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Abstract –In this paper, a new method for evaluating output frequency responses of nonlinear systems under multiple inputs is developed based on two theoretical results concerning the output frequency responses of nonlinear systems to multiple inputs and the determination of output frequencies of nonlinear systems subject to multiple input excitations. This method circumvents difficulties associated with the existing 'frequency mix vector' based approaches and can be applied to evaluate nonlinear output frequency responses to multiple inputs so as to investigate nonlinear behaviours of practical systems including electronic circuits at the system simulation and design stages.

Index Terms–nonlinear systems/circuits, output frequency responses, frequency domain analysis, sinusoidal inputs.

1. Introduction

Most engineering systems including electronic circuits are intrinsically nonlinear. Although measures such as differential configurations, feedback, inverse function cancellation, etc., are often taken to reduce nonlinear effects when practical design problems are addressed, nonlinear effects can not be actually cancelled out completely and it is therefore important to evaluate system behaviours to estimate how the residual nonlinearity degrades system performances.

Systems such as transistor amplifiers and OTA-C filters (operational transconductance amplifier-capacitor filters), which are designed to exhibit mainly linear characteristics but which still possess unavoidable residual nonlinearities, can be reasonably regarded as weakly nonlinear systems [1] and can be investigated in the frequency domain using the Volterra series theory of nonlinear systems [1] [2].

The frequency domain method of nonlinear systems based on the Volterra series theory was initially established in 1950's when the concept of Generalised Frequency Response Functions (GFRFs) of nonlinear systems was introduced [3]. GFRFs were defined as the multi-dimensional Fourier transformations of Volterra kernels in the Volterra series expansion of nonlinear systems which extend the frequency response function of linear systems to the nonlinear case. One of the important features of GFRFs is associated with the description of nonlinear system output responses in the frequency domain. The frequency domain output responses of practical systems are often directly related to system physical performances especially for electronic circuits. Therefore, analysis of the responses is important for examining system behaviours. Nonlinear effects which are likely to be expected in practice can be determined at the system design and simulation stage by evaluating and analysing the system output frequency responses.

In this analysis multiple inputs are defined as a summation of sinusoidal input signals with different frequencies, which can be used at the design and simulation stage and/or laboratory testing period of systems such as communication receivers [4] to excite the systems in order to examine the system output behaviours in the frequency domain. Analysis of nonlinear systems with multiple inputs has been an important topic in the frequency domain analysis of nonlinear systems using Volterra series theory since the 1950's. Many theories and methods have been developed to address problems associated with this topic [5] [1] and applications of the associated theories and methods to circuit analyses can be found in [4] [6] [2] [7].

The presently available theories underlying analysis of nonlinear systems with multiple inputs are almost all based on a concept called the 'frequency mix vector'. This concept reveals the manner by which intermodulation frequencies are generated in nonlinear systems. Intermodulation is an important nonlinear phenomenon which indicates that output frequency components of a nonlinear system could be much richer than the components in the input, while in the linear system case the possible output frequency components are exactly the same as the components in the corresponding input. Although the analyses using the 'frequency mix vector' can clearly interpret how output frequencies of nonlinear systems are produced by particular frequency mixes, the output frequency response components at frequencies of interest are generally difficult to evaluate in practice using associated methods. This is because the output component of a nonlinear system at a particular frequency actually depends on many different frequency mixes and it is generally hard to identify all frequency mix vectors associated with these different frequency mixes.

In this paper, the above problems associated with the frequency domain analyses of nonlinear systems under multiple inputs are addressed. After introducing preliminary knowledge concerning nonlinear system analyses under multiple inputs, an expression for the output frequency responses of nonlinear systems to multiple inputs is derived which can not only reveal how the nonlinear output frequency response is generated but can also be readily used to evaluate the result. Then, output frequencies of nonlinear systems under multiple inputs are analysed and an effective algorithm is developed to determine the output frequencies from the input frequencies and the system nonlinearities. This algorithm extends the concept regarding the relationship between the linear system input and output frequencies to the nonlinear case where systems are under arbitrary multiple input excitation. Finally, a new method is proposed to evaluate output frequency responses of nonlinear systems under multiple inputs. The method is an organic combination of the first and second results and provides an effective and practical means for evaluating nonlinear frequency domain effects of practical systems including electronic circuits at the system design and simulation stages.

2. Analysis of nonlinear systems under multiple inputs

Systems such as transistor amplifiers and OTA-capacitor filters which possess weak nonlinearities can be described by a Volterra series representation. The Volterra series representation of a nonlinear system can be generally written as

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{1}$$

where

$$y_n(t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_i \tag{2}$$

$u(t)$ and $y(t)$ are the system input and output, respectively, $h_n(\tau_1,\cdots,\tau_n)$, $1 \le n < N$, are the Volterra kernels, and N is the maximum order of system nonlinearities which is finite for a wide class of nonlinear systems and input excitations [8].

