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THE WAVELET TRANSFORM SPECTRUM AND RANDOM WAVELET SERIES WITH APPLICATIONS TO FRACTIONAL BROWNIAN MOTION

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The Wavelet Transform Spectrum and Random Wavelet Series with Applications to Fractional Brownian Motion

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***Abstract*—The concepts of the wavelet transform spectrum (WTS) and random wavelet series (RWS) are introduced based on the continuous and discrete wavelet transforms. The relationship between the WTS and the ordinary power spectral density (PSD) function is derived. It follows that the WTS and the PSD of a wide-sense stationary process can be related by the Fourier transform of the associated wavelet. The spectrum of the well-known fractional Brownian motion (fBm) is derived by means of the global wavelet transform spectrum (GWTS). The second-order statistical characteristics of the RWS of fBm is analysed by adopting the Haar wavelet.**

***Index Terms*—Wavelet transform. wavelet transform spectrum, random wavelet series, fractional Brownian motion**

I. Introduction

The wavelet transform provides a powerful tool for analyzing and synthesizing signals. Many applications of the wavelet transform can be found in random signal processing, including the statistical self-similarity of signals [1] and fractional Brownian motion [2]-[4].

Let f be a function defined in $L^2(R)$ .The continuous wavelet transform (CWT) with respect to the *mother wavelet* ψ is defined as [5].

$$W_f^{\psi}(b,a) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} f(t)\overline{\psi\left(\frac{t-b}{a}\right)}dt \qquad (1)$$

with the dilation (scale) parameter $a \in R^+$ and the shift (translation) parameter $b \in R$. The over-bar above the function $\psi(\cdot)$ indicates the complex conjugate.

In order to guarantee (1) is invertible so that f can be reconstructed from W_f^{ψ} , the following admissible condition is required

$$C_{\psi} = \int_0^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty \qquad (2)$$

where $\hat{\psi}$ is the Fourier transform of the function ψ .

The wavelet transform of a stochastic process $X(t)$, designated as $W_x^{\psi}(t,a)$, can be viewed as a random field on the upper half plane [6]. For a given scale parameter a , $W_x^{\psi}(t,a)$ can be thought of as the component of the original process at this given scale. Extensive research has been done to exploit the wavelet transform, to analyze, and to determine the characteristics of random processes [1]-[4][6][7].

In this paper, the concepts of the wavelet transform spectrum (WTS) and the global wavelet transform spectrum (GWTS) are introduced by means of the covariance function of the wavelet transform of a stochastic process. Then the relationship between the WTS and the ordinary power spectral density (PSD) function of a wide-sense stationary process is derived, and related by the Fourier transform of the associated wavelet. This is described in Section II. In Section III another concept, the random wavelet series (RWS), is introduced based on the discrete wavelet transform. Applications of the WTS and the RWS to fractional Brownian motion (fBm) are presented in Section IV, where the frequency behaviour of fBm, which obeys the so-called power-law, is derived by means of the global WTS. The second-order characteristics of the RWS of fBm is also analysed by employing the Haar basis.

II. Wavelet Transform Spectrum

Throughout this report let $X = \{X(t), t \in R\}$ be a second-order process defined on a probability space (Ω, F, P) , that is, X is jointly measurable and $X(t)$ is square integrable.

***Definition** 2.1 Let $\{X(t), t \in R\}$ be a zero-mean random process. The covariance of the wavelet transform of X at scale a_1 and a_2 is defined as*

$$R_x^\psi(t_1,t_2;a_1,a_2) = E[W_x^\psi(t_1,a_1)\overline{W_x^\psi(t_2,a_2)}] \quad (3)$$

that is,

$$R_x^\psi(t_1,t_2;a_1,a_2)$$
$$= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(\tau_1)\overline{X(\tau_2)}]\psi\left(\frac{\tau_1-t_1}{a_1}\right)\overline{\psi\left(\frac{\tau_2-t_2}{a_2}\right)}d\tau_1 d\tau_2 \quad (4)$$

where R_x is the covariance of the process X.

