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Limit Laws for Sums of Random Exponentials

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Abstract. We study the limiting distribution of the sum $S_N(t) = \sum_{i=1}^{N} e^{tX_i}$ as $t \to \infty$, $N \to \infty$, where (X_i) are i.i.d. random variables. Attention to such exponential sums has been motivated by various problems in random media theory. Examples include the quenched mean population size of a colony of branching processes with random branching rates and the partition function of Derrida's Random Energy Model. In this paper, the problem is considered under the assumption that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X_i > x\}$ is regularly varying at infinity with index $1 < \varrho < \infty$. An appropriate scale for the growth of N relative to t is of the form $e^{\lambda H_0(t)}$, where the rate function $H_0(t)$ is a certain asymptotic version of the cumulant generating function $H(t) = \log \mathsf{E}[e^{tX_i}]$ provided by Kasahara's exponential Tauberian theorem. We have found two critical points, $0 < \lambda_1 < \lambda_2 < \infty$, below which the Law of Large Numbers and the Central Limit Theorem, respectively, break down. Below λ_2, we impose a slightly stronger condition of normalized regular variation of h. The limit laws here appear to be stable, with characteristic exponent $\alpha = \alpha(\varrho, \lambda)$ ranging from 0 to 2 and with skewness parameter $\beta = 1$. A limit theorem for the maximal value of the sample $\{e^{tX_i},\ i = 1, \dots, N\}$ is also proved.

1. Introduction

In this work, we are concerned with the partial sums of exponentials of the form

$$S_N(t) = \sum_{i=1}^{N} e^{tX_i}, \tag{1}$$

where $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables and both t and N tend to infinity. Our goal is to study the limiting distribution of $S_N(t)$ and to explore possible 'phase transitions' due to various rates of growth of the parameters t and N.

In such analysis, two cases are naturally distinguished according to whether or not X_i are bounded above. In this paper, we assume that the random variables X_i are *unbounded above*; the opposite case can be treated similarly and will be considered elsewhere. One can also expect that the results will heavily depend on

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the structure of the distribution at infinity. In the present work, we focus on a fairly general class of distributions with the upper tail of the Weibull-type form

$$\mathsf{P}\{X_i > x\} \approx \exp(-cx^{\varrho}) \qquad (x \to +\infty), \tag{2}$$

where $\varrho > 1$. More precisely, we assume that the log-tail distribution function $h(x) = -\log \mathsf{P}\{X_i > x\}$ is regularly varying at infinity with index $1 < \varrho < \infty$. In particular, a normal distribution is contained in this class with $\varrho = 2$.

One motivation for our study is quite abstract and purely probabilistic. In fact, such a setting provides a natural tool to interpolate between the classical limit theorems concerning the bulk of the sample, i.e. the Law of Large Numbers (LLN) and the Central Limit Theorem (CLT), on the one hand, and limit theorems for extreme values, on the other hand. Indeed, it is clear that the limiting behavior of $S_N(t)$ is largely determined by the relationship between the parameters t and N. If, for instance, one lets N tend to infinity with t fixed or growing very slowly, then, under appropriate (exponential) moment conditions, the usual LLN and CLT should be valid. In contrast, if the growth rate of N is small enough as compared to t, then the asymptotic behavior of the sum $S_N(t)$ is dominated by its maximal term. We will see that when both t and N tend to infinity, a rich intermediate picture emerges made up of various limit regimes.

In this connection, let us mention a recent paper by Schlather (2001) who studied the asymptotic behavior of the l_p-norms of samples of positive i.i.d. random variables,

$$\|Y_{1n}\|_p = \left(\sum_{i=1}^{n} Y_i^p\right)^{1/p}, \tag{3}$$

where the norm order p grows together with the sample size n. The link with our setting becomes clear if one puts $Y_i = e^{X_i}$, so that (3) is expressed through an exponential sum of the form (1). Qualitatively speaking, Schlather (2001) has demonstrated that under a suitable parametrization of the functional relation between p and n, there is a 'homotopy' for the limiting distributions of the norms (3) extending from CLT to a limit law for the extreme value. However, his results basically refer to the case where the random variables Y_i are bounded above and, in the sense of extreme value theory, belong to the domain of attraction of the Weibull distribution Ψ_α with parameter $\alpha > 0$, with distribution function

$$\Psi_\alpha(x) = \exp\bigl(-(-x)^\alpha\bigr), \qquad x < 0.$$

Let us point out that our results are complementary to Schlather's findings, since for random variables X_i with the Weibull tails (2), the distribution of the maximum of $e^{X_1}, \dots, e^{X_n}$ can be shown to converge, as $n \to \infty$, to the Gumbel (double exponential) distribution Λ, with distribution function

$$\Lambda(x) = \exp\left(-e^{-x}\right), \qquad x \in \mathbb{R}.$$

The second motivation (and in fact the most important one) comes from problems related to the long-time dynamics in random media. In the simplest situation, sums of exponentials arise as the expected (quenched) total population size of a colony of non-interacting branching processes with random branching rates. Indeed, consider a collection of N branching processes $Z_i(t)$ driven by the binary branching rates $X_i = X_i(\omega)$ $(i = 1, \dots, N)$. More specifically, for a fixed environment ω (i.e., in a 'quenched' setting), each $Z_i(t)$ is a Markov continuous-time branching process evolving as follows: during infinitesimal time dt, a particle from the ith population, independently of other particles and the past history, with probability $|X_i|dt$ may split into two descendants (if $X_i > 0$) or die (if $X_i < 0$); otherwise, with probability $1 - |X_i|dt$, the particle survives over the time dt. Let $m_i(t) \equiv m_i(t, \omega)$ denote the expected number of particles in the ith population at time t. One can show that $m_i(t)$, as a function of t, satisfies the differential equation $m_i'(t) = X_i m_i(t)$ [see Athreya and Ney (1972), Chapter III, pages 106, 108]. Hence, assuming that $Z_i(0) = 1$ we obtain $m_i(t) = e^{tX_i}$, and therefore the quenched mean total population size is given by the sum (1).

