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Extension of Nonlocal Continuation and Boundedness Theory for Polynomial Systems

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ABSTRACT

In this paper we shall extend a number of results concerning the boundedness and nonlocal continuation of differential equations with sublinear bounds to ones with polynomial vector fields. We shall also show that the solution of many kinds of equations can be obtained as the limit of a sequence of time-varying linear approximations and use this to derive boundedness and stability results. These results directly generalise those of [1].
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1. Introduction

Nonlocal continuation and boundedness of solutions of nonlinear differential equations

$$\frac{dx_i}{dt} = f_i(t; x_1, \cdots, x_n) \ , \ t \in \mathbb{R}_+, x_i \in \mathbb{R}, \ 1 \le i \le n \tag{1.1}$$

has recently been studied in [1] by Yang in the case where f has a sublinear bound in terms of x. In fact, the following has been proved:
Theorem A Suppose that the functions f_i belong to $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ satisfy the condition

$$|f_i(t; u_1, \cdots, u_n)| \le \eta_i(t) + \sum_{j=1}^{n} \xi_{ij}(t)|u_j|, \ t \in \mathbb{R}_+, i = 1, \cdots, n \tag{1.2}$$

where $u_i \in \mathbb{R}$, η_i and ξ_{ij} belong to $C(\mathbb{R}_+, \mathbb{R}_+)$. Then every solution of (1.4) is continuable on $\mathbb{R}_+$ and satisfies the inequality

$$|x_i(t)| \le |x_i(0)| + \int_0^t \eta_i(s)ds + \int_0^t Q(s) \left[\exp \int_s^t P(\tau)d\tau\right] ds, \ t \in \mathbb{R}_+, i = 1, \cdots, n \tag{1.3}$$

where

$$P(t) \ : \ = \sum_{j=1}^{n} \max[\xi_{1j}(t), \cdots, \xi_{nj}(t)], \tag{1.4}$$

$$Q(t) \ : \ = \sum_{j=1}^{n} \left\{ \max[\xi_{1j}(t), \cdots, \xi_{nj}(t)] \left(|x_j(0)| + \int_0^t \eta_i(s)ds \right) \right\}$$

Using similar methods, n^{th} order equations of the form

$$y^{(n)} = F(t; y, y', \cdots, y^{(n-1)}) \ , \ t \in \mathbb{R}_+, y \in \mathbb{R} \tag{1.5}$$

are considered and the following result is obtained:
Theorem B Suppose that the function $F \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ satisfies the condition

$$|F(t; u_1, \cdots, u_n)| \leq g(t) + \sum_{j=1}^{n} h_j(t)|u_j|, \ t \in \mathbb{R}_+ \tag{1.6}$$

where $u_i \in \mathbb{R}$, g and h_j belong to $C(\mathbb{R}_+, \mathbb{R}_+)$. Then every solution of (1.4) is continuable on $\mathbb{R}_+$ and satisfies the inequalities

$$\begin{aligned} |y^{(k-1)}(t)| &\leq |y^{(k-1)}(0)| + \Phi(t), \ k = 1, 2, \cdots, n-2, \\ |y^{(n-1)}(t)| &\leq |y^{(n-1)}(0)| + \int_0^t g(s)ds + \Phi(t), \ t \in \mathbb{R}_+, \end{aligned}$$

where

$$\Phi(t) := \int_0^t \sum_{j=1}^{n} \rho_j(s) \left\{ \exp \int_s^t \left[\sum_{k=1}^{n-1} \rho_k(\tau)|y^{(k-1)}(0)| + \rho_n(\tau) \left(|y^{(n-1)}(0)| + \int_0^\tau g(v)dv \right) \right] d\tau \right\} ds \tag{1.7}$$

and

$$\rho_j(t) := \max[1, h_j(t)], \ t \in \mathbb{R}_+, \ 1 \leq j \leq n \tag{1.8}$$

The main restriction in these theorems is the global sublinear bound on the vector field, a condition which is never met for polynomial systems. In this paper we shall generalise the above results to the case of polynomial vector fields. Such vector fields are important in dynamical systems theory, including chaotic motion, isochronous systems etc. (see, for example, [2], [3]).

Another approach to nonlinear differential equations, stability and boundedness is via systems of the form

$$\dot{x} = A(x)x + f(t) \ , \ x(0) = x_0 \tag{1.9}$$

which have been considered extensively in [4], [5]. The main technical approaches have been to consider the Lie algebra generated by the matrices $A(x)$ as x varies in $\mathbb{R}^n$, or to think of these systems as perturbations of the system

$$\dot{x} = A(x)x \ , \ x(0) = x_0 \tag{1.10}$$

which is studied by introducing a sequence of linear, time-varying approximations

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) \ , \ x^{[i]}(0) = x_0 \tag{1.11}$$

In this paper we shall take the latter approach to obtain boundedness conditions on the solutions of (1.3) by showing that the sequence of functions $x^{[i]}(t)$ defined by (1.5) converges in $C([0, \infty); \mathbb{R}^n)$ and have uniform bounds in i.

