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OPTIMAL SLIDING SURFACE DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract: A method is introduced for optimal sliding surface design for a class of nonlinear systems. First, a nonlinear system is approximated as a linear time varying system and then an optimal sliding surface is designed for approximated system. The control input, which is designed by using approximated system, is then applied to the nonlinear system. It is shown that the approximated system's response converges to the nonlinear system's response.

Keywords: Optimal Control, Sliding Mode Control, Nonlinear Systems, Time Varying Sliding Surfaces.

I. INTRODUCTION

An optimal sliding surface design which minimises a desired performance criterion for linear time invariant systems has been studied and reported by various authors [6,7,8,9]. The LQR problem for linear time varying systems has also been investigated in terms of optimal sliding surface design [7]. Although different design methods, each of which uses linear or nonlinear switching surfaces, have been suggested for nonlinear systems [2,4,9], the optimal selection of the switching surface for a nonlinear system has not been investigated yet.

In this work, we shall suggest a method for choosing an optimal sliding surface for a class of nonlinear systems that can be represented by

$$\dot{x} = A(x)x + B(x)u \tag{1.1}$$

where $x \in R^n$, $u \in R^m$, $A(x) \in R^{n\times n}$ and $B(x) \in R^{n\times m}$. We shall present successive approximation approach to approximate the nonlinear system (1.1) by a time varying system, i.e.,

$$\dot{x}^{[i]} = A^{[i-1]}(t)x^{[i]} + B^{[i-1]}(t)u^{[i]} \tag{1.2}$$

We will use the result of [7], for optimal sliding surface design in our approximated system. Then we apply the control input, which is generated from the approximated system, to the original nonlinear equations.

The approximation theory is given in section 2. Section 3 briefly

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explains Sliding Mode Control (SMC) theory for linear time varying systems. In section 4, the optimal sliding surface design method is presented and the theory is applied to an example.

II. APPROXIMATION OF NONLINEAR SYSTEM

We shall refer to the recent work of Banks and McCaffrey [3] to explain the approximation theory. It has been shown that differential equations of the form

$$\dot{x}(t) = A(x(t))x(t) + f(t), x(t_0) = x_0 \tag{2.1}$$

may be represented by the following approximating sequence of linear differential equations

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + f(t) \tag{2.2}$$

where

$\dot{x}^{[1]}(t) = A(x_0)x^{[1]}(t) + f(t),\ x^{[1]}(t_0) = x_0$

We will give the following theorem without proof (see [3]).

Theorem 2.1. *Let* $A(x)$ *satisfy*

(A.1) $\mu(A(x)) \le \mu$

(A.2) $\|A(x) - A(y)\| \le \alpha\|x - y\|$

$\forall x, y \in R^n$ *and suppose that*

$$\alpha\left(\|x\| + \int_{t_0}^{T} \exp(-\mu(s - t_0)\|f(s)\|ds\right)$$

$(T - t_0)\exp(\mu(T - t_0)) < 1.$

Then the equation (2.1) has a unique solution on $[t_0, T]$ *which is given by the limit of the solutions of the approximating equations (2.2) on* $C([t_0, T], R^n)$.

III.SMC OF LINEAR TIME VARYING SYSTEMS

In this section, we shall briefly explain the SMC design procedure for linear time varying systems. We have, in general,

$$\dot{x} = A(t)x + B(t)u \tag{3.1}$$

where $x \in R^n, u \in R^m$, $A(t)$ and $B(t)$ are time varying matrices of proper dimensions. We shall assume that $(A(t), B(t))$ is controllable in the time interval $[t_1, t_2]$. The control function is designed to force the system to the $(n-m)$ dimensional time varying switching hyperplane

$$\sigma(x,t) = C(t)x \tag{3.2}$$

and the system is kept on the hyperplane by directing the system states towards it, i.e.,

$$\begin{aligned} \dot{\sigma}(x,t) > 0 \text{ if } \sigma(x,t) < 0 \\ \dot{\sigma}(x,t) < 0 \text{ if } \sigma(x,t) > 0 \end{aligned} \tag{3.3}$$