A completely different example is provided by the Random Energy Model (REM) introduced by Derrida (1980) as a simplified version of the mean-field Sherrington–Kirkpatrick model of a spin glass. The REM describes a system of size n with 2^n energy levels $E_i = \sqrt{n}\,X_i$ $(i = 1, \dots, 2^n)$, where (X_i) are i.i.d. random variables with standard normal distribution. Thermodynamics of the system is determined by the partition function

$$\mathcal{Z}_n(\beta) := \sum_{i=1}^{2^n} \exp(\beta\sqrt{n}\,X_i),$$

where $\beta > 0$ is the inverse temperature, which exemplifies the exponential sum (1) with $N = 2^n$, $t = \beta\sqrt{n}$. The free energy, first obtained by Derrida (1980) using heuristic arguments, is given by

$$F(\beta) := \lim_{n\to\infty} \frac{\log \mathcal{Z}_n(\beta)}{n} = \begin{cases} \beta^2/2 + \beta_{\mathrm{c}}^2/2 & \text{if } 0 < \beta \le \beta_{\mathrm{c}}, \\ \beta\beta_{\mathrm{c}} & \text{if } \beta \ge \beta_{\mathrm{c}}, \end{cases} \tag{4}$$

where $\beta_{\mathrm{c}} = \sqrt{2\log 2}$. Note that the function $F(\beta)$ is continuously differentiable but its second derivative is discontinuous at point β_{c} [a third-order phase transition, see Eisele (1983)]. Later on, Eisele (1983) and Olivieri and Picco (1984) rigorously derived the limit (4) (in probability and also with probability one) and also extended this result to the case where the random variables X_i have the Weibull-type upper tail (2).[1]

Recently, a detailed analysis of the limit laws for $\mathcal{Z}_n(\beta)$ in the Gaussian case has been accomplished by Bovier, Kurkova and Löwe (2002). In particular, they have shown that in addition to the first phase transition at the critical point β_{c}, manifested as the LLN breakdown for $\beta > \beta_{\mathrm{c}}$, within the high-temperature phase $\beta < \beta_{\mathrm{c}}$ there is the second phase transition at the critical point $\tilde\beta_{\mathrm{c}} = \sqrt{\log 2/2} = \frac{1}{2}\beta_{\mathrm{c}}$, in that for $\beta > \tilde\beta_{\mathrm{c}}$ the fluctuations of $\mathcal{Z}_n(\beta)$ become non-Gaussian. In our work, we extend these results to the class of distributions with the Weibull-type tails of the form (2). As compared to Bovier, Kurkova and Löwe (2002) who proceeded from the extreme value theory, we use methods of the theory of summation of independent random variables. Moreover, we show that the non-Gaussian limit laws are in fact stable.

[1] In fact, the class of distributions considered in these papers is subject to the condition $x^{-\varrho}h(x) \to c$ as $x \to \infty$, where $h(x) = -\log \mathsf{P}\{X_i > x\}$ and $1 < \varrho < \infty$ [see Eisele (1983), Theorem 2.3, page 130], which is more restrictive than our assumption of (normalized) regular variation of h.

Outline. The rest of the paper is organized as follows. In Section 2, we specify our regularity assumption on the distribution tail of the random variables X_i and formulate Kasahara's exponential Tauberian theorem, which relates the asymptotic behavior of the log-tail distribution function $h(x)$ to that of the cumulant generating function $H(t) = \log \mathsf{E}[e^{tX_i}]$. We then introduce the exponential scale $e^{\lambda H_0(t)}$ of growth of N relative to t, where H_0 is a certain asymptotic version of H provided by Kasahara's theorem, and define two critical values of the parameter λ, $0 < \lambda_1 < \lambda_2 < \infty$. In Section 3 we prove the LLN above λ_1 (Theorem 1) and the CLT above λ_2 (Theorem 2). In Section 4, we impose a slightly stronger condition of normalized regular variation of h, which leads to an important formula (Basic Identity, Lemma 3) characterizing precisely the relationship between h and H_0. Section 5 contains our main result (Theorem 3, $0 < \lambda < \lambda_2$) on convergence to a stable law, with characteristic exponent $\alpha = \alpha(\varrho, \lambda)$ ranging from 0 to 2 and with skewness parameter $\beta = 1$. We also address here the situation at the critical points λ_1 (LLN, Theorem 4) and λ_2 (CLT, Theorem 5). Section 6 is an illustration of our limit theorems in the model case of the Weibull distribution. In Section 7, the key points of the proofs of Theorems 3, 4 and 5 are briefly outlined. In particular, we explain the main idea of derivation of the Lévy–Khinchin spectral function, where the Basic Identity plays the crucial part. Finally, in Section 8 we obtain the limiting distribution of the maximum $M_N(t) = \max\{e^{tX_i},\ i = 1, \dots, N\}$ (Theorem 6) and discuss the relationship between $S_N(t)$ and $M_N(t)$ (Theorems 7 and 8).

General notations. We write := for 'is defined by' and =: for 'is denoted by'. The letter X is used for a generic representative of the random variables (X_i). The indicator of an event A is denoted by $\mathbf{1}_A$. Relation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$. Convergence in probability and in distribution is denoted by $\xrightarrow{p}$ and $\xrightarrow{d}$, respectively, and the symbol $o_p(1)$ denotes a random variable converging to zero in probability. By $\mathcal{N}(0, \sigma^2)$ we denote the normal distribution on $\mathbb{R}$ with zero mean and variance σ^2; in particular, $\mathcal{N}(0,1)$ stands for the standard normal distribution.