***Proposition** 2.1. Let $\{X(t), t\in R\}$ be a zero-mean, statistical self-similar random process with similarity parameter H. Then the wavelet transform covariance $R_x^\psi(t_1,t_2;a_1,a_2)$ is also self-similar with similarity parameter H+1/2, that is,* for $\forall\lambda>0$.

$$R_x^\psi(t_1,t_2;a_1,a_2) = \lambda^{-(2H+1)}R_x^\psi(\lambda t_1,\lambda t_2;\lambda a_1,\lambda a_2) \quad (5)$$

***Proposition** 2.2. Let $\{X(t), t\in R\}$ be a zero-mean, stationary random process, whose power spectral density function $S_x(\omega)$ exists, then*

$$R_x^\psi(t_1,t_2;a_1,a_2)$$
$$= \frac{\sqrt{a_1a_2}}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\hat{\psi}(a_1\omega)\overline{\hat{\psi}(a_2\omega)}e^{-i(t_1-t_2)\omega}d\omega \quad (6)$$

for any choice of wavelet ψ such that this integral is convergent.

Proof. Let $\psi_{(a,b)}(t) = a^{-1/2}\psi((t-b)/a)$. By (4)

$$R_x^\psi(t_1,t_2;a_1,a_2)$$
$$= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau_1,\tau_2)\psi\left(\frac{\tau_1-t_1}{a_1}\right)\overline{\psi\left(\frac{\tau_2-t_2}{a_2}\right)}d\tau_1 d\tau_2$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(\tau_2-\tau_1)\psi_{(a_1,t_1)}(\tau_1)d\tau_1\right]\overline{\psi_{(a_2,t_2)}(\tau_2)}d\tau_2$$
$$= \int_{-\infty}^{\infty}\left[(R_x * \psi_{(a_1,t_1)})(\tau_2)\right]\overline{\psi_{(a_2,t_2)}(\tau_2)}d\tau_2$$

Where the symbol "$*$" indicates the convolution of two functions. Applying Parseval's identity to the above, yields

$$R_x^\psi(t_1,t_2;a_1,a_2)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\hat{\psi}_{(a_1,t_1)}(\omega)\overline{\hat{\psi}_{(a_2,t_2)}(\omega)}d\omega$$
$$= \frac{\sqrt{a_1a_2}}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\hat{\psi}(a_1\omega)\overline{\hat{\psi}(a_2\omega)}e^{-i(t_1-t_2)\omega}d\omega$$

This shows that in the case of a zero-mean stationary random process, the wavelet transform covariance $R_x^\psi(t_1,t_2;a_1,a_2)$ is a function of t_1 and t_2 only through the difference (t_2-t_1). This enables an expression for the power spectral density to be obtained by taking the Fourier transform.

***Definition** 2.2 Let $\{X(t), t\in R\}$ be a zero-mean random stationary process. The wavelet transform spectrum(WTS) of X at scale a_1 and a_2 is defined as*

$$\underline{S}_x^\psi(\omega;a_1,a_2) = \int_{-\infty}^{\infty} R_x^\psi(\tau;a_1,a_2)e^{-i\omega\tau}d\tau \quad (7)$$

where $R_x^\psi(\tau;a_1,a_2) = R_x^\psi(t,t+\tau;a_1,a_2)$

***Proposition** 2.3. Let $\{X(t), t\in R\}$ be a zero-mean, stationary random process where the power spectral density function $S_x(\omega)$ exists, then*

$$\underline{S}_x^\psi(\omega;a_1,a_2) = \sqrt{a_1a_2}\,\hat{\psi}(a_1\omega)\overline{\hat{\psi}(a_2\omega)}S_x(\omega) \quad (8)$$

Proof. By (4),

$$R_x^\psi(t_1,t_2;a_1,a_2)$$
$$= \frac{1}{\sqrt{a_1a_2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau_2-\tau_1)\psi\left(\frac{\tau_1-t_1}{a_1}\right)\overline{\psi\left(\frac{\tau_2-t_2}{a_2}\right)}d\tau_1 d\tau_2$$
$$= \sqrt{a_1a_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x((t_2-t_1)+(a_2\beta-a_1\alpha))\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta$$
$$\underline{S}_x^\psi(\omega;a_1,a_2) = \int_{-\infty}^{\infty} R_x^\psi(\tau;a_1,a_2)e^{-i\omega\tau}d\tau$$
$$= \int_{-\infty}^{\infty}\left[\sqrt{a_1a_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau-a_1\alpha+a_2\beta)\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta\right]e^{-i\omega\tau}d\tau$$
$$= \sqrt{a_1a_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(\tau-a_1\alpha+a_2\beta)e^{-i\omega\tau}d\tau\right]\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta$$
$$= \sqrt{a_1a_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_x(\omega)\psi(\alpha)e^{-i(a_1\omega)\alpha}\overline{\psi(\beta)e^{-i(a_2\omega)\beta}}d\alpha d\beta$$
$$= \sqrt{a_1a_2}\,\hat{\psi}(a_1\omega)\overline{\hat{\psi}(a_2\omega)}S_x(\omega)$$