On the other hand, desired behaviour of linear time varying system is achieved by deliberate choice of $C(t)$. In most cases, some coordinate transformations put the problem in an easier form in the selection of $C(t)$. Thus, we define a time varying nonsingular coordinate transformation $z(t) = L(t)x(t)$, where

$$L(t)B(t) = \begin{bmatrix} 0 \\ B_2(t) \end{bmatrix}$$

and

$$L(t)A(t)L^{-1}(t) + \dot{L}(t)L^{-1}(t) = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}$$

The system in the new coordinates becomes

$$\begin{aligned} \dot{z}_1 &= A_{11}(t)z_1 + A_{12}(t)z_2 \\ \dot{z}_2 &= A_{21}(t)z_1 + A_{22}(t)z_2 + B_2(t)u \end{aligned} \tag{3.4}$$

where $z_1 \in R^{n-m}$, $z_2 \in R^m$ and $B_2(t)$ is a $m \times m$ nonsingular matrix. The time varying transformation simply lumps the control vector into the $z_2(t)$.

The switching hyperplane equation (3.3) can be rewritten in the new coordinates as

$$\sigma(z,t) = C_1(t)z_1 + C_2(t)z_2$$

Without loss of generality, we will assume that $C_2 = I$ and hence

$$\sigma(z,t) = C_1(t)z_1 + z_2 \qquad (3.5)$$

The system motion in the sliding mode, i.e., motion restricted to the switching surface $\sigma(z,t) = 0$, can be explained by means of the equivalent control method. The existence of a sliding mode implies $\sigma(z,t) = \dot{\sigma}(z,t) = 0$ for all $t \geq t_s$, where t_s is the time at which the sliding mode begins. Then the equivalent control is

$$u_{eq} = -B_2^{-1}[\{C_1A_{11} + A_{21} + \dot{C}_1\}z_1 + \{C_1A_{12} + A_{22}\}z_2]$$

which yields the so-called equivalent system as

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2$$

$$\dot{z}_2 = -\{C_1A_{11}(t) + \dot{C}_1\}z_1 - C_1A_{12}(t)z_2$$
$$= -C_1\dot{z}_1 - \dot{C}_1z_1$$

Note that the second equation represents the constraint equations, i.e., $\sigma(z,t) = C_1(t)z_1 + z_2 = 0$ and hence we can describe the system as a $(n-m)th$ reduced order system i.e.,

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2$$
$$\sigma(z,t) = C_1(t)z_1 + z_2 = 0$$

or alternatively,

$$\dot{z}_1 = A_{11}(t)z_1 + A_{12}(t)z_2 \qquad (3.6)$$

$$z_2 = -C_1(t)z_1 \qquad (3.7)$$

where z_2 plays the role of state feedback control. In order to use the standart LQR problem results, we need the following lemma which is a generalisation of lemma 1 of [4].

Lemma 3.1. *If the system in (3.1) is controllable in the time interval* $[t_1, t_2]$, *then the pair* $(A_{11}(t), A_{12}(t))$ *of the reduced order equivalent system (3.6) is also controllable in the time interval* $[t_1, t_2]$.

Proof: This follows by a simple extension of the proof of lemma 1 of [4]. □

The general control structure which satisfies the stability condition of sliding motion -eqn.'s (3.3)- can be formulated as

$$u = u_{eq} + ksgn(\sigma(z,t)) \qquad (3.8)$$

where $sgn(\cdot)$ is the signum function and k is a column vector of dimension m whose elements are $k_i < 0$.