2. Regularity and Kasahara's Tauberian theorem

Using the log-tail distribution function

$$h(x) := -\log \mathsf{P}\{X > x\}, \qquad x \in \mathbb{R}, \tag{5}$$

the upper distribution tail is represented in the form

$$\mathsf{P}\{X > x\} = e^{-h(x)}.$$

We now make our basic assumption on regularity of this tail.

Regularity Assumption 1. The function h is *regularly varying at infinity with index* $\varrho \in (1, \infty)$ (we write $h \in R_\varrho$). That is to say, for every constant $\kappa > 0$

$$\lim_{x\to\infty} \frac{h(\kappa x)}{h(x)} = \kappa^\varrho.$$

Since the random variable X has all exponential moments, the cumulant generating function

$$H(t) := \log \mathsf{E}[e^{tX}]$$

is well defined for all $t \geq 0$. The link between the asymptotic behavior of the functions h and H at infinity is characterized by Kasahara's exponential Tauberian theorem [Bingham, Goldie and Teugels (1989), Theorem 4.12.7, page 253]. Let the generalized inverse of a function f be defined by

$$f^{\leftarrow}(y) := \inf\{x : f(x) \geq y\},$$

with the convention that $\inf \varnothing = +\infty$ [see Resnick (1987), Section 0.2, pages 3–4]. One can show that $f \in R_\varrho$ if and only if $f^{\leftarrow} \in R_{1/\varrho}$ [see Bingham, Goldie and Teugels (1989), Theorem 1.5.12, page 28].

KASAHARA'S TAUBERIAN THEOREM. *Let $1 < \varrho < \infty$, and let $\varrho' > 1$ be such that*

$$\frac{1}{\varrho} + \frac{1}{\varrho'} = 1. \tag{6}$$

Suppose that $\varphi \in R_{1/\varrho}$ and put

$$\psi(u) := \frac{u}{\varphi(u)} \in R_{1/\varrho'}. \tag{7}$$

Then

$$h(x) \sim \frac{1}{\varrho}\, \varphi^{\leftarrow}(x) \qquad (x \to \infty) \tag{8}$$

if and only if

$$H(t) \sim \frac{1}{\varrho'}\, \psi^{\leftarrow}(t) =: H_0(t) \qquad (t \to \infty). \tag{9}$$

In particular, if $h \in R_\varrho$ then $H \in R_{\varrho'}$, and vice versa.

In the sequel, the following simple identity will be useful, which is just a rearrangement of the definition (6):

$$\frac{\varrho'}{\varrho} = \varrho' - 1. \tag{10}$$

Note that the expected value of the sum $S_N(t)$ is given by

$$\mathsf{E}[S_N(t)] = \sum_{i=1}^{N} \mathsf{E}[e^{tX_i}] = N e^{H(t)},$$

suggesting that the function $H(t)$ sets up an appropriate (exponential) scale of the form $e^{\lambda H(t)}$ for $N = N(t)$. In fact, it is technically more convenient to use $H_0(t)$ as a rate function [see (9)].[2] More precisely, denote

$$\lambda := \liminf_{t\to\infty} \frac{\log N}{H_0(t)} \tag{11}$$

and set

$$\lambda_1 := \varrho' - 1, \qquad \lambda_2 := 2^{\varrho'}(\varrho' - 1). \tag{12}$$

These two values prove to be critical ones with respect to the scale (11). Let us also introduce a new parameter,

$$\alpha \equiv \alpha(\varrho, \lambda) := \left(\frac{\varrho\lambda}{\varrho'}\right)^{1/\varrho'}. \tag{13}$$

[2] This makes no difference in the 'crude' Theorems 1 and 2 below, since $H_0(t) \sim H(t)$ as $t \to \infty$, but it will be crucial for the more delicate Theorems 3, 4 and 5.

Conversely, in view of formula (10) the parameter λ is expressed through α as

$$\lambda = \alpha^{\varrho'}(\varrho' - 1), \tag{14}$$

and from (12) it follows that the respective critical values of α are given by

$$\alpha_1 = 1, \qquad \alpha_2 = 2.$$

We will see that α plays the role of characteristic exponent in the limit laws.

3. 'Crude' limit theorems above the critical points

Our first theorem asserts that if N grows fast enough then $S_N(t)$ satisfies the Law of Large Numbers in its conventional form.

THEOREM 1 (LLN, $\lambda > \lambda_1$). *Suppose that $\lambda > \lambda_1$, and set*

$$S_N^*(t) := \frac{S_N(t)}{\mathsf{E}[S_N(t)]} = \frac{1}{N}\sum_{i=1}^{N} \exp\{tX_i - H(t)\}.$$

Then

$$S_N^*(t) \xrightarrow{p} 1 \qquad (t \to \infty).$$

PROOF. It suffices to prove that for some $r > 1$

$$\lim_{t\to\infty} \mathsf{E}\,|S_N^*(t) - 1|^r = 0.$$

By an inequality of von Bahr and Esseen [(1965), Theorem 2, page 301], for any $r \in [1, 2]$

$$\mathsf{E}\,|S_N^* - 1|^r \le 2N^{1-r}\,\mathsf{E}\,|e^{tX-H(t)} + 1|^r.$$

Furthermore, by the elementary inequality $(x+1)^r \le 2^{r-1}(x^r + 1)$ $(x \ge 0,\ r > 1)$, which easily follows from Jensen's inequality applied to x^r, we get

$$\mathsf{E}\,|S_N^* - 1|^r \le 2^r N^{1-r} e^{H(rt) - rH(t)} + O(N^{1-r}). \tag{15}$$

