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Non-commutative separate continuity and weakly almost periodicity for Hopf von Neumann algebras

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Abstract

For a compact Hausdorff space X, the space $SC(X \times X)$ of separately continuous complex valued functions on X can be viewed as a C^*-subalgebra of $C(X)^{**}\overline{\otimes}C(X)^{**}$, namely those elements which slice into $C(X)$. The analogous definition for a non-commutative C^*-algebra does not necessarily give an algebra, but we show that there is always a greatest C^*-subalgebra. This thus gives a non-commutative notion of separate continuity. The tools involved are multiplier algebras and row/column spaces, familiar from the theory of Operator Spaces. We make some study of morphisms and inclusions. There is a tight connection between separate continuity and the theory of weakly almost periodic functions on (semi)groups. We use our non-commutative tools to show that the collection of weakly almost periodic elements of a Hopf von Neumann algebra, while itself perhaps not a C^*-algebra, does always contain a greatest C^*-subalgebra. This allows us to give a notion of non-commutative, or quantum, semitopological semigroup, and to briefly develop a compactification theory in this context.

Keywords: Separate continuity, Hopf von Neumann algebra, C^*-bialgebra, weakly almost periodic function.

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1 Introduction

A semigroup which carries a topology is *semitopological* if the product is separately continuous. Natural examples arise when studying semigroup compactifications of groups, or from semigroups of operators on reflexive Banach spaces (indeed, these are linked, see [22, Theorem 4.6]). Indeed, for a locally compact group G, we say that $f \in C_b(G)$ is *weakly almost periodic* if the collection of (left, or equivalently, right) translates of f forms a relatively weakly compact subset of $C_b(G)$. The collection of all such functions, $\mathrm{wap}(G)$, forms a C^*-subalgebra of $C_b(G)$ with character space G^{wap} say. The product on G can be extended to G^{wap}, and the resulting semigroup is compact and semitopological. Furthermore, G^{wap} is the maximal compact semitopological semigroup to contain a dense homomorphic copy of G; see [4] and references therein.

We are interested in a "quantum group" approach to such questions. A compact semigroup S canonically gives rise to a C^*-bialgebra by considering $A = C(S)$ and the coproduct $\Delta : A \to A \otimes A \cong C(S \times S)$ defined by $\Delta(f)(s,t) = f(st)$. If S is only semitopological, then the natural codomain of Δ is now $SC(S \times S)$ the space of separately continuous functions. Given the tight connection between weakly almost periodic functions and separate continuity, it seems likely that to study a notion of "weakly almost periodicity" for, say, C^*-bialgebras, or Hopf von Neumann algebras, we will need a good notion of "separate continuity" for general C^*-algebras. Indeed, for commutative Hopf von Neumann algebras, similar connections were explored fruitfully in [9, Section 4]. This paper develops a suitable non-commutative framework to attack this problem.

The most important class of Hopf von Neumann algebras are those arising from locally compact quantum groups, [18], as these completely generalise the classical group algebras $L^1(G)$ and the "dual picture", the Fourier algebra $A(G)$, [14]. For a locally compact quantum group $\mathbb{G}$ we consider the Hopf von Neumann algebra $(L^\infty(\mathbb{G}),\Delta)$, and use Δ to turn the predual $L^1(\mathbb{G})$ into a (completely contractive) Banach algebra. There is a notion of a *weakly almost periodic* element of the dual of a Banach algebra (see Section 5 below) and thus one can talk of $\mathrm{wap}(L^1(\mathbb{G}))$. When applied to $L^1(G)$ this theory exactly recovers $\mathrm{wap}(G)$, see [30] for example. For the Fourier algebra, this was suggested as early as [15]; recent work for quantum groups can be found in [25, Section 4] and [17], for example. However, outside of the commutative situation, to our knowledge there has been little study in terms of compactifications (for different notions of compactification, compare [8, 27, 28]). In particular, it is unknown, except in a few cases (see Section 7 below) if $\mathrm{wap}(L^1(\mathbb{G}))$ is a C^*-algebra.

Using our non-commutative notion of separate continuity, we show that $\mathrm{wap}(L^1(\mathbb{G}))$ always contains a greatest C^*-subalgebra (that is, a C^*-subalgebra containing all others) which we denote by $\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$, to avoid confusion. In fact, $x\in\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$ if and only if $x\in\mathrm{wap}(L^1(\mathbb{G}))$ and also $x^*x, xx^*\in\mathrm{wap}(L^1(\mathbb{G}))$. (The reader should note that, using this definition, it is unclear why $x,y\in\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$ implies that $xy\in\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$; the equivalent properties established earlier in the paper make this clear.) We show that our notion of separate continuity is stable under completely bounded maps, and this is used to show that $\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$ is an $L^1(\mathbb{G})$-submodule.

Using the notion of non-commutative separate continuity, we also make a (tentative) definition of a quantum semitopological semigroup, and show that $\mathrm{wap}(L^\infty(\mathbb{G}),\Delta)$ fits into this framework, and can actually be interpreted as a "compactification" in this category. Thus we obtain a rather satisfactory theory, into which our recent notion of a C^*-Eberlein algebra, [7], fits. Whether we can extend other aspects of the C^*-Eberlein algebra theory, for example invariant means and decompositions, remains an open problem, see Section 7.

The paper is organised as follows: we make some preliminary remarks, mostly to fix notation and terminology. In Section 3 we motivate a tentative definition of "separate continuity" for an arbitrary C^*-algebra, show that this doesn't in general yield an algebra, but then give Theorem 3.1 which establishes equivalent conditions on elements which together form the maximal C^*-subalgebra. This gives the notion of $A\overset{\mathrm{sc}}{\otimes}B$, the notation chosen as $C(X)\overset{\mathrm{sc}}{\otimes}C(Y)\cong SC(X\times Y)$ for compact Hausdorff spaces X,Y. In Section 3.2 we simplify the theory in the case of von Neumann algebras. In Section 4 we study inclusions of C^*-algebras (and establish a very simple slice map property) and also show that $\overset{\mathrm{sc}}{\otimes}$ is stable under completely bounded maps. We then apply this theory to Hopf von Neumann algebras in Section 5. For a locally compact group, we have the choice as to work with $L^\infty(G)$ or perhaps $C_0(G)$ (and these give the same notion of weakly almost periodic function). Motivated by this, in Section 6 we look at C^*-bialgebras, and show how to consistently use the Hopf von Neumann theory here. We end with some open questions.

1.1 Acknowledgments

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2 Preliminaries

For a Hilbert space H we write the inner-product as $(\cdot|\cdot)$. We write $\mathcal{B}(H)$ for the bounded linear maps on H, and write $\mathcal{B}_0(H)$ for the ideal of compact operators. Thus $\mathcal{B}_0(H)$ is the closed span of the rank-one operators of the form $\theta_{\xi,\eta}:\alpha\mapsto(\alpha|\eta)\xi$. We denote by $\omega_{\xi,\eta}$ the normal functional $\mathcal{B}(H)\to\mathbb{C}; x\mapsto(x\xi|\eta)$.

We write $\otimes$ and $\overline{\otimes}$ for tensor products of C^*-algebras and von Neumann algebras, respectively. For a von Neumann algebra M we write M_* for its predual. The multiplier algebra of a C^*-algebra A is denoted by $M(A)$, and we use the notion of a non-degenerate $*$-homomorphism $\theta:A\to M(B)$; we always have the strictly continuous extension $\tilde\theta:M(A)\to M(B)$, see [21, Appendices] for example.

We work with the standard theory of Operator Spaces, [13, 24]. In particular, we write $\widehat{\otimes}$ for the operator space projective tensor product: the main result we use is that for von Neumann algebras M, N, the predual of $\mathsf{M}\overline{\otimes}\mathsf{N}$ is $\mathsf{M}_*\widehat{\otimes}\mathsf{N}_*$.

While we mostly work with arbitrary Hopf von Neumann algebras (or, later, C^*-bialgebras), the motivating examples come from the theory of locally compact quantum groups, [18, 19]. We follow the now standard notation that for a locally compact quantum group $\mathbb{G}$, we write $L^\infty(\mathbb{G})$ for the von Neumann algebraic version, $L^1(\mathbb{G})$ for its predual, and $C_0(\mathbb{G})$ for the C^*-algebra version.

Finally, we use standard properties of weakly compact operators between Banach spaces, see [6, Section 5, Chapter VI] for example: if $T : E \to F$ is a bounded linear map then T is weakly compact if and only if $T^* : F^* \to E^*$ is weakly compact, if and only if $T^{**} : E^{**} \to F^{**}$ maps into F.

Other terminology and theory will be introduced as needed.

3 Motivation, and separate continuity

We are ultimately interested in studying analogues of "weakly almost periodic" functions, in the context of locally compact quantum groups, or just Hopf von Neumann (or C^*-) algebras. When G is a locally compact group, we consider the group algebra $L^1(G)$ and turn $L^\infty(G)$ into an $L^1(G)$-bimodule in the usual way. The following are equivalent for a function $F \in L^\infty(G)$:

1. The orbit map $L^1(G) \to L^\infty(G), a \mapsto a \cdot F$ (or equivalently $F \cdot a$) is a weakly compact operator;
2. $F \in C_b(G)$ and the collection of left (or equivalently right) translates of F forms a relatively weakly compact subset of F.

Compare [30] for example. As explained in the introduction, the collection of such F forms a unital C^*-subalgebra of $C_b(G)$, and the character space is the compact semitopological semigroup G^{wap}.

This gives some high-level motivation for looking at separately continuous functions. Furthermore, in [9] for example, consideration of spaces of separately continuous functions proved to be a very useful technical tool. Let X be a compact Hausdorff space and denote by $SC(X \times X)$ the space of separately continuous functions $X \times X \to \mathbb{C}$. Following the clear presentation of [26, Section 2] (ultimately using work of Grothendieck) we have that for $f \in SC(X \times X)$ and $\mu, \lambda \in \mathfrak{M}(X)$, measures on X, if we define $(\mu \otimes \mathrm{id})f, (\mathrm{id} \otimes \lambda)f : X \to \mathbb{C}$ by

$$((\mu \otimes \mathrm{id})f)(x) = \int_X f(y,x)\ d\mu(y), \qquad ((\mathrm{id} \otimes \lambda)f)(x) = \int_X f(x,y)\ d\lambda(y) \qquad (x \in X),$$

then both $(\mu \otimes \mathrm{id})f, (\mathrm{id} \otimes \lambda)f$ are in $C(X)$, and we have a generalised Fubini-theorem,

$$\langle \lambda, (\mu \otimes \mathrm{id})f \rangle = \langle \mu, (\mathrm{id} \otimes \lambda)f \rangle.$$

Let us write $\mathcal{B}_{w^*}(\mathfrak{M}(X) \times \mathfrak{M}(X), \mathbb{C})$ for the space of separately weak*-continuous bilinear maps $\mathfrak{M}(X) \times \mathfrak{M}(X) \to \mathbb{C}$. Thus we have shown that $f \in SC(X \times X)$ induces Φ_f, say, in $\mathcal{B}_{w^*}(\mathfrak{M}(X) \times \mathfrak{M}(X), \mathbb{C})$. Furthermore, this establishes an isometric isomorphism between $SC(X \times X)$ and $\mathcal{B}_{w^*}(\mathfrak{M}(X) \times \mathfrak{M}(X), \mathbb{C})$, see [26, Proposition 2.5].