A special case of (8) is $a_1 = a_2 = a$, in such a case

$$\underline{S}_x^\psi(\omega;a) = a\left|\hat{\psi}(a\omega)\right|^2 S_x(\omega) \quad (9)$$

Consider the following stochastic process

$$\ddot{X}(t) + \alpha\dot{X}(t) + \beta X(t) = \varepsilon(t) \quad (10)$$

where $\alpha = \beta = 1$, $\varepsilon(t)$ is a random process whose integral is a Wiener process. Let the Mexican Hat wavelet (which is also referred to as the Marr or Bubble wavelet) be used, such that $\psi(t) = (1-t^2)e^{-t^2/2}$ and $\hat{\psi}(\omega) = \sqrt{2\pi}\omega^2 e^{-\omega^2/2}$. The power spectral density (PSD) function, $S_x(\omega)$, of the process (10) and the Fourier transform $\hat{\psi}(\omega)$ are shown in Fig. 1. The wavelet transform spectrum (WTS) of the process X is depicted in Fig.2, where the scale a is sampled with the period of 0.5 and the frequency ω is sampled with the period of 0.1 rad/s. Fig.2 shows that the WST is nonnegative and symmetrical about the the frequency ω.

Fig. 1.

Fig. 1. The power spectral density (PSD) function of the process X described by Eq. (10) and the Fourier transform of the Mexican Hat wavelet.

Fig. 2.

Fig. 2. The wavelet transform spectrum (WTS) of the process X described by Eq. (10).

***Definition** 2.3 Let $\{X(t), t \in R\}$ be a zero-mean random stationary process. Assume the spectrum $\underline{S}_x^{\psi}(\omega; a_1, a_2)$ of the wavelet transform of X exists. The global spectrum of the wavelet transform of X at scale a is defined as*

$$S_x^G(\omega) = \frac{1}{C_\psi}\int_0^\infty \underline{S}_x^{\psi}(\psi; a)\frac{da}{a^2} \tag{11}$$

***Proposition** 2.4. Let $\{X(t), t \in R\}$ be a zero-mean random stationary process. Assume that both the power spectrum of X and the wavelet transform spectrum of X exist. Then the global wavelet transform spectrum of X at scale a is equivalent to the power spectrum of X, that is*

$$S_x^G(\omega) = \frac{1}{C_\psi}\int_0^\infty \underline{S}_x^{\psi}(\psi; a)\frac{da}{a^2}$$

$$= \frac{1}{C_\psi}\int_0^\infty \frac{|\hat{\psi}(a\omega)|^2}{a} da = S_x(\omega) \tag{12}$$

III. Random Wavelet Series

A. Discrete Wavelet Transform

Due to heavy redundancy, the continuous wavelet transform (1) is usually sampled in the time-scale plane (b, a) with the dyadic grid $\{(2^m n, 2^m)\}_{m,n \in Z}$ to form an orthonormal wavelet representation. Based on this sampling scheme a signal $f(t)$ can be analyzed and synthesized by the following discrete wavelet transform (DWT) and the wavelet series (WS)

$$f(t) = \sum_m \sum_n d_n^m \psi_{m,n}(t)$$

$$= \sum_m \sum_n d_n^m 2^{-m/2}\psi(2^{-m}t - n) \tag{13}$$

$$d_n^m = \langle f, \psi_{m,n} \rangle = \int_{-\infty}^{\infty} f(t)\overline{\psi_{m,n}(t)}dt \tag{14}$$

The bases $\{\psi_{m,n}(t)\}_{m,n \in Z}$, formed by dilating and translating the basic mother wavelet function, are ideally suited for analysing self-similar signals.

Suppose that the signal $f(t)$ belongs to the subspace V_M spanned by the scaling function ϕ, which is associated with the wavelet ψ, that is

$$V_M = \overline{\underset{n}{span}}\{2^{-M/2}\phi(2^{-M}t - n)\} \tag{15}$$

Then $f(t)$ can be expressed as

$$f(t) = \sum_n a_M[n]2^{-M/2}\phi(2^{-M}t - n) \tag{16}$$

where

$$a_M[n] = \langle f, \phi_{M,n} \rangle = 2^{-M/2}\int_{-\infty}^{\infty} f(t)\overline{\phi(2^{-M}t - n)}dt$$

The multiresolution wavelet representation can be implemented by the following pyramidal algorithm

$$a_n^0 = a_M[n] \tag{17a}$$

$$a_n^m = \sum_k g[k - 2n]a_k^{m-1} \tag{17b}$$

$$d_n^m = \sum_k h[k - 2n]a_k^{m-1} \tag{17c}$$

where $g[n]$ and $h[n]$ are quadrature mirror filter(QMF) pairs, having roughly low-pass and high-pass discrete-time Fourier transforms, respectively.