Since $H \in R_{\varrho'}$ and also using (11) and the asymptotic equivalence $H(t) \sim H_0(t)$ [see (9)], we obtain

$$\begin{aligned}\liminf_{t\to\infty} &\left(\frac{(r-1)\log N}{H(t)} - \frac{H(rt)}{H(t)} + r\right)\\ &= \lambda(r-1) - r^{\varrho'} + r =: v_\lambda(r).\end{aligned} \tag{16}$$

Note that $v_\lambda(1) = 0$ and $v_\lambda'(1) > 0$ [due to the condition $\lambda > \lambda_1 = \varrho' - 1$, see (12)]. Hence, there exists $r > 1$ such that $v_\lambda(r) > 0$, which implies that the exponential term in (15) is bounded by $e^{-cH(t)} = o(1)$. $\square$

Our next result concerns the fluctuations of the sum $S_N(t)$ about the expected value. Note that

$$\mathsf{Var}[e^{tX}] = e^{H(2t)} - e^{2H(t)} \sim e^{H(2t)} \qquad (t \to \infty).$$

THEOREM 2 (CLT, $\lambda > \lambda_2$). *Suppose that $\lambda > \lambda_2$. Then*

$$\frac{S_N(t) - Ne^{H(t)}}{(Ne^{H(2t)})^{1/2}} \xrightarrow{d} \mathcal{N}(0,1) \qquad (t \to \infty).$$

PROOF. By the Lyapunov theorem [see Petrov (1995), Theorem 4.9, page 126], we only need to check that for an appropriate $r > 1$,

$$N^{-r} e^{-rH(2t)} \sum_{i=1}^{N} \mathsf{E}\,|e^{tX_i} - e^{H(t)}|^{2r} \to 0 \qquad (t \to \infty). \tag{17}$$

Arguing as in the proof of Theorem 1, one can show that the left-hand side of (17) is dominated by

$$2^{2r-1} N^{1-r} e^{-rH(2t)} \Big(e^{H(2rt)} + e^{2rH(t)} \Big) \sim 2^{2r-1} N^{1-r} e^{H(2rt) - rH(2t)}.$$

Furthermore, analogously to (16) we obtain

$$\begin{aligned} \liminf_{t\to\infty} \bigg(\frac{(r-1)\log N}{H(t)} - \frac{H(2rt)}{H(t)} + \frac{rH(2t)}{H(t)} \bigg) & \\ = \lambda(r-1) - (2r)^{\varrho'} + r2^{\varrho'} & \\ = 2^{\varrho'} \Big(2^{-\varrho'} \lambda(r-1) - r^{\varrho'} + r \Big) \equiv 2^{\varrho'} v_{\lambda'}(r), & \end{aligned}$$

where $\lambda' := 2^{-\varrho'}\lambda > 2^{-\varrho'}\lambda_2 = \varrho' - 1$ [see (12)] and the function $v_\lambda(\cdot)$ is defined on the right-hand side of (16). Similarly as above, there exists a number $r > 1$ such that $v_{\lambda'}(r) > 0$, and hence (17) follows. $\square$

4. Normalized regular variation

Below the critical points, the behavior of the sum $S_N(t)$ becomes increasingly sensitive to subtle details of the upper tail's structure. So to get enough control on these, we require slightly more regularity of the distribution tail.

REGULARITY ASSUMPTION 2. The log-tail distribution function h is *normalized regularly varying* (with index $1 < \varrho < \infty$). The latter means that for each $\varepsilon > 0$ the function $x^{-\varrho+\varepsilon} h(x)$ is ultimately increasing, whereas the function $x^{-\varrho-\varepsilon} h(x)$ is ultimately decreasing [cf. Bingham, Goldie and Teugels (1989), Section 1.3.2, page 15].

More insight into the property of normalized regular variation is given by the following lemma [cf. Bingham, Goldie and Teugels (1989), Section 1.3.2, page 15].

LEMMA 1. *A positive* (*measurable*) *function h is normalized regularly varying with index ϱ if and only if it is differentiable* (*a.e.*) *and*

$$\lim_{x\to\infty} \frac{x h'(x)}{h(x)} = \varrho. \tag{18}$$

If the log-tail distribution function h is known to be normalized regularly varying (with index ϱ), it follows that the function H_0, an asymptotic version of H provided by Kasahara's theorem [see (9)], is normalized regularly varying as well (with index ϱ'). Moreover, by comparing equations (8) and (9) via the relation (7) and using that the ordinary inverses of the functions $h(x)$ and $H_0(t)$ exist (for large enough x and t), one can show that for all t large enough, $H_0(t)$ is the unique solution of the equation

$$\frac{\varrho'}{\varrho} H_0(t) = h\Big(\frac{\varrho' H_0(t)}{t} \Big). \tag{19}$$

Let us consider two examples to illustrate the difference between the functions H and H_0.

Example 1 (Weibull's distribution). Let X have the Weibull distribution,

$$\mathsf{P}(X > x) = \exp(-x^\varrho/\varrho), \qquad x \ge 0, \tag{20}$$

with $\varrho > 1$. Then the log-tail distribution function reads $h(x) = x^\varrho/\varrho$ and the density function is of the form $f_X(x) = x^{\varrho-1}\exp(-x^\varrho/\varrho)$ $(x \ge 0)$. We have

$$\begin{aligned} e^{H(t)} = \mathsf{E}[e^{tX}] &= \int_0^\infty x^{\varrho-1}\exp(tx - x^\varrho/\varrho)\,dx \\ &= t^{\varrho'}\int_0^\infty y^{\varrho-1}\exp\{t^{\varrho'}(y - y^\varrho/\varrho)\}\,dy, \end{aligned} \tag{21}$$

where we used the substitution $x = t^{\varrho'-1}y$ and relation (10). Note that the function $g(y) = y - y^\varrho/\varrho$ has a unique regular maximum at point $y = 1$, with $g(1) = 1 - 1/\varrho = 1/\varrho'$, $g'(1) = 0$, $g''(1) = 1 - \varrho < 0$. Then the asymptotic Laplace method yields

$$e^{H(t)} \sim \left(\frac{2\pi}{\varrho - 1}\right)^{1/2} t^{\varrho'/2}\exp\left(\frac{t^{\varrho'}}{\varrho'}\right) \qquad (t \to \infty),$$

whence

$$H(t) = \frac{t^{\varrho'}}{\varrho'} + \frac{\varrho'}{2}\log t + \frac{1}{2}\log\left(\frac{2\pi}{\varrho - 1}\right) + o(1) \qquad (t \to \infty). \tag{22}$$