IV. OPTIMAL SLIDING SURFACE DESIGN

The dynamic optimization problem for time varying linear systems is well-defined, which is to minimise the functional

$$J = \int_0^{t_1} (x^TQ(t)x + u^TP(t)u)dt, t_1 \leq \infty \qquad (4.1)$$

subject to the system equations

$$\dot{x} = A(t)x + B(t)u \qquad (4.2)$$

Then, the optimal control is

$$u^o = -P^{-1}(t)B^T(t)R(t)x \qquad (4.3)$$

where $R(t)$ is the solution of matrix differential Riccati equation,

$$\dot{R}(t) + R(t)A(t) + A^T(t)R(t) - R(t)B(t)P^{-1}(t)B^T(t)R(t) + Q(t) = 0 \qquad (4.4)$$

with the boundary condition $R(t_1) = 0$.

Using the above derivation, we shall determine the equation of sliding surface over which the sliding motion is optimal with respect to the criterion

$$J = \int_{t_s}^{t_1} x^TQ(t)xdt, \quad Q(t) \geq 0 \qquad (4.5)$$

where t_s is the sliding motion starting time. We omit the control term in the functional since the sliding mode motion is control-independent and defined by the equation of discontinuity surfaces.

By applying time varying nonsingular transformation as described in section III, the criterion in eqn. (4.5) is written as

$$J = \int_{t_s}^{t_1} \{z_1^T Q_{11}(t) z_1 + 2 z_1^T Q_{12}(t) z_2 + z_2^T Q_{22}(t) z_2\} dt \tag{4.6}$$

where

$$L^{-1}(t)]^T Q(t) L^{-1}(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

By introducing a new variable ξ which relates z_1 and z_2;

$$\xi = z_2 + Q_{22}^{-1}(t) Q_{12}^T(t) z_1 \tag{4.7}$$

then, the reduced order system equation (3.6) and the criterion (4.6) become

$$\dot{z}_1 = \left(A_{11}(t) - A_{22}(t) Q_{22}^{-1}(t) Q_{12}^T(t)\right) z_1 + A_{12}(t)\xi \tag{4.8}$$

$$J = \int_{t_s}^{t_1} \{z_1^T \left(Q_{11}(t) - Q_{12}(t) Q_{22}^{-1}(t) Q_{12}^T(t)\right) z_1 + \xi^T Q_{22}(t) \xi\ \} dt$$

or,

$$J = \int_{t_s}^{t_1} \left(z_1^T \overline{Q}(t) z_1 + \xi^T \overline{P}(t) \xi\right) dt \tag{4.9}$$

where $\overline{Q}(t) = Q_{11}(t) - Q_{12}(t) Q_{22}^{-1}(t) Q_{12}^T(t)$ and $\overline{P}(t) = Q_{22}(t)$. Then, by virtue of eqn. (3), the optimum ξ is given by

$$\begin{aligned} \xi^o &= -\overline{P}^{-1}(t) A_{12}^T(t) R(t) z_1 \\ &= -Q_{22}^{-1}(t) A_{12}^T(t) R(t) z_1 \end{aligned} \tag{4.10}$$

where

$$\dot{R}(t) + R(t)\overline{A}(t) + \overline{A}^T(t) R(t) - R(t)\overline{B}(t)\overline{P}^{-1}(t)\overline{B}^T(t) R(t) + \overline{Q}(t) = 0$$

and

$\overline{A}(t) = A_{11}(t) - A_{22}(t) Q_{22}^{-1}(t) Q_{12}^T(t)$, $\overline{B}(t) = A_{12}(t)$. Using eqn.'s (4.7) and (4.10);

$$\begin{aligned} z_2 &= -C_1^o(t) z_1 \\ &= -Q_{22}^{-1}(t)\left(A_{12}^T(t) R(t) + Q_{12}^T(t)\right) z_1 \end{aligned} \tag{4.11}$$

or, the sliding hyperplane equation (eqn. 3.5) becomes

$$\sigma(z,t) = z_2 + C_1^o(t) z_1 \tag{4.12}$$

Now, we shall apply the above theory to an example.

Comment: Convergence of the controller sequence follows easily from Theorem 2.1 by chooping the time interval $[t_s, t_1]$ into small enough pieces.

