-A_{3j}\,\frac{\partial p}{\partial a_j} \;-\; g\,\rho_0(\mathbf{a}). \tag{3.2.3}$$

Equation (3.2.3) is the hydrostatic balance condition in Lagrangian form. The system (3.2.2), (3.2.3) along with thermodynamics relations is closed under suitable initial and boundary conditions because the positions $\xi_3(\mathbf{a},\tau)$ may be obtained from the partial differential equation (3.2.3): by differentiating (3.2.3) with respect to time τ and by using the boundary conditions to eliminate the terms $\partial\xi_3/\partial\tau$ at the boundary the appropriate boundary conditions for ξ_3 can be found.

The hydrostatic Lagrangian (3.2.1) is a singular Lagrangian just like the Lagrangian of the finite-dimensonal system encountered in Example 9. The succeeding more complicated application of Dirac's theory of constrained Hamiltonian dynamics (approach I) may be compared with the one in that finite-dimensional example.

The Hamiltonian formulation of this hydrostatically balanced system appears to be nontrivial, since the Lagrangian in (3.2.1) is *singular*. If the generalized momentum is defined as in (3.1.15), then the vertical momentum

$$\pi_3(\mathbf{a},\tau)=0. \tag{3.2.4}$$

In contrast, for the non-singular case the action principle (3.1.17) is uniquely defined in terms of the independent variations of the positions $\xi_i(\mathbf{a},\tau)$ and momenta $\pi_i(\mathbf{a},\tau)$ of the fluid parcels. In the singular case, however, the variations with respect to ξ_i and π_i are *not* independent, because the dynamics is constrained to the manifold defined by (3.2.4). The constrained variational principle (cf. (2.4.6)) with Lagrange multiplier $\lambda(\mathbf{a},\tau)$, which replaces (3.2.1), has the form

$$\delta\int_{\tau_0}^{\tau_1}\mathrm{d}\tau\Big\{\int_D\mathrm{d}\mathbf{a}\left\{\Big[\pi_i(\mathbf{a},\tau)+\rho_0(\mathbf{a})\,R_i\Big]\frac{\partial\xi_i(\mathbf{a},\tau)}{\partial\tau} \;-\; \mathcal{H}[\xi_h,\pi_h]\right\}+ \int_D\mathrm{d}\mathbf{a}\,\lambda(\mathbf{a},\tau)\,\pi_3(\mathbf{a},\tau)\Big\} \;=\; 0. \tag{3.2.5}$$

From (3.2.5) the following constrained dynamics may be derived:

$$\frac{d\mathcal{F}}{dt} \;\simeq\; [\mathcal{F},\mathcal{H}]^*=[\mathcal{F},\mathcal{H}]-\int_D\mathrm{d}\mathbf{a}\,\lambda(\mathbf{a},\tau)[\mathcal{F},\pi_3(\mathbf{a},\tau)], \tag{3.2.6}$$

where $[\cdot,\cdot]$ is defined by (3.1.19) and the Hamiltonian by

$$\mathcal{H}[\xi_h,\pi_h]=\int_D\mathrm{d}\mathbf{a}\,\rho_0(\mathbf{a})\left\{\frac{1}{2}\Big[\frac{\pi_h}{\rho_0(\mathbf{a})}\Big]^2+U(s_0,\rho)+g\,\xi_3\right\}. \tag{3.2.7}$$

Variation of λ in (3.2.5) gives (3.2.4) and variation of $\delta\pi_3$ gives

$$\frac{\partial \xi_3}{\partial \tau} \simeq -\lambda. \tag{3.2.8}$$

The Lagrange multiplier has thus taken over the rôle of minus the vertical velocity. (The notation $\simeq$ serves as a reminder that the brackets in (3.2.6) have to be calculated before the constraints are applied.) The bracket $[\cdot,\cdot]^*$ in (3.2.6) does not appear to be skew-symmetric, which is not surprising since one has to check if the constraint (3.2.4) is consistent with the dynamics (3.2.6), (3.2.7), such that the Lagrange multiplier may be solved. Dynamical consistency is reached when

$$0 = \frac{\partial \pi_3(\mathbf{a},\tau)}{\partial t} \simeq [\pi_3(\mathbf{a},\tau), \mathcal{H}]^* = [\pi_3(\mathbf{a},\tau), \mathcal{H}] \equiv \chi(\mathbf{a},\tau), \tag{3.2.9}$$

where (3.1.19) is used to calculate that $[\pi_3(\mathbf{a},\tau), \pi_3(\mathbf{a}',\tau)] = 0$. It leads to a so-called secondary constraint (Dirac 1958, 1964; Sudarshan and Mukunda 1974) $\chi(\mathbf{a},\tau) = 0$, which is the hydrostatic balance condition (3.2.3) we encountered earlier. Consequently, the multiplier $\lambda(\mathbf{a},\tau)$ may in principle be determined from the consistency requirement for $\chi(\mathbf{a},\tau)$, i.e. from

$$0 = \frac{\partial \chi(\mathbf{a}',\tau)}{\partial t} \simeq [\chi(\mathbf{a}',\tau), \mathcal{H}] - \int_D \mathrm{d}\mathbf{a}\ \lambda(\mathbf{a},\tau)\,[\chi(\mathbf{a}',\tau), \pi_3(\mathbf{a},\tau)]. \tag{3.2.10}$$

Equation (3.2.10) is an integral equation for λ and whether it has a general solution is a delicate question. One expects that $[\chi(\mathbf{a}',\tau), \pi_3(\mathbf{a},\tau)] \neq 0$, because of the dependence of $\chi(\mathbf{a})$ on $\boldsymbol{\xi}$. Considering (3.2.8) we may rephrase this question by asking whether we can find a solution for ξ_3 and $\partial\xi_3/\partial\tau$ by solving the system $\chi = 0$ and $\partial\chi/\partial\tau = 0$. Anticipating an extension of (2.4.18) to the infinite-dimensional case, we assume that one may find a solution for λ of the form

$$\lambda(\mathbf{a},\tau) = -\mathcal{I}\Big([\chi(\mathbf{a},\tau), \mathcal{H}]\Big) \tag{3.2.11}$$

for an appropriate integral operator $\mathcal{I}$, and the Dirac bracket becomes

$$[\mathcal{F},\mathcal{G}]_c = [\mathcal{F},\mathcal{G}] + \int_D \mathrm{d}\mathbf{a}\ [\mathcal{F}, \pi_3(\mathbf{a},\tau)]\,\mathcal{I}\Big([\chi(\mathbf{a},\tau), \mathcal{G}]\Big). \tag{3.2.12}$$

If one restricts the dynamics to the constrained manifold defined by (3.2.4) and (3.2.9), then a transformation of variables from $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ to ξ_1, ξ_2, π_1, π_2, χ, and π_3 reveals that the reference to functionals of χ and π_3 in the Dirac bracket (3.2.12) vanishes. This is possible because the Poisson bracket (3.1.19) is fairly simple, because the first constraint (3.2.4) depends on the momenta π_i only, and because the secondary constraint (3.2.9) depends on the positions ξ_i only. Conceptually at least, we may explicitly restrict the Hamiltonian to the constrained manifold defined by (3.2.4) and (3.2.9). However, an explicit solution of ξ_3 may not be available so instead we implicitly restrict the Hamiltonian to the constrained manifold. In particular, variation of $\mathcal{H}$ with respect to ξ_3 yields the secondary constraint χ, which is zero, and thus we do not need to work out the variation of ξ_3 with respect to ξ_h.

Consequently, the constrained dynamics for functionals of the form $\mathcal{F} = \mathcal{F}(\xi_h, \pi_h)$ takes the form

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F}, \mathcal{H}]_c, \tag{3.2.13}$$

with Dirac bracket

$$[\mathcal{F}, \mathcal{G}]_c = \int_D \mathrm{d}\mathbf{a} \left\{ \frac{\delta \mathcal{F}}{\delta \xi_h} \frac{\delta \mathcal{G}}{\delta \pi_h} - \frac{\delta \mathcal{F}}{\delta \pi_h} \frac{\delta \mathcal{G}}{\delta \xi_h} + \rho_0(\mathbf{a})\, \mathsf{f}(\xi_1, \xi_2)\, \epsilon_{ho} \frac{\delta \mathcal{F}}{\delta \pi_h} \frac{\delta \mathcal{G}}{\delta \pi_o} \right\}, \tag{3.2.14}$$

where $h = 1, 2$, together with Hamiltonian (3.2.7) restricted to the constrained manifold, thermodynamic relations, initial and boundary conditions, and constraints (3.2.4) and (3.2.9). The bracket (3.2.14) obeys (3.1.20) and (3.1.21), and is seen to be canonical when $\mathsf{f}(\xi_1, \xi_2) = 0$. Of course, the constrained Hamiltonian formulation (3.2.13), (3.2.14) could have been guessed immediately from the singular Lagrangian in (3.2.1) or found simply by substitution of $\pi_3 = 0$ into (3.1.17). The foregoing then merely illustrates Dirac's approach.

3.2.2 Eulerian Hamilton's principle in isentropic coordinates

The Lagrangian Hamilton's principle (3.2.1) for hydrostatic flows may be transformed into an isentropic Eulerian Hamilton's principle by a coordinate change from label coordinates a, b, $c = s$ and time τ to isentropic Eulerian coordinates x, y, entropy s and time $t \equiv \tau$ (for incompressible flow in isopycnal Eulerian coordinates see Holm (1996)) and a transformation from Lagrangian variables $\mathbf{x}(\mathbf{a}, s, \tau)$ to Eulerian variables $\mathbf{a}(\mathbf{x}, s, t) = (a(\mathbf{x}, s, t), b(\mathbf{x}, s, t))^T$. (This section is based on Bokhove 2000.) Consider the product of Jacobians $\partial(x, y, t, s)/\partial(a, b, \tau, s)$ and $\partial(a, b, \tau, s)/\partial(x, y, t, s)$, i.e.

$$\begin{pmatrix} x_a & x_b & x_\tau & x_s \\ y_a & y_b & y_\tau & y_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x & a_y & a_t & a_s \\ b_x & b_y & b_t & b_s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2.15}$$

where subscripts denote partial derivatives. From (3.2.15) one deduces that

$$\mathbf{v} \equiv \frac{\partial \mathbf{x}}{\partial \tau} = -\mathbf{\Gamma}^{-1} \frac{\partial \mathbf{a}}{\partial t} \Longleftrightarrow \frac{\partial a^i}{\partial t} + u^k \frac{\partial a^i}{\partial x^k} = 0, \tag{3.2.16}$$

which amounts physically to advection of fluid label $\mathbf{a}$ by horizontal fluid velocity $\mathbf{v} \equiv (u, v)^T$ on isentropic surfaces labelled by coordinate s. The tensor $\mathbf{\Gamma}$ is defined by $\Gamma^i_k = \partial a^i/\partial x^k$. All indices range from $i = 1, 2$ and care is taken in their placement. Hence, one finds

$$\begin{aligned} 0 = \delta S_c[\mathbf{a}] &= \delta \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\mathbf{x} \int_{s_B(\mathbf{x},t)}^{\infty} ds\, \sigma(\mathbf{x}, s, t) \\ &\left\{ \left(\frac{1}{2}\, u_h(\mathbf{x}, s, t) + R_h(\mathbf{x}) \right) u^h(\mathbf{x}, s, t) - U(s, \rho) - g\, z(\mathbf{x}, s, t) \right\} \end{aligned} \tag{3.2.17}$$

as Eulerian Hamilton's principle, in which velocity u^h is the economic shorthand defined in (3.2.16), in which pseudo density

$$\begin{aligned} \sigma(\mathbf{x}, s, t) &\equiv \sigma_0(\mathbf{a}, s)\, J(a, b) \\ &= -\frac{1}{g}\, \frac{\partial p(\mathbf{x}, s, t)}{\partial s} = \rho(\mathbf{x}, s, t)\, \frac{\partial z(\mathbf{x}, s, t)}{\partial s} \end{aligned} \tag{3.2.18}$$

with the horizontal Jacobian $J(a, b) \equiv \partial_x a\, \partial_y b - \partial_x b\, \partial_y a$, in which $u_h = \delta_{hj}\, u^j$, and in which variations are taken with respect to fluid parcel variables $\mathbf{a}$. The domain is a horizontally infinite, closed, or periodic domain, above a mountain range h_B, i.e. $z > h_B(\mathbf{x})$. In isentropic coordinates we have $s > s_B(\mathbf{x}, t)$ and the maximum horizontal extent of the domain is D_H. In the remainder of this section reference to the spatial and time dependence of the variables has generally been dropped.