A brief note on our notation is in order here. We shall use the following form for the Taylor series of a function $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ about some point $\bar{x}$:

$$f(x) = \sum_{|\mathbf{i}|=0}^{\infty} \frac{\alpha_{\mathbf{i}}}{\mathbf{i}!}(x - \bar{x})^{\mathbf{i}}$$

where $\mathbf{i} = (i_1, \cdots, i_n)$, $|\mathbf{i}| = i_1 + \cdots + i_n$, $(x - \bar{x})^{\mathbf{i}} = (x_1 - \bar{x}_1)^{i_1}(x_2 - \bar{x}_2)^{i_2} \cdots (x_n - \bar{x}_n)^{i_n}$ and

$$\alpha_{\mathbf{i}} = \frac{\partial^{|\mathbf{i}|}}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}} f(\bar{x}).$$

2. Polynomial Systems

Consider the nonlinear ordinary differential equation

$$\frac{dx_i}{dt} = f_i(t; x_1, \cdots, x_n) \ , \ t \in \mathbb{R}_+, x_i \in \mathbb{R}, \ 1 \le i \le n \tag{2.1}$$

where f_i is a polynomial function of x, which satisfies a global bound of the form

$$|f_i(t; x_1, \cdots, x_n)| \le \eta_i(t) + \sum_{|\mathrm{p}|>0}^{K} \zeta^i_{\mathrm{p}}(t)|x_1|^{p_1} \cdots |x_n|^{p_n} \tag{2.2}$$

for all $x \in \mathbb{R}^n$ and some $K > 0$. Denote the function on the right hand side (without the constant term $\eta_i(t)$) by $\Gamma_i(t; x_1, \cdots, x_n)$ i.e.

$$\Gamma_i(t; x_1, \cdots, x_n) = \sum_{|\mathrm{p}|>0}^{K} \zeta^i_{\mathrm{p}}(t)|x_1|^{p_1} \cdots |x_n|^{p_n} \ . \tag{2.3}$$

Given such a function and n positive real numbers $a_1, \cdots, a_n \in \mathbb{R}_+$, we can form a linear function of $|x_1|, \cdots, |x_n|$ by fixing all but one of the factors $|x_k|$ at a_k. For example, we can form the linear function

$$\begin{aligned} \ell_i(t; x_1, \cdots, x_n; a_1, \cdots, a_n) &= \sum_{j=1}^{n} \sum_{\substack{|\mathrm{p}|>0 \\ p_j>0}}^{K} \zeta^i_{\mathrm{p}}(t) a_1^{p_1} \cdots a_j^{p_j-1} \cdots a_n^{p_n} |x_j| \\ &\triangleq \sum_{j=1}^{n} \xi_{ij}(t, a)|x_j| \end{aligned} \tag{2.4}$$

where $\xi_{ij}(t, a) = \sum_{\substack{|\mathrm{p}|>0 \\ p_j>0}}^{K} \zeta^i_{\mathrm{p}}(t) a_1^{p_1} \cdots a_j^{p_j-1} \cdots a_n^{p_n}$. Let $S_{(a_1,\cdots,a_n)}$ be the set of all possible vectors $(\ell_1, \cdots, \ell_n)$ of linear functions of $|x_1|, \cdots, |x_n|$ associated with $\Gamma_i(t; x_1, \cdots, x_n), 1 \le i \le n$, in this way. Moreover, let $\bar{f}_i$ represent any of the corresponding functions associated with f_i in this way, i.e.

$$\bar{f}_i(t; x_1, \cdots, x_n) = f_i(t; 0, \cdots, 0) + \sum_{j=1}^{n} \sum_{\substack{|\mathrm{p}|>0 \\ p_j>0}}^{K} \zeta^i_{\mathrm{p}}(t) a_1^{p_1} \cdots a_j^{p_j-1} \cdots a_n^{p_n} x_j \tag{2.5}$$

We can now state the main result of this section:

Theorem 2.1 Suppose that the functions f_i belong to $C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R})$ and satisfy the condition

$$|f_i(t; x_1, \cdots, x_n)| \le \eta_i(t) + \sum_{|\mathrm{p}|>0}^{K} \zeta^i_{\mathrm{p}}(t)|x_1|^{p_1} \cdots |x_n|^{p_n} \ .$$

Moreover, suppose that the numbers $a_1, \cdots, a_n$ and the initial vector $x(0)$ satisfy the inequality

$$|x_i(0)| + \int_0^t \eta_i(t)ds + V(a_1, \cdots, a_n; t) < a_i \ , \ 1 \le i \le n \tag{2.6}$$

for all $t \in \mathbb{R}_+$, where

$$V(a_1, \cdots, a_n; t) = \max_{S_{(a_1,\cdots,a_n)}} \int_0^t Q(s,a) \left[\exp \int_s^t P(\tau, a) d\tau\right] ds$$

and

$$P(t,a) \triangleq \sum_{j=1}^{n} \max[\xi_{1j}(t,a), \cdots, \xi_{nj}(t,a)]$$

$$Q(t,a) \triangleq \sum_{j=1}^{n} \left\{ \max[\xi_{1j}(t,a), \cdots, \xi_{nj}(t,a)] \left(|x_j(0)| + \int_0^t \eta_j(t) ds\right)\right\}$$

then the solutions of the system (2.1) satisfy the inequality

$$|x_i(t)| < a_i, \tag{2.7}$$

for all $t \in \mathbb{R}_+$, $1 \le i \le n$, for any initial condition $x(0)$ which satisfies (2.6).