Linearising the bilinear map, we get the projective tensor product $\mathfrak{M}(X)\widehat{\otimes}\mathfrak{M}(X)$ which is the predual of the commutative von Neumann algebra $C(X)^{**}\overline{\otimes}C(X)^{**}$. Then the separately weak*-continuous members of $(\mathfrak{M}(X)\widehat{\otimes}\mathfrak{M}(X))^* \cong C(X)^{**}\overline{\otimes}C(X)^{**}$ are precisely those $x \in C(X)^{**}\overline{\otimes}C(X)^{**}$ which slice into $C(X)$. Hence we obtain that

$$SC(X \times X) \cong \{x \in C(X)^{**}\overline{\otimes}C(X)^{**} : (\mu \otimes \mathrm{id})x, (\mathrm{id} \otimes \mu)x \in C(X)\ (\mu \in \mathfrak{M}(X))\}.$$

This motivates, for a unital, but maybe noncommutative, C^*-algebra A, the definition that

$$SC(A \times A) = \{x \in A^{**}\overline{\otimes}A^{**} : (\mu \otimes \mathrm{id})x, (\mathrm{id} \otimes \mu)x \in A\ (\mu \in A^*)\}.$$

Unfortunately, this need not be an algebra in general, as hinted at in [7, Section 3]. For example, let H be an infinite-dimensional Hilbert space and set $A = \mathcal{B}_0(H)$ the compact operators on H. Then $A^{**} \cong \mathcal{B}(H)$ and $A^{**}\overline{\otimes}A^{**} \cong \mathcal{B}(H \times H)$. Consider $\Sigma \in \mathcal{B}(H \times H)$ the swap map. Then for $\omega_{\xi,\eta} \in \mathcal{B}(H)_*$ and $\alpha, \beta \in H$,

$$\big((\omega_{\xi,\eta} \otimes \mathrm{id})(\Sigma)\alpha\big|\beta\big) = \big(\alpha \otimes \xi\big|\eta \otimes \beta\big) = (\alpha|\eta)(\xi|\beta) = \big(\theta_{\xi,\eta}\alpha\big|\beta\big),$$

so $(\omega_{\xi,\eta} \otimes \mathrm{id})(\Sigma) = \theta_{\xi,\eta} \in \mathcal{B}_0(H)$ and similarly $(\mathrm{id} \otimes \omega_{\xi,\eta})(\Sigma) = \theta_{\xi,\eta} \in \mathcal{B}_0(H)$. By linearity and continuity, we conclude that $\Sigma \in SC(A \times A)$. However, of course $\Sigma^2 = 1_{H\otimes H}$ and slices of Σ^2 give all of $\mathbb{C}1_H \not\subseteq A$.

3.1 Non-commutative separate continuity

The following idea originates from a suggestion of Taka Ozawa. For C^*-algebras A, B we define, as above

$$SC(A \times B) = \{x \in A^{**}\overline{\otimes}B^{**} : (\mu \otimes \mathrm{id})x \in B, (\mathrm{id} \otimes \lambda)x \in A \ (\mu \in A^*, \lambda \in B^*)\}.$$

Our aim is to find a maximal C^*-subalgebra of $SC(A \times B)$, this will actually turn out to the "maximum" C^*-subalgebra.

We first recall some definitions from [2, Section 0]. For a C^*-algebra A and for J a (closed, two-sided) ideal in A, let $M(A; J) = \{x \in M(A) : xA + Ax \subseteq J\}$, which is a C^*-subalgebra of $M(A)$, and by restriction to J, is isomorphic to a C^*-subalgebra of $M(J)$. Let A^1 be the conditional unitisation of A, so for an auxiliary C^*-algebra B, we can consider $M(A^1 \otimes B; A \otimes B)$. It is easy to see that this algebra consists of those $x \in M(A^1 \otimes B)$ such that $x(1 \otimes b), (1 \otimes b)x \in A \otimes B$ for all $b \in B$.

In the following theorem we shall consider a C^*-algebra $B \subseteq \mathcal{B}(K)$ for a Hilbert space K. Using a fixed orthonormal basis $(e_j)_{j\in J}$ of K we can view members of $\mathcal{B}(K)$ as being $J \times J$ matrices, say $\mathcal{B}(K) \subseteq \mathbb{M}_J$. Given another C^*-algebra A we can extend this identification to view members of $A^{**}\overline{\otimes}\mathcal{B}(K)$ as being A^{**}-valued matrices, say $x \in A^{**}\overline{\otimes}\mathcal{B}(K)$ corresponds to $(x_{ij}) \in \mathbb{M}_J(A^{**})$.

Theorem 3.1. *Let A, B be C^*-algebras represented on Hilbert spaces H, K such that the induced maps $\mathcal{B}(H)_* \to A^*$ and $\mathcal{B}(K)_* \to B^*$ are both onto (for example, these could be the universal representations). We may then regard $A^{**}\overline{\otimes}B^{**}$ as a subalgebra of $\mathcal{B}(H\otimes K)$. For $x \in A^{**}\overline{\otimes}B^{**}$ the following are equivalent:*

1. *$x, x^*x, xx^* \in SC(A \times B)$;*
2. *$x \in M(A^1 \otimes \mathcal{B}_0(K); A \otimes \mathcal{B}_0(K)) \cap M(\mathcal{B}_0(H) \otimes B^1; \mathcal{B}_0(H) \otimes B)$;*
3. *we embed $A^{**}\overline{\otimes}B^{**}$ into $A^{**}\overline{\otimes}\mathcal{B}(K)$ and view $x = (x_{ij}) \in \mathbb{M}_J(A^{**})$ as above. We require that each $x_{ij} \in A$, and that both $\sum_j x_{ij}x_{ij}^*$ and $\sum_j x_{ji}^*x_{ji}$ are norm convergent sums, for each i. Similarly with the roles of A and B swapped.*

We now proceed to prove Theorem 3.1 with an aim to prove a little more than strictly necessary. Throughout the rest of this section, A, B will be C^*-algebras. Let $\pi_A : A \to \mathcal{B}(H), \pi_B : B \to \mathcal{B}(K)$ be arbitrary, non-degenerate, $*$-homomorphisms with normal extensions $\tilde{\pi}_A : A^{**} \to \mathcal{B}(H), \tilde{\pi}_B : B^{**} \to \mathcal{B}(K)$, see for example [29, Section 2, Chapter III]. Notice that under the identification of $x \in A^{**}\overline{\otimes}\mathcal{B}(K)$ with $(x_{ij}) \in \mathbb{M}_J(A^{**})$ we have that $x_{ij} = (\mathrm{id} \otimes \omega_{e_j,e_i})(x)$ (observe the order of indices). In particular, if $x = (\tilde{\pi}_A \otimes \tilde{\pi}_B)(y)$ for some $y \in SC(A \times B)$ then $x_{ij} \in \pi_A(A)$ for all i, j.

Proposition 3.2. *Let π_A, π_B be as above, and let $x \in \mathcal{B}(H \otimes K)$. The following are equivalent:*

1. *$x \in M(\pi_A(A^1) \otimes \mathcal{B}_0(K); \pi_A(A) \otimes \mathcal{B}_0(K))$;*
2. *with x identified with (x_{ij}) we have that $x_{ij} \in \pi_A(A)$ for all i, j, and both $\sum_j x_{ij}x_{ij}^*$ and $\sum_j x_{ji}^*x_{ji}$ are norm convergent sums in $\mathcal{B}(H)$, and hence in $\pi_A(A)$, for each i.*

Proof. Firstly, observe that (1) is equivalent to $x(1 \otimes \theta), (1 \otimes \theta)x \in \pi_A(A) \otimes \mathcal{B}_0(K)$ for all $\theta \in \mathcal{B}_0(K)$. Let us consider the case of $x(1 \otimes \theta)$. By linearity and continuity, we need only check that $x(1 \otimes \theta_{e_k,e_l}) \in \pi_A(A) \otimes \mathcal{B}_0(K)$ for all $k, l \in J$.

Fix $k, l \in J$ and let $z = x(1 \otimes \theta_{e_k,e_l})$. Considering z as a matrix in $\mathbb{M}_J(\mathcal{B}(H))$ we have that

$$z_{ij} = (\mathrm{id} \otimes \omega_{e_j,e_i})(x(1 \otimes \theta_{e_k,e_l})) = \delta_{l,j}(\mathrm{id} \otimes \omega_{e_k,e_i})(x) = \delta_{l,j} x_{ik} \in \pi_A(A).$$

Thus the matrix of z actually consists of one non-zero column.

Suppose that (2) holds, let $J_0 \subseteq J$ be a finite subset, and define

$$w_{ij} = \begin{cases} z_{ij} & : i \in J_0, \\ 0 & : \text{otherwise} \end{cases}$$

Thus $w = (w_{ij})$ is a finitely supported matrix and $w_{ij} \in \pi_A(A)$ for all i, j, so clearly $w \in \pi_A(A) \otimes \mathcal{B}_0(K)$. As both w and z are just single columns, we immediately see that

$$\|z - w\| = \Big\| \sum_{i \in J} (z_{il} - w_{il})^*(z_{il} - w_{il}) \Big\|^{1/2} = \Big\| \sum_{i \notin J_0} z_{il}^* z_{il} \Big\|^{1/2} = \Big\| \sum_{i \notin J_0} x_{ik}^* x_{ik} \Big\|^{1/2}.$$

By assumption, $\sum_i x_{ik}^* x_{ik}$ converges in norm, and so for any $\epsilon > 0$ there is J_0 finite with $\|z - w\| \leq \epsilon$. Thus $z \in \pi_A(A) \otimes \mathcal{B}_0(K)$.