Under the excitation of a multiple input defined by

$$u(t) = \sum_{i=1}^{R} |A_i| \cos(\omega_i t + \angle A_i) = \sum_{i=1}^{R} \left(\frac{A_i}{2} e^{j\omega_i t} + \frac{A_i^*}{2} e^{-j\omega_i t} \right) = \sum_{i=-R,\, i\neq 0}^{R} \frac{A_i}{2} e^{j\omega_i t} \qquad (3)$$

where A_i^* denotes the conjugate of A_i, $A_{-i} = A_i^*$, and $\omega_{-i} = -\omega_i$, the nth-order output response of the system (1) and (2) can be described, by substituting (3) into (2), as

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\ldots,\tau_n) \prod_{l=1}^{n} \sum_{i=-R,\, i\neq 0}^{R} \frac{A_i}{2} e^{j\omega_i (t-\tau_l)} d\tau_l$$

$$= \frac{1}{2^n} \sum_{i_1=-R,\, i_1\neq 0}^{R} \cdots \sum_{i_n=-R,\, i_n\neq 0}^{R} A_{i_1} \ldots A_{i_n} H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) e^{j(\omega_{i_1}+\cdots+\omega_{i_n})t} \qquad (4)$$

where

$$H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n) e^{-j(\omega_{i_1}\tau_1+\cdots+\omega_{i_n}\tau_n)} d\tau_1 \cdots d\tau_n \qquad (5)$$

is the nth-order GFRFs $H_n(j\omega_1,\cdots,j\omega_n)$ of the system evaluated at $\{\omega_1,\cdots,\omega_n\} = \{\omega_{i_1},\cdots,\omega_{i_n}\}$.

Equations (1) and (4) provide a general description for output responses of nonlinear systems under multiple inputs. Analysis of this response can, in most cases, be sufficiently performed based on equation (4) to investigate the nth-order portion of $y(t)$ and the total response is simply a summation of all $y_n(t)$'s from $n = 1$ to N.

Equation (4) indicates that when a sum of R sinusoids is applied to a nonlinear system, additional output frequencies are generated by the nth-order portion of the system output consisting of all possible combinations of the input frequencies

$-\omega_R,\cdots,-\omega_1,\omega_1,\cdots,\omega_R$

taken n at a time.

In order to illustrate this general expression, consider a practical example in [2] which is an OTA with capacitor load as shown in Fig.1 (a). Fig.1 (b) shows the nonlinear model of the OTA-C integrator with a nonlinear current source $i_o = f(v_a)$ where $f(.)$ is the characteristic function of the OTA which can be exactly determined from the circuit structure and parameters [2].

It can be shown from Fig. 1(b) that the circuit equations in the frequency domain are given by

$$\begin{cases} (V_a - V_s)G_s = 0 \\ (G_0 + j\omega C)V_o - F\{f[v_a(t)]\} = 0 \end{cases} \tag{6}$$

where $G_s = 1/R_s$, $G_0 = 1/R_0$, and V_s, V_a, V_o are Fourier transforms of $v_s(t)$, $v_a(t)$, and $v_o(t)$, respectively. Thus the block diagram description of the circuit can be shown as in Fig.2.

Representing $f[v_a(t)]$ by the Taylor series expansion about the operation point $v_a(t) = 0$ yields

$$f[v_a(t)] = \sum_{n=1}^{\infty} g_n v_a^n(t) \tag{7}$$

where $g_n = \dfrac{1}{n!}\dfrac{df[v_a(t)]}{v_a(t)}$. Then the nth-order GFRFs of the circuit can be obtained as [2]

$$H_n(j\omega_1,\cdots,j\omega_n) = \frac{g_n R_o}{j(\omega_1+,\cdots,+\omega_n)R_o C + 1}, \qquad n = 1,2,3,\ldots \tag{8}$$

Consider a case where the circuit is subject to a one-tone input, i.e., R=1, and examine the second order output response $y_2(t)$ of the circuit to this input. In this specific situation,

$$u(t) = \sum_{i=-R,\, i\neq 0}^{R} \frac{A_i}{2} e^{j\omega_i t} = \sum_{i=-1,\, i\neq 0}^{1} \frac{A_i}{2} e^{j\omega_i t} \tag{9}$$

and

$$\begin{aligned} y_2(t) &= \frac{1}{2^2} \sum_{i_1=-1,\, i_1\neq 0}^{1} \sum_{i_2=-1,\, i_2\neq 0}^{1} A_{i_1} A_{i_2} H_2(j\omega_{i_1}, j\omega_{i_2}) e^{j(\omega_{i_1}+\omega_{i_2})t} \\ &= \frac{(A_1^*)^2}{2^2} H_2(-j\omega_1, -j\omega_1) e^{-2j\omega_1 t} + \frac{|A_1|^2}{2^2} H_2(-j\omega_1, j\omega_1) e^{j(\omega_1-\omega_1)t} \\ &+ \frac{|A_1|^2}{2^2} H_2(j\omega_1, -j\omega_1) e^{j(\omega_1-\omega_1)t} + \frac{(A_1)^2}{2^2} H_2(j\omega_1, j\omega_1) e^{2j\omega_1 t} \end{aligned} \tag{10}$$

Moreover, substituting equation (8) into (10) for n=2 yields

$$y_2(t) = \frac{(A_1^*)^2}{2^2}\frac{g_2R_o}{1-2\omega_1R_oCj}e^{-2j\omega_1t} + \frac{|A_1|^2}{2^2}g_2R_o$$

$$+\frac{|A_1|^2}{2^2}g_2R_o + \frac{(A_1)^2}{2^2}\frac{g_2R_o}{1+2\omega_1R_oCj}e^{2j\omega_1t}$$

$$= \frac{1}{2}|A_1|^2 g_2R_o + \frac{1}{4}\frac{g_2R_o}{(2\omega_1R_oC)^2+1}\left\{(A_1^*)^2[1+2\omega_1R_oCj]e^{-2j\omega_1t} + (A_1)^2[1-2\omega_1R_oCj]e^{2j\omega_1t}\right\}$$

$$= \frac{1}{2}|A_1|^2 g_2R_o + \frac{g_2R_o|A_1|^2}{2\sqrt{(2\omega_1R_oC)^2+1}}\cos\left[2\omega_1t + \angle A_1^2 - \tan^{-1}2\omega_1R_oC\right] \quad (11)$$