Proof From (2.6) we clearly have $|x_i(0)| < a_i, 1 \le i \le n$. If (2.7) does not hold for all $t \ge 0$, then let T be the smallest time at which $|x_j(t)| = a_j$ for some $j \in \{1, \cdots, n\}$. For $t \in [0, T]$ we have $|x_i(t)| \le a_i$ for each i, so we have

$$|f_i(t; x_1, \cdots, x_n)| \le |\bar{f}_i(t; x_1, \cdots, x_n)|$$

for any choice of $\bar{f}_i$ on this interval and so by theorem 1 of [1], we have

$$|x_i(t)| \le |x_i(0)| + \int_0^t \eta_i(t) ds + \int_0^t Q(s,a) \left[\exp \int_s^t P(\tau, a) d\tau\right] ds \tag{2.8}$$

for all $t \in [0, T]$, where we have used any fixed functions $\ell_i, 1 \le i \le n$ in $S_{(a_1,\cdots,a_n)}$. However, by (2.6) each right hand side of (2.8) is strictly less than a_i, which gives a contradiction.□

Example 2.2 Consider the quadratic system of equations

$$\begin{aligned}
\dot{x}_1 &= \eta_1(t) + \mu_{10}^1(t)x_1 + \mu_{01}^1(t)x_2 + \mu_{20}^1(t)x_1^2 + \mu_{11}^1(t)x_1x_2 + \mu_{02}^1(t)x_2^2 \\
\dot{x}_2 &= \eta_2(t) + \mu_{10}^2(t)x_1 + \mu_{01}^2(t)x_2 + \mu_{20}^2(t)x_1^2 + \mu_{11}^2(t)x_1x_2 + \mu_{02}^2(t)x_2^2
\end{aligned}$$

i.e.

$$\begin{aligned}
\dot{x}_1 &= \eta_1(t) + \sum_{i=0}^{2}\sum_{j=0}^{2} \mu_{ij}^1(t) x_1^i x_2^j \\
\dot{x}_2 &= \eta_2(t) + \sum_{i=0}^{2}\sum_{j=0}^{2} \mu_{ij}^2(t) x_1^i x_2^j
\end{aligned}$$

Given (a_1, a_2), the set $S_{(a_1,a_2)}$ contains four pairs of linear functions; i.e.

$$\begin{aligned}
&(|\mu_{10}^1(t)|x_1 + |\mu_{01}^1(t)|x_2 + a_1|\mu_{20}^1(t)|x_1 + a_1|\mu_{11}^1(t)|x_2 + a_2|\mu_{02}^1(t)|x_2, \\
&\quad |\mu_{10}^2(t)|x_1 + |\mu_{01}^2(t)|x_2 + a_1|\mu_{20}^2(t)|x_1 + a_1|\mu_{11}^2(t)|x_2 + a_2|\mu_{02}^2(t)|x_2) \\
&(|\mu_{10}^1(t)|x_1 + |\mu_{01}^1(t)|x_2 + a_1|\mu_{20}^1(t)|x_1 + a_2|\mu_{11}^1(t)|x_1 + a_2|\mu_{02}^1(t)|x_2, \\
&\quad |\mu_{10}^2(t)|x_1 + |\mu_{01}^2(t)|x_2 + a_1|\mu_{20}^2(t)|x_1 + a_1|\mu_{11}^2(t)|x_2 + a_2|\mu_{02}^2(t)|x_2) \\
&(|\mu_{10}^1(t)|x_1 + |\mu_{01}^1(t)|x_2 + a_1|\mu_{20}^1(t)|x_1 + a_1|\mu_{11}^1(t)|x_2 + a_2|\mu_{02}^1(t)|x_2, \\
&\quad |\mu_{10}^2(t)|x_1 + |\mu_{01}^2(t)|x_2 + a_1|\mu_{20}^2(t)|x_1 + a_2|\mu_{11}^2(t)|x_1 + a_2|\mu_{02}^2(t)|x_2) \\
&(|\mu_{10}^1(t)|x_1 + |\mu_{01}^1(t)|x_2 + a_1|\mu_{20}^1(t)|x_1 + a_2|\mu_{11}^1(t)|x_1 + a_2|\mu_{02}^1(t)|x_2, \\
&\quad |\mu_{10}^2(t)|x_1 + |\mu_{01}^2(t)|x_2 + a_1|\mu_{20}^2(t)|x_1 + a_2|\mu_{11}^2(t)|x_1 + a_2|\mu_{02}^2(t)|x_2)
\end{aligned}$$

i.e.