Conversely, suppose (1) holds, and let z be as before, now known to be a member of $\pi_A(A) \otimes \mathcal{B}_0(K)$. As $\pi_A(A) \otimes \mathcal{B}_0(K) \subseteq \mathbb{M}_J(\mathcal{B}(H))$ is the norm closure of finite matrices with entries in $\pi_A(A)$, for each $\epsilon > 0$ there is such a finite matrix $w = (w_{ij})$ with $\|z - w\| \leq \epsilon$. As the operation of projecting onto the lth column is (completely) contractive, we may suppose without loss of generality that $w_{ij} = \delta_{l,j} w_{ij}$ for all i, j. Thus $(z - w)$ has matrix consisting of just a non-zero column, and so

$$\epsilon \geq \|z - w\| = \Big\| \sum_i (z_{il} - w_{il})^*(z_{il} - w_{il}) \Big\|^{1/2} = \Big\| \sum_i (x_{ik} - w_{il})^*(x_{ik} - w_{il}) \Big\|^{1/2}.$$

If $w_{il} = 0$ for $i \notin J_0$ with J_0 finite, then in particular

$$\Big\| \sum_{i \notin J_0} x_{ik}^* x_{ik} \Big\|^{1/2} \leq \epsilon,$$

as required.

We have hence shown that $x(1 \otimes \theta) \in \pi_A(A) \otimes \mathcal{B}_0(K)$ for all $\theta \in \mathcal{B}_0(K)$, if and only if $\sum_j x_{ji}^* x_{ji}$ is norm convergent for each i. The other case follows similarly, or from simply replacing x by x^*. □

Proof of Theorem 3.1. That (2) and (3) are equivalent follows from Proposition 3.2.

Suppose that (2) holds. Let $\xi, \eta \in K$, let $i, j \in J$, and set $y = x(1 \otimes \theta_{e_i,\eta}), z = x(1 \otimes \theta_{e_j,\xi})$. Then

$$z^* y = (1 \otimes \theta_{\xi,e_j}) x^* x (1 \otimes \theta_{e_i,\eta}) = (\mathrm{id} \otimes \omega_{e_i,e_j})(x^* x) \otimes \theta_{\xi,\eta}.$$

By assumption, $y, z \in A \otimes \mathcal{B}_0(K)$, and so $(\mathrm{id} \otimes \omega_{e_i,e_j})(x^* x) \in A$. By linearity and continuity, and using that $\mathcal{B}(K)_* \to B^*$ is onto, it follows that $(\mathrm{id} \otimes \mu)(x^* x) \in A$ for all $\mu \in B^*$. Similarly, $(\mathrm{id} \otimes \mu)(x x^*) \in A$ for all $\mu \in B^*$. Furthermore, $(\mathrm{id} \otimes \omega_{\xi,e_j})(y) = (\xi|\eta)(\mathrm{id} \otimes \omega_{e_i,e_j})(x) \in A$ and so by a suitable choice of ξ, η, and again by linearity and continuity, we conclude that $(\mathrm{id} \otimes \mu)(x) \in A$ for all $\mu \in B^*$. Repeating the argument with the roles of A and B swapped shows that (1) holds.

Conversely, suppose that (1) holds. By the discussion above, the matrix (x_{ij}) does consist of elements of A. The matrix representation of $x^* x$ is $(x^* x)_{ij} = \sum_k x_{ki}^* x_{kj}$, the sum converging strongly in $\mathcal{B}(H)$, for example. By assumption, $(x^*x)_{ij} \in A$. For each positive $\mu \in A^*$ choose $\omega \in \mathcal{B}(H)_*$ with $\omega|_A = \mu$. Writing $J_0 \subset\subset J$ to indicate that J_0 is a finite subset of J, we see that

$$\begin{aligned}\sup_{J_0 \subset\subset J} \langle \mu, \sum_{k \in J_0} x_{ki}^* x_{kj} \rangle &= \sup_{J_0 \subset\subset J} \langle \sum_{k \in J_0} x_{ki}^* x_{kj}, \omega \rangle = \lim_{J_0 \subset\subset J} \langle \sum_{k \in J_0} x_{ki}^* x_{kj}, \omega \rangle \\ &= \sum_{k \in J} \langle x_{ki}^* x_{kj}, \omega \rangle = \langle (x^*x)_{ij}, \omega \rangle = \langle \mu, (x^*x)_{ij} \rangle.\end{aligned}$$

So we have an increasing net in A^+ which converges, against elements of the unit ball of A^*_+, to an element of A^+. By an application of Dini's Theorem, compare [19, Lemma A.3], it follows that $\sum_k x_{ki}^* x_{kj} = (x^*x)_{ij}$ with convergence in norm in A. Applying a similar argument to xx^*, and then swapping the roles of A and B, shows that (3) holds. ∎

Let us now define the object which we shall study for the rest of the paper.

Definition 3.3. *For C^*-algebras A and B define*

$$A \overset{\mathrm{sc}}{\otimes} B = \{x \in SC(A \times B) : x^*x, xx^* \in SC(A \times B)\}.$$

Theorem 3.4. *For C^*-algebras A, B we have that $A \overset{\mathrm{sc}}{\otimes} B$ is a C^*-subalgebra of $A^{**}\overline{\otimes}B^{**}$ and every $*$-algebra contained in $SC(A \times B)$ is contained in $A \overset{\mathrm{sc}}{\otimes} B$.*

Proof. The first claim follows immediately from the equivalence of (1) and (2) above, as (2) is stated in terms of the intersection of two C^*-algebras. The second claim is immediate. ∎

3.2 For von Neumann algebras

We now aim to apply this construction to von Neumann algebras M, N. By definition, this would involve working in $\mathsf{M}^{**}\overline{\otimes}\mathsf{N}^{**}$, but the constructions in this section allow us to work with $\mathsf{M}\overline{\otimes}\mathsf{N}$ instead. When M, N are commutative, similar (but less general) ideas are explored in [9, Section 4].

Fix von Neumann algebras M, N with preduals $\mathsf{M}_*, \mathsf{N}_*$. Consider the canonical map from a Banach space to its bidual $\kappa = \kappa_{\mathsf{M}_*} : \mathsf{M}_* \to \mathsf{M}^*$. Then $\kappa^* : \mathsf{M}^{**} \to \mathsf{M}$ is a $*$-homomorphism, normal by construction. In fact, $\kappa(\mathsf{M}_*)$ is 1-complemented in M^*, see [29, Section 2, Chapter III]. Thus $\kappa^* \otimes \kappa^*$ is a normal $*$-homomorphism $\mathsf{M}^{**}\overline{\otimes}\mathsf{N}^{**} \to \mathsf{M}\overline{\otimes}\mathsf{N}$. Let θ_{sc} be the restriction of this map to $SC(\mathsf{M} \times \mathsf{N})$, so θ_{sc} further restricted to $\mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{N}$ is a $*$-homomorphism, separately weak*-continuous.

By analogy with the Banach algebra situation (see Section 5 below, or, if one prefers, a completely bounded analogue of Arens's original work, [1]) define $\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ to be those $x \in \mathsf{M}\overline{\otimes}\mathsf{N}$ such that the maps

$$L_x : \mathsf{M}_* \to \mathsf{N}, \omega \mapsto (\omega \otimes \mathrm{id})(x), \qquad R_x : \mathsf{N}_* \to \mathsf{M}, \omega \mapsto (\mathrm{id} \otimes \omega)(x)$$

are weakly compact. Notice that $L_x^* \circ \kappa_{\mathsf{N}_*} = R_x$ and $R_x^* \circ \kappa_{\mathsf{M}_*} = L_x$, and so L_x is weakly compact if and only if R_x is.

Lemma 3.5. *Let $x \in SC(\mathsf{M} \times \mathsf{N})$ and set $y = \theta_{sc}(x)$. Then $y \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$.*

Proof. For $\omega \in \mathsf{N}_*$ and $\tau \in \mathsf{M}_*$ we have that

$$\langle R_y(\omega), \tau \rangle = \langle x, \kappa(\tau) \otimes \kappa(\omega) \rangle = \langle (\mathrm{id} \otimes \kappa(\omega))(x), \kappa(\tau) \rangle = \langle (\mathrm{id} \otimes \kappa(\omega))(x), \tau \rangle,$$

as by assumption, $(\mathrm{id} \otimes \kappa(\omega))(x) \in \mathsf{M} \subseteq \mathsf{M}^{**}$.

Let (ω_α) be a bounded net in N_* and by moving to a subnet if necessary, suppose that $\omega_\alpha \to \mu \in N^*$ weak* in N^*. Then for $\lambda \in \mathsf{M}^*$,

$$\lim_\alpha \langle \lambda, R_y(\omega_\alpha) \rangle = \lim_\alpha \langle \lambda, (\mathrm{id} \otimes \kappa(\omega_\alpha))(x) \rangle = \lim_\alpha \langle x, \lambda \otimes \kappa(\omega_\alpha) \rangle = \lim_\alpha \langle (\lambda \otimes \mathrm{id})(x), \kappa(\omega_\alpha) \rangle.$$

As $(\lambda\otimes\mathrm{id})(x)\in N\subseteq N^{**}$ this limit is equal to

$$\langle(\lambda\otimes\mathrm{id})(x),\mu\rangle=\langle\lambda,(\mathrm{id}\otimes\mu)(x)\rangle,$$

where $(\mathrm{id}\otimes\mu)(x)\in\mathsf{M}$. Thus $R_y(\omega_\alpha)\to(\mathrm{id}\otimes\mu)(x)\in\mathsf{M}$ weakly. This establishes that R_y is weakly compact, as required. □

We will now proceed to show that the map $\theta_{sc}:SC(\mathsf{M}\times\mathsf{N})\to\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ is actually a bijection. Firstly we show it is onto, for which a further idea of Arens is required; we follow the notation of [10, Section 3], adapted to the von Neumann algebra situation. Given $x\in\mathsf{M}\overline{\otimes}\mathsf{N}$, for $\mu\in\mathsf{M}^*$ we define $(\mu\otimes\mathrm{id})(x)\in\mathsf{N}$ by

$$\langle(\mu\otimes\mathrm{id})(x),\omega\rangle=\langle\mu,(\mathrm{id}\otimes\omega)(x)\rangle\qquad(\omega\in\mathsf{N}_*).$$

Similarly define $(\mathrm{id}\otimes\lambda)(x)\in\mathsf{M}$ for $\lambda\in\mathsf{N}^*$ Then we define two completely contractive maps $(\mathsf{M}^{**}\overline{\otimes}\mathsf{N}^{**})_*=\mathsf{M}^*\widehat{\otimes}\mathsf{N}^*\to(\mathsf{M}_*\widehat{\otimes}\mathsf{N}_*)^{**}=(\mathsf{M}\overline{\otimes}\mathsf{N})^*$ by

$$\mu\otimes\lambda\mapsto\mu\otimes_\Box\lambda,\quad\mu\otimes_\Diamond\lambda,$$

and extending by linearity and continuity, where we define

$$\langle\mu\otimes_\Box\lambda,x\rangle=\langle\mu,(\mathrm{id}\otimes\lambda)(x)\rangle,\qquad\langle\mu\otimes_\Diamond\lambda,x\rangle=\langle\lambda,(\mu\otimes\mathrm{id})(x)\rangle.$$

(We note that the definition of $\otimes_\Diamond$ in [10, Page 16] is wrong, or at least inconsistent; one should swap Φ,Ψ in the formula on page 16.)