Equation (11) indicates that the second order response of the circuit to the one-tone input is composed of two frequency components $\omega = 0$ and $\omega = 2\omega_1$, which are the absolute values of the summations of the input frequencies $-\omega_1$ and ω_1 taken two at a time, i.e.,

$$|\omega_1 - \omega_1| = |-\omega_1 + \omega_1| = 0, \quad |-\omega_1 - \omega_1| = 2\omega_1, \text{ and } |\omega_1 + \omega_1| = 2\omega_1$$

The specific case above is a very simple example where the output frequencies composed of all possible combinations of the input frequencies can be easily identified. For general cases where systems are subject to arbitrary multiple inputs where R could be any integer, the frequency domain analysis of nonlinear systems under multiple inputs is more complicated and is usually carried out based on a concept called the 'frequency mix vector' [1].

Because, under a multiple input, output frequencies generated by the nth-order system nonlinearity consist of all possible combinations of the input frequencies $-\omega_R,\cdots,-\omega_1,\omega_1,\cdots,\omega_R$ taken n at a time, let m_i denote the number of times the frequency ω_i appears in a particular frequency mix, the frequency mix can then be represented by the vector

$$m = (m_{-R},\cdots,m_{-1},m_1,\ldots,m_R) \quad (12)$$

where the m_i's obey the constraint

$$m_{-R}+\cdots+m_{-1}+m_1+\cdots+m_R = n \quad (13)$$

Vector m is referred to as the nth-order frequency mix vector and the corresponding output frequency is given by

$$\omega_m = (m_1 - m_{-1})\omega_1 + \cdots + (m_R - m_{-R})\omega_R \tag{14}$$

Therefore, the output frequencies in $y_n(t)$ given by equation (4) can be interpreted as those frequencies that can be generated by all possible choices of the m_i's such that (13) is satisfied.

It has been shown that the output component in (4) which corresponds to a particular frequency mix m is given by [1]

$$y_n(t,m) = \frac{n!}{2^n}\left(\prod_{i=-R,\, i\neq 0}^{R} \frac{A_i^{m_i}}{m_i\,!}\right) H_n\left(m_{-R}\{j\omega_{-R}\},\cdots,m_{-1}\{j\omega_{-1}\},m_1\{j\omega_1\},\cdots,m_R\{j\omega_R\}\right)e^{j\omega_m t} \tag{15}$$

where the GFRF $H_n(.)$ is assumed to be symmetric , i.e.

$$H_2(j\omega_1, j\omega_2) = H_2(j\omega_2, j\omega_1)$$

in the n=2 case, for example, and $m_i\{j\omega_i\}$ denotes m_i consecutive arguments in $H_n(.)$ having the same frequency $j\omega_i$. Thus the nth-order portion of $y(t)$ can be written as

$$y_n(t) = \sum_m y_n(t,m) \tag{16}$$

where the summation over m is defined to be

$$\sum_m = \sum_{\substack{m_{-R}=0 \\ (13)}}^{n} \cdots \sum_{m_R=0}^{n}$$

and the equation number (13) appended below the summation signs indicates that only terms for which the indices sum to n are included in the 2R-fold summation.

The above analysis for nth-order nonlinear output responses to multiple inputs clearly reflects how an output frequency component is produced by a particular frequency mix and how the component can be evaluated using the associated frequency mix vector.

Consider the above circuit example again but under a two-tone input. The frequencies in the 2nd-order output response of the circuit are

$$\omega_m = (m_1 - m_{-1})\omega_1 + (m_2 - m_{-2})\omega_2$$

with the m_i's obeying the constraint

$$m_{-2} + m_{-1} + m_1 + m_2 = 2$$

The output component corresponding to a particular $m = \{m_{-2}, m_{-1}, m_1, m_2\}$ can be determined using equation (15) where the GFRF is defined by equation (8) for n=2.

In this case, it is not difficult to show the associated frequency mix vectors are

$$\{1,1,0,0\},\ \{0,1,1,0\},\ \{0,0,1,1\},\ \{1,0,0,1\},\ \{1,0,1,0\},\ \{0,1,0,1\},\ \{2,0,0,0\},\ \{0,2,0,0\},\ \{0,0,2,0\},\ \{0,0,0,2\}$$

and $y_2(t)$ is therefore the result of the summation of $y_n(t.m)$ given by (15) over all these frequency mix vectors.

Output frequencies corresponding to these vectors can be easily obtained as

$$-\omega_1 - \omega_2,\ -\omega_1 + \omega_1 = 0,\ \omega_1 + \omega_2,\ -\omega_2 + \omega_2 = 0,\ -\omega_2 + \omega_1,\ -\omega_1 + \omega_2,$$
$$-2\omega_2,\ -2\omega_1,\ 2\omega_1,\ 2\omega_2$$

Therefore the practical output frequencies, which are the nonnegative results of the above frequencies, are

$$\omega_1 + \omega_2,\quad \omega_2 - \omega_1,\quad 2\omega_1,\quad 2\omega_2,\quad \text{and } 0$$

Although, as shown above, the frequency mix vector is very helpful in nonlinear frequency domain response analyses under multiple inputs, an important defect with this concept is that distinct frequency mix vectors of the same order may give rise to the same output frequency. For example, when $R=3$, $\{\omega_1, \omega_2, \omega_3\} = \{1,2,4\}$, and $n=2$, the frequency mix vector $m = (0,0,-1,0,0,1)$ yields an output frequency $\omega_m = \omega_3 - \omega_1 = 3$, while the frequency mix vector $m = (0,0,0,1,1,0)$ also yields $\omega_m = \omega_2 + \omega_1 = 3$. So, in general, equation (15) can not be used to represent the frequency response of the system nth-order nonlinear output. Based on the concept of 'frequency mix vector', this response can only be represented as

$$y_{n\omega}(t) = \sum_{\substack{\text{all possible m} \\ \text{such that } \omega_m = \omega}} y_n(t:m) \qquad (17)$$

where $y_{n\omega}(t)$ denotes the total nth-order output response at frequency ω.