$$(\xi_{11}(k)x_1 + \xi_{12}(k)x_2, \xi_{21}(k)x_1 + \xi_{22}(k)x_2) \ , \ 1 \leq k \leq 4$$

where

$$\begin{aligned}
\xi_{11}(1) &= |\mu^1_{10}(t)| + a_1|\mu^1_{20}(t)|, \xi_{12}(1) = |\mu^1_{01}(t)| + a_1|\mu^1_{11}(t)| + a_2|\mu^1_{02}(t)| \\
\xi_{21}(1) &= |\mu^2_{10}(t)| + a_1|\mu^2_{20}(t)|, \xi_{22}(1) = |\mu^2_{01}(t)| + a_1|\mu^2_{11}(t)| + a_2|\mu^2_{02}(t)|
\end{aligned}$$

with similar expressions for $\xi_{ij}(k), 2 \leq k \leq 4$. Let

$$\begin{aligned}
P(t,a;k) &= \sum_{j=1}^{2} \max[\xi_{1j}(k), \xi_{2j}(k)] \ , \ 1 \leq k \leq 4 \\
Q(t,a;k) &= \sum_{j=1}^{2} \left\{ \max[\xi_{1j}(k), \xi_{2j}(k)] \left(|x_j(0)| + \int_0^t \eta_j(t)ds \right) \right\}
\end{aligned}$$

Then, if

$$V(a,t) = \max_{1 \leq k \leq 4} \int_0^t Q(s,a;k) \left[\exp \int_s^t P(\tau,a;k)d\tau \right] ds$$

and $x(0)$ satisfies

$$|x_i(0)| + \int_0^t \eta_i(t)ds + V(a,t) < a_i \ , \ i = 1,2$$

we have

$$|x_i(t)| < a_i \ , \ i = 1,2.$$

In particular, consider the equation

$$\begin{aligned}
\dot{x}_1 &= \frac{\alpha}{1+t^2}\left(1 + x_1 + 2x_2 + 6x_2^2\right) \\
\dot{x}_2 &= \alpha e^{-t}\left(1 + 2x_1 - 4x_2 + 3x_1^2\right)
\end{aligned} \tag{2.9}$$

In this case the set $S(a_1, a_2)$ has only one element and the corresponding linear equation is

$$\begin{aligned}
\dot{x}_1 &= \frac{\alpha}{1+t^2}\left(1 + x_1 + 2x_2 + 6x_2 a_2\right) \\
\dot{x}_2 &= \alpha e^{-t}\left(1 + 2x_1 - 4x_2 + 3x_1 a_1\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\xi_{11} &= \frac{\alpha}{1+t^2} \ , \ \xi_{12} = \frac{\alpha}{1+t^2}(2 + 6a_2) \\
\xi_{21} &= \alpha e^{-t}(2 + 3a_1) \ , \ \xi_{22} = 4\alpha e^{-t}
\end{aligned}$$

and so

$$\begin{aligned}
P(t,a) &\leq \frac{\alpha}{1+t^2}(2 + 3a_1) + \frac{\alpha}{1+t^2}(4 + 6a_2) \\
&= \frac{\alpha}{1+t^2}(6 + 3a_1 + 6a_2)
\end{aligned}$$

Also,

$$
\begin{aligned}
Q(t,a) &\le \frac{\alpha}{1+t^2}(2+3a_1)\left(|x_1(0)| + \int_0^t \frac{1}{1+s^2}ds\right) \\
&+\frac{\alpha}{1+t^2}(4+6a_2)\left(|x_2(0)| + \int_0^t e^{-s}ds\right) \\
&= \frac{\alpha}{1+t^2}\left[(2+3a_1)\left(|x_1(0)| + \arctan t\right) + (4+6a_2)\left(|x_2(0)| + 1 - e^{-t}\right)\right] \\
&\le \frac{\alpha}{1+t^2}\left[(2+3a_1)\left(|x_1(0)| + \frac{\pi}{2}\right) + (4+6a_2)\left(|x_2(0)| + 1\right)\right]
\end{aligned}
$$

Finally,

$$
\begin{aligned}
V(a,t) &= \int_0^t Q(s,a)\left[\exp \int_s^t P(\tau,a)d\tau\right] ds \\
&\le \alpha\left(\frac{\pi}{2}\right)\exp\left[(6+3a_1+6a_2)\alpha\left(\frac{\pi}{2}\right)\right]\times \\
&\left[(2+3a_1)\left(|x_1(0)| + \frac{\pi}{2}\right) + (4+6a_2)\left(|x_2(0)| + 1\right)\right]
\end{aligned}
$$

Hence, to satisfy (2.6), we must have

$$
\begin{aligned}
&|x_1(0)| + \alpha\left(\frac{\pi}{2}\right) + \alpha\left(\frac{\pi}{2}\right)\exp\left[(6+3a_1+6a_2)\alpha\left(\frac{\pi}{2}\right)\right]\times \\
&\left[(2+3a_1)\left(|x_1(0)| + \frac{\pi}{2}\right) + (4+6a_2)\left(|x_2(0)| + 1\right)\right] < a_1
\end{aligned}
$$

$$
\begin{aligned}
&|x_2(0)| + \alpha + \alpha\left(\frac{\pi}{2}\right)\exp\left[(6+3a_1+6a_2)\alpha\left(\frac{\pi}{2}\right)\right]\times \\
&\left[(2+3a_1)\left(|x_1(0)| + \frac{\pi}{2}\right) + (4+6a_2)\left(|x_2(0)| + 1\right)\right] < a_2
\end{aligned}
$$

This will hold, for example, if $\alpha \le 0.005, a_1 = a_2 = 1$ and $|x_1(0)| \le 1/2, |x_2(0)| \le 1/2$. Hence, under these conditions, the solutions $x_1(t), x_2(t)$ are bounded by 1.