The following again goes back to Arens, but we include a proof for reference and motivation.

Lemma 3.6. *We have that $x\in\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ if and only if $\langle\mu\otimes_\Box\lambda,x\rangle=\langle\mu\otimes_\Diamond\lambda,x\rangle$ for all μ,λ.*

Proof. To show "if", let (ω_α) be a bounded net in M_* converging weak* to $\mu\in\mathsf{M}^*$. For $\lambda\in\mathsf{N}^*$ we have that

$$\lim_\alpha\langle\lambda,L_x(\omega_\alpha)\rangle=\lim_\alpha\langle(\mathrm{id}\otimes\lambda)(x),\omega_\alpha\rangle=\langle\mu\otimes_\Box\lambda,x\rangle=\langle\mu\otimes_\Diamond\lambda,x\rangle=\langle\lambda,(\mu\otimes\mathrm{id})(x)\rangle,$$

and so $L_x(\omega_\alpha)\to(\mu\otimes\mathrm{id})(x)\in\mathsf{N}$ weakly. As in the proof of Lemma 3.5, it follows that $x\in\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$.

Conversely, let (τ_β) in N_* converge weak* to $\lambda\in\mathsf{N}^*$, and let (ω_α) as before. Assuming that L_x is weakly compact, we may assume that $L_x(\omega_\alpha)\to x_0\in\mathsf{M}$, say, weakly. Then

$$\begin{aligned}\langle\mu\otimes_\Box\lambda,x\rangle&=\lim_\alpha\langle(\mathrm{id}\otimes\lambda)(x),\omega_\alpha\rangle=\lim_\alpha\langle\lambda,L_x(\omega_\alpha)\rangle=\langle\lambda,x_0\rangle=\lim_\beta\langle x_0,\tau_\beta\rangle=\lim_\beta\lim_\alpha\langle L_x(\omega_\alpha),\tau_\beta\rangle\\&=\lim_\beta\lim_\alpha\langle R_x(\tau_\beta),\omega_\alpha\rangle=\lim_\beta\langle\mu,R_x(\tau_\beta)\rangle=\langle\lambda,(\mu\otimes\mathrm{id})(x)\rangle=\langle\mu\otimes_\Diamond\lambda,x\rangle,\end{aligned}$$

as required. □

Proposition 3.7. *Let $x\in\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$, and define $y\in\mathsf{M}^{**}\overline{\otimes}\mathsf{N}^{**}=(\mathsf{M}^*\widehat{\otimes}\mathsf{N}^*)^*$ by $\langle y,\mu\otimes\lambda\rangle=\langle\mu\otimes_\Box\lambda,x\rangle=\langle\mu\otimes_\Diamond\lambda,x\rangle$. Then $y\in SC(\mathsf{M}\times\mathsf{N})$; indeed, $(\mu\otimes\mathrm{id})(y)=(\mu\otimes\mathrm{id})(x)\in\mathsf{N}$ and $(\mathrm{id}\otimes\lambda)(y)=(\mathrm{id}\otimes\lambda)(x)\in\mathsf{M}$.*

Conversely, if $y\in SC(\mathsf{M}\times\mathsf{N})$ and we set $x=\theta_{sc}(y)$ then $(\mu\otimes\mathrm{id})(x)=(\mu\otimes\mathrm{id})(y)\in\mathsf{N}$ and $(\mathrm{id}\otimes\lambda)(x)=(\mathrm{id}\otimes\lambda)(y)\in\mathsf{M}$. As such, the map $\theta_{sc}:SC(\mathsf{M}\times\mathsf{N})\to\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ is an isomorphism.

Proof. The first claim follows from a simple calculation. Now let $y\in SC(\mathsf{M}\times\mathsf{N})$ and set $x=\theta_{sc}(y)$. Consider the biadjoint $R_x^{**}:\mathsf{N}^*\to\mathsf{M}^{**}$, which satisfies that if $\mu\in\mathsf{N}^*$ is the weak*-limit of $(\omega_\alpha)\subseteq\mathsf{N}_*$ then $R_x^{**}(\mu)$ is the weak*-limit, in M^{**}, of the net $(R_x(\omega_\alpha))$. As R_x is weakly compact, this actually converges weakly in M, and the proof of Lemma 3.6 shows that $R_x^{**}(\mu)=(\mu\otimes\mathrm{id})(x)$. However, from the proof of Lemma 3.5, this net converges to $(\mathrm{id}\otimes\mu)(y)\in\mathsf{M}$, and so the result follows. □

Theorem 3.8. *The map $\theta_{sc}:\mathsf{M}\overset{\mathrm{sc}}{\otimes}\mathsf{N}\to\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ is an injective $*$-homomorphism, separately normal, and has image those $x\in\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ such that also $x^*x,xx^*\in\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$.*

Proof. It follows from Proposition 3.7 that θ_{sc} is injective, and it is separately normal by construction. If $y \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{N}$ then also $y^*y, yy^* \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{N}$. Thus if $x = \theta_{sc}(y)$ then $x^*x = \theta_{sc}(y^*y) \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$, and similarly $xx^* \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$. Conversely, if $x \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ with $x^*x, xx^* \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ then let $y \in SC(\mathsf{M} \times \mathsf{N})$ with $\theta_{sc}(y) = x$. There is $z \in SC(\mathsf{M} \times \mathsf{N})$ with $\theta_{sc}(z) = x^*x$. As θ_{sc} is the restriction of $\kappa^* \otimes \kappa^*$, which is a $*$-homomorphism to $\mathsf{M}\overline{\otimes}\mathsf{N}$,

$$(\kappa^* \otimes \kappa^*)(z) = x^*x = (\kappa^* \otimes \kappa^*)(y^*y),$$

and so $z = y^*y$. Thus y^*y, and similarly yy^*, are members of $SC(\mathsf{M} \times \mathsf{N})$, and so by definition, $y \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{N}$. $\square$

Hence $\mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{N}$ is the maximum C^*-subalgebra of $\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$. However, again the space $\mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{N})$ need not be an algebra in general. An example of M and $t \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{M})$ such that $t^*t \notin \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{M})$ may be constructed as follows. Let $\mathsf{M} = \mathcal{B}(H)$ for a separable, infinite-dimensional Hilbert space H. It is easy to find a positive $x \in \mathcal{B}(H \otimes H) = \mathsf{M}\overline{\otimes}\mathsf{M}$ such that the map $\mathsf{M}_* \to \mathsf{M}; \omega \mapsto (\omega \otimes \mathrm{id})(x)$ is not weakly compact. Now let $u : H \to H \otimes H$ be a unitary, and fix a unit vector $\xi_0 \in H$. Then $t \in \mathcal{B}(H \otimes H)$ defined by $t(\xi) = u^*x^{1/2}(\xi) \otimes \xi_0$ has the required property. This follows as $t^*t = x \notin \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{M})$, while one can show that the maps $\mathsf{M}_* \to \mathsf{M}; \omega \mapsto (\omega \otimes \mathrm{id})(t)$ and $\omega \mapsto (\mathrm{id} \otimes \omega)(t)$ both factor through a Hilbert space, and so are weakly compact.

4 Morphisms and inclusions

In this section we study stability properties of $\overset{\mathrm{sc}}{\otimes}$. We start by considering inclusions. Let A be a C^*-algebra and let $A_0 \subseteq A$ be a C^*-subalgebra. Then the inclusion $\iota : A_0 \to A$ induces the inclusion $\iota^{**} : A_0^{**} \to A^{**}$ which is a normal $*$-homomorphism. Indeed, if we identify A_0 with a subalgebra of A, then the restriction map $A^* \to A_0^*$ is a quotient map, with kernel $A_0^\perp = \{\mu \in A^* : \langle \mu, a \rangle = 0 \ (a \in A_0)\}$. Then $A_0^{**} = (A_0^*)^* \cong (A^*/A_0^\perp)^* \cong A_0^{\perp\perp}$. If also $B_0 \subseteq B$ is an inclusion of C^*-algebras, then we have the chain of isometric inclusions

$$A_0 \overset{\mathrm{sc}}{\otimes} B_0 \subseteq SC(A_0 \times B_0) \subseteq A_0^{**}\overline{\otimes}B_0^{**} \subseteq A^{**}\overline{\otimes}B^{**}.$$

The following result gives a simple "slice map" criteria to determine membership of $A_0 \overset{\mathrm{sc}}{\otimes} B_0$.

Theorem 4.1. *For $x \in A^{**}\overline{\otimes}B^{**}$ the following are equivalent:*

1. *x is in (the image of) $A_0 \overset{\mathrm{sc}}{\otimes} B_0$;*
2. *$x \in A \overset{\mathrm{sc}}{\otimes} B$ and x is in (the image of) $SC(A_0 \times B_0)$ (that is, $(\mu \otimes \mathrm{id})(x) \in B_0, (\mathrm{id} \otimes \lambda)(x) \in A_0$ for $\mu \in A^*, \lambda \in B^*$).*

Proof. (1) $\implies$ (2): For $x \in A_0 \overset{\mathrm{sc}}{\otimes} B_0$ and for $\mu \in B^*$, letting $\mu_0 \in B_0^*$ be the restriction, we have that $(\mathrm{id} \otimes \mu)(x) = (\mathrm{id} \otimes \mu_0)(x) \in A_0 \subseteq A$. Similarly $(\mathrm{id} \otimes \mu)(x^*x) \in A$ and $(\mathrm{id} \otimes \mu)(xx^*) \in A$. Analogously, right slices of x, x^*x, xx^* are in B, and so $x \in A \overset{\mathrm{sc}}{\otimes} B$. Clearly also $x \in SC(A_0 \times B_0)$.

(2) $\implies$ (1): Let $B \subseteq \mathcal{B}(K)$ be the universal representation and let $K \cong \ell^2(J)$, so we can regard $A^{**}\overline{\otimes}B^{**} \subseteq \mathbb{M}_J(A^{**})$ again. From Theorem 3.1 we know that if $x = (x_{ij})$ then the sums $\sum_j x_{ij}x_{ij}^*$ and $\sum_j x_{ji}^*x_{ji}$ converge in norm. That $x \in SC(A_0 \times B_0)$ tells us that each $x_{ij} \in A_0$, and hence that the sums actually converge in A_0. We apply similar arguments with the roles and A and B swapped, and then by Theorem 3.1 again we conclude that $x \in A_0 \overset{\mathrm{sc}}{\otimes} B_0$. $\square$

Let us make the following simple remark. As $A \subseteq A^{**}$ and $B \subseteq B^{**}$ we have the inclusion $A \overset{\mathrm{sc}}{\otimes} B \subseteq A^{**} \overset{\mathrm{sc}}{\otimes} B^{**}$. As A^{**}, B^{**} are von Neumann algebras, we can apply Theorem 3.8 to identify $A^{**} \overset{\mathrm{sc}}{\otimes} B^{**}$ as a subalgebra of $A^{**}\overline{\otimes}B^{**}$. The composition gives an inclusion $A \overset{\mathrm{sc}}{\otimes} B \to A^{**}\overline{\otimes}B^{**}$, and this is nothing but the canonical inclusion.