To initialize the above pyramidal algorithm (17), it is usual to let $a_n^0 = x[n]$ ($M = 0$) for a discrete-time sequence. For an analogue signal $f(t)$, some algorithms have been proposed for arbitrary M [8]-[10].

B. Random Wavelet Series

If the signal $f(t)$ in (13) is a zero mean random process, the wavelet series (14) is itself a zero-mean random sequence with respect to the two indices m and n. This makes it possible to introduce the following concept of a random wavelet series (RWS).

***Definition** 3.2.1 Let $\{X(t), t \in R\}$ be a zero-mean random process which can be expressed as (14). The random wavelet series of X is defined as*

$$\{d_{m,n}^{\psi,x}\}_{m,n\in Z} = \left\{2^{-m/2}\int_{-\infty}^{\infty} X(t)\overline{\psi(2^{-m}t-n)}dt\right\}_{m,n\in Z} \quad (18)$$

provided the path integral in (18) is defined with probability one.

***Definition** 3.2.2 Let $\{X(t), t\in R\}$ be a zero-mean random process. The covariance of a random wavelet series of X with respect to scale indices m_1, m_2 and translation indices n_1, n_2 is defined as*

$$R_x^{\psi}[n_1,n_2;m_1,m_2] = E[d_{m_1,n_1}^{\psi,x} d_{m_2,n_2}^{\psi,x}] \quad (19)$$

Random wavelet series can be viewed as a special class of a random field produced by random wavelet transform, on which several results have been proposed [1][6][11]-[13].

***Proposition** 3.2.1 Let $\{X(t), t\in R\}$ be a zero-mean statistical self-similar random process with parameter H. Then for a fixed scale index m, the covariance of the random wavelet series $\{d_{m,n}^{\psi,x}\}$ can be expressed as*

$$R_x^{\psi}[n_1,n_2;m] = E[d_{m,n_2}^{\psi,x} d_{m,n_2}^{\psi,x}]$$

$$= 2^{m(2H+1)} E[d_{0,n_2}^{\psi,x} d_{0,n_2}^{\psi,x}] = 2^{m(2H+1)} R_x^{\psi}[n_1,n_2;0] \quad (20)$$

***Proposition** 3.2.2. Let $\{X(t), t\in R\}$ be a zero-mean, stationary random process and the power spectral density function $S_x(\omega)$ exists. Then the covariance of the random wavelet series of X is*

$$R_x^{\psi}[n_1,n_2;m_1,m_2] = \frac{2^{(m_1+m_2)/2}}{2\pi}$$

$$\times\int_{-\infty}^{\infty} S_x(\omega)\hat{\psi}(2^{m_1}\omega)\overline{\hat{\psi}(2^{m_2}\omega)}\exp(-i(2^{m_1}n_1 - 2^{m_2}n_2))d\omega \quad (21)$$

for any choice of wavelet ψ such that this integral is convergent.

Proposition 3.2.2 is a direct result from proposition 2.2 by setting $a_1 = 2^{m_1}, a_2 = 2^{m_2}$, $t_1 = 2^{m_1}n_1$ and $t_2 = 2^{m_2}n_2$ in (6).

IV. APPLICATIONS TO FRACTIONAL BROWNIAN MOTION

Fractional Brownian motion (fBm) is a natural generalization of ordinary Brownian motion [14]. Although fBm is itself highly nonstationay, the first order increment of this process is stationary. Denote $B_H = \{B_H(t), 0 < H < 1, t\in R\}$ for fBm, then some characteristics of B_H can be briefly summarized as follows:

1) The increment of fBm is a zero-mean, strict-sense stationary random process, and the variance of the increment is

$$Var[B_H(t_2) - B_H(t_1)] = V_H|t_2 - t_1|^{2H}, \ \forall t_1,t_2 \in R \quad (22)$$

where

$$V_H = \text{var}[B_H(1)] = \Gamma(1-2H)\frac{\cos(\pi H)}{\pi H}$$

2) The increment of fBm is a stationary self-similar process with parameter H, in the sense that for any $a > 0$ and $t_0, \tau \in R$

$$\{B_H(t_0+\tau) - B_H(t_0)\} \overset{d}{=} \{a^{-H}[B_H(t_0+a\tau) - B_H(t_0)]\} \quad (23)$$