In the evaluation of (3.2.17) the following expressions, or variations or time-derivatives thereof, are useful

$$(\Gamma^{-1})^i_k\, \Gamma^k_j = \delta^i_j, \tag{3.2.19}$$

$$\frac{\partial\left(\sigma(\Gamma^{-1})^m_k\right)}{\partial t} + \frac{\partial\left(\sigma u^j\, (\Gamma^{-1})^m_k\right)}{\partial x^j} = \sigma\, (\Gamma^{-1})^n_k\, \frac{\partial u^m}{\partial x^n}, \tag{3.2.20}$$

$$\begin{aligned} -\int_{D_H} d\mathbf{x} \int_{s_B}^{\infty} \mathrm{d}s\, \sigma\, \delta(U + g\, z) &= \\ -\int_{D_H} d\mathbf{x} \int_{s_B}^{\infty} \mathrm{d}s\, \frac{p}{\rho}\, \delta\sigma &+ \int_{D_H} d\mathbf{x}\, (\sigma\, \frac{p}{\rho})\Big|_{s_B} \delta s_B. \end{aligned} \tag{3.2.21}$$

Subscripts B imply evaluation at the boundary B at $z = h_B(\mathbf{x})$.

The continuity equation expressed in terms of the pseudo density appears directly from definitions (3.2.16) and (3.2.18)

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \sigma_0}{\partial a^k}\, \frac{\partial a^k}{\partial t}\, J(a, b) + \sigma_0\, \epsilon^{ij}\, \epsilon_{mn} \left(\frac{\partial^2 a^m}{\partial x^i \partial t} \right) \frac{\partial a^n}{\partial x^j} = -u^j\, \frac{\partial \sigma}{\partial x^j} - \sigma\, \frac{\partial u^j}{\partial x^j}. \tag{3.2.22}$$

The variation of (3.2.17) with respect to δa^k and $(\delta a^k)_B$ yields

$$\begin{aligned} 0 = \delta S_c[\mathbf{a}] &= \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\mathbf{x} \int_{s_B}^{\infty} \mathrm{d}s\, \sigma (\Gamma^{-1})^n_k \left\{ \frac{\partial u_n}{\partial t} + u^j\, \frac{\partial u_n}{\partial x^j} + u^j\, \frac{\partial R_n}{\partial x^j} - \right. \\ &\left. u^j\, \frac{\partial R_j}{\partial x^n} + \frac{\partial M}{\partial x^n} \right\} \delta a^k + \int_{t_0}^{t_1} \mathrm{d}t \int_{D_H} \mathrm{d}\mathbf{x} \left\{ \left((\Gamma^{-1})^n_k \frac{\partial s_B}{\partial x^n}\, (\delta a^k)_B - \delta s_B \right) B_M\, \sigma \right. \\ &\left. -p \left((\delta\, z)_B + \frac{\partial z}{\partial s}\, \delta s_B \right) - \sigma (\Gamma^{-1})^n_k\, (u_n + R_n) \left(\frac{\partial s_B}{\partial t} + u^j\, \frac{\partial s_B}{\partial x^j} \right) (\delta a^k)_B \right\} \Big|_{s_B}, \end{aligned} \tag{3.2.23}$$

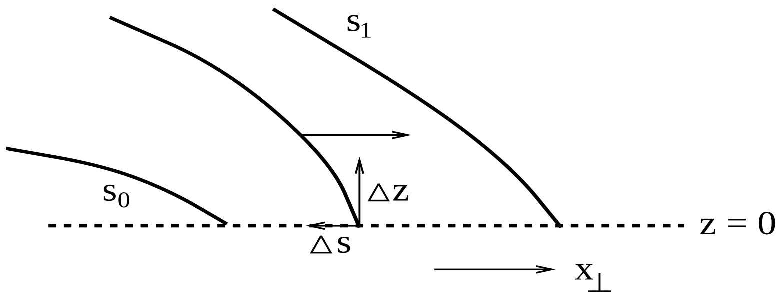

Figure 2 Sketch of the relation between geopotential $\phi \equiv g\, z$ and entropy s variations at an earth's surface $z = 0$. $s_{0,1}$ with $s_0 < s_1$ are two values of an isentropic surface. The resulting relations for other geometries follow likewise.

where a function

$$B_M \equiv (\frac{1}{2}\, u_h + R_h)\, u^h - M \tag{3.2.24}$$

has been defined with Montgomery potential

$$M \equiv E + g\, z \tag{3.2.25}$$

and enthalpy $E \equiv U + p/\rho$.

The first two boundary terms in (3.2.23) cancel one another with the help of the following relations

$$\delta s_B = \frac{\partial s_B}{\partial (a^m_B)}\, \delta(a^m_B) \quad \text{and} \quad \delta(a^m_B) \;=\; (\delta\, a^m)_B + \Big(\frac{\partial a^m}{\partial s}\Big)_B \delta s_B, \tag{3.2.26}$$

in which the variation of $a^m_B \equiv a^m(\mathbf{x}_h, s = s_B(\mathbf{x}, t), t)$ is not equal to the boundary value of the variation of $a^m(\mathbf{x}, s, t)$. The third and fourth boundary terms vanish when one uses the relation

$$0 = \delta\, (z\,|_{z=h_B}) = \frac{\partial z(\mathbf{x}, s, t)}{\partial s}\Big|_{z_B} \delta s_B + \Big(\delta z(\mathbf{x}, s, t)\Big)\Big|_{s_B}; \tag{3.2.27}$$

the first equality in (3.2.27) emerges since $z(\mathbf{x}, s_B(\mathbf{x}, t), t) = h_B(\mathbf{x})$ and hence

$$\delta\, z(\mathbf{x}, s_B(\mathbf{x}, t), t) = \delta\, h_B(\mathbf{x}) = 0$$

and the second one in (3.2.27) follows geometrically from Fig. 3.2.2.

The equations of motion which arise from these variations are thus horizontal advection of the boundary entropy at $z = h_B(\mathbf{x})$

$$(\delta a^k)_B : \qquad \frac{\partial s_B}{\partial t} + u^k\, \frac{\partial s_B}{\partial x^k} = 0 \tag{3.2.28}$$

and the horizontal momentum equations

$$(\delta a^k): \qquad \frac{\partial u_m}{\partial t} + u^k \frac{\partial u_m}{\partial x^k} + u^k \frac{\partial R_m}{\partial x^k} - u^k \frac{\partial R_k}{\partial x^m} + \frac{\partial M}{\partial x^m} = 0. \tag{3.2.29}$$

Since u_m is defined by (3.2.16) the momentum equations are second order in time for the fluid labels. It may come as a surprise that the advection of boundary entropy does not involve the mountain h_B. However, at the boundary one finds in isentropic coordinates that

$$\begin{aligned} \frac{D(z-h_B)}{Dt}\bigg|_{s_B} &= -\left(\frac{\partial z}{\partial s}\right)\bigg|_{s_B} \frac{Ds_B}{Dt}\bigg|_{s_B} + \left(\mathbf{v}\cdot\nabla\bigg|_s z(x,y,s)\right)_{s_B} - \\ &\left(\mathbf{v}\cdot\nabla\bigg|_s h_B(\mathbf{x})\right)_{s_B} = 0 \quad \Longrightarrow \frac{Ds_B}{Dt}\bigg|_{s_B} = 0, \end{aligned} \tag{3.2.30}$$

while in Cartesian coordinates one finds that

$$\left(\frac{\partial s(\mathbf{x},z,t)}{\partial t} + (\mathbf{v}\cdot\nabla)\, s(\mathbf{x},z,t) + (\mathbf{v}\cdot\nabla)\, h_B(\mathbf{x})\, \frac{\partial s(\mathbf{x},z,t)}{\partial z}\right)_{z=h_B} = 0. \tag{3.2.31}$$

Equations of motion (3.2.28) and (3.2.29) need to be complemented with the first law of thermodynamics (3.1.7) and the definition of the pseudo density (3.2.18). The latter two with (3.2.25) imply that

$$T = \frac{\partial M}{\partial s}. \tag{3.2.32}$$

With the ideal gas law $p = \rho R T$, in which $R = c_p - c_v$ is the gas constant or difference between specific heat c_p at constant pressure and specific heat c_v at constant volume, equations (3.1.7), (3.2.18) and (3.2.32) can be reduced to the elliptic equation

$$\sigma = -\frac{p_{00}}{g}\frac{\partial}{\partial s}\left[\left(\frac{\partial M}{\partial s}\right)^{c_p/R} e^{-(s-s_{00})/R}\right] \tag{3.2.33}$$

with reference pressure p_{00} at 1000 mb and reference entropy s_{00}. The lower boundary condition at $s = s_B(\mathbf{x}, t)$ is

$$M = c_p \frac{\partial M}{\partial s} + g\, h_B(\mathbf{x}) \tag{3.2.34}$$

and the upper one as $z, s \to \infty$ is $T \to \infty$. (Alternatively, an upper stratospheric boundary condition of prescribed pressure at $s = s_U$ may be specified as

$$p \equiv p_{00}\, (M_s)^{c_p/R}\, e^{-(s-s_{00})/R} \equiv p_U.) \tag{3.2.35}$$

With the ideal gas law (3.2.25) becomes $M = c_p T + g\, z$.

A Hamiltonian formulation of the hydrostatic primitive equations in isentropic coordinates may be derived via a reduction to a new, smaller set of variables $\mathbf{v}, \sigma$ and s_B because the Hamiltonian is invariant under fluid particle relabelling, just like in the compressible non-hydrostatic case, and because the bracket is closed under the reduction process. More details can be found in Bokhove (1996).

Via Noether's theorem conservation of energy may be derived by considering the invariance of the Lagrangian under transformation $t' = t + \delta\, t(t)$, and conservation of potential vorticity may be derived likewise under particle relabelling transformations which preserve pseudo density σ (e.g. Salmon 1983 and Padheye and Morrison 1996). Thus one has $d\mathcal{H}/dt = 0$. Variation of this Hamiltonian invariant

$$\mathcal{H} \;=\; \int_{D_H}\int_{s_B}^{\infty} \mathrm{d}\mathbf{x}\,\mathrm{d}s\,\sigma \left\{\frac{1}{2}\,|\mathbf{v}|^2 + U(s,\rho) + g\,z\right\}, \tag{3.2.36}$$

the Eulerian version of (3.2.7), yields

$$\frac{\delta\mathcal{H}}{\delta\mathbf{v}} \;=\; \sigma\,\mathbf{v}, \quad \frac{\delta\mathcal{H}}{\delta\sigma} = \frac{1}{2}|\mathbf{v}|^2 + M, \tag{3.2.37}$$

$$\frac{\delta\mathcal{H}}{\delta s_B} \;=\; -\left[\frac{1}{2}\,\sigma\,|\mathbf{v}|^2 + \sigma\,M\right]\Big|_{s=s_B}. \tag{3.2.38}$$

The derivation of Casimir invariants from the potential vorticity equation

$$\frac{\partial q}{\partial t} + \mathbf{v}\cdot\nabla q = 0, \tag{3.2.39}$$

with two-dimensional gradient ∇ and potential vorticity

$$q = \frac{\mathsf{f} + \hat{\mathbf{z}}\cdot\nabla_3\times\mathbf{u}}{\sigma}, \tag{3.2.40}$$

and continuity equation (3.2.22) needs some care due to the time dependence in the endpoints of the integral over the entropy coordinate. These Casimir invariants have the form

$$\mathcal{C} = \int_{D_H}\int_{s_B}^{\infty} \mathrm{d}\mathbf{x}\,\mathrm{d}s\,\sigma\,C(q,s). \tag{3.2.41}$$