4.1 Completely bounded maps

In this section we show that $\overset{\mathrm{sc}}{\otimes}$ is stable under completely bounded maps.

Theorem 4.2. *Let A, B, C be C^*-algebras and let $\phi : A \to B$ be a completely bounded map. Then $\phi^{**} \otimes id : A^{**}\overline{\otimes}C^{**} \to B^{**}\overline{\otimes}C^{**}$ is completely bounded, and restricts to a map $\phi \overset{\mathrm{sc}}{\otimes} \mathrm{id} : A \overset{\mathrm{sc}}{\otimes} C \to B \overset{\mathrm{sc}}{\otimes} C$.*

From the theory of operator spaces, we know that $\phi^* : B^* \to A^*$ is completely bounded and hence also $\phi^* \otimes \mathrm{id} : B^*\widehat{\otimes}C^* \to A^*\widehat{\otimes}C^*$ is completely bounded, all with equal norms. Hence we do obtain $\phi^{**} \otimes \mathrm{id} : A^{**}\overline{\otimes}C^{**} \to B^{**}\overline{\otimes}C^{**}$.

Alternatively, from the structure theory of completely bounded maps (see [23, Theorem 8.4] for example) there is a Hilbert space L, a non-degenerate $*$-representation $\pi : A \to \mathcal{B}(L)$ and $T, S \in \mathcal{B}(H', L)$ with $\phi(a) = T^*\pi(a)S \in B \subseteq \mathcal{B}(H')$. Here we may, and will, choose $B \subseteq \mathcal{B}(H')$ to be the universal representation. Let $\tilde{\pi} : A^{**} \to \mathcal{B}(L)$ be the normal extension. Then a simple calculation shows that $\phi^{**} : A^{**} \to B^{**}$ has the form $\phi^{**}(x) = T^*\tilde{\pi}(x)S \in B^{**} \cong B'' \subseteq \mathcal{B}(H')$. Let $A \subseteq \mathcal{B}(H)$ be the universal representation, so again $A^{**} \cong A'' \subseteq \mathcal{B}(H)$. By the structure theory for normal $*$-representations, [29, Theorem 5.5, Chapter IV], there is an auxiliary Hilbert space L', a projection $p \in A'\overline{\otimes}\mathcal{B}(L') \subseteq \mathcal{B}(H \otimes L')$ and a unitary $U : p(H \otimes L') \to L$ such that

$$\tilde{\pi}(x) = Up(x \otimes 1_{L'})U^* \qquad (x \in A^{**}).$$

Consequently, by enlarging the original L if necessary, we may actually assume that $L = H \otimes L'$, and that $\pi(a) = a \otimes 1$. Thus $T, S \in \mathcal{B}(H', H \otimes L')$ and

$$\phi(a) = T^*(a \otimes 1)S, \qquad \phi^{**}(x) = T^*(x \otimes 1)S \qquad (a \in A, x \in A^{**}).$$

We remark that, even in this setting, we can always choose S, T with $\|S\|\|T\| = \|\phi\|_{cb}$. If also $C \subseteq \mathcal{B}(K)$ is the universal representation, then $A^{**}\overline{\otimes}C^{**} \subseteq \mathcal{B}(H)\overline{\otimes}\mathcal{B}(K) = \mathcal{B}(H \otimes K)$ and similarly $B^{**}\overline{\otimes}C^{**} \subseteq \mathcal{B}(H' \otimes K)$. Then

$$(\phi^{**} \otimes \mathrm{id})(x) = (T \otimes 1_K)^* x_{13}(S \otimes 1_K) \qquad (x \in A^{**}\overline{\otimes}C^{**}).$$

Here we use the "leg numbering notation", so x_{13} is $x \in \mathcal{B}(H \otimes K)$ acting on the 1st and 3rd components of $H \otimes L' \otimes K$.

Proof of Theorem 4.2. Let $x \in A \overset{\mathrm{sc}}{\otimes} C \subseteq A^{**}\overline{\otimes}C^{**}$. We shall verify condition (2) of Theorem 3.1 for $(\phi^{**} \otimes \mathrm{id})(x)$. For $\theta \in \mathcal{B}_0(K)$, from the above discussion,

$$\begin{aligned}(\phi^{**} \otimes \mathrm{id})(x)(1 \otimes \theta) &= (T \otimes 1)^* x_{13}(S \otimes \theta) = (T \otimes 1)^*(x(1 \otimes \theta))_{13}(S \otimes 1)\\ &\in (T \otimes 1)^*\big(A \otimes 1 \otimes \mathcal{B}_0(K)\big)(S \otimes 1) \subseteq T^*(A \otimes 1)S \otimes \mathcal{B}_0(K).\end{aligned}$$

However, as $\phi(a) = T^*(a \otimes 1)S \in B$ for all $a \in A$, it follows that $(\phi^{**} \otimes \mathrm{id})(x)(1 \otimes \theta) \in B \otimes \mathcal{B}_0(K)$ as required. Similarly, $(1 \otimes \theta)(\phi^{**} \otimes \mathrm{id})(x) \in B \otimes \mathcal{B}_0(K)$.

Now let $\theta \in \mathcal{B}_0(H')$ and consider

$$(\phi^{**} \otimes \mathrm{id})(x)(\theta \otimes 1) = (T \otimes 1)^* x_{13}(S\theta \otimes 1).$$

Notice that $S\theta \in \mathcal{B}_0(H', H \otimes L')$. To simplify the proof, notice that we are always free to replace L' by $L' \otimes \ell^2(X)$ for any index set X, if we also replace T by $T \otimes 1_{\ell^2(X)}$ and similarly for S. That is, we are free to assume that there is some isometry $V : H' \to H \otimes L'$. Let $R = S\theta V^* \in \mathcal{B}_0(H \otimes L')$ so that $S\theta = RV$ and hence

$$(\phi^{**} \otimes \mathrm{id})(x)(\theta \otimes 1) = (T \otimes 1)^* x_{13}(RV \otimes 1) = (T \otimes 1)^* x_{13}(R \otimes 1)(V \otimes 1).$$

For $\theta_1 \in \mathcal{B}_0(H), \theta_2 \in \mathcal{B}_0(L')$ we have that

$$x_{13}(\theta_1 \otimes \theta_2 \otimes 1) = (x(\theta_1 \otimes 1))_{13}(1 \otimes \theta_2 \otimes 1) \in \mathcal{B}_0(H) \otimes \mathcal{B}_0(L') \otimes B,$$

as $x \in A \overset{\text{sc}}{\otimes} B$. As $\mathcal{B}_0(H \otimes L') \cong \mathcal{B}_0(H) \otimes \mathcal{B}_0(L')$ it follows that $x_{13}(R \otimes 1) \in \mathcal{B}_0(H) \otimes \mathcal{B}_0(L') \otimes B$ and so

$$(\phi^{**} \otimes \mathrm{id})(x)(\theta \otimes 1) \in (T \otimes 1)^* \big(\mathcal{B}_0(H) \otimes \mathcal{B}_0(L') \otimes B\big)(V \otimes 1) \subseteq \mathcal{B}_0(H') \otimes B,$$

as required. Similarly $(\theta \otimes 1)(\phi^{**} \otimes \mathrm{id})(x) \in \mathcal{B}_0(H') \otimes B$ and so $(\phi^{**} \otimes \mathrm{id})(x) \in B \overset{\text{sc}}{\otimes} C$ as claimed. $\square$

While we stated this result only in the "one-sided" case, it obviously holds for maps of the form $\phi_1 \overset{\text{sc}}{\otimes} \phi_2$.

5 Weakly almost periodic functionals

We now come to our principle application, that of studying "weakly almost periodic functionals" on locally compact quantum groups, or more generally Hopf von Neumann algebras. Let (M, Δ) be a Hopf von Neumann algebra, so M is a von Neumann algebra and $\Delta : \mathsf{M} \to \mathsf{M}\overline{\otimes}\mathsf{M}$ is a normal unital injective $*$-homomorphism, coassociative in the sense that $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$. Then the preadjoint of Δ, say $\Delta_* : \mathsf{M}_* \widehat{\otimes} \mathsf{M}_* \to \mathsf{M}_*$, turns M_* into a completely contractive Banach algebra. We shall write $\star$ for the product in M_* and for the module action of M_* on M (and denote the module action of M on M_* simply by juxtaposition).

We can thus import the normal Banach algebraic definition: $\mathrm{wap}(\mathsf{M}_*)$ consists of those $x \in \mathsf{M}$ such that the orbit map $\mathsf{M}_* \to \mathsf{M}; \omega \mapsto \omega \star x = (\mathrm{id} \otimes \omega)\Delta(x)$ is weakly compact. Equivalently, $x \in \mathrm{wap}(\mathsf{M}_*)$ if and only if $\Delta(x) \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{M})$, using the notation we introduced in Section 3.2.

Theorem 5.1. *Let* $\mathrm{wap}(\mathsf{M}, \Delta)$ *be the collection of those* $x \in \mathrm{wap}(\mathsf{M}_*)$ *such that* $x^*x, xx^* \in \mathrm{wap}(\mathsf{M}_*)$*. Then* $\mathrm{wap}(\mathsf{M}, \Delta)$ *is a unital C^*-subalgebra of* M*, and any $*$-subalgebra of* $\mathrm{wap}(\mathsf{M}_*)$ *is contained in* $\mathrm{wap}(\mathsf{M}, \Delta)$*.*

Proof. As Δ is a $*$-homomorphism, $x \in \mathrm{wap}(\mathsf{M}, \Delta)$ if and only if $\Delta(x) \in \mathrm{wap}(\mathsf{M}\overline{\otimes}\mathsf{M})$. The result now follows from Theorem 3.8. As $\Delta(1) = 1 \otimes 1$ clearly $\mathrm{wap}(\mathsf{M}, \Delta)$ is unital. $\square$

Notice that, by definition, $x \in \mathrm{wap}(\mathsf{M}, \Delta)$ if and only if $\Delta(x) \in \mathsf{M} \overset{\text{sc}}{\otimes} \mathsf{M} \subseteq \mathsf{M}\overline{\otimes}\mathsf{M}$.