On the other hand, equation (19) can be easily solved to obtain [cf. (22)]

$$H_0(t) = \frac{t^{\varrho'}}{\varrho'}\,, \qquad t \ge 0. \tag{23}$$

Example 2 (Normal distribution). Let X have the standard normal distribution $\mathcal{N}(0,1)$. Here $\varrho = \varrho' = 2$, $\lambda_1 = 1$, $\lambda_2 = 4$, and $\alpha = \sqrt{\lambda}$. The function h is given by

$$\begin{aligned} h(x) &= -\log\left(\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}\,dy\right) \\ &= \frac{x^2}{2} + \log x + \frac{\log(2\pi)}{2} + o(1) \qquad (x \to \infty) \end{aligned}$$

and can be shown to be normalized regularly varying. Note that for each $t \in \mathbb{R}$

$$\mathsf{E}[e^{tX}] = e^{t^2/2},$$

whence

$$H(t) = \log \mathsf{E}[e^{tX}] = \frac{t^2}{2}\,, \qquad t \in \mathbb{R}.$$

Equation (19) for H_0 can be solved asymptotically. For $\lambda \notin \{\lambda_1, \lambda_2\}$, one only needs to find $H_0(t)$ to within $o(1)$,

$$H_0(t) = \frac{t^2}{2} - \log t - \frac{1}{2}\log(2\pi) + o(1) \qquad (t \to \infty).$$

The case of the critical points is more subtle but is perfectly tractable as well.

Let us now derive the most important implication of normalized regular variation, that is an *exact identity* relating the functions h and H_0. For $x > 0$, set

$$\eta_x(t) := \frac{\log x}{t} + \frac{\mu(t) H_0(t)}{t}\,, \tag{24}$$

where $\mu = \mu(t)$ is a (unique) solution of the equation

$$h\Big(\frac{\mu H_0(t)}{t}\Big) = \frac{\varrho\lambda}{\varrho'}\, h\Big(\frac{\varrho' H_0(t)}{t}\Big). \tag{25}$$

In particular, for $x = 1$ we have

$$\eta_1(t) = \frac{\mu(t) H_0(t)}{t}\,. \tag{26}$$

Using that $h \in R_\varrho$ and comparing the asymptotics of both parts of equation (25) as $t \to \infty$, with the help of relations (6) and (13) we arrive at the following assertion.

LEMMA 2. *The function $\mu(t)$ has the limit*

$$\lim_{t\to\infty} \mu(t) = \frac{\varrho\lambda}{\alpha}\,. \tag{27}$$

Note that equation (25) combined with (19) yields

$$h\Big(\frac{\mu(t) H_0(t)}{t}\Big) = \lambda H_0(t).$$

Hence, recalling the relation (26) we obtain our main result in this section.

LEMMA 3 (Basic Identity). *For all t large enough, the following identity is true*:

$$h(\eta_1(t)) \equiv \lambda H_0(t). \tag{28}$$

How the function $\eta_x(t)$ emerges and the role of the Basic Identity will be explained later on.

5. Limit theorems below the critical points

In addition to regularity, more accuracy is now needed in specifying the rate of growth of N. Henceforth, we impose the following

SCALING ASSUMPTION. The number $N = N(t)$ of terms in the sum $S_N(t)$ satisfies the condition

$$\lim_{t\to\infty} N \exp\big\{-\lambda H_0(t)\big\} = 1, \tag{29}$$

where λ is a parameter such that $0 < \lambda < \infty$.

THEOREM 3 (Convergence to a stable law, $0 < \lambda < \lambda_2$). *Suppose that $0 < \lambda < \lambda_2$, i.e. $0 < \alpha < 2$. Set*

$$B(t) := \exp\big\{\mu(t) H_0(t)\big\} \tag{30}$$

and

$$A(t) := \begin{cases} \dfrac{N e^{H(t)}}{B(t)} & \textit{if} \quad \lambda_1 < \lambda < \lambda_2 \quad (1 < \alpha < 2), \\[2mm] \dfrac{N B_1(t)}{B(t)} & \textit{if} \quad \lambda = \lambda_1 \qquad\qquad (\alpha = 1), \\[2mm] 0 & \textit{if} \quad 0 < \lambda < \lambda_1 \quad (0 < \alpha < 1), \end{cases} \tag{31}$$

where $B_1(t)$ *is a truncated exponential moment,*

$$B_1(t) := \mathsf{E}\big[e^{tX}\mathbf{1}_{\{X\le\eta_1(t)\}}\big]. \tag{32}$$

Then

$$\frac{S_N(t)}{B(t)} - A(t) \xrightarrow{d} \mathcal{F}_\alpha,$$

where $\mathcal{F}_\alpha$ *is a stable law with characteristic exponent* $\alpha = \alpha(\varrho,\lambda)$ *defined in* (13) *and with skewness parameter* $\beta = 1$. *The characteristic function* ϕ_α *of the law* $\mathcal{F}_\alpha$ *is given by*

$$\log\phi_\alpha(u) = \begin{cases} -\Gamma(1-\alpha)|u|^\alpha \exp\left(-\dfrac{i\pi\alpha}{2}\operatorname{sgn} u\right) & (\alpha\ne 1)\\[2ex] iu(1-\gamma) - \dfrac{\pi}{2}|u|\left(1 + i\operatorname{sgn} u\cdot\dfrac{2}{\pi}\log|u|\right) & (\alpha = 1)\end{cases}$$

where $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ *is the gamma-function,* $\operatorname{sgn} u := u/|u|$ *for* $u\ne 0$ *and* $\operatorname{sgn} 0 := 0$, *and* $\gamma = 0.5772\ldots$ *is the Euler constant.*[3]