Preservation in time is shown as follows

$$\begin{aligned}
\frac{d\mathcal{C}}{dt} \;&=\; \int \mathrm{d}\mathbf{x}\int_{s_B}^{\infty} \mathrm{d}s\left(\frac{\partial\sigma}{\partial t}\,C + \sigma\,C_q\,\frac{\partial q}{\partial t}\right) - \int \mathrm{d}\mathbf{x}\,(\sigma\,C)|_{s_B}\,\frac{\partial s_B}{\partial t}\\
&=\; -\int \mathrm{d}\mathbf{x}\int_{s_B}^{\infty} \mathrm{d}s\nabla\cdot(\sigma\,\mathbf{v}\,C) + \int \mathrm{d}\mathbf{x}\,(\sigma\,C\,\mathbf{v})|_{s_B}\cdot\nabla s_B\\
&=\; -\int \mathrm{d}\mathbf{x}\,\nabla\cdot\int_{s_B}^{\infty}\mathrm{d}s\,\sigma\,\mathbf{v}\,C\\
&=\; -\int_{\partial D_H}\mathrm{d}l\,\mathbf{n}\cdot\int_{s_B}^{\infty}\mathrm{d}s\,\sigma\,\mathbf{v}\,C(q,s) = 0,
\end{aligned} \tag{3.2.42}$$

where $\mathrm{d}l$ is an infinitesimal line element at the extreme horizontal limits ∂D_H. The last boundary contribution is seen to cancel when there are walls, either because at vertical boundaries $\mathbf{n}\cdot\mathbf{v} = 0$ or because the integral over s has identical limits when there are non-vertical boundaries, or for vanishing or cancelling flows at infinity. Without loss of generality Casimir invariants may be split into general and circulation components as follows,

$$\mathcal{C} = \int \mathrm{d}\mathbf{x} \int_{s_B}^{\infty} \mathrm{d}s\,\sigma \Big\{ C(q,s) - \lambda(s)\,q \Big\} \tag{3.2.43}$$

in which the last term may be transformed to yield the circulation at the boundaries.

(The hydrostatic equations of motion in Cartesian coordinates may also be derived via postulation of a constrained bracket from the Eulerian Hamiltonian formulation (3.1.22)–(3.1.24) of the non-hydrostatic compressible equations — cf. section 2.4.3.)

3.2.3 Formal stability

From the equations of motion (3.2.22), (3.2.28) and (3.2.29) non-resting basic states

$$\mathbf{v} = \mathbf{U}(\mathbf{x}),\ \sigma = \Sigma(\mathbf{x}),\ B = \bar{B}(\mathbf{x}),\ M = \bar{M}(\mathbf{x}),\ q = \bar{Q}(\mathbf{x}), \quad \text{and } s_B = S_B(\mathbf{x}),$$

with Bernoulli function $B \equiv (1/2)\,|\mathbf{v}|^2 + M$, are solutions of the system

$$\begin{aligned} 0 &= \mathbf{U}\cdot\nabla S_B \quad \text{at} \quad s = S_B(\mathbf{x}),\\ 0 &= \bar{Q}\,\Sigma\,\hat{\mathbf{z}}\times\mathbf{U} + \nabla\Big(\frac{1}{2}\,|\mathbf{U}|^2 + \bar{M}\Big),\\ 0 &= \nabla\cdot(\Sigma\,\mathbf{U}) \end{aligned} \tag{3.2.44}$$

and its accompanying thermodynamic relations. After introducing a transport streamfunction $\hat{\mathbf{z}}\times\nabla\Psi = \Sigma\,\mathbf{U}$ it follows that

$$\bar{Q}\,\nabla\Psi = \nabla\,\bar{B} \tag{3.2.45}$$

and that $\bar{B}, \bar{Q}, S_B$ are constant along streamlines, i.e. are functions of Ψ.

The Casimir function $C(q,s)$ and parameter $\lambda(s)$ in (3.2.43) may be determined such that the first variation of pseudo energy $\mathcal{A} = \mathcal{H}[\mathsf{u}] + \mathcal{C}[\mathsf{u}] - \mathcal{H}[\mathsf{U}] - \mathcal{C}[\mathsf{U}]$ with state variable $\mathsf{u} = \{\mathbf{v}, \sigma, s_B\}$ and basic state $\mathsf{U} = \{\mathbf{U}, \Sigma, S_B\}$ vanishes at this basic state (cf. (2.7.3)). With Hamiltonian (3.2.36) this first variation is

$$\begin{aligned} \delta\mathcal{A} \;=\;& \int_{D_H}\int_{s_B}^{\infty} \mathrm{d}\mathbf{x}\,\mathrm{d}s \Big\{ (B + C - q\,C_q)\,\delta\sigma + \\ & (\nabla C_q \times\hat{\mathbf{z}} + \sigma\,\mathbf{v})\cdot\delta\,\mathbf{v} \Big\} - \int_{D_H} \mathrm{d}\mathbf{x} \Big\{ \sigma\,(B + C - \lambda\,q)\,\delta\,s_B + \\ (C_q - \lambda)\,\delta\mathbf{v}\times\hat{\mathbf{z}}\cdot\nabla\,s_B \Big\}\Big|_{s=s_B} &+ \int_{\partial D_H} \mathrm{d}l\,\mathbf{n}\cdot\int_{s_B}^{\infty} \mathrm{d}s\,(C_q - \lambda)\,\delta\mathbf{v}\times\hat{\mathbf{z}}. \end{aligned} \tag{3.2.46}$$

Hence one finds that

$$\begin{aligned}\bar{B}(\Psi) &= -C(\bar{Q},s) + \bar{Q}\,C_{\bar{Q}}(\bar{Q},s),\\ \nabla C_{\bar{Q}}(\bar{Q},s)\times\hat{\mathbf{z}} &= -\Sigma\,\mathbf{U} = \nabla\Psi\times\hat{\mathbf{z}},\\ \lambda(s)|_{(s=S_B,\partial D_H)} &= C_{\bar{Q}}(\bar{Q},s)|_{(s=S_B,\partial D_H)}.\end{aligned} \tag{3.2.47}$$

Requirement (3.2.47) may be satisfied at $s = s_B$ since contours of entropy and potential vorticity coincide there for the basic state flow, and at ∂D_H because no-normal flow implies that the basic-state streamfunction there is a function of s only. Alternatively, in a semi-infinite domain potential vorticity may become a function of s only. From (3.2.47) it follows that a solution for $C(\nu, s)$ is

$$C(\nu,s) = \nu\Big(\int_0^\nu \mathrm{d}\gamma\,\frac{K(\gamma,s)}{\gamma^2} + k(s)\Big) \tag{3.2.48}$$

with $\bar{B} \equiv K(\bar{Q}, s)$ and for arbitrary $k(s)$. Defining disturbance quantities in the usual way like $\sigma = \Sigma + \sigma'$, $p = \Pi + p'$, $T = \bar{T} + T'$, etc. one derives the following pseudo energy

$$\begin{aligned}\mathcal{A} = &\int_{D_H}\int_{S_B}^{\infty}\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\{(\Sigma+\sigma')\int_0^{q'}\mathrm{d}\gamma\Big[C_\gamma(\bar{Q}+\gamma,s) - C_{\bar{Q}}(\bar{Q})\Big] +\\ &\frac{1}{2}(\Sigma+\sigma')\,|\mathbf{v}'|^2 + \sigma'\,\mathbf{U}\cdot\mathbf{v}'\Big\} + \int_{D_H}\int_{S_B+s_B'}^{\infty}\mathrm{d}\mathbf{x}\,\mathrm{d}s\,(\Sigma+\sigma')\,E(\Sigma+\sigma',s) -\\ &\int_{D_H}\int_{S_B}^{\infty}\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\{\Sigma\,E(\Sigma,s) + \bar{M}\,\sigma'\Big\} +\\ &\int_{D_H}\mathrm{d}\mathbf{x}\Big\{(p\,z)\Big|_{S_B+s_B{}'} - \Pi_B\,Z_B + \Sigma_B\,g\,Z_B\,s_B{}'\Big\} -\\ &\int_{D_H}\int_{S_B}^{S_B+s_B{}'}\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\{\frac{1}{2}\Sigma\,|\mathbf{U}|^2 + \Sigma_B\,g\,Z_B + \Sigma\,C(\bar{Q},s) - \lambda(s)\,\bar{Q}\,\Sigma\Big\} -\\ &\int\int_{S_B}^{S_B+s_B{}'}\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\{(\Sigma+\sigma')\Big[\mathbf{U}\cdot\mathbf{v}' + \frac{1}{2}|\mathbf{v}'|^2 + \Sigma\,C(Q+q',s) -\\ &\qquad\lambda(s)\,q'\Big] + \frac{1}{2}\sigma'\,|\mathbf{U}|^2 - \Sigma\,C(\bar{Q},s) - \lambda(s)\,\bar{Q}\,\sigma'\Big\},\end{aligned} \tag{3.2.49}$$

in which the subscript in Π_B denotes evaluation of Π at the steady state boundary $s = S_B$ and so forth except in $s_B{}'$. For simplicity only the small-amplitude limit $\mathcal{A}_2$ of (3.2.49) is considered for which the boundary coincides with an isentrope s_0. The result is

$$\begin{aligned}\mathcal{A}_2 = &\int_{D_H}\int_{s_0}^{\infty}\mathrm{d}\mathbf{x}\,\mathrm{d}s\Big\{\frac{1}{2(\Sigma+\sigma')}\Big|(\Sigma+\sigma')\,\mathbf{v}' + \mathbf{U}\,\sigma'\Big|^2 -\\ &\frac{1}{2}\frac{|\mathbf{U}|^2}{(\Sigma+\sigma')}\sigma'^2 + \frac{1}{\Sigma}\frac{\partial Z}{\partial s}\sigma'\,p' + \frac{1}{2}\Big(\frac{R}{c_p}-1\Big)\frac{1}{\Pi}\frac{\partial Z}{\partial s}p'^2 + \frac{1}{2}C''(\bar{Q})\,q'^2\Big\}\end{aligned} \tag{3.2.50}$$

(use has been made of the ideal gas law). The appearance of terms proportional to $\sigma'\,p'$ and p'^2 in (3.2.50) prevents the derivation of formal stability criteria for general moving basic states. Holm and Long (1989) derived formal stability criteria for hydrostatic, incompressible Boussinesq flows expressed in isopycnal coordinates by introducing an effective local wavenumber, which in the isentropic coordinates used here would amount to σ'/p', but their criteria are conditional in that they are dependent on the nature of the perturbations rather than only on the nature of the steady state.

For a resting basic state formal stability criteria follow from (3.2.50) as

$$\frac{g\,\bar{\rho}}{c_p\,\Sigma} = N^2 > 0 \tag{3.2.51}$$

with positive $\bar{\rho}$ for flows (basic state plus disturbance) with a lower isentrope coinciding with the lower boundary. Condition (3.2.51) ensures static stability. Formal stability criteria for flows with boundary-intersecting isentropes and with resting basic states may also be derived (see Bokhove 2000).

3.3 Incompressible hydrodynamics

3.3.1 Slaving principle and Mach number perturbations

We will exemplify the leading-order Hamiltonian perturbation approach V first by considering a leading-order Mach-number expansion of the three-dimensional compressible fluid equations with a barotropic equation of state $p \equiv p(\rho)$. The starting point in our systematic derivation of a Hamiltonian formulation of the three-dimensional homogeneous Euler equations is the Hamiltonian formulation of the inviscid and unforced three-dimensional compressible, barotropic equations.