Let us recall some of the theory of Arens products on Banach algebras, for example see [10, Section 3], [11, Section 2], and references therein. Let A be a Banach algebra, $X \subseteq A^*$ a closed A-submodule, and suppose that X is "introverted", meaning that if we define

$$\langle \Phi \cdot \mu, a \rangle = \langle \Phi, \mu \cdot a \rangle, \quad \langle \mu \cdot \Phi, a \rangle = \langle \Phi, a \cdot \mu \rangle \qquad (a \in A, \mu \in A^*, \Phi \in A^{**})$$

then $\Phi \cdot \mu, \mu \cdot \Phi \in X$ for all $\mu \in X, \Phi \in A^{**}$. In this case, we can define products (the first and second Arens products) on X^* by

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle, \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \qquad (\Phi, \Psi \in X^*, \mu \in X).$$

If $X \subseteq \mathrm{wap}(A)$ then automatically X is introverted, [20, Lemma 1.2]. In fact, for either $\Box$ or $\Diamond$, X^* becomes a "dual Banach algebra" (that is, the product is separately weak*-continuous) if and only if $X \subseteq \mathrm{wap}(A)$, see [11, Proposition 2.4].

When $A = \mathsf{M}_*$ we have, using the notation of Section 3.2, that

$$\langle \mu \Box \lambda, x \rangle = \langle \mu \otimes_\Box \lambda, \Delta(x) \rangle, \quad \langle \mu \Diamond \lambda, x \rangle = \langle \mu \otimes_\Diamond \lambda, \Delta(x) \rangle \qquad (x \in X \subseteq \mathsf{M}, \mu, \lambda \in X^*).$$

We remark that, by the Hahn-Banach Theorem, it does not matter if we work with X^* or M^*. Then Lemma 3.6 immediately shows that if $X \subseteq \mathrm{wap}(\mathsf{M}_*)$, then $\Box = \Diamond$ on X^*.

Theorem 5.2. *For any Hopf von Neumann algebra,* $\mathrm{wap}(\mathsf{M}, \Delta)$ *is an* M_**-submodule of* M*. As such,* $\mathrm{wap}(\mathsf{M}, \Delta)^*$ *becomes a dual Banach algebra for either Arens product (which agree).*

Proof. Let $x \in \operatorname{wap}(\mathsf{M}, \Delta)$, so by definition, $\Delta(x) \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M} \subseteq \mathsf{M}\overline{\otimes}\mathsf{M}$. To be careful, let $y \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$ be the image of $\Delta(x)$. Let $\omega \in \mathsf{M}_*$ and consider

$$\begin{aligned}\Delta\big(\omega \star x\big) &= \Delta\big((\mathrm{id} \otimes \omega)\Delta(x)\big) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(\Delta \otimes \mathrm{id})\Delta(x) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(\mathrm{id} \otimes \Delta)\Delta(x) = (\mathrm{id} \otimes \phi)\Delta(x),\end{aligned}$$

where $\phi : \mathsf{M} \to \mathsf{M}$ is the (normal) completely bounded map $z \mapsto (\mathrm{id} \otimes \omega)\Delta(z)$. Let $y' = (\mathrm{id} \overset{\mathrm{sc}}{\otimes} \phi)(y) \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$ thanks to Theorem 4.2. For $\omega_1, \omega_2 \in \mathsf{M}_*$ and with $\kappa = \kappa_{\mathsf{M}_*} : \mathsf{M}_* \to \mathsf{M}^*$, we have that

$$\begin{aligned}\langle y', \kappa(\omega_1) \otimes \kappa(\omega_2)\rangle &= \langle y, \kappa(\omega_1) \otimes \phi^*\kappa(\omega_2)\rangle = \langle y, \kappa(\omega_1) \otimes \kappa\phi_*(\omega_2)\rangle \\ &= \langle \Delta(x), \omega_1 \otimes \phi_*(\omega_2)\rangle.\end{aligned}$$

Here we used the embedding of $\mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$ into $\mathsf{M}\overline{\otimes}\mathsf{M}$, the definition of $\mathrm{id} \overset{\mathrm{sc}}{\otimes} \phi$ and that ϕ is normal with preadjoint $\phi_* : \mathsf{M}_* \to \mathsf{M}_*; \omega_2 \mapsto \omega_2 \star \omega$. Hence

$$\langle y', \kappa(\omega_1) \otimes \kappa(\omega_2)\rangle = \langle x, \omega_1 \star (\omega_2 \star \omega)\rangle = \langle \Delta(\omega \star x), \omega_1 \otimes \omega_2\rangle.$$

Thus the image of $y' \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$ in $\mathsf{M}\overline{\otimes}\mathsf{M}$ is simply $\Delta(\omega \star x)$ and so $\omega \star x \in \operatorname{wap}(\mathsf{M}, \Delta)$.

Analogously, to show that $x \star \omega \in \operatorname{wap}(\mathsf{M}, \Delta)$ we show that $\Delta(x \star \omega) \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$. As $\Delta(x \star \omega) = (\phi' \otimes \mathrm{id})\Delta(x)$ where $\phi'(z) = (\omega \otimes \mathrm{id})\Delta(z)$ this will follow in the same way. ☐

5.1 In the language of compactifications

When G is a locally compact group, $\operatorname{wap}(G) = \operatorname{wap}(L^1(G)) \subseteq L^\infty(G)$ is a commutative C^*-algebra with character space G^{wap} which becomes a compact semitopological semigroup. In fact, G^{wap} is "maximal" in the sense that if S is a compact semitopological semigroup and $\phi : G \to S$ a continuous (semi)group homomorphism, then there is a semigroup homomorphism $\phi_0 : G^{\mathrm{wap}} \to S$ factoring ϕ.

We can turn this into a statement about algebras and coproducts in the usual way (compare [8, 28]). However, in this setting, we would need a good notion of a "non-commutative" or "quantum" semitopological semigroup. The following is now an obvious, but tentative, definition.

Definition 5.3. *A compact quantum semitopological semigroup is a pair (A, Δ_A) where A is a unital C^*-algebra and $\Delta_A : A \to A \overset{\mathrm{sc}}{\otimes} A$ is a $*$-homomorphism, "coassociative" in the sense that the induced product on A^* is associative.*

As $A \overset{\mathrm{sc}}{\otimes} A \subseteq A^{**}\overline{\otimes}A^{**}$ by definition, the product on A^* is simply

$$\langle \mu \star \lambda, a\rangle = \langle \Delta_A(a), \mu \otimes \lambda\rangle \qquad (a \in A, \mu, \lambda \in A^*).$$

The "C^*-Eberlein algebras" explored in [7] fit into this framework, thanks to [7, Definition 3.6] and [7, Section 3.3].

Theorem 5.4. *Let (M, Δ) be a Hopf von Neumann algebra and let* $\operatorname{wap} = \operatorname{wap}(M, \Delta)$ *be as in Theorem 5.1. Viewing* $\operatorname{wap} \overset{\mathrm{sc}}{\otimes} \operatorname{wap}$ *as a subspace of $M \overset{\mathrm{sc}}{\otimes} M$, which in turn is a subspace of $M\overline{\otimes}M$, we have that Δ restricts to a map* $\Delta_{\mathrm{wap}} : \operatorname{wap} \to \operatorname{wap} \overset{\mathrm{sc}}{\otimes} \operatorname{wap}$.

Proof. Let $x \in \operatorname{wap}$ and let y be the image of $\Delta(x)$ in $\mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$. By Theorem 4.1 we need to show that $y \in SC(\operatorname{wap} \times \operatorname{wap})$, that is, that $(\mu \otimes \mathrm{id})y, (\mathrm{id} \otimes \mu)y \in \operatorname{wap}$ for all $\mu \in \mathsf{M}^*$.

Let $\mu \in \mathsf{M}^*$ and choose a bounded net (ω_α) in M_* converging weak* to μ. For $\omega \in \mathsf{M}_*$ we have that

$$\langle (\mu \otimes \mathrm{id})y, \omega\rangle = \langle \mu, (\mathrm{id} \otimes \omega)\Delta(x)\rangle = \lim_\alpha \langle x, \omega_\alpha \star \omega\rangle = \lim_\alpha \langle x \star \omega_\alpha, \omega\rangle.$$

As $x \in \operatorname{wap}(\mathsf{M}_*)$ the map $\mathsf{M}_* \to \mathsf{M}; \tau \mapsto x \star \tau$ is weakly compact, and so we may assume that $(x \star \omega_\alpha)$ converges weakly. By Theorem 5.2, this net is contained in wap which is a norm closed subspace, hence weakly closed. We conclude that $(x \star \omega_\alpha)$ converges to a member of wap and hence $(\mu \otimes \mathrm{id})y \in$ wap. Analogously, $(\mathrm{id} \otimes \mu)y \in$ wap, as required. □

Combining this result with Theorem 5.2 we see that $(\mathrm{wap}, \Delta_{\mathrm{wap}})$ is a compact quantum semitopological semigroup.

We can now show that $\operatorname{wap}(\mathsf{M}, \Delta)$ has the required universal property to be a "compactification". Given (A, Δ_A) a compact quantum semitopological semigroup, let $\theta : A \to \mathsf{M}$ be a $*$-homomorphism, and let $\tilde\theta : A^{**} \to \mathsf{M}$ be the normal extension. As Δ_A maps into $A \overset{\mathrm{sc}}{\otimes} A \subseteq A^{**}\overline{\otimes}A^{**}$, the map $(\tilde\theta \otimes \tilde\theta)\Delta_A : A \to \mathsf{M}\overline{\otimes}\mathsf{M}$ makes sense. If $(\tilde\theta \otimes \tilde\theta)\Delta_A = \Delta\theta$ then we shall say that θ is a *morphism*. This is equivalent to the restriction of θ^* to M_* being a Banach algebra homomorphism $\mathsf{M}_* \to A^*$.

Theorem 5.5. *Let (M, Δ) be a Hopf von Neumann algebra. Let (A, Δ_A) be a compact quantum semitopological semigroup and let $\theta : A \to M$ be a morphism. Then $\theta(A) \subseteq \operatorname{wap}(M, \Delta)$, and $\operatorname{wap}(M, \Delta)$ is the union of the images of all such θ. Furthermore, there is a $*$-homomorphism, intertwining the coproducts, $\theta_0 : A \to \mathrm{wap}$ which factors θ.*

Proof. Notice that we simply define θ_0 to be the corestriction of θ, assuming that θ does map into wap, and that as $(\mathrm{wap}, \Delta_{\mathrm{wap}})$ is itself a compact quantum semitopological semigroup, the inclusion map shows that wap is the union of images of suitable θ.