Example: We have a second order nonlinear system

$$\dot{x} = A(x)x + b(x)u$$

where

$$A(x) = \begin{bmatrix} -0.5x_1 - x_2^2 & x_1 x_2 - 2x_2^3 + 2 \\ 2x_2 - 3x_1^4 & -2x_1^2 + 3x_2 \end{bmatrix}$$

and

$$b(x) = \begin{bmatrix} 0 \\ 1 + (x_1 x_2)^2 \end{bmatrix}$$

Each linear time-varying approximation of the nonlinear system is represented as

$$\dot{x}^{[i]} = A^{[i-1]}(t) x^{[i]} + b^{[i-1]}(t) u^{[i]}$$

for i=1,..,5

where $A^{[0]}(t) = A^{[0]} = A(x_0)$ and $b^{[0]}(t) = b^{[0]} = b(x_0)$. The initial position vector is $x_0 = [1, 0.5]^T$. For each approximation, we design sliding mode controller which minimise the following criterion

$$J = \int_{t_s}^{t_1} \{x_1^T q_{11} x_1 + x_2^T q_{22} x_2\} dt$$

where $q_{11} = 1$ and $q_{22} = 5$. The approximated control input is then applied to the original nonlinear system.

Figure 1 shows the response of $\dot{x}^{[1]} = A^{[0]}x^{[1]} + b^{[0]}u^{[1]}$. The designed control is then applied to the nonlinear system. Figure 2 shows this result.

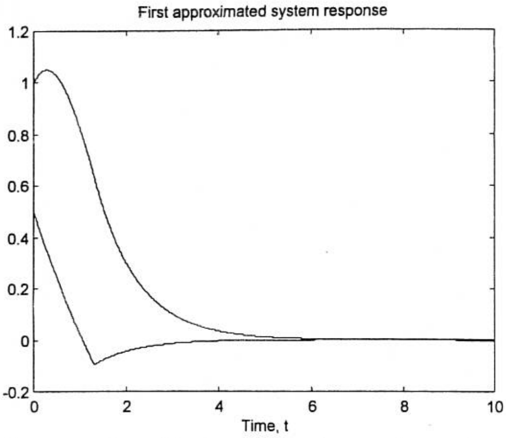

Figure 1- First approximated system $(x^{[1]})$response.

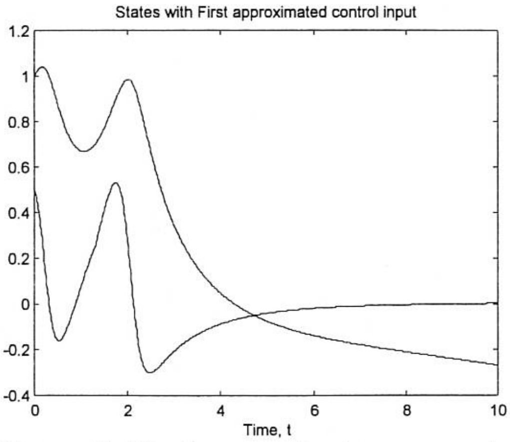

Figure 2- Nonlinear system's response to the first appoximated control input.

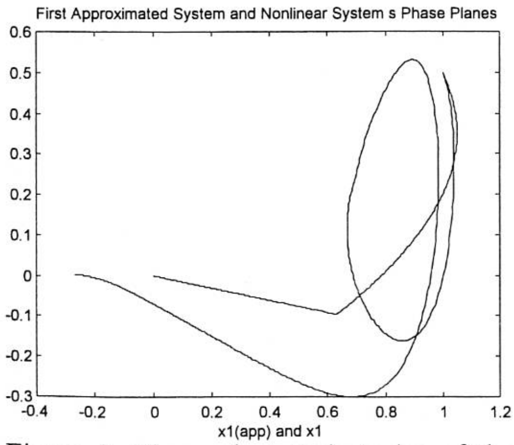

Figure 3- Phase plane trajectories of the first approximated system and nonlinear system.