The Hamiltonian formulation of the three-dimensional compressible, barotropic fluid equations is (3.1.18) with Hamiltonian (3.1.22) and bracket (3.1.24) for a *constant* value of entropy s. The dynamical variables are thus $\mathbf{u}$ and ρ. Hereafter we will assume that the domain Ω is closed for simplicity. In particular, variations of the Hamiltonian $\mathcal{H}$ are now

$$\delta\mathcal{H} = \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z \left\{ \rho\,\mathbf{u}\cdot\delta\mathbf{u} + \left(\frac{1}{2}\,|\mathbf{u}|^2 + U(\rho) + \frac{p}{\rho}\right)\delta\rho \right\}, \tag{3.3.1}$$

where one has $\delta U = (p/\rho^2)\,\delta\rho$ from the second law of thermodynamics for fixed entropy.

The equations of motion arise from (3.1.18), (3.1.24) and (3.3.1) as

$$\frac{\partial\rho}{\partial t} = -\nabla_3\cdot(\rho\mathbf{u}), \tag{3.3.2}$$

$$\frac{\partial\mathbf{u}}{\partial t} = \mathbf{u}\times\boldsymbol{\omega_a} - \nabla_3\left(\frac{1}{2}\mathbf{u}^2\right) - \frac{1}{\rho}\nabla_3\,p(\rho), \tag{3.3.3}$$

with $\boldsymbol{\omega_a} \equiv \nabla_3 \times (\mathbf{u}+\mathbf{R})$ the sum of planetary and relative vorticity, for suitable boundary condition $\mathbf{u}\cdot\mathbf{n}=0$ on the boundary $\partial\Omega$ with normal $\mathbf{n}$, in addition to the admissibility condition

$$\mathbf{n}\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{u}}=0, \qquad \text{on } \partial\Omega, \tag{3.3.4}$$

for an otherwise arbitrary functional $\mathcal{G}$.

The equations (3.3.3) will be scaled by

$$\begin{aligned} \mathbf{u} &= V\,\mathbf{u}', \quad t=(L/V)\,t', \quad \mathbf{x}=L\,\mathbf{x}', \quad U=V\,U' \\ \rho &= \rho_0\,\rho_T'=\rho_0\,(1+Ma\,\rho'), \quad \frac{\partial p}{\partial\rho}=c_0^2\,\frac{\partial p'}{\partial\rho'}, \end{aligned} \tag{3.3.5}$$

where L is a characteristic length scale, V a velocity scale, ρ_0 the basic state density and ρ_T' the total density. In terms of the vorticity, three-dimensional divergence $D=\nabla_3\cdot\mathbf{u}$ and perturbation density ρ (dropping the primes), the dimensionless equations read

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = -\mathbf{u}\cdot\nabla_3\boldsymbol{\omega}-\boldsymbol{\omega_a}D+\boldsymbol{\omega_a}\cdot\nabla_3\mathbf{u} \tag{3.3.6}$$

$$\frac{\partial D}{\partial t}+\frac{1}{Ma^2}\nabla_3^2\,\Lambda(\rho_T) = -\nabla_3\cdot(\mathbf{u}\cdot\nabla_3\mathbf{u}) \tag{3.3.7}$$

$$\frac{\partial\rho}{\partial t}+\frac{1}{Ma}D = -\nabla_3\cdot(\rho\,\mathbf{u}), \tag{3.3.8}$$

where $\Lambda\equiv\int^{\rho_T}\mathrm{d}\gamma\,(1/\gamma)\,(\partial p/\partial\gamma)$ and $Ma\equiv V/c_0$ is the Mach number. Using the expansion

$$\nabla_3\,\Lambda(\rho_T)=Ma\,\nabla_3\,\rho+\mathcal{O}(Ma^2) \tag{3.3.9}$$

in (3.3.8), we find that the equations of motion have the form (2.3.1), (2.3.2) with

$$\Gamma\equiv\begin{pmatrix} 0 & {\nabla_3}^2 \\ 1 & 0 \end{pmatrix}, \tag{3.3.10}$$

when we identify slow $s\equiv\boldsymbol{\omega}$ and fast $f\equiv\{D,\rho\}$ variables. The linearized fast equations (3.3.7), (3.3.8) yield the familiar wave equation of gas dynamics. When we transform the (dimensional) bracket (3.1.24) to a non-dimensional one in terms of the new variables $\boldsymbol{\omega}$, D and ρ the bracket takes the prototypical form (2.5.2)

$$\begin{aligned} [\mathcal{F},\mathcal{G}] = \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\Big\{\frac{\boldsymbol{\omega_a}}{1+Ma\,\rho}\cdot\Big[\Big(\nabla_3\times\frac{\delta\mathcal{F}}{\delta\boldsymbol{\omega}}\Big)\times\Big(\nabla_3\times\frac{\delta\mathcal{G}}{\delta\boldsymbol{\omega}}\Big)+ \\ \Big(\nabla_3\,\frac{\delta\mathcal{F}}{\delta D}\Big)\times\Big(\nabla_3\,\frac{\delta\mathcal{G}}{\delta D}\Big)-\Big(\nabla_3\,\frac{\delta\mathcal{F}}{\delta D}\Big)\times\Big(\nabla_3\times\frac{\delta\mathcal{G}}{\delta\boldsymbol{\omega}}\Big)+ \\ \Big(\nabla_3\,\frac{\delta\mathcal{G}}{\delta D}\Big)\times\Big(\nabla_3\times\frac{\delta\mathcal{F}}{\delta\boldsymbol{\omega}}\Big)\Big]+\frac{1}{Ma}\,(\nabla_3^2\,\frac{\delta\mathcal{G}}{\delta D})\,\frac{\delta\mathcal{F}}{\delta\rho}-\frac{1}{Ma}\,(\nabla_3^2\,\frac{\delta\mathcal{F}}{\delta D})\,\frac{\delta\mathcal{G}}{\delta\rho}\Big\} \end{aligned} \tag{3.3.11}$$

plus boundary terms. To obtain (3.3.11), one has to relate the functional derivatives with respect to $\{\mathbf{u}, \rho\}$ to ones with respect to $\{\boldsymbol{\omega}, D, \rho\}$ and substitute these results into (3.1.24), just like the transformation in Example 6.

The Hamiltonian is

$$\mathcal{H} = \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z\,(1 + Ma\,\rho)\Big\{\frac{1}{2}|\mathbf{u}|^2 + U(\rho_T)\Big\} \tag{3.3.12}$$

and variations of $\mathcal{H}$ with respect to the fast variables D and ρ do not seem to be linear in D and ρ, i.e. do not meet the assumptions introduced in section 2.5. We may, however, use the Casimir invariant $\int \mathrm{d}\mathbf{x}\,K_c\,\rho_T$ with constant K_c to replace $\mathcal{H}$ by a pseudo energy invariant $\mathcal{A}$ which satisfies these assumptions. With dimensional (denoted by asterisks) $K_c^* = -p(\rho_0)/\rho_0 - U^*(\rho_0)$ the non-dimensional pseudo energy invariant becomes

$$\mathcal{A} \;=\; \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\Big\{\frac{1}{2}\,(1 + Ma\,\rho)\,|\mathbf{u}|^2 + \frac{1}{2}\,\rho^2 + O(\rho^3)\Big\}. \tag{3.3.13}$$

Variation of (3.3.13) gives

$$\delta\mathcal{A} = \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\Big\{\boldsymbol{\Psi}\cdot\delta\boldsymbol{\omega} - \chi\,\delta D + (\rho + \frac{1}{2}\,Ma\,|\mathbf{u}|^2)\,\delta\rho + O(\rho^2)\Big\}, \tag{3.3.14}$$

which shows that variations of $\mathcal{A}$ with respect to D and ρ are linear in D and ρ. In calculating (3.3.14) we have used the following decomposition

$$\rho_T\mathbf{u} = (1 + Ma\,\rho)\,\mathbf{u} = \nabla_3\times\boldsymbol{\Psi} + \nabla_3\chi \tag{3.3.15}$$

such that

$$\nabla_3\times(\rho_T\mathbf{u}) = \nabla_3\times(\nabla_3\times\boldsymbol{\Psi}), \qquad \nabla_3\cdot(\rho_T\mathbf{u}) = \nabla_3^2\chi; \tag{3.3.16}$$

the decomposition of the velocity field into a potential part and a remainder is unique. The velocity streamfunction vector $\boldsymbol{\Psi}$ has been gauged by taking $\nabla_3\cdot\boldsymbol{\Psi} = 0$ (following Benjamin 1984). Boundary integrals arising in the variation of (3.3.13) cancel after using $\mathbf{n}\cdot\delta\mathbf{u} = 0, \mathbf{n}\cdot\nabla_3\chi = 0$ and $\boldsymbol{\Psi} = 0$ at $\partial\Omega$. The corresponding admissibility condition $\partial\mathcal{G}/\partial\boldsymbol{\omega} = 0$ at $\partial\Omega$ can now be used to eliminate boundary contributions in (3.3.11). These boundary conditions may be overly restrictive.

3.3.2 Leading-order Hamiltonian perturbations

The Hamiltonian formulation of the leading-order Mach-number perturbation of the compressible barotropic equations appears when we compare (3.3.11) with (2.5.2) and use the results of section 2.6. One obtains

$$\frac{d\mathcal{F}}{dt} \;=\; [\mathcal{F}, \mathcal{A}_0]_0, \tag{3.3.17}$$

$$\mathcal{A}_0 \;=\; \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\frac{1}{2}|\mathbf{u}^{(0)}|^2, \tag{3.3.18}$$

where $\mathcal{A}_0$ is the $O(1)$ part of pseudo energy (3.3.13), with bracket

$$\begin{aligned}[\mathcal{F},\mathcal{G}]_0 &\equiv \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z \left\{\boldsymbol{\omega_a}\cdot\left(\nabla_3\times\frac{\delta\mathcal{F}}{\delta\boldsymbol{\omega}}\right)\times\left(\nabla_3\times\frac{\delta\mathcal{G}}{\delta\boldsymbol{\omega}}\right)\right\}\\ &= \int_\Omega \mathrm{d}\mathbf{x}\,\mathrm{d}z \left\{-\frac{\delta\mathcal{F}}{\delta\boldsymbol{\omega}}\cdot\nabla_3\times\left[\boldsymbol{\omega_a}\times\left(\nabla_3\times\frac{\delta\mathcal{G}}{\delta\boldsymbol{\omega}}\right)\right]\right\} \qquad (3.3.19)\end{aligned}$$

in addition to the constraints

$$D^{(0)} = \nabla_3\cdot\mathbf{u}^{(0)} = 0, \qquad \rho^{(0)} = 0. \qquad (3.3.20)$$

Boundary terms arising in integration by parts of the first line in (3.3.19) vanish after using the admissibility condition. The bracket (3.3.19) is skew-symmetric and bilinear and by construction satisfies Jacobi's identity. A direct proof of these mathematical requirements, which guarantee the Hamiltonian structure, can be found in Olver (1982, 1986). The equations of motion resulting from (3.3.17)–(3.3.19) with

$$\frac{\delta\mathcal{H}_0}{\delta\boldsymbol{\omega}} = \boldsymbol{\Psi}^{(0)} \qquad (3.3.21)$$

are

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = -\nabla_3\times(\boldsymbol{\omega_a}\times\nabla_3\times\boldsymbol{\Psi}^{(0)}), \qquad (3.3.22)$$

where $\mathbf{u}^{(0)} = \nabla_3\times\boldsymbol{\Psi}^{(0)}$, $\nabla_3\cdot\boldsymbol{\Psi}^{(0)} = 0$, $\boldsymbol{\omega} = -\nabla_3^2\boldsymbol{\Psi}^{(0)}$, and $\boldsymbol{\Psi}^{(0)} = 0$ on $\partial\Omega$.