3. Pseudo-Linear Systems

In this section we shall consider a system of equations of the form

$$\dot{x} = \eta(t) + A(x)x \ , \ x(0) = x_0 \tag{3.1}$$

and introduce a sequence of (linear) approximations of the form

$$\dot{x}^{[i]}(t) = \eta(t) + A(x^{[i-1]}(t))x^{[i]}(t) \ , \ x^{[i]}(0) = x_0 \ , \ i \ge 2 \tag{3.2}$$

with

$$\dot{x}^{[1]}(t) = \eta(t) + A(y)x^{[1]}(t) \ , \ x^{[1]}(0) = x_0 \tag{3.3}$$

for any fixed $y \in \mathbb{R}^n$. We first consider the convergence of the system (3.2),(3.3) in $C([0,T];\mathbb{R}^n)$, for some $T > 0$. First, let $\Phi^{[i-1]}(t,t_0)$ denote the transition matrix generated by $A(x^{[i-1]}(t))$. It is known (see [6]) that

$$\left\|\Phi^{[i-1]}(t,t_0)\right\| \leq \exp\left[\int_{t_0}^{t} \mu\left(A\left(x^{[i-1]}(\tau)\right)\right) d\tau\right]$$

where $\mu(A)$ is the logarithmic norm of A. The next result gives an estimate for $\Phi^{[i-1]} - \Phi^{[i-2]}$. We shall assume, initially, that $A(x)$ satisfies a global Lipschitz condition of the form

$$\|A(x) - A(y)\| \leq \alpha \|x - y\| \ , \ \forall x, y \in \mathbb{R}^n \tag{3.4}$$

for some $\alpha > 0$ and relax this later to a local condition.

Lemma 3.1 Suppose that $\mu(A(x)) \leq \mu$ for some constant μ and all x and suppose also that A satisfies the global Lipschitz condition (3.4). Then,

$$\left\|\Phi^{[i-1]}(t,t_0) - \Phi^{[i-2]}(t,t_0)\right\| \leq \alpha e^{\mu(t-t_0)}(t-t_0) \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|$$

Proof $\Phi^{[i-1]}, \Phi^{[i-2]}$ are solutions of the respective equations

$$\begin{aligned} \dot{z} &= A(x^{[i-1]}(t))z \ , \ z(t_0) = I \\ \dot{w} &= A(x^{[i-2]}(t))w \ , \ w(t_0) = I \end{aligned}$$

Hence,

$$\frac{d}{dt}(z-w) = A(x^{[i-1]}(t))(z-w) + \left[A(x^{[i-1]}(t)) - A(x^{[i-2]}(t))\right] w$$

and so

$$z - w = \int_{t_0}^{t} \Phi^{[i-1]}(t,s)\left[A(x^{[i-1]}(s)) - A(x^{[i-2]}(s))\right] w(s)ds$$

i.e.

$$\begin{aligned} \|z-w\| &\leq \int_{t_0}^{t} \exp\left(\int_{s}^{t} \mu(A(x^{[i-1]}(\tau)))d\tau\right) \exp\left(\int_{t_0}^{s} \mu(A(x^{[i-2]}(\tau)))d\tau\right) \\ &\qquad \times\alpha \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\| ds \\ &\leq \exp(\mu(t-t_0))\alpha(t-t_0) \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\| . \ \square \end{aligned}$$

Now, from (3.2), we have

$$x^{[i]}(t) = \Phi^{[i-1]}(t,t_0)x_0 + \int_0^t \Phi^{[i-1]}(t,s)\eta(s)ds.$$

Let

$$\xi^{[i]}(t) = \sup_{s\in[0,t]} \left\|x^{[i]}(s) - x^{[i-1]}(s)\right\| .$$

Then,

$$x^{[i]}(t) - x^{[i-1]}(t) = \left(\Phi^{[i-1]}(t,0) - \Phi^{[i-2]}(t,0)\right) x_0 + \int_0^t \left(\Phi^{[i-1]}(t,s) - \Phi^{[i-2]}(t,s)\right) \eta(s) ds$$

and so

$$\begin{aligned} \xi^{[i]}(t) &\leq \alpha \exp(\mu t) t \xi^{[i-1]}(t) \|x_0\| + \alpha \int_0^t \exp(\mu(t-s))(t-s)\xi^{[i-1]}(s) \|\eta(s)\| ds \\ &\leq \alpha \exp(\mu t) t \xi^{[i-1]}(t) \left[\|x_0\| + \int_0^t \exp(-\mu s) \|\eta(s)\| ds \right] . \end{aligned}$$

Suppose that $\exp(-\mu t) \|\eta(t)\| \in L^1[0,T]$ for some $0 < T \leq \infty$, and let

$$K = \int_0^T \exp(-\mu s) \|\eta(s)\| d \tag{3.5}$$

Then we have

$$\xi^{[i]}(t) \leq \alpha(\|x_0\| + K) t \exp(\mu t) \xi^{[i-1]}(t) \tag{3.6}$$

for $t \in [0,T]$; i.e.