3) The covariance of fBm is

$$R_{B_H}(t,s) = E[B_H(t)B_H(s)]$$

$$= \frac{V_H}{2}\left[|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right] \quad (24)$$

***Proposition** 4.1. The wavelet transform spectrum of fBm at scale a is*

$$S_{B_H}^{\psi}(\omega) = \frac{a|\hat{\psi}(a\omega)|^2}{|\omega|^{2H+1}} \quad (25)$$

Proof. By (3), the covariance of the wavelet transform of fBm at scale a is

$$R_{B_H}^{\psi}(t,s;a) = E[W_{B_H}^{\psi}(t,a)\overline{W_{B_H}^{\psi}(s,a)}]$$

$$= \frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{B_H}(\tau_1,\tau_2)\psi(\frac{\tau_1-t}{a})\overline{\psi(\frac{\tau_2-s}{a})}d\tau_1 d\tau_2$$

$$= \frac{V_H}{2}\frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[|\tau_1|^{2H} + |\tau_2|^{2H} - |\tau_1-\tau_2|^{2H}\right]$$

$$\times\psi(\frac{\tau_1-t}{a})\overline{\psi(\frac{\tau_2-s}{a})}d\tau_1 d\tau_2$$

$$= -\frac{V_H}{2}\frac{1}{a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\tau_1-\tau_2|^{2H}\psi(\frac{\tau_1-t}{a})\overline{\psi(\frac{\tau_2-s}{a})}d\tau_1 d\tau_2$$

$$= -\frac{V_H}{2}a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|a(\alpha-\beta)+(t-s)|^{2H}\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta$$

It can be seen that although fBm is itself nonstationary, the covariance of the wavelet transform of fBm behaves as a stationary process. Therefore, the definition 2.2 can be extended to fBm and the spectrum of the wavelet of it can be introduced as follows

$$\begin{aligned}
S^{\psi}_{B_H}(\omega;a) &= \int_{-\infty}^{\infty} R^{\psi}_{B_H}(\tau;a)e^{-i\omega\tau}d\tau \\
&= -\frac{V_H}{2}a\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\tau + a(\alpha-\beta)\right|^{2H}\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta\right]e^{-i\omega\tau}d\tau \\
&= -\frac{V_H}{2}a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left|\tau + a(\alpha-\beta)\right|^{2H}e^{-i\omega\tau}d\tau\right]\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta \\
&= \frac{V_H}{2}\frac{\Gamma(1+2H)\sin(\pi H)}{|\omega|^{2H+1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ia\omega(\alpha-\beta)}\psi(\alpha)\overline{\psi(\beta)}d\alpha d\beta \\
&= \frac{a|\hat{\psi}(\omega)|^2}{|\omega|^{2H+1}}
\end{aligned}$$

***Proposition** 4.2. The global wavelet transform spectrum of fBm is*

$$S^{G}_{B_H}(\omega) = \frac{1}{|\omega|^{2H+1}} \tag{26}$$

Recall Proposition 2.4, (26) means that the spectrum of fBm obeys the power law of fractional order, the desired behavior of fBm. This result agrees with that obtained by applying the Wigner-Ville time-frequency method [2] and other spectral representation approaches [See, e.g., 15].

***Proposition** 4.3. Assume that the Haar wavelet is used as the basis function in the orthonormal wavelet transformation. Then the covariance of the random wavelet series of fBm with respect to scale indices m, m' and translation indices n, n' can be expressed as*

$$\begin{aligned}
R^{\psi}_{B_H}[n,n';m,m'] &= E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m',n'}] \\
&= \frac{V_H}{2}\frac{2^{-(m+m')/2}}{(2H+1)(2H+2)}\xi(n,n';m,m')
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
&\xi(n,n';m,m') \\
&= \left|2^m n - 2^{m'}(n'+1)\right|^{2H+2} - 2\left|2^m n - 2^{m'}(n'+1/2)\right|^{2H+2} \\
&+ \left|2^m n - 2^{m'} n'\right|^{2H+2} - 2\left|2^m(n+1/2) - 2^{m'}(n'+1)\right|^{2H+2} \\
&+ 4\left|2^m(n+1/2) - 2^{m'}(n'+1/2)\right|^{2H+2} \\
&- 2\left|2^m(n+1/2) - 2^{m'}n'\right|^{2H+2} + \left|2^m(n+1) - 2^{m'}(n'+1)\right|^{2H+2} \\
&- 2\left|2^m(n+1) - 2^{m'}(n'+1/2)\right|^{2H+2} + \left|2^m(n+1) - 2^{m'}n'\right|^{2H+2}
\end{aligned} \tag{28}$$