3.3.3 Formal stability

Although helicity $\int \mathrm{d}\mathbf{x}\,\mathrm{d}z\,\boldsymbol{\omega}\cdot(\mathbf{u}+2\,\mathbf{R})$ is a Casimir invariant, satisfying $[\mathcal{C},\mathcal{G}] = 0$, it generally does not allow us to remove the first variation of the pseudo energy at the basic state. To circumvent this problem, Vladimirov (1987) uses material conservation of a tracer or particle label to establish an extremum variational principle. He could, however, only establish stability criteria for flows in which basic state and disturbance are both planar or axially or helically symmetric. The additional symmetry imposed on the flow, basic state plus disturbance, often allows additional Casimir invariants to be introduced which may then be used to derive formal and nonlinear stability criteria for these symmetric flows (e.g. Shepherd 1991). For flows in a horizontal plane the streamfunction $\Psi(\bar{\Omega})$ becomes a function of basic state vertical vorticity $\bar{\Omega}$ and the formal stability criterion reads

$$\frac{\nabla\Psi}{\nabla\bar{\Omega}} > 0. \qquad (3.3.23)$$

This criterion corresponds to Arnol'd's (1966) first stability theorem.

3.4 Shallow-water equations

3.4.1 Lagrangian and Eulerian Hamilton's principle

Hamilton's principle for the rotating shallow-water equations is

$$0 = \delta \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \, \mathcal{L} \tag{3.4.1}$$

(e.g. Salmon 1985), where Lagrangian

$$\mathcal{L} = \int_\Omega \mathrm{d}\mathbf{a} \, h_0(\mathbf{a}) \left\{ \left(\frac{1}{2} \frac{\partial x_i}{\partial \tau} + R_i(\mathbf{x}) \right) \frac{\partial x_i}{\partial \tau} - g \left(\frac{1}{2} h + h_B \right) \right\} \tag{3.4.2}$$

is subject to independent variations of the positions $\mathbf{x}(\mathbf{a}, \tau)$ of parcel labels $\mathbf{a} = (a, b)^T$ and of time τ in a domain Ω. These equations model the dynamics of a thin homogeneous, hydrostatic layer of rotating fluid in which the dependence on the vertical, label c or spatial z, coordinate has been neglected or averaged over the depth $h \equiv h_0(\mathbf{a})/J^a(x_1, x_2)$, with $J^a(A, B) \equiv \partial_a A \, \partial_b B - \partial_b A \, \partial_a B$. The topography is at $z = h_B(\mathbf{x}(\mathbf{a}, \tau))$ and the free surface is at $z = h_B + h$. An Eulerian Hamilton's principle in terms of spatial coordinates x, y and time t and fluid labels $\mathbf{a}(\mathbf{x}, t)$ as variables may be derived via a transformation of (3.4.1), (3.4.2). Such a transformation is similar to the one described in detail in section 3.2.2 and yields

$$\begin{aligned} 0 = \delta S_{swe}[\mathbf{a}] &= \delta \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \, h(\mathbf{x}, t) \\ &\left\{ \left(\frac{1}{2} u_h(\mathbf{x}, t) + R_h(\mathbf{x}) \right) u^h(\mathbf{x}, t) - g \left(\frac{1}{2} h(\mathbf{x}, t) + h_B(\mathbf{x}) \right) \right\}, \end{aligned} \tag{3.4.3}$$

in which velocity u^h is the economic shorthand defined by

$$\mathbf{v} \equiv \frac{\partial \mathbf{x}}{\partial \tau} = -\mathbf{\Gamma}^{-1} \frac{\partial \mathbf{a}}{\partial t} \Longleftrightarrow \frac{\partial a^i}{\partial t} + u^k \frac{\partial a^i}{\partial x^k} = 0, \tag{3.4.4}$$

in which the depth h of the water column can be rewritten as

$$h = h_0(a, b) \ \det |\partial(a, b)/\partial(x, y)| \tag{3.4.5}$$

and in which variations are taken with respect to fluid parcel variables $\mathbf{a}(\mathbf{x}, t)$. (Hamilton's principle (3.4.3) can also be derived from a hydrostatic Eulerian action principle by direct substitution (approach II) of the constraint $\rho =$ constant and by vertical integration of the z-independent velocity vector.)

3.4.2 Eulerian action principle

The generalized momentum corresponding to (3.4.3) (cf. (2.1.8)) is

$$\begin{aligned} \pi_k^*(\mathbf{x}, t) = \frac{\delta \mathcal{L}[\mathbf{a}]}{\delta \left[\dfrac{\partial a^k}{\partial t} \right]} &= h \, (\Gamma^{-1})_k^m \left(\delta_{mn} \, (\Gamma^{-1})_j^n \, \frac{\partial a^j}{\partial t} - R_m \right) \\ &= -h \, (\Gamma^{-1})_k^m \left(u_m + R_m \right). \end{aligned} \tag{3.4.6}$$

An Eulerian action principle (cf. (2.1.10)) follows after a Legendre transform (cf. (2.1.9)), and may be rewritten in terms of $\boldsymbol{\pi}^*$ and $\mathbf{a}$ or in terms of $\mathbf{v}$ and $\mathbf{a}$. One finds either

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_\Omega \mathrm{d}\mathbf{x}\, \pi_k^* \frac{\partial a^k}{\partial t} - \mathcal{H}[\pi_i^*, a^i] \right\} \tag{3.4.7}$$

or

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \left\{ \int_\Omega \mathrm{d}\mathbf{x} \left[-h\,(u_i + R_i)\,(\Gamma^{-1})_k^i \frac{\partial a^k}{\partial t} \right] - \mathcal{H}[u_i, a^i] \right\} \tag{3.4.8}$$

with the Hamiltonian as Legendre transform

$$\mathcal{H}[\pi_i^*, a^i] = \mathcal{H}[u_i, a^i] = \int_\Omega \mathrm{d}\mathbf{x} \left\{ \frac{1}{2} \left(h\, u_i\, u^i + g\, h^2 \right) + g\, h\, h_B \right\}. \tag{3.4.9}$$

Variations of (3.4.8) are taken with respect to fluid parcels $\mathbf{a}$ and velocity $\mathbf{v}(\mathbf{x}, t)$ and yield the shallow-water momentum

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathrm{f}\, \hat{\mathbf{z}} \times \mathbf{v} \;=\; -g\, \nabla\, (h + h_B) \tag{3.4.10}$$

and parcel advection equations (3.4.4), respectively. The continuity equation

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\, \mathbf{v}) \;=\; 0 \tag{3.4.11}$$

follows by taking the time derivative of (3.4.5) and use of (3.4.4). All these manipulations are analogous to the ones for the hydrostatic equations in section 3.2.2, however, variations of the potential energy $g\,((1/2)\, h^2 + h\, h_B)$ in the shallow-water equations are simpler than those of the internal plus potential energy, $U + g\, z$, in the stratified hydrostatic equations.

3.4.3 Hamiltonian formulation

The Hamiltonian formulation of the shallow-water equations (e.g. Shepherd 1990) may now be derived systematically from (3.4.8), (3.4.9) (Sudarshan and Mukunda 1974, paragraphs starting at pages 132 and 424). The following brackets emerge in terms of $\boldsymbol{\pi}^*$ and $\mathbf{a}$

$$[\mathcal{F}, \mathcal{G}] = \int_\Omega \mathrm{d}\mathbf{x} \left\{ \frac{\delta \mathcal{F}}{\delta \mathbf{a}} \frac{\delta \mathcal{G}}{\delta \boldsymbol{\pi}^*} - \frac{\delta \mathcal{F}}{\delta \boldsymbol{\pi}^*} \frac{\delta \mathcal{G}}{\delta \mathbf{a}} \right\} \tag{3.4.12}$$

or in terms of $\mathbf{v}$ and $\mathbf{a}$

$$[\mathcal{F}, \mathcal{G}] = \int_\Omega \mathrm{d}\mathbf{x} \left\{ q\, \hat{\mathbf{z}} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \times \frac{\delta \mathcal{G}}{\delta \mathbf{v}} + \frac{\delta \mathcal{G}}{\delta \mathbf{a}} \cdot \left(\frac{1}{h} \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \cdot \nabla \right) \mathbf{a} - \frac{\delta \mathcal{F}}{\delta \mathbf{a}} \cdot \left(\frac{1}{h} \frac{\delta \mathcal{G}}{\delta \mathbf{v}} \cdot \nabla \right) \mathbf{a} \right\}. \tag{3.4.13}$$

Via a reduction or transformation of variables from $\{\boldsymbol{\pi}^*, \mathbf{a}\}$ or $\{\mathbf{a}, \mathbf{v}\}$ to $\{\mathbf{v}, h\}$ (similar to the transformation in Example 6), we find the Hamiltonian formulation (3.1.18) with Hamiltonian (3.4.9), Poisson bracket

$$[\mathcal{F}, \mathcal{G}] = \int_\Omega \mathrm{d}\mathbf{x} \left\{ q\, \hat{\mathbf{z}} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \times \frac{\delta \mathcal{G}}{\delta \mathbf{v}} - \frac{\delta \mathcal{F}}{\delta h} \left(\nabla \cdot \frac{\delta \mathcal{G}}{\delta \mathbf{v}} \right) + \frac{\delta \mathcal{G}}{\delta h} \left(\nabla \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \right) \right\} \tag{3.4.14}$$

and potential vorticity

$$q = \frac{\mathsf{f} + \hat{\mathbf{z}} \cdot \nabla_3 \times \mathbf{u}}{h}. \tag{3.4.15}$$

Variations of the Hamiltonian (3.4.9) give

$$\delta \mathcal{H} = \int_\Omega \mathrm{d}\mathbf{x} \left\{ h\, \mathbf{v} \cdot \delta \mathbf{v} + \left(\frac{1}{2} |\mathbf{v}|^2 + g\,(h + h_B) \right) \delta h \right\} \tag{3.4.16}$$

and the equations of motion (3.4.10) and (3.4.11) follow from (3.1.18), (3.4.14) and (3.4.16).

Conservation of the Hamiltonian in time is assured since $d\mathcal{H}/dt = [\mathcal{H}, \mathcal{H}] = 0$ by virtue of the skew-symmetric nature of the bracket and the lack of explicit time dependence of $\mathcal{H}$. Casimir invariants of the shallow-water equations

$$\mathcal{C} = \int_\Omega \mathrm{d}\mathbf{x}\, h\, C(q) \tag{3.4.17}$$

with arbitrary function $C(\cdot)$ are solutions of $[\mathcal{C}, \mathcal{G}] = 0$ for arbitrary functionals $\mathcal{G}$, but their invariance is also readily checked from the equations of motion by first deriving the potential vorticity equation $\partial\, q/\partial t + \mathbf{v} \cdot \nabla q = 0$.