From (17) it is hard to evaluate $y_{n\omega}(t)$ practically. This is because, given a frequency of interest ω, it is generally a very difficult job to identify all possible m's such that $\omega_m = \omega$, however determining these m's is necessary if $y_{n\omega}(t)$ is to be evaluated from (17). In [5], a general algorithm was proposed to address this problem which transformed the problem of identifying all possible m's to the problem of sorting out all possible integers $\rho's$ such that

$$\rho_1\omega_1 +,\cdots,+\rho_R\omega_R = \omega$$

Obviously the difficulties with the original method of identifying all possible m's can not be bypassed when using the algorithm in [5].

Motivated by the attempt to completely resolve the problem above with existing methods, a new method is proposed in Section 5 to provide a practical and effective strategy to evaluate the nonlinear frequency responses to multiple inputs and therefore to investigate possible nonlinear behaviours of systems in the frequency domain. The derivations and analyses in Sections 3 and 4 establish the important and necessary theoretical basis for this new method.

3. Expression for output frequency responses of nonlinear systems under multiple inputs

When a nonlinear system described by equations (1) and (2) is excited by a multiple input (3), the system nth-order nonlinear output is generally given by equation (4) which can be rewritten as

$$y_n(t) = \frac{1}{2^n} \sum_{i_1=-R,\, i_1\neq 0}^{R} \cdots \sum_{i_n=-R,\, i_n\neq 0}^{R} A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1},\cdots,j\omega_{i_n}) e^{j(\omega_{i_1}+\ldots+\omega_{i_n})t} \quad (18)$$

where $A(.)$ is defined by

$$A(\omega) = \begin{cases} A_i & \text{if } \omega \in \{\omega_i,\ i = \pm 1,\ldots,\pm R\} \\ 0 & \text{otherwise} \end{cases} \quad (19)$$

In order to obtain a more transparent frequency domain to time domain relationship, considering

$$A(-\omega_1)\ldots A(-\omega_n)H_n(-j\omega_1,\ldots,-j\omega_n)=[A(\omega_1)\ldots A(\omega_n)H_n(j\omega_1,\ldots,j\omega_n)]^* \tag{20}$$

where the * denotes conjugation, write equation (18) as

$$y_n(t)=\frac{1}{2^n}\sum_{\text{all possible }\omega>0}\left[\begin{array}{l}\sum\limits_{\omega_{i_1}+\ldots+\omega_{i_n}=\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})e^{j\omega t}\\ +\sum\limits_{\omega_{i_1}+\ldots+\omega_{i_n}=-\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})e^{-j\omega t}\end{array}\right]$$

$$+\frac{1}{2^n}\sum_{\omega_{i_1}+\ldots+\omega_{i_n}=0}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})e^{j0t}$$

$$=\frac{2}{2^n}\sum_{\text{all possible }\omega>0}\mathrm{Re}[\sum_{\omega_{i_1}+\ldots+\omega_{i_n}=\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})e^{j\omega t}]$$

$$+\frac{1}{2^n}\sum_{\omega_{i_1}+\ldots+\omega_{i_n}=0}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})$$

$$=\sum_{\text{all possible }\omega\geq 0}\left|\overline{\mathbf{Y}}_n(j\omega)\right|\cos[\omega t+\angle\overline{\mathbf{Y}}_n(j\omega)] \tag{21}$$

where

$$\overline{\mathbf{Y}}_n(j\omega)=\begin{cases}\dfrac{1}{2^{n-1}}\sum\limits_{\omega_{i_1}+\ldots+\omega_{i_n}=\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) & \omega>0\\ \dfrac{1}{2^n}\sum\limits_{\omega_{i_1}+\ldots+\omega_{i_n}=\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) & \omega=0\end{cases} \tag{22}$$

In (21) and (22),

$$\sum_{\omega_{i_1}+\ldots+\omega_{i_n}=\omega}A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})$$

denotes the summation of

$$A(\omega_{i_1})\cdots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})$$

over all the $\omega_{i_1},\ldots,\omega_{i_n}$, $\omega_{i_l}\in\{-\omega_R,\ldots,-\omega_1,\omega_1,\ldots,\omega_R\}$, $l=1,\ldots,n$, which satisfy the constraint

$\omega_{i_1}+,\ldots,+\omega_{i_n}=\omega$.

$\overline{\mathbf{Y}}_n(j\omega)$ defined by (22) is the nth-order output frequency response of the system (1) and (2) to the input (3) in terms of nonnegative frequencies, which represents the contribution of the nth-order system nonlinearity to the output frequency component ω.