REMARK 1. The scaling relation $N\sim\exp\{\lambda H_0(t)\}$ implies

$$B(t) = \exp\{\mu(t)H_0(t)\} \sim N^{\mu(t)/\lambda}.$$

By Lemma 2, we have

$$\mu(t)\sim\frac{\varrho\lambda}{\alpha} \qquad (t\to\infty).$$

Hence, N is being raised to the power

$$\frac{\mu(t)}{\lambda}\sim\frac{\varrho}{\alpha} > \frac{1}{\alpha}.$$

This should be compared to the classical results in the i.i.d. case [see, e.g., Ibragimov and Linnik (1971), Theorem 2.1.1, pages 37, 46], where the normalization is essentially of the form $N^{1/\alpha}$. As we see, the sums of random exponentials (1) have the limit distribution by virtue of a non-classical (heavier) normalization.

Let us now describe what happens at the critical points. In fact, the Law of Large Numbers and the Central Limit Theorem prove to be valid at the critical points λ_1 and λ_2, respectively; however the constants now require some truncation.

THEOREM 4 (LLN, $\lambda=\lambda_1$). *If* $\lambda=\lambda_1$ $(\alpha=1)$ *then*

$$\frac{S_N(t)}{NB_1(t)} \xrightarrow{p} 1,$$

where $B_1(t)$ *is given by* (32).

THEOREM 5 (CLT, $\lambda=\lambda_2$). *If* $\lambda=\lambda_2$ $(\alpha=2)$ *then*

$$\frac{S_N(t)-\mathsf{E}[S_N(t)]}{(NB_2(t))^{1/2}} \xrightarrow{d} \mathcal{N}(0,1),$$

where $B_2(t)$ *is a truncated exponential moment of 'second order',*

$$B_2(t) := \mathsf{E}\big[e^{2tX}\mathbf{1}_{\{X\le\eta_1(t)\}}\big]. \tag{33}$$

[3] See Gradshteyn and Ryzhik (1994), 8.367, page 955.

6. Model example: Weibull's distribution

For illustration purposes, let us give more explicit versions of the above limit theorems in the particular situation where X has the Weibull distribution (20).

Example 3 (Weibull's distribution revisited). As shown in Example 1, in the Weibull case we have $H_0(t) = t^{\varrho'}/\varrho'$ [see (23)]. It is then easy to verify that the function $\mu(t)$, the root of equation (25), is given by

$$\mu(t) \equiv \frac{\varrho\lambda}{\alpha}$$

[cf. (27)]. So, from (30) using (13) we get

$$B(t) = \exp\bigl(\alpha^{\varrho'-1} t^{\varrho'}\bigr). \tag{34}$$

Furthermore, according to (26) we have

$$\eta_1(t) = (\alpha t)^{\varrho'-1}. \tag{35}$$

If $\alpha = \alpha_1 = 1$ then (35) yields $\eta_1(t) = t^{\varrho'-1}$, so for the function $B_1(t)$ defined in (32) we obtain similarly to (21)

$$\begin{aligned} B_1(t) &= \int_0^{t^{\varrho'-1}} x^{\varrho-1} \exp\bigl(tx - x^\varrho/\varrho\bigr)\, dx \\ &= t^{\varrho'} \int_0^1 y^{\varrho-1} \exp\bigl\{t^{\varrho'}(y - y^\varrho/\varrho)\bigr\}\, dy. \end{aligned} \tag{36}$$

As shown in Example 1, the function $g(y) = y - y^\varrho/\varrho$ has a regular maximum at point $y = 1$, which happens to be the right endpoint of the integration interval in (36). Hence, the Laplace method implies that, asymptotically, $B_1(t)$ makes up exactly one-half of the full integral [cf. (21)]

$$t^{\varrho'} \int_0^\infty y^{\varrho-1} \exp\bigl\{t^{\varrho'}(y - y^\varrho/\varrho)\bigr\}\, dy = \mathsf{E}[e^{tX}],$$

that is to say,

$$B_1(t) \sim \frac{1}{2}\,\mathsf{E}[e^{tX}] \qquad (t \to \infty).$$

Similarly, from (35) with $\alpha = \alpha_2 = 2$ we have $\eta_1(t) = (2t)^{\varrho'-1}$. Hence, the function $B_2(t)$ defined in (33) is represented as

$$\begin{aligned} B_2(t) &= \int_0^{(2t)^{\varrho'-1}} x^{\varrho-1} \exp\bigl(2tx - x^\varrho/\varrho\bigr)\, dx \\ &= (2t)^{\varrho'} \int_0^1 y^{\varrho-1} \exp\bigl\{(2t)^{\varrho'}(y - y^\varrho/\varrho)\bigr\}\, dy \end{aligned} \tag{37}$$

[via the substitution $x = (2t)^{\varrho'-1} y$], and exactly the same argument as before shows that

$$B_2(t) \sim \frac{1}{2}\,\mathsf{E}[e^{2tX}] \sim \frac{1}{2}\,\mathsf{Var}[e^{tX}].$$