In the next section a dimensionless formulation is needed based on a characteristic velocity scale V, a length scale L, and a mean depth H. For simplicity we also take $\mathsf{f} = 1$ and consider a doubly-periodic domain Ω; problems concerning boundary-trapped Kelvin waves (e.g. Gill 1982), which can be slow as well as fast in domains large relative to the Rossby radius of deformation, are thus avoided. With $\mathbf{v} = V\,\mathbf{v}'$, $\mathbf{x} = L\,\mathbf{x}'$ and $h = H\,h'$ one finds, after dropping the primes and taking $h_B = 0$, (3.1.18) with Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int_\Omega \mathrm{d}\mathbf{x} \left\{ (1 + \epsilon\, F_r\, \eta)\, |\mathbf{v}|^2 + F_r\, \eta^2 \right\} \tag{3.4.18}$$

and Poisson bracket

$$[\mathcal{F}, \mathcal{G}] = \int_\Omega \mathrm{d}\mathbf{x} \left\{ q^*\, \mathbf{k} \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \times \frac{\delta \mathcal{G}}{\delta \mathbf{v}} + \frac{1}{\epsilon\, F_r} \frac{\delta \mathcal{G}}{\delta \mathbf{v}} \cdot \nabla \frac{\delta \mathcal{F}}{\delta \eta} - \frac{1}{\epsilon\, F_r} \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \cdot \nabla \frac{\delta \mathcal{G}}{\delta \eta} \right\}, \tag{3.4.19}$$

where $\epsilon \equiv V/\mathsf{f}L$ is the Rossby number, $F_r \equiv \mathsf{f}^2 L^2/gH$ the rotational Froude number, $q^* \equiv (\hat{\mathbf{z}} \cdot \nabla \times \mathbf{v} + 1/\epsilon)/h$, and $\eta \equiv (h-1)/(\epsilon\, F_r)$ the departure from the mean (constant) depth of the fluid. The dimensionless equations of motion are

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\epsilon}\, \hat{\mathbf{z}} \times \mathbf{v} &= -\frac{1}{\epsilon} \nabla \eta \\ \frac{\partial \eta}{\partial t} + \nabla \cdot (\eta\, \mathbf{v}) + \frac{1}{\epsilon\, F_r} \nabla \cdot \mathbf{v} &= 0. \end{aligned} \tag{3.4.20}$$

3.4.4 Formal stability

Formal stability criteria have been derived by Ripa (1983). Steady state solutions $\mathbf{v} = \mathbf{U}(\mathbf{x}), h = H(\mathbf{x})$ satisfy

$$\nabla \cdot (H\,\mathbf{U}) = 0, \quad Q_s \nabla \Psi_s = \nabla B, \tag{3.4.21}$$

where $H\,\mathbf{U} = \hat{z} \times \nabla \Psi_s$, $B = \frac{1}{2}|\mathbf{U}|^2 + g\,H$ is the Bernoulli function and Q_s is the steady state potential vorticity satisfying $\mathbf{U} \cdot \nabla Q_s = 0$. Q_s and B are thus all constant along streamlines $\Psi_s =$ constant. Linear stability is ensured when the following formal stability criteria are satisfied

$$|\mathbf{U}|^2 < g\,H, \quad \frac{d^2C}{dQ_s{}^2} = \frac{\nabla \Psi_s}{\nabla Q_s} > 0, \tag{3.4.22}$$

and these conditions are derived in a fashion similar to the ones in Example 11. The first criterion corresponds to a subsonic condition in the sense that the basic state flow speed needs to be less than the minimim gravity wave speed, and the second corresponds to that of Arnol'd's (1966) first stability theorem and is a sufficient condition for stability for the equivalent barotropic quasi-geostrophic system which will be derived in the next section.

3.5 Equivalent barotropic quasi-geostrophic equations

3.5.1 Slaving principle and Rossby number perturbations

The Hamiltonian formulation of the equivalent barotropic ("barotropic" now refers to the lack of stratification, "equivalent" refers to the inclusion of free-surface effects) quasi-geostrophic equations may be derived in systematic fashion from the shallow-water one by the leading-order Hamiltonian perturbation approach V. The approach entails a modified Rossby-number expansion of the shallow-water Hamiltonian formulation, and it is a Hamiltonian version of the quasi-geostrophic system rederived in Warn et al. (1995).

The prototypical singular form (2.3.1), (2.3.2) of the shallow-water equations (3.4.20) arises via a transformation of the variables $\mathbf{v}$ and h to: the slow variable quasi-geostrophic potential vorticity Q, and the fast variables divergence D and geostrophic imbalance Υ, defined by

$$Q \equiv \nabla^2 \psi - F_r\,\eta, \tag{3.5.1}$$
$$D \equiv \nabla \cdot \mathbf{v} = \nabla^2 \chi, \tag{3.5.2}$$
$$\Upsilon \equiv \nabla^2 \psi - \nabla^2 \eta, \tag{3.5.3}$$

respectively. The streamfunction ψ and velocity potential χ arise in the Helmholtz decomposition of the horizontal velocity

$$\mathbf{v} \equiv \hat{\mathbf{z}} \times \nabla \psi + \nabla \chi. \tag{3.5.4}$$

The corresponding equations in prototypical form (cf. (2.3.1), (2.3.2)) are now

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= -J(\psi, Q) - \nabla\chi\cdot\nabla Q - Q\,D, \\
\frac{\partial D}{\partial t} - \frac{1}{\epsilon}\Upsilon &= -J(\chi, \nabla^2\psi) - \frac{1}{2}\nabla_H^2|\nabla\chi|^2 + \nabla^2 J(\chi,\psi) + \\
&\quad 2J(\psi_x, \psi_y), \\
\frac{\partial \Upsilon}{\partial t} - \frac{\mathcal{P}\,D}{\epsilon} &= -J(\psi, \nabla^2\psi) - \nabla\cdot(\nabla^2\psi\,\nabla\chi) + \\
&\quad \nabla^2\Big\{J(\psi,\eta) + \nabla\cdot(\eta\nabla\chi)\Big\},
\end{aligned} \tag{3.5.5}$$

where $\mathcal{P} \equiv \nabla^2/Fr - 1$ is the Helmholtz operator. After linearising (3.5.5) and using a normal-mode ansatz the fast variables may be associated with gravity-wave motion of frequency $O(1/\epsilon)$, while the slow variable is associated with zero-frequency motion. A transformation of the bracket (3.4.19) in terms of the new variables yields the prototypical form (cf. (2.5.2))

$$\begin{aligned}
[\mathcal{F},\mathcal{G}] &= \int_\Omega \mathrm{d}\mathbf{x}\Big\{\frac{Q}{1+\epsilon\,F_r\,\eta}\Big[J\Big(\frac{\delta\mathcal{F}}{\delta D},\frac{\delta\mathcal{F}}{\delta D}\Big) + J\Big(\frac{\delta\mathcal{F}}{\delta\Upsilon}+\frac{\delta\mathcal{F}}{\delta Q},\frac{\delta\mathcal{G}}{\delta\Upsilon}+\frac{\delta\mathcal{G}}{\delta Q}\Big) \\
&\quad +\Big(\nabla\frac{\delta\mathcal{F}}{\delta D}\Big)\cdot\nabla\Big(\frac{\delta\mathcal{G}}{\delta\Upsilon}+\frac{\delta\mathcal{G}}{\delta Q}\Big) - \Big(\nabla\frac{\delta\mathcal{G}}{\delta D}\Big)\cdot\nabla\Big(\frac{\delta\mathcal{F}}{\delta\Upsilon}+\frac{\delta\mathcal{F}}{\delta Q}\Big)\Big] \\
&\quad +\frac{1}{\epsilon}\Big[\Big(\nabla^2\frac{\delta\mathcal{F}}{\delta D}\Big)\mathcal{P}\Big(\frac{\delta\mathcal{G}}{\delta\Upsilon}\Big) - \Big(\nabla^2\frac{\delta\mathcal{G}}{\delta D}\Big)\mathcal{P}\Big(\frac{\delta\mathcal{F}}{\delta\Upsilon}\Big)\Big]\Big\},
\end{aligned} \tag{3.5.6}$$

where the boundary terms that arise after several integrations by parts cancel after usage of doubly-periodic boundary conditions or quiescense at infinity.

3.5.2 Leading-order Hamiltonian perturbations

The approach presented in section 2.6 immediately gives the Hamiltonian formulation of the barotropic quasi-geostrophic equations. After identifying (2.6.6) one finds

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F},\mathcal{H}_0]_0, \tag{3.5.7}$$

$$[\mathcal{F},\mathcal{G}]_0 \equiv \int_\Omega \mathrm{d}\mathbf{x}\,Q\,J\Big(\frac{\delta\mathcal{F}}{\delta Q},\frac{\delta\mathcal{G}}{\delta Q}\Big) \tag{3.5.8}$$

$$\mathcal{H}_0 = \frac{1}{2}\int_D \mathrm{d}\mathbf{x}\Big\{|\nabla\psi^{(0)}|^2 + F_r\,{\psi^{(0)}}^2\Big\} \tag{3.5.9}$$

in addition to the constraints

$$D^{(0)} = \nabla\cdot\mathbf{v}^{(0)} = 0 \Rightarrow \chi^{(0)} = 0, \tag{3.5.10}$$

$$\Upsilon^{(0)} = 0 \Rightarrow \psi^{(0)} = \eta^{(0)}, \tag{3.5.11}$$

$$Q = \nabla^2\psi^{(0)} - F_r\,\psi^{(0)} \tag{3.5.12}$$

(e.g. Warn et al. 1995). We recall that the slow variable Q, now clearly seen to be the quasi-geostrophic potential vorticity, is not expanded while all the other (fast) variables are expanded in a power series of ϵ. The streamfunction is also expanded because it projects on both fast and slow variables except at leading order. The quasi-geostrophic equations

$$\frac{\partial Q}{\partial t} + J(\psi^{(0)}, Q) = 0 \tag{3.5.13}$$

now readily follow from (3.5.7)–(3.5.12). Although the leading-order Hamiltonian approach provides an elegant derivation of the quasi-geostrophic Hamiltonian formulation, the question how to include low-frequency boundary Kelvin waves remains unanswered. The reason is that the assumed time-scale separation in the approach is invalid for large (partly) bounded domains because low-frequency boundary Kelvin gravity waves then appear.

The Hamiltonian $\mathcal{H}_0$ is again conserved in time and Casimir invariants of the equivalent barotropic quasi-geostrophic equations have the form

$$\mathcal{C} = \int_\Omega \mathrm{d}\mathbf{x}\, C(Q). \tag{3.5.14}$$

3.5.3 Formal stability

Consider a steady state solution $J(\Psi^{(0)}, Q_s) = 0$ of (3.5.13). With the energy-Casimir method formal stability criteria are derived to be

$$\frac{d^2C}{d{Q_s}^2} = \frac{\nabla\, \Psi^{(0)}}{\nabla\, Q_s} > 0 \tag{3.5.15}$$

(cf. McIntyre and Shepherd 1987 and references therein).

3.6 Ageostrophic barotropic equations

A Hamiltonian formulation of higher-order balanced models will be derived in two ways: (IV) by postulating Dirac's Poisson bracket for a velocity constraint, and (II) by direct substitution of a velocity constraint into an Eulerian variational principle of the shallow-water equations. These models have formal accuracy beyond the quasi-geostrophic model and will be denoted as ageostrophic barotropic equations. In particular, the equations of motion will be derived for a velocity constraint one order beyond geostrophy, which is based on a slaving approach. Both approaches find their roots in Salmon's (1985, 1988b) work and are variations on recent work by Allen and Holm (1996) and McIntyre and Roulstone (1996) (see also Bokhove and Shepherd (1996) in the context of low-order models). The imposed velocity constraint is systematically derived by applying the modified Rossby-number expansions to a velocity slaving principle. A systematic derivation of conservative balanced models is now achieved by *combining* constrained Hamiltonian methods with perturbation (or alternatively iteration) methods based on slaving principles.