Phase plane trajectories of the first approximated system and nonlinear system are given in figure 3. Figure 4 shows the optimal sliding surface slope for the first approximated system.

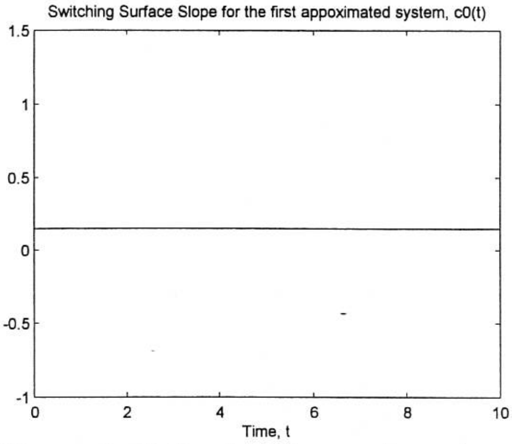

Figure 4- Optimal sliding surface slope for the first approximated system.

Inserting the previous approximation results into the nonlinear system, we have, in general,

$$\dot{x}^{[i]} = \begin{bmatrix} a_{11}^{[i-1]}(t) & a_{12}^{[i-1]}(t) \\ a_{21}^{[i-1]}(t) & a_{22}^{[i-1]}(t) \end{bmatrix} x^{[i]} + \begin{bmatrix} 0 \\ b_2^{[i-1]}(t) \end{bmatrix} u$$

for i=2,..,5

We have taken the sliding motion starting time of first approximation, t_s, as the general starting time of sliding motion. The final time, t_1, is 10 and the boundary condition for differential Riccati equation is $R(t_1) = R(10) = 0$. The k value in eqn. (3.8) is -0.5 for all approximations.

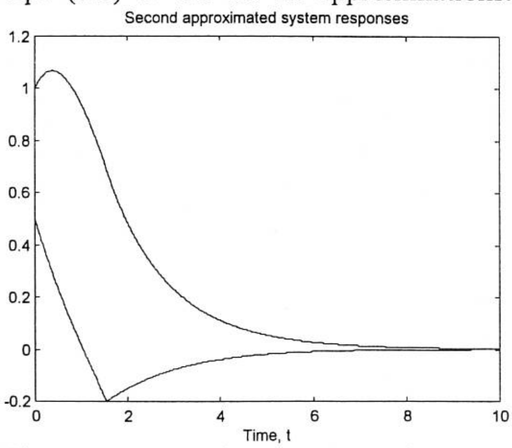

Figure 5- Second approximated system $(x^{[2]})$response.

Figures 5,6,7, and 8 show the results of second approximation. The results of third approximation are given in figures 9,10,11, and 12.

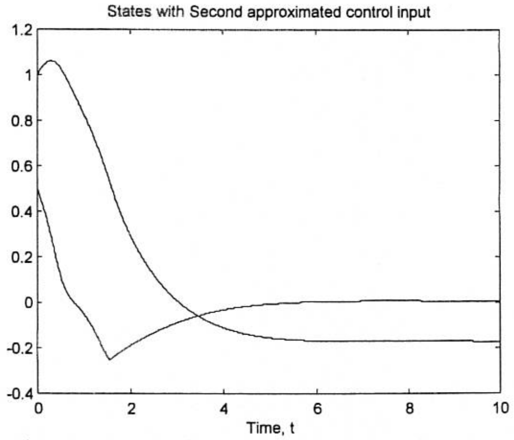

Figure 6- Nonlinear system's response to the second appoximated control input.