As a result, we can combine the LLN of Theorems 1 and 4 as follows: *If the random variables X_i have the Weibull distribution* (20) *then, as* $t \to \infty$,

$$\frac{S_N(t)}{\mathsf{E}[S_N(t)]} \xrightarrow{p} \begin{cases} 1 & \text{if} \quad \lambda > \lambda_1 \qquad (\alpha > 1), \\ \frac{1}{2} & \text{if} \quad \lambda = \lambda_1 \qquad (\alpha = 1), \\ 0 & \text{if} \quad 0 < \lambda < \lambda_1 \quad (0 < \alpha < 1). \end{cases} \tag{38}$$

The last statement in (38) (for $0 < \lambda < \lambda_1$) readily follows from Theorem 3 for $0 < \alpha < 1$ [see also (39) and (40) below] using the fact that $B(t)/\,\mathsf{E}[S_N(t)] \to 0$ as $t \to \infty$. Indeed, note that

$$\frac{B(t)}{\mathsf{E}[S_N(t)]} = \exp\Big\{\alpha^{\varrho'-1} t^{\varrho'} - \log N - H(t)\Big\}$$

and, by the Scaling Assumption (29) and regularity of the functions $H(t) \sim H_0(t)$ with index ϱ',

$$\lim_{t\to\infty} \frac{\alpha^{\varrho'-1} t^{\varrho'} - \log N - H(t)}{H_0(t)} = \varrho' \alpha^{\varrho'-1} - \lambda - 1 < 0,$$

where the last inequality follows, by the substitution (14), from the elementary inequality [see Hardy, Littlewood and Pólya (1952), Section 2.15, Theorem 41, page 39]

$$1 - \alpha^{\varrho'} > \varrho' \alpha^{\varrho'-1}(1-\alpha) \qquad (0 < \alpha < 1,\ \ \varrho' > 1).$$

Analogously, Theorems 2 and 5 yield the following united assertion: *If the random variables X_i have the Weibull distribution* (20) *then, as* $t \to \infty$,

$$\frac{S_N(t) - \mathsf{E}[S_N(t)]}{\big(\mathsf{Var}[S_N(t)]\big)^{1/2}} \xrightarrow{d} \begin{cases} \mathcal{N}(0,1) & \text{if} \quad \lambda > \lambda_2 \quad (\alpha > 2), \\ \mathcal{N}\big(0, \frac{1}{2}\big) & \text{if} \quad \lambda = \lambda_2 \quad (\alpha = 2). \end{cases}$$

Finally, Theorem 3 takes the following form: *If X_i have the Weibull distribution* (20) *then, as* $t \to \infty$,

$$\frac{S_N(t) - \tilde{A}(t)}{\exp\big(\alpha^{\varrho'-1} t^{\varrho'}\big)} \xrightarrow{d} \mathcal{F}_\alpha, \tag{39}$$

where the stable law $\mathcal{F}_\alpha$ is described in Theorem 3 *and $\tilde{A}(t)$ is of the form*

$$\tilde{A}(t) := \begin{cases} \mathsf{E}[S_N(t)] & \text{if} \quad \lambda_1 < \lambda < \lambda_2 \quad (1 < \alpha < 2), \\ N B_1(t) & \text{if} \quad \lambda = \lambda_1 \qquad (\alpha = 1), \\ 0 & \text{if} \quad 0 < \lambda < \lambda_1 \quad (0 < \alpha < 1), \end{cases} \tag{40}$$

with $B_1(t)$ given by (36).

7. Sketch of the proofs

Theorems 3, 4 and 5 can be proved using the known methods for sums of independent random variables [see Gnedenko and Kolmogorov (1968) and Petrov (1975)]. However, the actual proofs are technically quite involved, because we have imposed only very minimal smoothness conditions on the distribution of X. So the full details are not given here, but rather will be published elsewhere. Nevertheless, it is not difficult to explain the main points behind the calculations. In particular, it is important to clarify the central role and power of the Basic Identity (28).

The key step in the proofs is the evaluation of the tail probability [cf. Petrov (1975), Chapter IV, § 1, 2]

$$
\begin{aligned}
\mathsf{P}\big\{e^{tX} > xB(t)\big\} &= \mathsf{P}\Big\{e^{tX} > x\exp\big[\mu(t)H_0(t)\big]\Big\} \\
&= \mathsf{P}\Big\{X > \frac{\log x}{t} + \frac{\mu(t)H_0(t)}{t}\Big\} \\
&= \mathsf{P}\{X > \eta_x(t)\} \\
&= \exp\big[-h(\eta_x(t))\big],
\end{aligned} \tag{41}
$$

where we used the notations (5) and (24). This expression needs to be compared to the sample size, $N \sim e^{\lambda H_0(t)}$, and therefore we have to relate the function $h(\eta_x(t))$ to the canonical scale determined by the rate function $H_0(t)$. In so doing, equation (28) plays the major role, as well as the following lemma.