***Proof**.* See Appendix.

Some remarks on Proposition 4.3 are summarized below:

1) For a given scale index m

$$\begin{aligned}
R^{\psi}_{B_H}[k;m] &= E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m,n+k}] \\
&= \frac{V_H}{2}\frac{2^{m(2H+1)}}{(2H+1)(2H+2)}\eta[k]
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\eta[k] &= |k-1|^{2H+2} - 4|k-1/2|^{2H+2} + 6|k|^{2H+2} \\
&= -4|k+1/2|^{2H+2} + |k+1|^{2H+2}
\end{aligned} \tag{30}$$

This can be easily obtained by setting $m' = m, n' = n+k$ in (27) and (29). Fig. 3 shows the graph of the covariance $R^{\psi}_{B_H}[k;m]$ with respect to k for $m=1$ and $H=0.2$, 0.4, 0.6 and 0.8.

2) The variance of the of the random wavelet series of fBm is

$$\begin{aligned}
Var[d^{\psi,B_H}_{m,n}] &= E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m,n}] \\
&= \frac{V_H}{2}\frac{(1-2^{-2H})2^{m(2H+1)}}{(H+1)(2H+1)}
\end{aligned} \tag{31}$$

The covariance $R^{\psi}_{B_H}[k;m]$ defined by (29) can now be normalized by means of (31), such that

$$r^{\psi}_{B_H}[k;m] = \frac{V^{\psi}_{B_H}[k;m]}{Var[d^{\psi,B_H}_{m,n}]} = \frac{\eta[k]}{\eta[0]} = \frac{\eta[k]}{2-2^{1-2H}} \tag{32}$$

In Fig. 4., the normalized covariance $r^{\psi}_{B_H}[k;m]$ is plotted with respect to k and H, where the Haar wavelet is adopted.

3) For a given scale index m, it is easy to verify from (30) that for a large k

$$\eta[k] \approx \frac{3}{2}|k|^{2H-2} \tag{33}$$

Therefore,

$$R^{\psi}_{B_H}[k;m] = E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m,n+k}] \sim O(|k|^{2(H-1)}) \tag{34}$$

For an arbitrary orthonormal wavelet (other than the Haar wavelet)with Kth vanishing moment, (34) still holds in the sense that

$$R^{\psi}_{B_H}[k;m] = E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m,n+k}] \sim O(|k|^{2(H-K)}) \tag{35}$$

Furthermore, for fixed scale indexes m, m', Tewfik and Kim [4] verified that $R^{\psi}_{B_H}[n,n';m,m']$ decays as $O(|2^m n - 2^{m'} n'|^{2(H-K)})$ for all n, n', that is

$$R^{\psi}_{B_H}[n,n';m,m'] = E[d^{\psi,B_H}_{m,n}\, d^{\psi,B_H}_{m',n'}]$$

$$\sim O(\left|2^{m} n - 2^{m'} n'\right|^{2(H-K)}) \quad (36)$$

4) For ordinary Brownian motion ($H = 1/2$),

$$R_{B_H}^{\psi}[n, n'; m, m'] = E[d_{m,n}^{\psi,B_H} d_{m',n'}^{\psi,B_H}] = 0 \quad (37)$$

under the assumption that $2^{m-m'} n \le n' \le 2^{m-m'}(n+1) - 1$ and $m \ge m'$. For a fixed scale index m,

$$R_{B_H}^{\psi}[k; m] = E[d_{m,n}^{\psi,B_H} d_{m,n+k}^{\psi,B_H}]$$

$$= Var[d_{m,n}^{\psi,B_H}]\delta_k = \frac{2^{2m} V_H}{12}\delta_k \quad (38)$$

Fig. 3.

Fig. 3. The covariance of the random wavelet series of fBm defined by (29). The Haar wavelet is used and the scale index m=1.

Fig.4.

Fig. 4. The normalized covariance of the random wavelet series of fBm defined by (32). The Haar wavelet system is adopted here.

V. CONCLUSIONS

The concepts of the wavelet transform spectrum (WTS) and the random wavelet series (RWS) have been introduced. One of the main results of this work says that, for a stationary stochastic process whose power spectrum density (PSD) exits, the global WTS is equal to the PSD. By extending the concept of WTS to fractional Brownian motion (fBm), the spectrum of fBm was then directly derived. The second-order statistical behaviour of the RWS of fBm was also analysed using the Haar wavelet basis. The result on the RWS of fBm in this work (Proposition 4.3) is parallel to that of Kaplan and Kuo proposed in [16], where the case of discrete fractional Gaussian noise (DFGN) was considered.