Substituting (21) into (1) gives

$$y(t)=\sum_{n=1}^{N}y_n(t)=\sum_{n=1}^{N}\sum_{\text{all possible }\omega\geq 0}\left|\overline{\mathbf{Y}}_n(j\omega)\right|\cos[\omega t+\angle\overline{\mathbf{Y}}_n(j\omega)]$$

$$=\sum_{\text{all possible}\omega\geq 0}\mathrm{Re}[\sum_{n=1}^{N}\overline{\mathbf{Y}}_n(j\omega)e^{j\omega t}]=\sum_{\text{all possible}\omega\geq 0}\mathrm{Re}[\overline{\mathbf{Y}}(j\omega)e^{j\omega t}]=\sum_{\text{all possible}\omega\geq 0}\left|\overline{\mathbf{Y}}(j\omega)\right|\cos[\omega t+\angle\overline{\mathbf{Y}}(j\omega)] \tag{23}$$

where

$$\overline{\mathbf{Y}}(j\omega)=\sum_{n=1}^{N}\overline{\mathbf{Y}}_n(j\omega) \tag{24}$$

So, the output frequency response of a nonlinear system under a multiple input is given, in terms of nonnegative frequencies, by

$$\begin{cases}\overline{\mathbf{Y}}(j\omega)=\sum_{n=1}^{N}\overline{\mathbf{Y}}_n(j\omega) \qquad \omega\geq 0\\ \overline{\mathbf{Y}}_n(j\omega)=\begin{cases}\frac{1}{2^{n-1}}\sum_{\omega_{i_1}+,\ldots,+\omega_{i_n}=\omega}A(\omega_{i_1})\ldots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) & \omega>0\\ \frac{1}{2^{n}}\sum_{\omega_{i_1}+,\ldots,+\omega_{i_n}=\omega}A(\omega_{i_1})\ldots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n}) & \omega=0\end{cases}\\ \omega_{i_l}\in\{-\omega_R,\ldots,-\omega_1,\omega_1,\ldots,\omega_R\},\ l=1,\ldots,n\end{cases} \tag{25}$$

Notice that the relationship between the system output frequency spectrum $\mathbf{Y}(j\omega)$ and $\overline{\mathbf{Y}}(j\omega)$ is

$$\mathbf{Y}(j\omega)=\begin{cases}\frac{\overline{\mathbf{Y}}(j\omega)}{2} & \omega\neq 0\\ \overline{\mathbf{Y}}(j\omega) & \omega=0\end{cases} \tag{26}$$

Because of equation (23), $\overline{\mathbf{Y}}(j\omega)$ can be more easily related to the system time domain response.

It can be observed from equation (25) that the possible output frequencies in the nth-order nonlinear output $y_n(t)$ are

$$\omega = \omega_{i_1} +, \cdots, +\omega_{i_n}$$

with $\omega_{i_l} \in \{-\omega_R, \ldots, -\omega_1, \omega_1, \ldots, \omega_R\}$, $l = 1, \ldots, n$. This clearly reflects how the output frequencies are composed in this situation. In addition, from the definition of

$$\sum_{\omega_{i_1}+,\ldots,+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \ldots, j\omega_{i_n})$$

the terms which compose the nth-order output frequency component $\overline{\mathbf{Y}}_n(j\omega)$ can be readily identified. Therefore, the evaluation of $\overline{\mathbf{Y}}_n(j\omega)$ and, moreover, of the total frequency response $\overline{\mathbf{Y}}(j\omega)$ can easily be achieved using equation (25). This is because the evaluation of $\overline{\mathbf{Y}}_n(j\omega)$ can be simply implemented in the following way

$$\begin{aligned} \overline{\mathbf{Y}}_n(j\omega) &= \frac{1}{2^n} \sum_{\omega_1+\ldots+\omega_n=\omega} A(\omega_{i_1})\ldots A(\omega_{i_n}) H_n(j\omega_{i_1}, \ldots, j\omega_{i_n}) \\ &= \frac{1}{2^n} \sum_{i_1=-R,\, i_1\neq 0}^{R} \cdots\cdots \sum_{i_{n-1}=-R,\, i_{n-1}\neq 0}^{R} A(\omega_{i_1})\ldots A(\omega_{i_{n-1}}) A(\omega-\omega_{i_1}-\ldots-\omega_{i_{n-1}}) H_n[j\omega_{i_1}, \ldots, j\omega_{i_{n-1}}, j(\omega-\omega_{i_1}, \ldots, -\omega_{i_{n-1}})] \end{aligned} \quad (27)$$

where $n_1 = n$ for $\omega=0$, $n_1 = n-1$, for $\omega>0$, and the terms in which

$$\omega - \omega_{i_1} - \ldots - \omega_{i_{n-1}} \neq \omega_l \quad l \in \{-R, \ldots, -1, 1, \ldots, R\}$$

are zeros according to the definition of A(.) given by equation (19), and, moreover, $\overline{\mathbf{Y}}(j\omega)$ can be obtained by just making a summation of the results determined from (27) from n=1 to N.

In order to illustrate how to evaluate $\overline{\mathbf{Y}}_n(j\omega)$ using the above idea, consider an example where the OTA-C circuit in Section 2 is excited by a two-tone input

$$u(t) = \cos 2t + \cos 3t$$

and the second order output frequency response $\overline{\mathbf{Y}}_2(j\omega)$ of the circuit at frequencies $\omega = 1$ and $\omega = 3$ is to be examined.