3.6.1 Velocity-to-height slaving principle

Consider the scaled shallow-water equations

$$\epsilon^2\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}\right] + \epsilon\,\mathsf{f}\,\hat{\mathbf{z}}\times\mathbf{v} = -\frac{1}{F_r}\nabla(h+h_B), \tag{3.6.1}$$

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\,\mathbf{v}) = 0, \tag{3.6.2}$$

where $h + h_B = 1 + \epsilon\,F_r\,\eta$. Substitution of a velocity slaving ansatz $\mathbf{v} = \mathbf{v}^C[h+h_B] \equiv \mathbf{v}^C[h]$ into (3.6.1), (3.6.2) and manipulation yields the following superbalance equations (e.g. (2.3.3), (2.3.4))

$$\epsilon\left[\frac{\delta\mathbf{v}^C[h]}{\delta h}\left[-\nabla\cdot(h\,\mathbf{v}^C[h])\right] + (\mathbf{v}^C[h]\cdot\nabla)\,\mathbf{v}^C[h]\right] + \mathsf{f}\,\hat{\mathbf{z}}\times\mathbf{v}^C[h] \;=\; -\nabla\,\eta, \tag{3.6.3}$$

where $\delta\,\mathbf{v}^C[h]/\delta\,h = 1/(\epsilon\,F_r)\,\delta\,\mathbf{v}^C[h]/\delta\,\eta$ acts as an operator $\mathcal{O}_c$ on $-\nabla\cdot(h\,\mathbf{v}^C)$. Consider, for example, $\mathbf{v}^C[h] = \mathcal{O}[h] = h\,h_x$ then

$$\frac{\partial \mathbf{v}^C[h]}{\partial t} = -h_x\,\nabla\cdot(h\,\mathbf{v}^C[h]) - h\,\partial_x\left[-\nabla\cdot(h\,\mathbf{v}^C[h])\right] \equiv \mathcal{O}_c[-\nabla\cdot(h\,\mathbf{v}^C[h])].$$

Expanding $\mathbf{v}^C = \mathbf{v}^{(0)} + \epsilon\,\mathbf{v}^{(1)} + \ldots$, one finds geostrophic balance at leading order

$$\mathbf{v}^{(0)}[h] = \frac{1}{\epsilon\,F_r\,\mathsf{f}}\,\hat{\mathbf{z}}\times\nabla(h+h_B) = \frac{1}{\mathsf{f}}\,\hat{\mathbf{z}}\times\nabla\,\eta. \tag{3.6.4}$$

At the next order one finds

$$\mathcal{L}\mathbf{v}^{(1)} \equiv \frac{1}{F_r}\nabla\Big(\nabla\cdot\mathbf{v}^{(1)}\Big) - \mathsf{f}^2\,\mathbf{v}^{(1)} \;=\; -\mathsf{f}\,(\mathbf{v}^{(0)}\cdot\nabla)\,\hat{\mathbf{z}}\times\mathbf{v}^{(0)}. \tag{3.6.5}$$

Given boundary conditions for $\mathbf{v}^{(1)}$ these elliptic equations may be solved. The slaving approach, however, does not seem to provide such boundary conditions. On physical grounds one expects $\mathbf{v}^C\cdot\mathbf{n} = 0$ at solid boundaries and to ensure consistency one may take $\mathbf{n}\cdot$(3.6.1) at $O(\epsilon)$, i.e.

$$\mathbf{n}\cdot(\mathbf{v}^{(0)}\cdot\nabla)\,\mathbf{v}^{(0)} - \mathsf{f}\,\mathbf{v}^{(1)}\cdot\mathbf{t} = 0, \tag{3.6.6}$$

at a solid boundary with $\mathbf{n}$ the corresponding outward normal and $\mathbf{t}$ the unit vector tangent to the boundary. Without regard to conservation properties the constraint velocity $\mathbf{v}^{(0)}+\epsilon\,\mathbf{v}^{(1)}$ and the divergence $\nabla\cdot\mathbf{v}^{(2)}$ need to be determined and may be substituted in the continuity equation in order to yield a balanced equation one order higher than quasi-geostrophy. As becomes clear in section 3.6.2 it suffices to impose a constrained velocity $\mathbf{v}^{(0)} + \epsilon\,\mathbf{v}^{(1)}$ in a constrained variational or Hamiltonian approach since the Lagrange multiplier or particle velocity that will appear in the momentum and continuity equations is seen to be of the required order in ϵ. For unit (non-dimensional)

$\mathsf{f} = F_r = 1$ and constant topography, constraints (3.6.5) may be rewritten in terms of $\psi^{(1)}$ and $\chi^{(1)}$ as

$$(\nabla^2 - 1)\,\nabla\cdot\mathbf{v}^{(1)} \equiv (\nabla^2 - 1)\,\nabla^2\chi^{(1)} = J(\eta, \nabla^2\eta), \tag{3.6.7}$$

$$\hat{\mathbf{z}}\cdot\nabla\times\mathbf{v}^{(1)} \equiv \nabla^2\,\psi^{(1)} = -2\,J(\eta_x, \eta_y), \tag{3.6.8}$$

which appeared in Warn et al. (1995). Similar expressions emerge for space dependent h_B and f. The total constraint velocity may now be written in the form

$$\begin{aligned}\mathbf{v}^C[h] &\equiv \mathbf{v}^{(0)}[h] + \epsilon\,\mathbf{v}^{(1)}[h] \\ &= \frac{1}{\epsilon\,F_r\,\mathsf{f}}\,\hat{\mathbf{z}}\times\nabla(h+h_B) - \epsilon\,\mathcal{L}^{-1}\Big\{\mathsf{f}\,\hat{\mathbf{z}}\times(\mathbf{v}^{(0)}\cdot\nabla)\,\mathbf{v}^{(0)}\Big\}\end{aligned} \tag{3.6.9}$$

where $\mathcal{L}^{-1}$ is the inverse of $\mathcal{L}$ in (3.6.5) for a given set of boundary conditions for $\mathbf{v}^{(1)}$ (or alternatively for $\mathbf{v}^C$).

3.6.2 Postulation of constrained dynamics

For a given set of velocity constraints $f_i \equiv v_i = v_i^C$ $(i = 1, 2)$, balanced dynamics can be formulated by postulating the constrained dynamics

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F}, \mathcal{H}^*], \tag{3.6.10}$$

for the shallow-water Poisson bracket (cf. (3.4.19))

$$[\mathcal{F}, \mathcal{G}] \equiv \int_\Omega \mathrm{d}\mathbf{x}\Big\{q\,\hat{\mathbf{z}}\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}}\times\frac{\delta\mathcal{G}}{\delta\mathbf{v}} - \frac{\delta\mathcal{F}}{\delta h}\nabla\cdot\frac{\delta\mathcal{G}}{\delta\mathbf{v}} + \frac{\delta\mathcal{G}}{\delta h}\nabla\cdot\frac{\delta\mathcal{F}}{\delta\mathbf{v}}\Big\}, \tag{3.6.11}$$

and Hamiltonian

$$\mathcal{H}^* = \int_\Omega \mathrm{d}\mathbf{x}\,\Big\{\frac{1}{2}\,h\,|\mathbf{v}^C|^2 + \frac{1}{\epsilon^2\,F_r}\Big(\frac{1}{2}\,h^2 + h\,h_B\Big)\Big\} + \int_\Omega d\mathbf{x}\,\boldsymbol{\lambda}\cdot\Big(\mathbf{v} - \mathbf{v}^C(h)\Big) \tag{3.6.12}$$

(cf. (2.4.13)), in which the bracket calculations have to be performed before the constraints are ultimately substituted in the equations of motion.

Variation of $\mathcal{H}^*$ yields functional derivatives

$$\begin{aligned}\frac{\delta\mathcal{H}^*}{\delta h} &= \frac{1}{2}\,|\mathbf{v}^C[h]|^2 + \frac{1}{\epsilon^2\,F_r}\,(h+h_B) + \frac{\delta\mathbf{v}^C[h]}{\delta h}\cdot(h\,\mathbf{v}^C[h] - \boldsymbol{\lambda}) \equiv B^T \\ \frac{\delta\mathcal{H}^*}{\delta\mathbf{v}} &= \boldsymbol{\lambda} \equiv h\,\mathbf{v}^P.\end{aligned} \tag{3.6.13}$$

These may be substituted into (3.6.10), (3.6.11) to give the balanced equations, while enforcing $\mathbf{v} = \mathbf{v}^C$ afterwards,

$$\frac{\partial\mathbf{v}^C}{\partial t} = -q^C\,h\,\hat{\mathbf{z}}\times\mathbf{v}^P - \nabla B^T, \tag{3.6.14}$$

$$\frac{\partial h}{\partial t} = -\nabla\cdot(h\,\mathbf{v}^P), \tag{3.6.15}$$

where B^T is the total Bernoulli function, $\mathbf{v}^P$ is the advective or particle velocity directly related to the Lagrange multiplier, and q^C is the constrained potential vorticity

$$q^C = \frac{\mathsf{f}/\epsilon + \mathbf{z} \cdot \nabla \times \mathbf{v}^C}{h}. \tag{3.6.16}$$

Comparison of the balanced momentum equations (3.6.14) with (3.6.3) shows that particle velocity $\mathbf{v}^P$ is formally accurate to $O(\epsilon^2)$. Since (3.6.10) involves the original shallow-water bracket it follows immediately that the shallow-water Casimir invariants remain invariant. The energy $\mathcal{H}^*$ is conserved by the skew-symmetry of the bracket and so are $\mathcal{C}[q^C, h]$, $\mathcal{H}[\mathbf{v}^C, h]$ after the constraints are applied. Direct manipulation of (3.6.14), (3.6.15) to prove (material) conservation of q^C and $\mathcal{C}[q^C[h], h]$ is the same as for the parent shallow-water equations.

The above method and the following one do result in velocity splitting, which means that the constraint velocity $\mathbf{v}^C$ is different from the particle velocity $\mathbf{v} \equiv \mathbf{v}^P$ which clearly plays the rôle of a Lagrange (vector) multiplier. Extra boundary conditions on the ("ageostrophic") difference velocity $\mathbf{v}^{AC} \equiv \mathbf{v}^P - \mathbf{v}^C$ in addition to the usual ones on $\mathbf{v}$, like no normal flow at fixed walls, arise in the variation of the Hamiltonian $\mathcal{H}^*$. In summary, boundary conditions on $\mathbf{v}^C$ are needed to derive higher-order velocity constraints from a slaving principle, while boundary conditions on the diagnostic advective velocity $\mathbf{v}^P$ or rather $\mathbf{v}^{AC}$ are needed to eliminate boundary contributions in a variational or Hamiltonian formulation.

3.6.3 Direct substitution in Eulerian action principle

Direct substitution of the velocity constraint $\mathbf{v} = \mathbf{v}^C[h]$ into a dimensionless version of (3.4.8) yields a constrained action principle

$$0 = \delta \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left\{ -\epsilon\, h\, (\epsilon\, u_i^C[h] + R_i)\, (\Gamma^{-1})^i_k \,\frac{\partial a^k}{\partial t} - H(u_i^C[h], a^i) \right\} \tag{3.6.17}$$

in terms of variations δa^k only, in which $\mathbf{v}^C[h]$ acts again as a (non-local) operator on h and in which $H(u_i^C[h], a^i)$ is the Hamiltonian density.

Variation of (3.6.17) with suitable boundary conditions yields

$$\begin{aligned} 0 \;=\; & \delta \int_{t_0}^{t_1} \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x}\, h\, (\Gamma^{-1})^i_k \Big\{ \epsilon^2 \frac{\partial u_i^C}{\partial t} + \\ & \epsilon \Big[\Big(\frac{\partial R_i}{\partial x^m} - \frac{\partial R_m}{\partial x^i} \Big) + \epsilon \Big(\frac{\partial u_i^C}{\partial x^m} - \frac{\partial u_m^C}{\partial x^i} \Big) \Big]\, u^{Pm} + \\ & \frac{\partial}{\partial x^i} \Big[\frac{1}{2} \epsilon^2\, u_m^C\, u^{Cm} + \frac{1}{F_r}\,(h + h_B) + \epsilon^2\, (u^{Cm} - u^{Pm}) \frac{\delta u_m^C}{\delta h} \Big] \Big\}\, \delta a^k \end{aligned} \tag{3.6.18}$$

and thus the momentum equations (3.6.14). The continuity equation (3.6.15) follows from the definition of $h = h_0(\mathbf{a})\, J(a, b)$ and (3.4.4).