APPENDIX A

Proof of proposition 4.3

$$E[d_{m,n}^{\psi,B_H} d_{m',n'}^{\psi,B_H}] = 2^{-(m+m')/2}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[B_H(t)B_H(s)]\psi(2^{-m}t - n)\overline{\psi(2^{-m'}s - n')}dtds$$

$$= \frac{V_H}{2} 2^{-(m+m')/2}$$

$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right] \psi(2^{-m}t - n)\psi(2^{-m'}s - n')dtds$$

$$= -\frac{V_H}{2} 2^{-(m+m')/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |t-s|^{2H} \psi(2^{-m}t - n)\psi(2^{-m'}s - n')dtds$$

$$= -\frac{V_H}{2} 2^{(m+m')/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|2^{m}(t+n) - 2^{m'}(s+n')\right|^{2H} \psi(t)\psi(s)dtds$$

$$= -\frac{V_H}{2} 2^{(m+m')/2} 2^{2m'H} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|2^{m-m'}(t+n) - n' - s\right|^{2H} \psi(t)\psi(s)dtds$$

$$= \frac{V_H}{2} 2^{(m+m')/2} 2^{2m'H} \int_{-\infty}^{\infty} \psi(t)\left[-\int_{-\infty}^{\infty} \psi(s)\left|s - 2^{m-m'}(t+n) - n'\right|^{2H} ds\right]dt$$

$$= \frac{V_H}{2} 2^{(m+m')/2} 2^{2m'H} \int_{-\infty}^{\infty} \psi(t)\Theta_{\psi}(\lambda(t), H)dt \quad (A1)$$

where

$$\lambda(t) = 2^{m-m'}(t+n) - n'$$

$$\Theta_{\psi}(\lambda(t), H)$$

$$= -\int_{-\infty}^{\infty} \psi(\tau)|\tau - \lambda(t)|^{2H} d\tau = -\left[\int_{0}^{1/2} |\tau - \lambda(t)|^{2H} d\tau - \int_{1/2}^{1} |\tau - \lambda(t)|^{2H} d\tau\right] = \int_{1/2-\lambda(t)}^{1-\lambda(t)} |\tau|^{2H} d\tau - \int_{1/2-\lambda(t)}^{1-\lambda(t)} |\tau|^{2H} d\tau$$

$$= \begin{cases} \frac{1}{(2H+1)}\left[(1-\lambda(t))^{2H+1} - 2(1/2-\lambda(t))^{2H+1} + (-\lambda(t))^{2H+1}\right], & \lambda(t) < 0 \\ \frac{1}{(2H+1)}\left[(1-\lambda(t))^{2H+1} - 2(1/2-\lambda(t))^{2H+1} + (\lambda(t))^{2H+1}\right], & 0 \le \lambda(t) < 1/2 \\ \frac{1}{(2H+1)}\left[(1-\lambda(t))^{2H+1} + 2(\lambda(t)-1/2)^{2H+1} - (\lambda(t))^{2H+1}\right], & 1/2 \le \lambda(t) < 1 \\ \frac{1}{(2H+1)}\left[-(\lambda(t)-1)^{2H+1} + 2(\lambda(t)-1/2)^{2H+1} - (\lambda(t))^{2H+1}\right], & \lambda(t) \ge 1 \end{cases} \quad (A2)$$

Therefore

$$\int_{-\infty}^{\infty} \psi(t)\Theta_{\psi}(\lambda(t), H)dt$$

$$= \int_{0}^{1/2} \Theta_{\psi}(\lambda(t), H)dt - \int_{1/2}^{1} \Theta_{\psi}(\lambda(t), H)dt$$

$$= 2^{m'-m}\left[\int_{\lambda(0)}^{\lambda(1/2)} \Theta_{\psi}(s, H)ds - \int_{\lambda(1/2)}^{\lambda(1)} \Theta_{\psi}(s, H)ds\right]$$

$$= \frac{2^{m'-m}}{(2H+1)(2H+2)}$$

$$\times\Big\{ |\lambda(0)-1|^{2H+2} - 2|\lambda(0)-1/2|^{2H+2} + |\lambda(0)|^{2H+2}$$

$$-2|\lambda(1/2)-1|^{2H+2} + 4|\lambda(1/2)-1/2|^{2H+2} - 2|\lambda(1/2)|^{2H+2}$$

$$|\lambda(1)-1|^{2H+2} - 2|\lambda(1)-1/2|^{2H+2} + |\lambda(1)|^{2H+2} \Big\} \quad \text{(A3)}$$

Inserting (A3) into (A1), (27) is produced.