In this case, $R = 2,\ A_1 = A_2 = 1,\ \omega_1 = 2,\ \omega_2 = 3$,

$$A(\omega) = \begin{cases} A_i = 1 & if\ \omega \in \{\omega_i,\ i = \pm 1, \pm 2\} = \{-3, -2, 2, 3\} \\ 0 & \text{otherwise} \end{cases}$$

and the second order GFRF

$$H_2(j\omega_1, j\omega_2) = \frac{g_2 R_o}{j(\omega_1 + \omega_2) R_o C + 1}$$

Therefore

$$\overline{\mathbf{Y}}_2(j\omega) = \frac{1}{2^{n_1}} \sum_{i_1=-2,\, i_1 \neq 0}^{2} A(\omega_{i_1}) A(\omega - \omega_{i_1}) H_2[j\omega_{i_1}, j(\omega - \omega_{i_1})]$$

$$= \frac{1}{2^{n_1}} \{A(\omega_{-2}) A(\omega - \omega_{-2}) H_2[j\omega_{-2}, j(\omega - \omega_{-2})]$$

$$+ A(\omega_{-1}) A(\omega - \omega_{-1}) H_2[j\omega_{-1}, j(\omega - \omega_{-1})]$$

$$+ A(\omega_1) A(\omega - \omega_1) H_2[j\omega_1, j(\omega - \omega_1)]$$

$$+ A(\omega_2) A(\omega - \omega_2) H_2[j\omega_2, j(\omega - \omega_2)]\}$$

$$= \frac{1}{2^{n_1}} \{A(-3) A(\omega + 3) H_2[-3j, (\omega + 3)j] + A(-2) A(\omega + 2) H_2[-2j, (\omega + 2)j]$$

$$+ A(2) A(\omega - 2) H_2[2j, (\omega - 2)j] + A(3) A(\omega - 3) H_2[3j, (\omega - 3)j]\}$$

$$= \frac{1}{2^{n_1}} \{A(\omega + 3) + A(\omega + 2) + A(\omega - 2) + A(\omega - 3)\} \frac{g_2 R_o}{j\omega R_o C + 1} \tag{28}$$

Thus $\overline{\mathbf{Y}}_2(j1)$ and $\overline{\mathbf{Y}}_2(j3)$ can be immediately obtained as

$$\overline{\mathbf{Y}}_2(j1) = \frac{1}{2^{2-1}} \{A(4) + A(3) + A(-1) + A(-2)\} \frac{g_2 R_o}{jR_o C + 1} = \frac{1}{2^{2-1}} \{0 + 1 + 0 + 1\} \frac{g_2 R_o}{jR_o C + 1} = \frac{g_2 R_o}{jR_o C + 1}$$

and

$$\overline{\mathbf{Y}}_2(j3) = \frac{1}{2^{2-1}} \{A(6) + A(5) + A(1) + A(0)\} \frac{g_2 R_o}{j3R_o C + 1} = \frac{1}{2^{2-1}} \{0 + 0 + 0 + 0\} \frac{g_2 R_o}{j3R_o C + 1} = 0$$

and the corresponding output components are therefore

$$y_{2(\omega=1)}(t)=\frac{|g_2|R_o}{\sqrt{(R_oC)^2+1}}\cos(t-\tan^{-1}R_oC)$$

and

$$y_{2(\omega=3)}(t)=0$$

Clearly, compared to the evaluation of an nth-order output response at a particular frequency ω using equation (17), the computation of this response based on equation (25) is much more straightforward and the difficulty with determining all possible frequency mix vectors for a specific frequency, which is necessary when equation (17) is used, is circumvented. Notice that the expression for $\overline{\mathbf{Y}}_2(j\omega)$ given in equation (25) or (27) accommodates all possible terms which could have contributions to frequency ω and, when given a specific value of ω, the terms which actually have no effect on the response at the specific frequency automatically become zero due to the definition of $A(\omega)$.

The analyses and examples above indicate that based on equation (25) output frequency responses of nonlinear systems under multiple inputs can easily be evaluated at any frequencies of interest. However it is obviously unnecessary to evaluate the response components at frequencies which are beyond the possible output frequencies since these components are definitely zero. To address this issue involves determining possible output frequencies of nonlinear systems subject to multiple input excitations.

4. Determination of output frequencies of nonlinear systems under multiple inputs

For linear systems it is well known that the possible output frequencies are exactly the same as the frequencies in the corresponding input. However, this property does not hold if the system is nonlinear. When a nonlinear system is subject to a multiple input, it has been shown from the analyses in previous sections that output frequencies generated by the nth-order system nonlinearity consist of all possible combinations of the input frequencies $-\omega_R,\cdots,-\omega_1,\omega_1,\cdots,\omega_R$ taken n at a time. This result can be analytically described as a set given by

$$\left\{\omega \mid \omega=\omega_{i_1}+,\ldots,+\omega_{i_n},\ \omega_{i_l}\in\{-\omega_R,,\ldots,-\omega_1,\omega_1,\ldots,\omega_R\},\ l=1,\ldots,n\right\} \qquad (29)$$

The problem to be addressed initially here is to develop an algorithm to determine the frequencies composed of the nonnegative part of the result given by (29).

For the simplest case of n=1, it is obvious that these frequencies are

$$\{\omega_1, \ldots, \omega_R\}$$

which can be rewritten in a vector form as

$$W_1 = \begin{bmatrix} \left|\sum \overline{W}_1(1,:)\right| \\ \vdots \\ \left|\sum \overline{W}_1(R,:)\right| \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_R \end{bmatrix} \tag{30}$$

where

$$\overline{W_1} = \begin{bmatrix} \omega_1 & \cdots & \omega_R \end{bmatrix}^T \tag{31}$$

and $\sum \overline{W_1}(l,:)$, $1 \le l \le R$, denotes the summation of the elements in the lth - row of matrix $\overline{W_1}$.