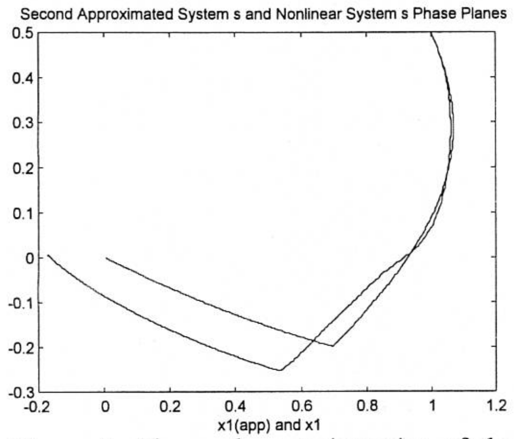

Figure 7- Phase plane trajectories of the second approximated system and nonlinear system.

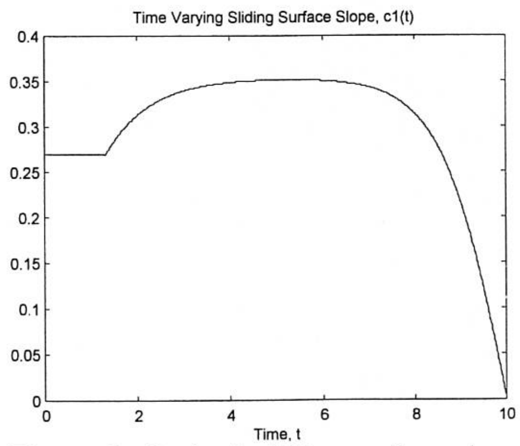

Figure 8- Optimal sliding surface slope for the second approximated system.

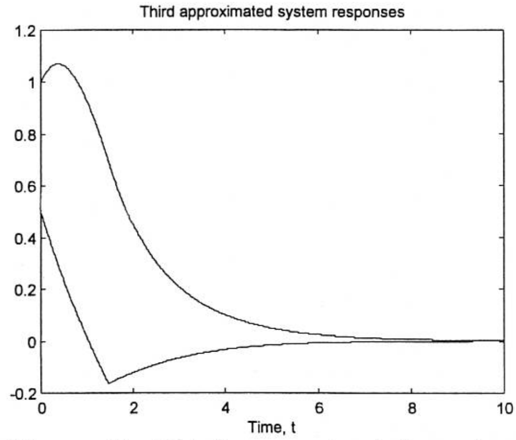

Figure 9- Third approximated system ($x^{[3]}$)response.

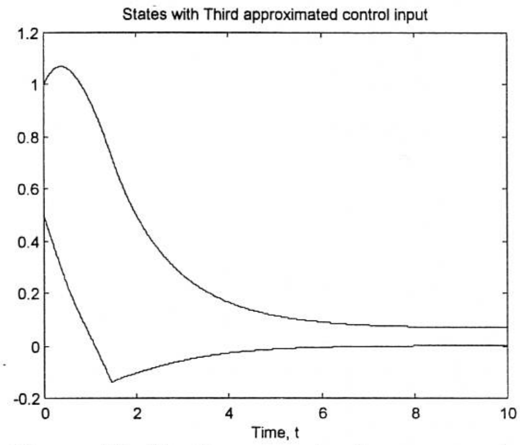

Figure 10- Nonlinear system's response to the third appoximated control input.

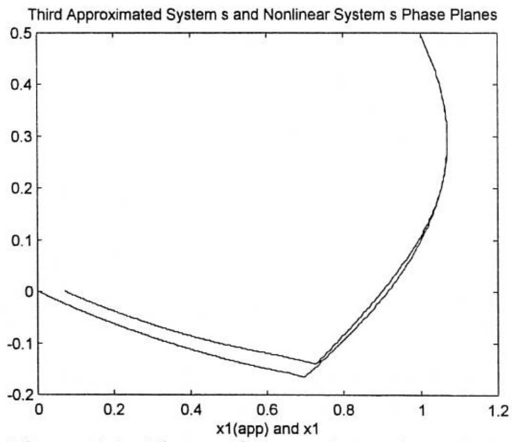

Figure 11- Phase plane trajectories of the third approximated system and nonlinear system.

Fourth and fifth approximations results are given in figures 13,14,15 and 16 and figures 17,18,19, and 20 respectively.