$$\xi^{[i]}(T) \leq \nu \xi^{[i-1]}(T) \tag{3.7}$$

where

$$\nu = \sup_{s \in [0,t]} \alpha(\|x_0\| + K) t \exp(\mu t) . \tag{3.8}$$

Thus,

$$\xi^{[i]}(T) \leq \nu^{i-2} \xi^{[2]}(T) . \tag{3.9}$$

We can now state

Theorem 3.2 If $A(x)$ satisfies

$$\begin{aligned} \mu(A(x)) &\leq \mu \; \forall x \in \mathbb{R}^n \\ \|A(x) - A(y)\| &\leq \alpha \|x - y\| \; , \; \forall x, y \in \mathbb{R}^n \end{aligned}$$

and

$$\nu \triangleq \sup_{s \in [0,t]} \alpha(\|x_0\| + K) t \exp(\mu t) \; < 1$$

then the equation (3.1) has a unique solution on $[0,T]$ which is the limit of the sequence $x^{[i]}(t)$ in $C([0,T];\mathbb{R}^n)$.

Proof This follows from the fact that $x^{[i]}(t)$ is a Cauchy sequence in $C([0,T];\mathbb{R}^n)$, since for $i > j$,

$$\begin{aligned} \sup_{t \in [0,T]} \left\| x^{[i]}(t) - x^{[j]}(t) \right\| &\leq \sum_{k=j+1}^{i} \sup_{t \in [0,T]} \left\| x^{[k]}(t) - x^{[k-1]}(t) \right\| \\ &\leq \sum_{k=j+1}^{i} \nu^{k-2} \xi^{[2]}(T) \\ &= \nu^{j-2} \left(\frac{1 - \nu^{i-j}}{1 - \nu} \right) \xi^{[2]}(T) \end{aligned}$$

by (3.9). □
Corollary 3.3 If $\mu < 0$ and

$$\alpha \left(\|x_0\| + \int_0^\infty \exp(-\mu s) \|\eta(s)\| \, ds \right) \left(-\frac{1}{\mu} \right) e^{-1} < 1$$

then the sequence converges for all $t \geq 0$ to the unique solution of (3.1). □

We can find a bound on $\|x(\cdot)\| = \lim_{i\to\infty} x^{[i]}(\cdot)$ (where the limit is taken in $C([0,T];\mathbb{R}^n)$) in the following way:

$$\begin{aligned} \sup_{t\in[0,T]} \left\| x^{[i]}(t) \right\| &= \sup_{t\in[0,T]} \left\| x^{[i]}(t) - x^{[i-1]}(t) + x^{[i-1]}(t) - \cdots - x^{[1]}(t) + x^{[1]}(t) \right\| \\ &\leq \sum_{j=2}^{i} \xi^{[j]}(t) + \sup_{t\in[0,T]} \left\| x^{[1]}(t) \right\| \\ &= \frac{1-\nu^{i-1}}{1-\nu} \xi^{[2]}(t) + \sup_{t\in[0,T]} \left\| x^{[1]}(t) \right\| \end{aligned} \tag{3.10}$$

and so, letting $i \to \infty$,

$$\sup_{t\in[0,T]} \left\| x^{[i]}(t) \right\| \leq \frac{1}{1-\nu} \xi^{[2]}(t) + \sup_{t\in[0,T]} \left\| x^{[1]}(t) \right\| \tag{3.11}$$

We can bound $x^{[1]}(t)$ and $\xi^{[2]}(t)$ in a similar way to that used in lemma 3.1 In fact, we have
Lemma 3.4 Bounds on $x^{[1]}(t)$ and $\xi^{[2]}(t)$ are given by

$$\sup_{t\in[0,T]} \left\| x^{[i]}(t) \right\| \leq \sup_{t\in[0,T]} e^{t\bar{\mu}} \left(\|x_0\| + \int_0^T e^{-s\bar{\mu}} \|\eta(s)\| \, ds \right)$$

where $\bar{\mu} = \mu(A(y))$ and

$$\begin{aligned} \xi^{[2]}(t) \leq\ & \alpha \sup_{t\in[0,T]} \left\| x^{[i]}(t) \right\| \frac{1}{\mu} \left\{ \sup_{t\in[0,T]} \left(e^{\mu t} - 1 \right) \right. \\ & \left. \left[e^{t\bar{\mu}} \|x_0 - y\| + e^{t\bar{\mu}} \int_0^T e^{-s\bar{\mu}} \|\eta(s)\| \, ds + \|A(y)\| \cdot \|y\| \frac{1}{\mu} \left(e^{\bar{\mu}t} - 1 \right) \right] \right\} \end{aligned}$$

Proof The first estimate is trivial. For the second, note that

$$\begin{aligned} \dot{x}^{[2]}(t) - \dot{x}^{[1]}(t) &= A(x^{[1]}(t)) x^{[2]}(t) - A(y) x^{[1]}(t) \\ &= A(x^{[1]}(t)) \left(x^{[2]}(t) - x^{[1]}(t) \right) + \left(A(x^{[1]}(t)) - A(y) \right) x^{[1]}(t) \end{aligned}$$

and so

$$\begin{aligned} \left\| x^{[2]}(t) - x^{[1]}(t) \right\| &\leq \int_0^t \left\| \Phi^{[1]}(t,s) \right\| \cdot \left\| A(x^{[1]}(s)) - A(y) \right\| \cdot \left\| x^{[1]}(s) \right\| ds \\ &\leq \alpha \sup_{t\in[0,T]} \left\| x^{[1]}(t) \right\| \int_0^t e^{\mu(t-s)} \left\| x^{[1]}(s) - y \right\| ds \, . \end{aligned}$$