The output frequencies in the case of n=2 can be determined from

$$\begin{cases} |\omega_1 + \omega_l|, l = -R, \ldots, -1, 1, \ldots, R \\ \quad\vdots \quad \vdots \quad \vdots \\ |\omega_R + \omega_l|, l = -R, \ldots, -1, 1, \ldots, R \end{cases} \tag{32}$$

Define two vectors

$$I = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Big\} 2R \tag{33}$$

and

$$W = \begin{bmatrix} -\omega_R & \cdots & -\omega_1 & \omega_1 & \cdots & \omega_R \end{bmatrix}^T \tag{34}$$

to express equation (32) in terms of a vector as

$$W_2 = \begin{bmatrix} |\omega_1 - \omega_R| \\ \vdots \\ |\omega_1 - \omega_1| \\ |\omega_1 + \omega_1| \\ \vdots \\ |\omega_1 + \omega_R| \\ \vdots \\ \vdots \\ |\omega_R - \omega_R| \\ \cdots \\ |\omega_R - \omega_1| \\ |\omega_R + \omega_1| \\ \cdots \\ |\omega_R + \omega_R| \end{bmatrix} = \begin{bmatrix} \left|\sum \overline{W_2}(1,:)\right| \\ \vdots \\ \vdots \\ \left|\sum \overline{W_2}(2R^2,:)\right| \end{bmatrix} \tag{35}$$

where

$$\overline{W_2} = \begin{bmatrix} I\overline{W_1}(1,:) & W \\ \vdots & \vdots \\ I\overline{W_1}(R,:) & W \end{bmatrix} = \begin{bmatrix} \omega_1 & -\omega_R \\ \vdots & \vdots \\ \vdots & -\omega_1 \\ \vdots & \omega_1 \\ \vdots & \vdots \\ \omega_1 & \omega_R \\ \vdots & \vdots \\ \vdots & \vdots \\ \omega_R & -\omega_R \\ \vdots & \vdots \\ \vdots & -\omega_1 \\ \vdots & \omega_1 \\ \vdots & \vdots \\ \omega_R & \omega_R \end{bmatrix} \tag{36}$$

For n=3, it is easy to show that the vector representing the output frequencies produced by the third order nonlinearity is

$$W_3 = \begin{bmatrix} \left|\sum \overline{W_3}(1,:)\right| \\ \vdots \\ \vdots \\ \left|\sum \overline{W_3}(R(2R)^2,:)\right| \end{bmatrix} \tag{37}$$

where

$$\overline{W}_3 = \begin{bmatrix} I\overline{W}_2(1,:) & W \\ \vdots & \vdots \\ I\overline{W}_2(2R^2,:) & W \end{bmatrix} \tag{38}$$

Consequently the algorithm for computing the vector representing the (nonnegative) frequencies in the nth-order nonlinear output is given by

$$\begin{cases} W_n = \begin{bmatrix} \left|\sum \overline{W}_n(1,:)\right| \\ \vdots \\ \vdots \\ \left|\sum \overline{W}_n[R(2R)^{n-1},:]\right| \end{bmatrix} \\ \overline{W}_n = \begin{bmatrix} I\overline{W}_{n-1}(1,:) & W \\ \vdots & \vdots \\ I\overline{W}_{n-1}(R(2R)^{n-2},:) & W \end{bmatrix} \quad n \geq 2 \quad \overline{W}_1 = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_R \end{bmatrix} \end{cases} \tag{39}$$

Many of the elements in W_n may be the same. Therefore the final result of this algorithm is a set composed of all different elements of W_n. Denote this set as Ω_n then

$$\Omega_n = \{\{ W_n \}\} \tag{40}$$

where $\{\{X\}\}$ means a set composed of all the different elements of vector X.

In order to illustrate the application of this algorithm, consider an example where

$\omega_1 = 1,\ \omega_2 = 3,$

and the frequencies in the second order nonlinear output is to be determined.

In this case,

$R = 2,\ n = 2,\ W_1 = [\omega_1, .., \omega_R]^T = [1,3]^T,$

$I = [1,1,1,1]^T,\ W = [-3,-1,1,3]^T,\ \overline{W}_1 = W_1,$

$$\overline{W}_2 = \begin{bmatrix} I\overline{W}_1(1,:) & W \\ \vdots & \vdots \\ I\overline{W}_1(R,:) & W \end{bmatrix} = \begin{bmatrix} \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \times 1 & \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \\ \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \times 3 & \begin{bmatrix}-3\\-1\\1\\3\end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \\ 3 & -3 \\ 3 & -1 \\ 3 & 1 \\ 3 & 3 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} \left|\sum \overline{W_2}(1,:)\right| \\ \vdots \\ \left|\sum \overline{W_2}(2(2\times 2)^1,:)\right| \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \\ 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$$

therefore

$$\Omega_2 = \{\{\ W_2\ \}\} = \{0,2,4,6\}.$$

From the above algorithm Ω_n can be determined for any n. Therefore, the frequencies in the system output represented by Ω can be obtained as

$$\Omega = \bigcup_{n=1}^{N} \Omega_n$$

However, there is actually no need to obtain all Ω_n's and then to determine Ω as shown above. This is because for any n there is a deterministic relationship between the frequencies in the nth-order nonlinear output and the frequencies in the (n+2)th-order nonlinear output.