So it remains to show that for $a \in A$, we do have that $\theta(a) \in$ wap, that is, that $\Delta(\theta(a)) \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$. However, $\Delta(\theta(a)) = (\tilde\theta \otimes \tilde\theta)\Delta_A(a)$. By Theorem 4.2 we have that $x = (\theta \overset{\mathrm{sc}}{\otimes} \theta)\Delta_A(a) \in \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$. Let y be the image of x in $\mathsf{M}\overline{\otimes}\mathsf{M}$. Let $\kappa = \kappa_{\mathsf{M}_*} : \mathsf{M}_* \to \mathsf{M}^*$ and recall that actually $\tilde\theta = (\theta^*\kappa)^* = \kappa^*\theta^{**}$. Then, by definition of the various maps,

$$\langle y, \omega_1 \otimes \omega_2 \rangle = \langle x, \kappa(\omega_1) \otimes \kappa(\omega_2) \rangle = \langle (\theta^{**} \otimes \theta^{**})\Delta_A(a), \kappa(\omega_1) \otimes \kappa(\omega_2) \rangle = \langle (\tilde\theta \otimes \tilde\theta)\Delta_A(a), \omega_1 \otimes \omega_2 \rangle.$$

Thus, as required, $\Delta(\theta(a))$ is in the image of $\mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M}$ in $\mathsf{M}\overline{\otimes}\mathsf{M}$. □

6 Continuous analogues

So far we have worked with Hopf von Neumann algebras, non-commutative generalisations of measure spaces. By analogy, there should be "continuous" version of the theory, namely one which works with C^*-bialgebras. Recall that a C^*-bialgebra is a pair (A, Δ_A) where A is a C^*-algebra and $\Delta_A : A \to M(A \otimes A)$ is a non-degenerate $*$-homomorphism which is coassociative.

In this section, we wish to treat abstract C^*-bialgebras, but also those which arise from locally compact quantum groups, where we have more structure, and in particular good interaction with the Hopf von Neumann theory. We hence proceed with a little generality.

Fix a C^*-bialgebra (A, Δ_A). Let (M, Δ) be a Hopf von Neumann algebra and suppose we have an injective $*$-homomorphism $\theta : A \to \mathsf{M}$ which is non-degenerate in the sense that if (e_α) is a bounded approximate identity for A then $(\theta(e_\alpha))$ converges weak* to 1 in M. Then θ extends to $\tilde\theta : M(A) \to \mathsf{M}$ which is also injective, identifying $M(A)$ with $\{x \in \mathsf{M} : x\theta(a), \theta(a)x \in \theta(A)\ (a \in A)\}$. We also denote by $\tilde\theta$ the normal extension $A^{**} \to \mathsf{M}$. These maps are compatible in the sense that if we view $M(A)$ as being $\{x \in A^{**} : xA, Ax \subseteq A\}$ then $\tilde\theta$ restricted to $M(A)$ agrees with the extension of θ from A to $M(A)$.

We shall then make the further assumption that $(\tilde\theta \otimes \tilde\theta)\Delta_A(a) = \Delta_{\mathsf{M}}(\theta(a))$ for all $a \in A$; this implies the same for all $a \in M(A)$. Again, we can either interpret this formula as meaning

$$(\theta \otimes \theta)\big(\Delta_A(a)(b \otimes c)\big) = \Delta_{\mathsf{M}}(\theta(a))(\theta(b) \otimes \theta(c)) \qquad (a, b, c \in A),$$

or in terms of extensions to biduals, that is, including $M(A \otimes A)$ into $A^{**}\overline{\otimes}A^{**}$.

- If (A, Δ_A) is an abstract C^*-bialgebra, then, for example, we may take $\mathsf{M} = A^{**}$, and then form $\Delta_{\mathsf{M}} : A^{**} \to A^{**}\overline{\otimes}A^{**}$ by first considering $\Delta_A : A \to M(A \otimes A) \subseteq A^{**}\overline{\otimes}A^{**}$ and then forming the normal extension. Then $\theta = \kappa_A : A \to A^{**}$ is the canonical map.
- If $A = C_0(\mathbb{G})$ arises from a locally compact quantum group, then the most natural choice is to take $\mathsf{M} = L^\infty(\mathbb{G})$ with its usual coproduct. Then θ is the inclusion.

Notice that if $\mathsf{M}, \Delta_{\mathsf{M}}, \theta$ is any choice, then we always have a quotient map $\phi : A^{**} \to \mathsf{M}$ such that $\phi\kappa_A = \theta$. Indeed, if $\mathsf{M} \subseteq \mathcal{B}(H)$ then ϕ will map onto $\theta(A)'' = \mathsf{M}$ by our assumption on θ, compare [29, Section 2, Chapter III]. Then ϕ will intertwine the coproducts $\Delta_{A^{**}}$ and Δ_{M}. We shall verify, as we go along, that whether we work in A^{**} or in M is unimportant.

As motivation for the following, consider a locally compact group G and set $A = C_0(G), \mathsf{M} = L^\infty(G)$ with θ the inclusion. We wish to know when $f \in C_b(G) = M(C_0(G))$ is in $\mathrm{wap}(G)$. One abstract approach would be to try to embed $M(A\otimes A)$ into $M(A)^{**}\overline{\otimes}M(A)^{**}$ (so as to ask when we land in $M(A) \overset{\mathrm{sc}}{\otimes} M(A)$). However, the comment after Lemma 6.2 shows that this cannot work. Instead, we map the problem into M, and work with $\mathrm{wap}(\mathsf{M}, \Delta)$. Our task then is to show that this is independent of the choice of M (which it is!)

Lemma 6.1. *The image of $M(A)$ in M is an M_*-submodule.*

Proof. For $x \in M(A)$ and $\omega \in \mathsf{M}_*$ we will show that $\tilde\theta(x) \star \omega \in M(A) \subseteq \mathsf{M}$. As θ is non-degenerate, by Cohen-Factorisation (see [5, Section 11] or [21, Proposition A2], for example) there exists $b \in A, \omega' \in \mathsf{M}_*$ with $\omega = \theta(b)\omega'$. Then, for $c \in A, \omega'' \in \mathsf{M}_*$,

$$\begin{aligned}
\langle (\omega \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde\theta(x))\theta(c), \omega''\rangle &= \langle \tilde\theta(x), \theta(b)\omega' \star \theta(c)\omega''\rangle = \langle b\theta^*(\omega') \star c\theta^*(\omega''), x\rangle \\
&= \langle \theta^*(\omega') \otimes \theta^*(\omega''), \tilde\Delta_A(x)(b \otimes c)\rangle \\
&= \langle \theta(d), \omega''\rangle,
\end{aligned}$$

where $d = (\theta^*(\omega') \otimes \mathrm{id})(\tilde\Delta_A(x)(b \otimes c)) \in A$ as $\tilde\Delta_A(x)(b \otimes c) \in A \otimes A$. Similar remarks apply to slicing on the other side. $\square$

Lemma 6.2. *Let $\mu \in M(A)^*$ and let $\mu_0 \in \mathsf{M}^*$ be a Hahn-Banach extension (that is, $\mu_0 \circ \tilde\theta = \mu$). For $x \in M(A)$, both $(\mathrm{id} \otimes \mu_0)\Delta_{\mathsf{M}}(\tilde\theta(x))$ and $(\mu_0 \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde\theta(x))$ depend only on μ.*

Proof. Considering $(\mathrm{id} \otimes \mu_0)\Delta_{\mathsf{M}}(\tilde\theta(x))$, our claim will follow if $(\omega \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde\theta(x))$ is a member of $\tilde\theta(M(A))$ for each $\omega \in \mathsf{M}_*$. However, this follows from the previous lemma. $\square$

Unfortunately, there is no good reason why $(\mu \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde\theta(x))$ should be a member of $\tilde\theta(M(A))$. We instead look to work more directly with M.

Proposition 6.3. *Let $x \in M(A)$. Then $\tilde\theta(x) \in \mathrm{wap}(\mathsf{M}_*)$ if and only if $x \in \mathrm{wap}(A^*)$ where A^* is considered as the predual of the Hopf von Neumann algebra $(A^{**}, \Delta_{A^{**}})$.*

Proof. Let $T : A^* \to A^{**}$ be the map $T(\mu) = (\mu \otimes \mathrm{id})\Delta_{A^{**}}(x)$. By Lemma 6.1 applied with $\mathsf{M} = A^{**}$, we see that T maps into $M(A) \subseteq A^{**}$ and so $x \in \mathrm{wap}(A^*)$ if and only if T is weakly compact, if and only if the corestriction $T : A^* \to M(A)$ is weakly compact. Similarly let $S : \mathsf{M}_* \to \mathsf{M}$ be $S(\omega) = (\omega \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde\theta(x))$ so that $\tilde\theta(x) \in \mathrm{wap}(\mathsf{M}_*)$ if and only if S is weakly compact.

With $\phi : A^{**} \to \mathsf{M}$ as above, we have the commutative diagram

$$\begin{tikzcd}
A^* \arrow[r, "T"] & M(A) \arrow[r, hook] & A^{**} \arrow[dl, "\phi"] \\
\mathsf{M}_* \arrow[u, "\theta^*|_{\mathsf{M}_*}"] \arrow[r, "S"] & \mathsf{M} &
\end{tikzcd}$$

Thus, if T is weakly compact, then so is S.

Suppose now that S is weakly compact. Let $\mu \in A^*$ with $\|\mu\| < 1$, so again we can find $a \in A, \mu' \in A^*$ with $\mu = a\mu'$ and $\|a\|\|\mu'\| < 1$. Choose a net (ω'_α) in the unit ball of M_* with $\theta^*(\omega'_\alpha) \to \mu'$ weak* in A^*. For each α set $\omega_\alpha = \theta(a)\omega'_\alpha$. For $\omega = \theta(c)\omega' \in \mathsf{M}_*$, we have that

$$\begin{aligned}\langle \phi T(\mu), \omega\rangle &= \langle \theta\big((\mu' \otimes \mathrm{id})(\Delta_{A^{**}}(x)(a \otimes c))\big), \omega'\rangle \\ &= \lim_\alpha \langle (\theta \otimes \theta)(\tilde{\Delta}_A(x)(a \otimes c)), \omega'_\alpha \otimes \omega'\rangle \\ &= \lim_\alpha \langle \Delta_{\mathsf{M}}(\tilde{\theta}(x))(\theta(a) \otimes \theta(c)), \omega'_\alpha \otimes \omega'\rangle \\ &= \lim_\alpha \langle S(\omega_\alpha), \omega\rangle.\end{aligned}$$

As S is weakly compact, $X = \{S(\tau) : \tau \in \mathsf{M}_*, \|\tau\| \leq 1\}$ is relatively weakly compact in M. The above shows that $\phi T(\mu)$ is in the weak* closure of X, but as X is relatively weakly compact and convex, this agrees with the norm closure of X, which is a weakly compact set. We conclude that ϕT maps the unit ball of A^* into a relatively weakly compact subset of M, that is, ϕT is weakly compact. As T actually maps into $M(A)$ and ϕ restricted to $M(A)$ is an isometry, it follows that T is weakly compact, as required. $\square$

Definition 6.4. *Let* $\mathrm{wap}(A, \Delta_A) = \{x \in M(A) : \tilde{\theta}(x) \in \mathrm{wap}(\mathsf{M}, \Delta_{\mathsf{M}})\}$*, a C^*-subalgebra of $M(A)$. By the proposition, this space depends only on (A, Δ_A).*

The following is the analogue of Theorem 5.4. It can again be shown that the construction is independent of the choice of M.