In deriving the above ageostrophic Hamiltonian balanced equations boundary contributions emerge due to integration by parts. To eliminate these, the physical boundary conditions for the velocity in the shallow-water equations are used for the particle velocity, but extra boundary conditions are required in the evaluation of

$$\mathcal{R} \;=\; \int_\Omega \mathrm{d}\mathbf{x}\, h\,\mathbf{v}^{AC}\cdot\delta\mathbf{v}^C[h] = \int_\Omega \mathrm{d}\mathbf{x}\, h\,\mathbf{v}^{AC}\cdot\frac{\delta\mathbf{v}^C[h]}{\delta h}\,\delta h. \qquad (3.6.19)$$

For higher-order constraint velocities that involve non-local operators acting on the height, it is anticipated that an abstract treatment in terms of Green's functions is required. For convenience only periodic boundary conditions will be considered here, leaving the important more realistic boundary considerations for future work. With periodic boundary conditions, for which $\mathsf{f} = 1$, one finds that

$$\begin{aligned}\mathcal{R} \;=\; & \int_\Omega \mathrm{d}\mathbf{x}\left\{-\frac{1}{\epsilon\,F_r}\,\mathbf{z}\cdot\nabla\times(h\,\mathbf{v}^{AC}) + \frac{1}{\epsilon\,F_r^2}\,J\Big[\nabla(h+h_B),\mathcal{L}^{-1}(h\,\mathbf{v}^{AC})\Big] + \right.\\ & \left.\frac{1}{\epsilon\,F_r^2}\,\nabla\cdot J\Big[h+h_B,\mathcal{L}^{-1}(h\,\mathbf{v}^{AC})\Big]\right\}\delta h = -\int_\Omega \mathrm{d}\mathbf{x}\, B^{AC}\delta h \qquad (3.6.20)\end{aligned}$$

where $B^{AC} \equiv (\delta\mathbf{v}^C/\delta h)\cdot h\,(\mathbf{v}^C - \mathbf{v}^P) \equiv B^T - B^C$. The momentum equations (3.6.14) with explicit expressions for constraint velocity $\mathbf{v}^C$ are found to be

$$\begin{aligned}\frac{\partial\mathbf{v}^C}{\partial t} \;=\; & -q^C\,h\,\hat{\mathbf{z}}\times\mathbf{v}^P - \nabla\left\{\frac{1}{\epsilon^2\,F_r}\,(h+h_B) + \frac{1}{2}\,|\mathbf{v}^C|^2 + \right.\\ & \frac{1}{\epsilon\,F_r}\,\hat{\mathbf{z}}\cdot\nabla\times(h\,\mathbf{v}^{AC}) - \frac{1}{\epsilon\,F_r^2}\,J\Big[\nabla(h+h_B),\mathcal{L}^{-1}(h\,\mathbf{v}^{AC})\Big] - \\ & \left.\frac{1}{\epsilon\,F_r^2}\,\nabla\cdot J\Big[h+h_B,\mathcal{L}^{-1}(h\,\mathbf{v}^{AC})\Big]\right\}. \qquad (3.6.21)\end{aligned}$$

Consistency relations are required to obtain solutions for $\mathbf{v}^P$. Combining momentum (3.6.21) and continuity equations (3.6.15) one may eliminate all time derivatives and arrive at a coupled set of linear elliptic equations for $\mathbf{v}^P$. Even with periodic boundary conditions it is not straightforward to find solvability conditions that guarantee existence and uniqeness, and I leave this as an open question. (Ren and Shepherd (1997) found for Salmon's (1985) L1-dynamics, i.e. for constrained shallow-water flows with geostrophic constraint velocity $\mathbf{v}^{(0)}$, that the condition $q^C > 0$ with $\mathbf{v}^C = \mathbf{v}^{(0)}$ guarantees a unique solution in a closed domain.)

3.6.4 Formal stability

Ren and Shepherd (1997) derived formal and nonlinear stability criteria for Salmon's (1985) L1-dynamics. They also outlined how to derive stability criteria for higher-order balanced constraints and noted that complications may arise due to the non-local nature of the constraint velocity $\mathbf{v}^C$. These complications will be largely circumvented here by considering formal stability criteria for flows with periodic boundaries only.

Given the balanced system (3.6.14), (3.6.15), steady state solutions

$$\mathbf{v}^P = \mathbf{U}^P(\mathbf{x}),\ \mathbf{v}^C = \mathbf{U}^C(\mathbf{x}),\ h = H(\mathbf{x}),\ q = Q_s^C(\mathbf{x})\ , B^T = B_s^T(\mathbf{x}),$$

similar to the shallow-water ones (3.4.21), are found to be

$$\nabla \cdot (H\,\mathbf{U}^P) = 0, \quad Q_s^C\,\nabla\,\Psi^P = \nabla\,B_s^T, \tag{3.6.22}$$

where $H\,\mathbf{U}^P \equiv \hat{\mathbf{z}} \times \nabla\Psi^P$,

$$B_s^T = \frac{1}{2}\,|\mathbf{U}^C|^2 + \frac{1}{\epsilon^2\,F_r}\,(H + h_B) - \frac{\delta \mathbf{U}^C}{\delta H} \cdot H\,\mathbf{U}^{AC} \tag{3.6.23}$$

is the total Bernoulli function, and Q_s^C is the steady state constrained potential vorticity satisfying $\mathbf{U}^P \cdot \nabla\,Q_s^C = 0$. Hence, Q_s^C and B_s^T are all constant along streamlines Ψ^P = constant. The first-order variation of the pseudo-energy invariant

$$\mathcal{A} \equiv (\mathcal{H} + \mathcal{C})[h] - (\mathcal{H} + \mathcal{C})[H] \tag{3.6.24}$$

with respect to h, is now required to vanish at the basic state, thus determining the unknown function $C(\gamma)$ in the Casimir invariant $\int \mathrm{d}\mathbf{x}\,h\,C(q^C)$. Defining perturbations $h = H + h'$, etc., one finds at first order in the perturbation

$$\begin{aligned}\delta\mathcal{A} \;\approx\; & \int_\Omega \mathrm{d}\mathbf{x}\left\{\Big(B_s^T + C(Q_s^C) - Q_s^C\,C'(Q_s^C)\Big)\,\delta h' + \right.\\ &\left.\Big(H\,\mathbf{U}^P + \nabla C'(Q_s^C) \times \hat{\mathbf{z}}\Big) \cdot \mathbf{v}^{C'} - B^{AC}\,\delta h' - H\,\mathbf{U}^{AC} \cdot \delta\mathbf{v}^{AC'}\right\}.\end{aligned} \tag{3.6.25}$$

Elimination of the linear perturbations in $\mathcal{A}$ determines (cf. Ren and Shepherd 1997) $C(\nu)$ to be

$$C(\nu) \equiv \nu\left(\int_0^\nu \frac{K(\gamma)}{\gamma^2}\,\mathrm{d}\gamma + \text{constant}\right) \tag{3.6.26}$$

with $B_s^T(\mathbf{x}) \equiv K(Q_s^C(\mathbf{x}))$ provided that the last two terms in (3.6.25) vanish, which follows from the calculations leading to (3.6.20) (albeit checked here for periodic boundary conditions only).

Following Ripa (1983) a finite-amplitude pseudo energy may now be derived which will be quadratic in the limit of small-amplitude disturbances. The following dimensional formal stability criteria (cf. Ren and Shepherd 1997)

$$|\mathbf{U}^C|^2 < g\,H, \quad \frac{d^2C}{d(Q_s^C)^2} = \frac{\nabla\Psi^P}{\nabla\,Q_s^C} > 0 \tag{3.6.27}$$

are required to ensure that $\mathcal{A}$ is positive definite for small-amplitude disturbances. The first criterion corresponds to a "subsonic" condition in the sense that the constrained basic state flow speed needs to be less than the minimum gravity wave speed, and the second one corresponds to that of Arnol'd's (1966) first stability theorem.

Further investigation is required to see whether Ripa's formal stability criteria can be refined by exploiting the relationship between $\mathbf{v}^C$ and h, and whether nonlinear norm stability can be proven for the above ageostrophic balanced system.

4 Summary and discussion

The problem of preservation of variational and Hamiltonian structure under singular perturbations in a small parameter has been considered.

In the first part of this chapter a general framework was presented for finite-dimensional systems. In particular, singular perturbation methods based on a slaving ansatz were shown to yield constraints on the dynamics (section 2.3) and these constraints were subsequently imposed on the original variational or Hamiltonian formulation of the "parent " dynamics either via (I) Dirac's theory, (II) direct substitution, (III) a Lagrange multiplier method or via (IV) postulation of a constrained bracket (section 2.4). While this two-step method considers the derivation of constraints and their imposition separately, it has also been shown that a constrained Dirac bracket arises automatically, in one step (approach V), from a slaving ansatz when a particular prototypical Hamiltonian system (singular in small parameter ϵ) is considered (section 2.5). Unfortunately the resulting Dirac bracket is often too complicated or impossible to calculate for continuous systems and only a regular leading-order perturbation of the Dirac bracket has led to a viable, but only leading-order, singular Hamiltonian perturbation theory (section 2.6). Except for this leading-order theory, the various ways to impose given constraints are in essence similar although specific applications may favor one implementation or another.

In the second part, these general approaches were applied to fluids and shown to yield several known and novel (sections 3.2 and 3.6) variational or Hamiltonian formulations of balanced geophysical fluid systems. (These reduced systems are balanced in the sense of being approximations, in which particular high-frequency wave types have been eliminated, to the complete compressible Euler equations of motion.) The application of the leading-order Hamiltonian perturbation theory provides alternative derivations of the well known Hamiltonian formulations of three-dimensional homogeneous vorticity dynamics and the equivalent barotropic quasi-geostrophic equations (sections 3.3 and 3.5). Many other known and new balanced models in geophysical fluid dynamics may be derived in a unified way with these variational or

Equations of motion	Basic state	Formal stability criteria		
		static	subsonic	vortical
Compressible	resting	$N^2, c_0^2 > 0$ $U_{\rho\, s}\|_0 > 0$		
Hydrostatic	resting	$N^2 > 0$		
Shallow-water	general		$\|\mathbf{U}\|^2 < g\, H$	$\frac{\nabla\Psi_s}{\nabla Q_s} > 0$
Barotropic ageostrophic	general		$\|\mathbf{U}^C\|^2 < g\, H$	$\frac{\nabla\Psi^P}{\nabla Q_s^C} > 0$
Planar incompr. homogeneous	planar flows			$\frac{\nabla\Psi}{\nabla\Omega} > 0$
Barotropic quasi-geostrophic	general			$\frac{\nabla\Psi^{(0)}}{\nabla Q_s} > 0$
Three-component vortex model			$\bar{v}^2/(\bar{h} + h') <$ $\left(1 + \frac{[C'(\bar{q})]^2}{\bar{h}^2\, C''(\bar{q})}\right)$	$C''(\bar{q}) > 0$

Table 1. Summary of the formal stability criteria of fluid equations considered in section 3. Note that for the stability criteria of incompressible three-dimensional flows considered here, both basic state *and* disturbance need to be planar.

Hamiltonian perturbation approaches.

Formal stability criteria for steady states were given for all fluid dynamical examples considered here. These criteria consisted of only Arnol'd's first stability condition on vortical components of flows in quasi-geostrophic dynamics, plus a subsonic condition in shallow-water and ageostrophic barotropic dynamics, plus a static stability condition for hydrostatic stratified flows in isentropic coordinates and three-dimensional compressible dynamics, although for the last two systems formal stability conditions are only available for resting steady states. It may seem puzzling that the ageostrophic barotropic equations lead to a subsonic stability criterion when interior surface gravity waves are absent. However, formal stability criteria analogous to Ripa's theorem may be further analyzed when the relationship between constrained velocity and depth of the shallow layer of fluid is taken into account.

A summary of the formal stability criteria considered in this chapter is given in Table 1. Although formal stability criteria guarantee linear stability, their violation is just a necessary and not sufficient condition for instability.

The observed reduction of the number of stability conditions relates in reverse order to the number of imposed balance constraints which, in effect, eliminate gravity- or acoustic-wave dynamics. The shallow-water formal stability criteria for the finite-dimensional vortex model in Example 11, the full, barotropic quasi-geostrophic, and ageostrophic shallow-water equations are clearly related. Moreover, formal stability criteria for Hamiltonian balanced models based on velocity constraints will yield Ripa's shallow-water criteria with the velocity replaced by the constrained velocity. For Ripa's criteria boundary conditions and consistency relations for $\mathbf{v}^{AC}$ may be necessary to make the first variation of pseudo energy extremal but in the second variation no boundary contributions will emerge (cf. section 3.6). We thus see that formal stability criteria analogous to Ripa's theorem do not distinguish boundary-trapped Kelvin from Poincaré gravity waves. In contrast, this distinction will become apparent when the actual velocity constraints are taken into account in the derivation of formal or nonlinear stability criteria or in particular linear stability calculations, since quadratic boundary contributions are then expected to emerge in the pseudo energy. (The actual velocity constraint was taken into account for L1-dynamics by Ren and Shepherd (1997) and for semi-geostrophic dynamics by Kushner, McIntyre and Shepherd (1998)).

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