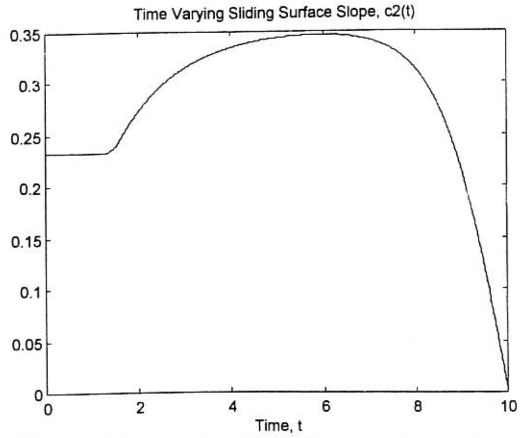

Figure 12- Optimal sliding surface slope for the third approximated system.

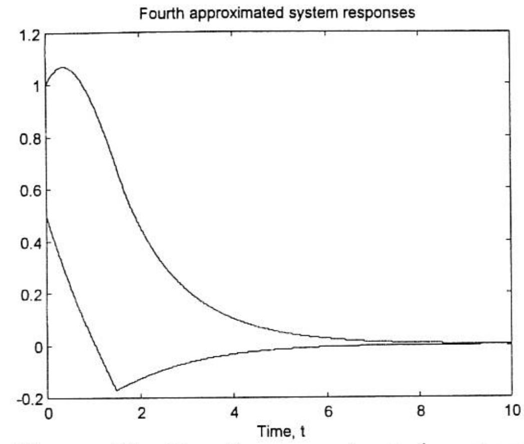

Figure 13- Fourth approximated system ($x^{[4]}$)response.

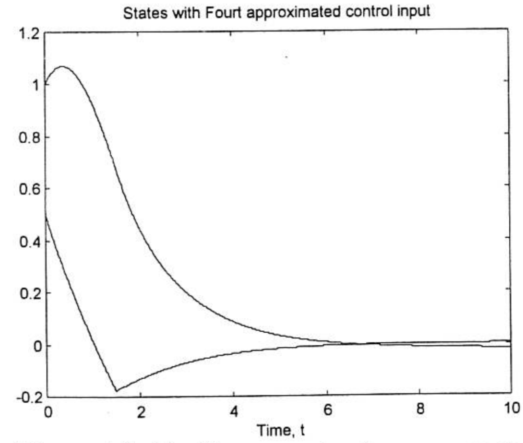

Figure 14- Nonlinear system's response to the fourth appoximated control input.

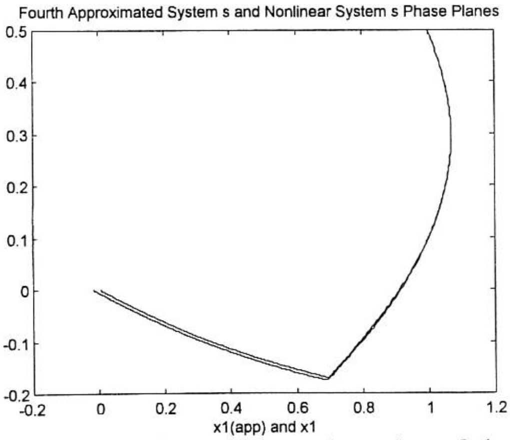

Figure 15- Phase plane trajectories of the fourth approximated system and nonlinear system.

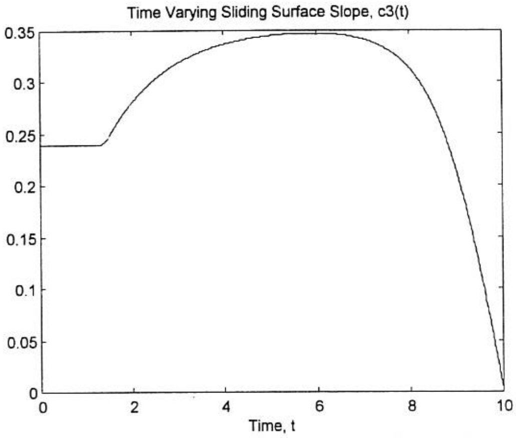

Figure 16- Optimal sliding surface slope for the fourth approximated system.