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List of Figures

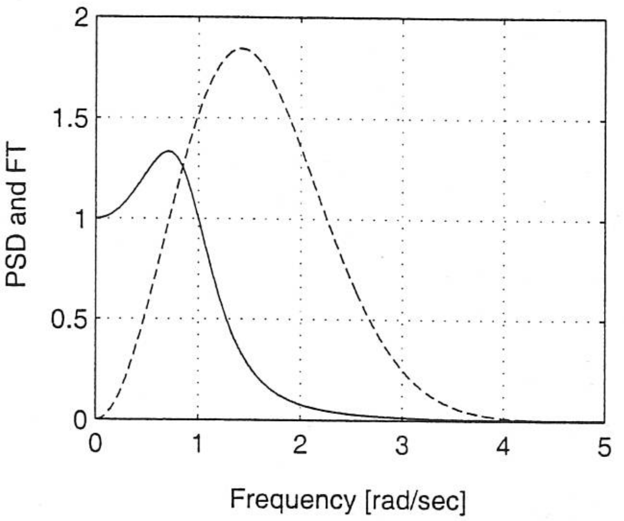

Fig 1

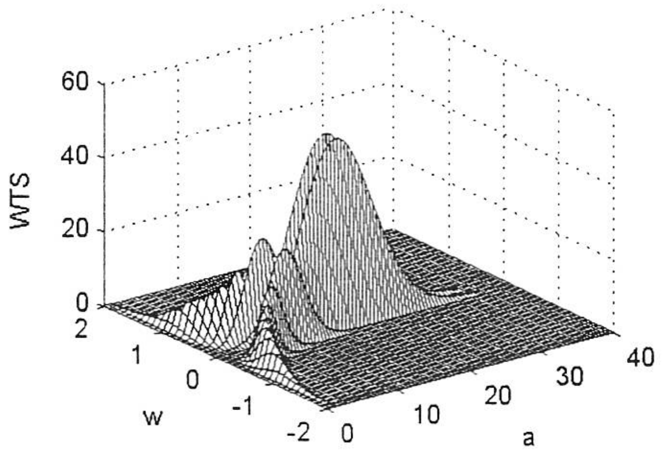

Fig. 2

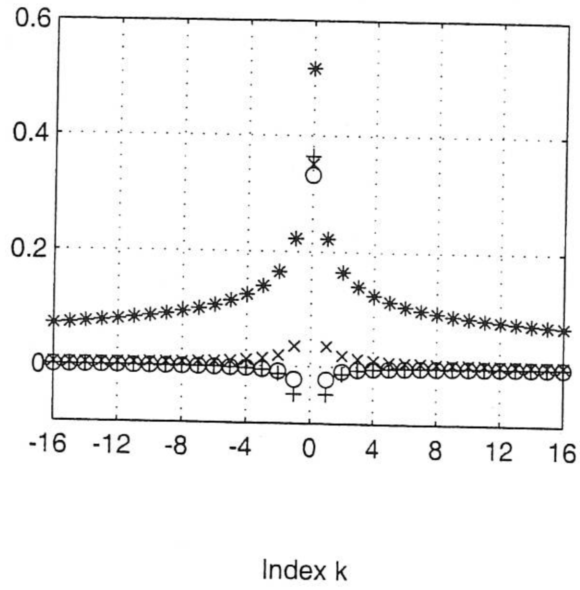

Fig 3

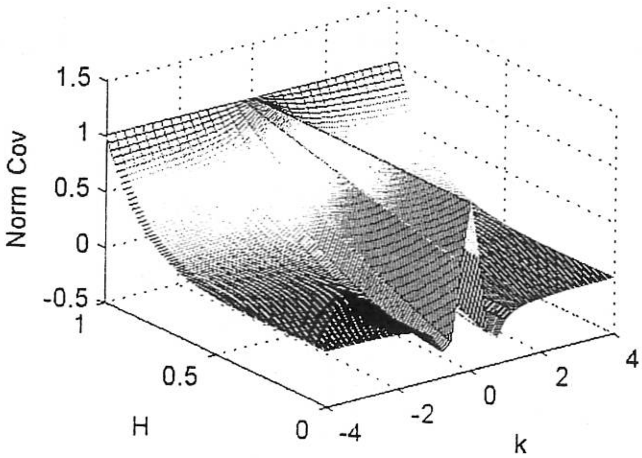

Fig. 4