Now, $x^{[1]}(t) - y$ satisfies the equation

$$\dot{x}^{[1]}(t) - \dot{y} = \eta(t) + A(y)\left(x^{[1]}(t) - y\right) + A(y)y$$

and so

$$x^{[1]}(t) - y = e^{A(y)t}(x_0 - y) + \int_0^t e^{A(y)(t-s)}(\eta(s) + A(y)y)\, ds$$

from which the result follows. □

Remark 3.5 We have shown that, under the above assumptions, $x(t)$ satisfies a bound of the form (3.11) on the interval $[0, T]$, say

$$\sup_{t\in[0,T]} \|x(t)\| \le B = B(\nu, \mu, \bar{\mu}, \eta(.), x_0, \alpha).$$

Hence, by (3.10) and (3.11) we only require the Lipschitz condition (3.4) for $x \in \{z : \|z\| \le B\} \triangleq \Delta$, i.e.

$$\|A(x) - A(y)\| \le \alpha \|x - y\| \ , \ \forall x, y \in \Delta.$$

If $\mu < 0$ and $K = \int_0^\infty \exp(-\mu s)\|\eta(s)\|\, ds < \infty$,

$$\nu_\infty = \sup_{t\in[0,\infty)} \alpha\left(\|x_0\| + K\right) t \exp(\mu t) < 1$$

then the solutions exist for all $t \ge 0$ provided the Lipschitz condition on $A(x)$ holds for all $x, y \in \{z : \|z\| \le B_\infty\}$ where

$$B_\infty = B_\infty(\nu_\infty, \mu, \bar{\mu}, \eta(.), x_0, \alpha) = \sup_{t\in[0,\infty)} \|x(t)\| . \square$$

Remark 3.6 The above results show that if $A(x)$ satisfies a local Lipschitz condition, then for T small enough, the sequence of approximations (3.2) and (3.3) converges on $[0, T]$. □

4. Sublinear Approximations

We shal finally consider pseudo-linear systems of the form

$$\dot{x} = \tilde{\eta}(t) + A(x)x \ , \ x(0) = x_0 \tag{4.1}$$

in the spirit of section 2 and [1]. Again, we introduce a sequence of linear approximations to (4.1) in the form

$$\dot{x}^{[1]}(t) = \tilde{\eta}(t) + A(y)x^{[1]}(t) \ , \ x^{[1]}(0) = x_0 \tag{4.2}$$

and

$$\dot{x}^{[i]}(t) = \tilde{\eta}(t) + A(x^{[i-1]}(t))x^{[i]}(t) \ , \ x^{[1]}(0) = x_0 \ , \ i \ge 2. \tag{4.3}$$

If $f_j^{[i]}(t; x_1, \cdots, x_n) = \tilde{\eta}_j(t) + \left[A(x^{[i-1]}(t))x(t)\right]_j$, then $f_j^{[i]}$ satisfies the inequality

$$\left|f_j^{[i]}(t; u_1, \cdots, u_n)\right| \le \eta_j(t) + \sum_{k=1}^{n} \xi_{jk}^{[i]}(t)\, |u_k|$$

where

$$\eta_j(t) = |\tilde{\eta}_j(t)| \ , \ \xi_{jk}^{[i]}(t) = \left|a_{jk}(x^{[i-1]}(t))\right|$$

and

$$A(x^{[i-1]}(t)) = \left(a_{jk}(x^{[i-1]}(t))\right) \ .$$

We shall assume that

$$\xi_{jk}^{[i]}(t) = \left|a_{jk}(x^{[i-1]}(t))\right| \leq \rho_{jk}(\left|x^{[i-1]}(t)\right|)$$

for some polynomial functions $\rho_{jk}(z)$ (where $|x|$ denotes the vector norm $|x| = \max\left(|x_1|, \cdots, |x_n|\right)$. Then by theroem 1 in [1], we have

$$\left|x_j^{[i]}(t)\right| \leq |x_j(0)| + \int_0^t \eta_j(s)ds + \int_0^t Q(s)\left[\exp\int_s^t P(\tau)d\tau\right]ds \ , \ t \in \mathbb{R}^+, \ i = 1, \cdots, n \qquad (4.4)$$

where

$$\begin{aligned} P(t) \quad & \triangleq \sum_{j=1}^{n} \max\left[\xi_{1j}^{[i]}(t), \xi_{2j}^{[i]}(t), \cdots, \xi_{nj}^{[i]}(t)\right] \\ & \leq \sum_{j=1}^{n} \max\left[\rho_{1j}(\left|x^{[i-1]}(t)\right|), \rho_{2j}(\left|x^{[i-1]}(t)\right|), \cdots, \rho_{nj}(\left|x^{[i-1]}(t)\right|)\right] \end{aligned}$$

and

$$\begin{aligned} Q(t) \quad & \triangleq \sum_{j=1}^{n}\left\{\max\left[\xi_{1j}^{[i]}(t), \xi_{2j}^{[i]}(t), \cdots, \xi_{nj}^{[i]}(t)\right]\left(|x_j(0)| + \int_0^t \eta_j(s)ds\right)\right\} \\ & \leq \sum_{j=1}^{n}\left\{\max\left[\rho_{1j}(\left|x^{[i-1]}(t)\right|), \rho_{2j}(\left|x^{[i-1]}(t)\right|), \cdots, \rho_{nj}(\left|x^{[i-1]}(t)\right|)\right]\left(|x_j(0)| + \int_0^t \eta_j(s)ds\right)\right\} . \end{aligned}$$