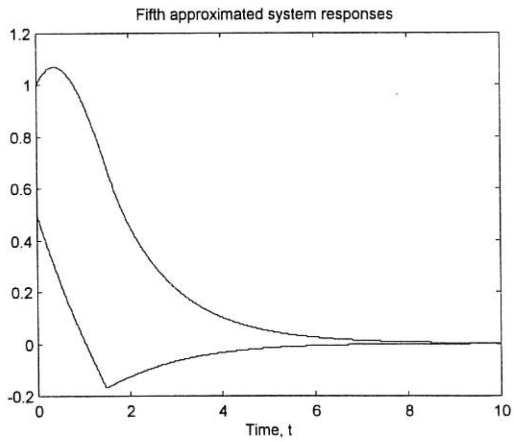

Figure 17- Fifth approximated system ($x^{[5]}$)response.

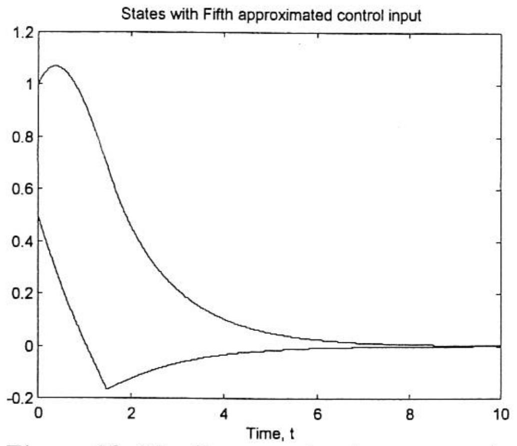

Figure 18- Nonlinear system's response to the fifth appoximated control input.

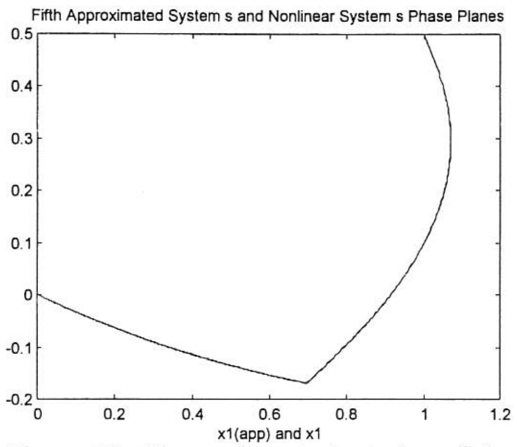

Figure 19- Phase plane trajectories of the fifth approximated system and nonlinear system.

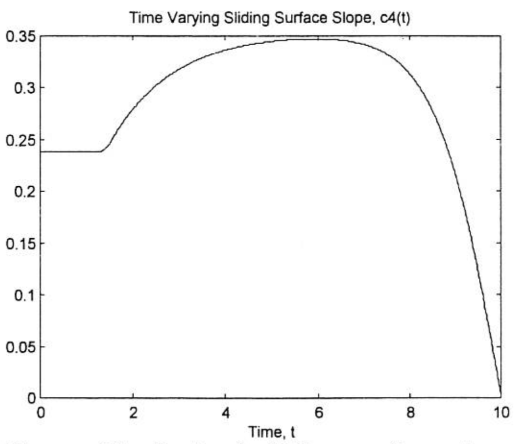

Figure 20- Optimal sliding surface slope for the fifth approximated system.

V. CONCLUSIONS

In this work, we have presented a new method for optimal sliding surface design of a class of nonlinear systems. Our approach is based on optimal sliding surface design of linear time varying systems. We first approximate our nonlinear system as time varying linear systems and then use the optimal sliding surface design method for linear time varying systems. The simulation results show the clear convergence and success of the method.

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