LEMMA 4. *For each* $x > 0$,

$$\lim_{t\to\infty} \big[h(\eta_x(t)) - h(\eta_1(t))\big] = \alpha \log x.$$

PROOF. Note that, as $t \to \infty$,

$$\eta_x(t) - \eta_1(t) = \frac{\log x}{t} \to 0$$

and

$$\eta_1(t) = \frac{\mu(t)H_0(t)}{t} \to \infty.$$

By Taylor's formula and normalized regular variation of the function h [see (18)] we have, as $t \to \infty$,

$$h(\eta_x(t)) - h(\eta_1(t)) \sim h'(\eta_1(t))\big[\eta_x(t) - \eta_1(t)\big] \sim \frac{\varrho h(\eta_1(t))}{\eta_1(t)} \cdot \frac{\log x}{t}\,.$$

Using the Basic Identity (28), the right-hand side can be rewritten as

$$\frac{\varrho\lambda}{\mu(t)} \log x \to \alpha \log x \qquad (t \to \infty),$$

according to (27), and the lemma is proved. □

Let us now obtain the main ingredient of the limiting infinitely divisible law — the *Lévy–Khinchin spectral function* $\mathcal{L}$, using the formula

$$
\mathcal{L}(x) = \begin{cases} \lim\limits_{t\to\infty} N\,\mathsf{P}\{e^{tX} \le xB(t)\}, & x < 0, \\ -\lim\limits_{t\to\infty} N\,\mathsf{P}\{e^{tX} > xB(t)\}, & x > 0 \end{cases} \tag{42}
$$

[see Petrov (1975), Chapter IV, § 2, Theorem 8, pages 81–82]. First of all, note that $\mathcal{L}(x) = 0$ if $x < 0$. For $x > 0$, using the Scaling Assumption (29) and formula (41) we have

$$N\,\mathsf{P}\big\{e^{tX} > xB(t)\big\} \sim \exp\big\{\lambda H_0(t) - h(\eta_x(t))\big\}. \tag{43}$$

The Basic Identity (28) and Lemma 4 imply

$$
\begin{aligned}
\lambda H_0(t) - h(\eta_x(t)) &= h(\eta_1(t)) - h(\eta_x(t)) \\
&\to -\alpha \log x \qquad (t \to \infty).
\end{aligned} \tag{44}
$$

Hence, returning to (42) we obtain

$$\begin{aligned}-\mathcal{L}(x) &= \lim_{t\to\infty} \exp\bigl\{\lambda H_0(t) - h(\eta_x(t))\bigr\}\\ &= \exp\{-\alpha \log x\}\\ &= x^{-\alpha},\end{aligned}$$

and therefore α is indeed the characteristic exponent of the limiting law.

8. Limit distribution of the maximum

Consider the partial maximum

$$M_N(t) := \max\bigl\{e^{tX_i},\ i=1,\dots,N\bigr\} = \exp\bigl(tX_{1,N}\bigr),$$

where

$$X_{1,N} := \max\{X_i,\ i=1,\dots,N\}.$$

Recall the notation (30),

$$B(t) = \exp\bigl\{\mu(t)H_0(t)\bigr\}.$$

THEOREM 6. *For all $\lambda>0$, as $t\to\infty$,*

$$\frac{M_N(t)}{B(t)} \xrightarrow{d} \Phi_\alpha,$$

where Φ_α is the Fréchet distribution, with distribution function

$$\Phi_\alpha(x) = \begin{cases} \exp(-x^{-\alpha}) & \text{if } x>0,\\ 0 & \text{otherwise.}\end{cases}$$

PROOF. As before [cf. (41)], for $x>0$ we have

$$\begin{aligned}\mathsf{P}\{M_N \le xB(t)\} &= \mathsf{P}\{X_{1,N}\le \eta_x(t)\}\\ &= \Bigl(1-\exp\bigl\{-h(\eta_x(t))\bigr\}\Bigr)^N\\ &\sim \exp\Bigl(-N\exp\bigl\{-h(\eta_x(t))\bigr\}\Bigr)\\ &\to \exp(-x^{-\alpha}) \qquad (t\to\infty),\end{aligned}$$

as shown in (43) and (44). □

REMARK 2. The Fréchet distribution Φ_α is one of the three types of possible weak limits for maxima of i.i.d. random variables [see Galambos (1978), Sections 2.1 and 2.4]. However, the known general theorems about convergence of the maximum to Φ_α are not directly applicable in our case.

From Theorem 6, it is easy to derive a logarithmic Law of Large Numbers for the maximum.

THEOREM 7. *For all $\lambda>0$, as $t\to\infty$,*

$$\frac{\log M_N(t)}{H_0(t)} \xrightarrow{p} \varrho'\left(\frac{\varrho\lambda}{\varrho'}\right)^{1/\varrho}.$$

PROOF. Taking the logarithm of $M_N(t)$ and dividing by $H_0(t) \to \infty$, from Theorem 6 we deduce that

$$\frac{\log M_N(t)}{H_0(t)} - \mu(t) = o_p(1) \qquad (t \to \infty),$$

whence our claim follows. $\square$

It is interesting to compare the maximal term $M_N(t)$ with the entire sum $S_N(t)$. In fact, Theorem 3 implies the following Law of Large Numbers for $\log S_N(t)$, which can be seen as providing an analogue of the limiting free energy $F(\beta)$ in the Random Energy Model [see (4)].

THEOREM 8. *For all* $\lambda > 0$, *as* $t \to \infty$,

$$\frac{\log S_N(t)}{H_0(t)} \xrightarrow{p} \begin{cases} \varrho' \left(\dfrac{\varrho\lambda}{\varrho'}\right)^{1/\varrho} & \text{if } \ 0 < \lambda \le \lambda_1, \\ \lambda + 1 & \text{if } \ \lambda \ge \lambda_1. \end{cases}$$

Comparing Theorems 7 and 8, we note that in the case $0 < \lambda \le \lambda_1$

$$\frac{\log M_N(t)}{\log S_N(t)} \xrightarrow{p} 1 \qquad (t \to \infty), \tag{45}$$

which indicates that the contribution of the maximal term M_N to the sum S_N is logarithmically equivalent to the whole sum. In the opposite case where $\lambda > \lambda_1$, the limit in (45) is strictly less than 1, so that the maximum $M_N(t)$ is negligible as compared to the sum $S_N(t)$. This observation is supported by the LLN being valid for $\lambda \ge \lambda_1$ (see Theorems 1 and 4).

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