Now suppose that we have shown that $\left|x^{[i-1]}(t)\right| \leq K$, $t \in [0, T]$ for some bound K and some $T > 0$. Let

$$R(K) = \sup_{z\in[0,K]} \sum_{j=1}^{n} \max\left[\rho_{1j}(z), \rho_{2j}(z), \cdots, \rho_{nj}(z)\right] \ .$$

Clearly, $R(K) < \infty$ and $\sup_{t\in[0,T]} P(t) \leq R(K)$. Also,

$$\sup_{t\in[0,T]} Q(t) \leq R(K) \cdot \left(|x(0)| + \left|\int_0^T \eta(s)ds\right|\right) .$$

Hence, from (4.4) we have

$$\sup_{t\in[0,T]} \left|x^{[i]}(t)\right| \leq |x(0)| + \left|\int_0^T \eta(s)ds\right| + T\left[R(k) \cdot \left(|x(0)| + \left|\int_0^T \eta(s)ds\right|\right)\exp(T \cdot R(K))\right] . \quad (4.5)$$

Also, from (4.2), we have

$$x(t) = e^{A(y)t}x_0 + \int_0^t e^{A(y)(t-s)}\tilde{\eta}(s)ds$$

so

$$\sup_{t\in[0,T]} |x(t)| \leq M(T,n) \sup_{t\in[0,T]} e^{\mu t} \left(|x_0| + \left| \int_0^T \eta(s)ds \right| \right) \tag{4.6}$$

for some constants M (depending on T and n) and μ. We then have

Theorem 4.1 Suppose that

$$|x(0)| + \left| \int_0^T \eta(s)ds \right| + T \left[R(k) \cdot \left(|x(0)| + \left| \int_0^T \eta(s)ds \right| \right) \exp(T \cdot R(K)) \right] \leq K$$

and

$$M(T,n) \sup_{t\in[0,T]} e^{\mu t} \left(|x_0| + \left| \int_0^T \eta(s)ds \right| \right) \leq K$$

for any given $K > 0$. Then the sequence of approximations (4.2), (4.3) has a convergent subsequence which converges in $C([0,T];\mathbb{R}^n)$ to the solution of (4.1) and, moreover, $x(t)$ satisfies the bound

$$|x(t)| \leq K \ , \ t \in [0,T].$$

Proof By (4.5) and (4.6) we have shown that under the conditions of the theorem, $x^{[i]}(t)$ is uniformly bounded in $C([0,T];\mathbb{R}^n)$ (by K). A similar argument shows that the sequence $x^{[i]}(t)$ is equicontinuous on $[0,T]$ and so the result follows from the Arzela-Ascoli theorem, by standard arguments. □

Remark 4.2 Since K is arbitrary in this theorem, it gives us a nonlocal bound on the solution of (4.1). If K is large then the theorem gives conditions on the length of time the solution exists relative to the size of the initial condition.

Remark 4.3 Just as in [1], we can easily consider higher-order systems of the form

$$y^{(n)} = F(t; y, y', \cdots, y^{(n-1)})$$

by the usual phase plane trick.

5. Conclusions

In this paper we have extended the results of [1] to the case of polynomial vector fields. We have used two approaches to the problem; in the first we fix all but one variable in each polynomial right hand side of the defining equations so that a sublinear estimate can be obtained. The results of [1] are then applied and it is shown that the solution will be bounded on some time interval depending on the bound on the fixed variables. In the second method we derive a sequence of linear (time-varying) approximations to the original equation to which the results of [1] can again be applied. The sequence is shown to converge in an appropriate space. It is also clear that the methods apply to systems with only local sublinear bounds, rather than the global ones in [1].

References

[1] E-H. Yang. 'Nonlocal Continuation and Boundedness Results for Nonlinear Differential Equations', Nonlin. Anal., Theory , Methods , Applics.,**28**, 1827–1833,1997.

[2] P. Mardesic, C. Rousseau and B.Toni, 'Linearization of Ischronous Centres', J. Diff. Eqns., **121**, 67-108, 1995.

[3] S.P. Banks and D. McCaffrey. 'Lie Algebras, Structure of Nonlinear Systems and Chaos', Preprint.

[4] S.P. Banks and S.K. Al-Jurani : 'Lie Algebras and the Stability of Nonlinear Systems', Int. J. Control, **60**, 315-329, 1994.

[5] S.P. Banks and S.K. Al-Jurani : 'Pseudo-Linear Systems, Lie Algebras and Stability', IMA J. Math. Cont. Inf., **13**, 385-401, 1996.

[6] F.Brauer , 'Perturbations of nonlinear systems of differential equations', II, J. Math. Anal. App., **17**, 418-434, 1967.