Theorem 6.5. *Let* $\mathrm{wap} = \mathrm{wap}(A, \Delta_A)$*. For* $x \in \mathrm{wap}$*, we have that* $\Delta_{\mathsf{M}}(\tilde{\theta}(x))$ *is in* $\mathrm{wap} \overset{\mathrm{sc}}{\otimes} \mathrm{wap} \subseteq M(A) \overset{\mathrm{sc}}{\otimes} M(A) \subseteq \mathsf{M} \overset{\mathrm{sc}}{\otimes} \mathsf{M} \subseteq \mathsf{M}\overline{\otimes}\mathsf{M}$*. As such,* Δ_{M} *restricts to a map* $\Delta_{\mathrm{wap}} : \mathrm{wap} \to \mathrm{wap} \overset{\mathrm{sc}}{\otimes} \mathrm{wap}$*.*

Proof. Exactly as in the proof of Theorem 5.4, this will follow if we can show that $(\mu \otimes \mathrm{id})\Delta_{\mathsf{M}}(\tilde{\theta}(x)) \in \mathrm{wap}$ for $\mu \in \mathsf{M}^*$ (and analoguously for $\mathrm{id} \otimes \mu$). By weak compactness, it suffices to show this for $\mu \in \mathsf{M}_*$, that is, that $\mathrm{wap} \subseteq \mathsf{M}$ is an M_*-submodule. However, $\mathrm{wap} = M(A) \cap \mathrm{wap}(\mathsf{M}, \Delta_{\mathsf{M}})$ and we know that $\mathrm{wap}(\mathsf{M}, \Delta_{\mathsf{M}})$ is an M_*-submodule, so the result follows from Lemma 6.1. $\square$

We could now continue to prove an analogue of Theorem 5.5 in this setting. We leave the details to the reader.

6.1 For locally compact quantum groups

For G a locally compact group, the classical theory tells us that $\mathrm{wap}(L^\infty(G)) = \mathrm{wap}(C_b(G))$, see for example [30]. We now make some remarks in this direction in the setting of locally compact quantum groups.

We recall the notion of a locally compact quantum group $\mathbb{G}$ being *coamenable*, [3]. In our setting, the most useful equivalent definition is that $\mathbb{G}$ is coamenable if and only if $L^1(\mathbb{G})$ has a bounded approximate identity.

Theorem 6.6. *Let $\mathbb{G}$ be coamenable. Then* $\mathrm{wap}(L^\infty(\mathbb{G}), \Delta)$ *is contained in* $M(C_0(\mathbb{G}))$ *and so agrees with* $\mathrm{wap}(C_0(\mathbb{G}), \Delta)$*.*

Proof. If $x \in \mathrm{wap}(L^\infty(\mathbb{G}), \Delta)$ then $x \in \mathrm{wap}(L^1(\mathbb{G}))$, and a bounded approximate identity argument shows that x is contained in the norm closure of $\{\omega \star x : \omega \in L^1(\mathbb{G})\}$ (as in the classical case, compare [30]). Then use that $\omega \star x \in M(C_0(\mathbb{G}))$ for any $x \in L^\infty(\mathbb{G}), \omega \in L^1(\mathbb{G})$, see [25, Theorem 2.4]; or see directly [25, Remark 4.5]. $\square$

We remark that similarly [25, Theorem 4.4] immediately implies that $C_0(\mathbb{G}) \subseteq \mathrm{wap}(C_0(\mathbb{G}), \Delta) \subseteq \mathrm{wap}(L^\infty(\mathbb{G}), \Delta)$, for any $\mathbb{G}$.

7 Questions for further study

We wrote $\overset{\text{sc}}{\otimes}$ by analogy with the theory of tensor products (compare, for example, the extended Haagerup tensor product, [12]). It is easy to see that $A \overset{\text{sc}}{\otimes} B$ is isometrically isomorphic to $B \overset{\text{sc}}{\otimes} A$. Is $\overset{\text{sc}}{\otimes}$ "associative"? Firstly, $(A \overset{\text{sc}}{\otimes} B) \overset{\text{sc}}{\otimes} C \subseteq (A \overset{\text{sc}}{\otimes} B)^{**}\overline{\otimes}C^{**}$ and $A \overset{\text{sc}}{\otimes} (B \overset{\text{sc}}{\otimes} C) \subseteq A^{**}\overline{\otimes}(B \overset{\text{sc}}{\otimes} C)^{**}$, and we cannot directly compare these, but we can embed both spaces into $A^{**}\overline{\otimes}B^{**}\overline{\otimes}C^{**}$ using the ideas of Section 3.2. However, we have been unable to decide if the two embedded spaces agree.

Condition (3) of Theorem 3.1 is stated in terms of "rows" and "columns" of operator matrices. Such notions are prominent in the theory of operator spaces. Is there perhaps a way these ideas could be made to work profitably for general operator spaces, not just C^*-algebras?

The theory as applied to locally compact quantum groups gives maybe three main questions:

- Is Theorem 6.6 true without the coamenable hypothesis?
- Does $\mathrm{wap}(L^\infty(\mathbb{G}), \Delta)$ always (or sometimes!) have an invariant mean?
- When does $\mathrm{wap}(L^\infty(\mathbb{G}), \Delta) = \mathrm{wap}(L^1(\mathbb{G}))$? For an amenable discrete group G, this is true for $\widehat{G}$, that is, $\mathrm{wap}(A(G)) = \mathrm{wap}(VN(G), \Delta)$. This follows from [15, Proposition 2], showing that $\mathrm{wap}(A(G)) = UCB(\widehat{G})$, and [16, Proposition 2], which shows that $UCB(\widehat{G})$ is a C^*-algebra. As far as we are aware, our question is open for $\widehat{\mathbb{F}_2}$, for example.

Finally, we studied compactifications of C^*-Eberlein algebras in [7]: these are generated by coefficients of certain special unitary corepresentations of $\mathbb{G}$. Is it true that the coefficients of any unitary corepresentation of $\mathbb{G}$ live in $\mathrm{wap}(C_0(\mathbb{G}), \Delta)$?

References

[1] R. Arens, "The adjoint of a bilinear operation", *Proc. Amer. Math. Soc.* 2 (1951) 839–848.

[2] S. Baaj, G. Skandalis, "Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres", *Ann. Sci. École Norm. Sup. (4)* 26 (1993) 425–488.

[3] E. Bédos, L. Tuset, "Amenability and co-amenability for locally compact quantum groups", *Internat. J. Math.* 14 (2003) 865–884.

[4] J. Berglund, H. Junghenn, P. Milnes, "Analysis on semigroups. Function spaces, compactifications, representations", (John Wiley & Sons, Inc., New York, 1989).

[5] F. F. Bonsall, J. Duncan, "Complete normed algebras", (Springer-Verlag, New York-Heidelberg, 1973).

[6] J. B. Conway, A course in functional analysis. Second edition. (Springer-Verlag, New York, 1990)

[7] B. Das, M. Daws, "Quantum Eberlein compactifications and invariant means", to appear in *Indiana Univ. Math. J.*, see arXiv:1406.1109v1 [math.FA]

[8] M. Daws, "Remarks on the Quantum Bohr Compactification", *Illinois J. Math.* 57 (2013) 1131–1171.

[9] M. Daws, "Characterising weakly almost periodic functionals on the measure algebra", *Studia Math.* 204 (2011) 213–234.

[10] M. Daws, "Multipliers, self-induced and dual Banach algebras", *Dissertationes Math.* 470 (2010) 62 pp.

[11] M. Daws, "Dual Banach algebras: representations and injectivity", *Studia Math.* 178 (2007) 231–275.

[12] E. G. Effros, Z.-J. Ruan, "Operator space tensor products and Hopf convolution algebras", *J. Operator Theory* 50 (2003) 131–156.

[13] E. G. Effros, Z.-J. Ruan, Operator spaces. (Oxford University Press, New York, 2000).

[14] P. Eymard, "L'algèbre de Fourier d'un groupe localement compact", *Bull. Soc. Math. France* 92 (1964) 181–236.

[15] E. E. Granirer, "Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group", *Trans. Amer. Math. Soc.* 189 (1974) 371–382.

[16] E. E. Granirer, "Density theorems for some linear subspaces and some C^*-subalgebras of $\mathrm{VN}(G)$", in Symposia Mathematica, Vol. XXII (Convegno sull'Analisi Armonica e Spazi di Funzioni su Gruppi Localmente Compatti, INDAM, Rome, 1976) (1977) 61–70.

[17] Z. Hu, M. Neufang, Z.-J. Ruan, "Module maps over locally compact quantum groups", *Studia Math.* 211 (2012) 111–145.

[18] J. Kustermans, S. Vaes, "Locally compact quantum groups in the von Neumann algebraic setting", *Math. Scand.* 92 (2003) 68–92.

[19] J. Kustermans, S. Vaes, "Locally compact quantum groups", *Ann. Sci. École Norm. Sup. (4)* 33 (2000) 837–934.

[20] A. T.-M. Lau, R. J. Loy, "Weak amenability of Banach algebras on locally compact groups", *J. Funct. Anal.* 145 (1997) 175–204.

[21] T. Masuda, Y. Nakagami, S. L. Woronowicz, "A C^*-algebraic framework for quantum groups", *Internat. J. Math.* 14 (2003) 903–1001.

[22] M. Megrelishvili, "Fragmentability and representations of flows", *Topology Proc.* 2003 (27) 497–544.

[23] V. I. Paulsen, Completely bounded maps and operator algebras. (Cambridge University Press, Cambridge, 2002).

[24] G. Pisier, Introduction to operator space theory. (Cambridge University Press, Cambridge, 2003).

[25] V. Runde, "Uniform continuity over locally compact quantum groups", *J. Lond. Math. Soc. (2)* 80 (2009) 55–71.

[26] V. Runde, "Connes-amenability and normal, virtual diagonals for measure algebras. I", *J. London Math. Soc. (2)* 67 (2003) 643–656.

[27] P. Salmi, "Quantum semigroup compactifications and uniform continuity on locally compact quantum groups", *Illinois J. Math.* 54 (2010) 469–483.

[28] P. Sołtan, "Quantum Bohr compactification", *Illinois J. Math.* 49 (2005) 1245–1270.

[29] M. Takesaki, Theory of operator algebras. I. (Springer-Verlag, Berlin, 2002).

[30] A. Ülger, "Continuity of weakly almost periodic functionals on $L^1(G)$", *Quart. J. Math. Oxford Ser. (2)* 148 (1986) 495–497.

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