It can be shown from (39) that

$$\overline{W}_{n+2} = \begin{bmatrix} I\overline{W}_{n+1}(1,:) & W \\ \vdots & \vdots \\ I\overline{W}_{n+1}(R(2R)^n,:) & W \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \begin{bmatrix} \overline{W}_{n-1}(1,:) \\ \vdots \\ \vdots \\ \overline{W}_{n+1}(R(2R)^n,:) \end{bmatrix} & \begin{matrix} W \\ \vdots \\ \vdots \\ W \end{matrix} \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \overline{W}_{n+1} & \begin{matrix} W \\ \vdots \\ \vdots \\ W \end{matrix} \end{bmatrix} \quad (41)$$

and similarly

$$\overline{W}_{n+1} = \begin{bmatrix} \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^{n-1}} \overline{W}_n & \begin{matrix} W \\ \vdots \\ \vdots \\ W \end{matrix} \end{bmatrix} \quad (42)$$

Substituting (42) into (41) yields

$$\overline{W}_{n-2} = \left[\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \left[\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^{n-1}} \overline{W}_n , \quad \left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^{n-1} \right] , \quad \left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^n \right]$$

$$= \left[\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^{n-1}} \overline{W}_n , \quad \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^{n-1} , \quad \left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^n \right]$$

(43)

In the matrix given by (43), the first matrix block takes the form

$$\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^{n-1}} \overline{W}_n = \left[\underbrace{\overline{W}_n(1,:)^T \ \cdots \ \overline{W}_n(1,:)^T}_{(2R)^2} \cdots \cdots \underbrace{\overline{W}_n(R(2R)^{n-1},:)^T \ \cdots \ \overline{W}_n(R(2R)^{n-1},:)^T}_{(2R)^2} \right]^T$$

(44)

the second matrix block becomes

$$\underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}}_{R(2R)^n} \left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^{n-1} = \left[\underbrace{\underbrace{[\omega_{-R} \cdots \omega_{-R}}_{2R} \cdots\cdots \underbrace{\omega_{R} \cdots \omega_{R}]}_{2R} \cdots\cdots\cdots \underbrace{[\omega_{-R} \cdots \omega_{-R}}_{2R} \cdots\cdots \underbrace{\omega_{R} \cdots \omega_{R}]}_{2R}}_{R(2R)^{n-1}} \right]^T$$

(45)

and the third one becomes

$$\left.\begin{bmatrix} W \\ \vdots \\ \vdots \\ W \end{bmatrix}\right\} R(2R)^n = \left[\underbrace{\underbrace{[\omega_{-R}\cdots\omega_R]\cdots\cdots[\omega_{-R}\cdots\omega_R]}_{2R} \quad \cdots\cdots\cdots \quad \underbrace{[\omega_{-R}\cdots\omega_R]\cdots\cdots[\omega_{-R}\cdots\omega_R]}_{2R}}_{R(2R)^{n-1}} \right]^T \tag{46}$$

It is not difficult to observe from (44), (45) and (46) that elements of vector

$$W_{n+2} = \begin{bmatrix} \left|\sum \overline{W}_{n+2}(1,:)\right| \\ \vdots \\ \vdots \\ \left|\sum \overline{W}_{n+2}[R(2R)^{n+1},:]\right| \end{bmatrix}$$

include all elements of the vector

$$W_n = \begin{bmatrix} \left|\sum \overline{W}_n(1,:)\right| \\ \vdots \\ \vdots \\ \left|\sum \overline{W}_n[R(2R)^{n-1},:]\right| \end{bmatrix}$$

This implies that

$$\{\{W_n\}\} \in \{\{W_{n+2}\}\}$$

that is

$$\Omega_n \in \Omega_{n+2} \tag{47}$$

or all frequencies in the nth-order nonlinear output are present in the (n+2)th-order nonlinear output.

This conclusion was proved before [5] under the assumption that $\omega_1,\cdots,\omega_R$ form a frequency base which means there does not exist a set of rational numbers $\{r_1,\cdots,r_R\}$ (not all zero) such that

$$r_1\omega_1+,\cdots,+r_R\omega_R = 0$$

Since no assumptions are made on $\omega_1,\cdots,\omega_R$ in the above derivation, the conclusion has now been established for arbitrary input frequencies.

It is straightforward from (47) that the frequencies in the system output

$$\Omega = \Omega_N \bigcup \Omega_{N-(2p^*-1)} \tag{48}$$

where the value to be taken by p^* could be $1,2,\ldots,[N/2]$ where $[.]$ denotes to take the integer part. The specific value of p^* depends on the system nonlinearities. If the system GFRFs $H_{N-(2i-1)}(.) = 0$, for $i = 1,\ldots,q-1$, and $H_{N-(2q-1)}(.) \neq 0$, then $p^* = q$.

In the example above where a nonlinear system is subject to a two tone input with $\omega_1 = 1$, $\omega_2 = 3$, assume that the maximum nonlinear order N=2 and the first order frequency response function $H_1(.) \neq 0$. It is straightforward to show that in this case $p^* = 1$. Therefore, the output frequencies of the system can be determined using (48) as

$$\Omega = \Omega_N \bigcup \Omega_{N-(2p^*-1)} = \Omega_2 \bigcup \Omega_1 = \{0,2,4,6\} \bigcup \{1,3\} = \{0,1,2,3,4,6\}$$

Equations (39), (40), and (48) compose an algorithm for determining output frequencies of nonlinear systems under multiple inputs. This result actually extends the relationship between the frequencies in the input and output of linear systems to the nonlinear case when the systems are subject to multiple inputs and is therefore also of theoretical significance.

5. New method for evaluating nonlinear output frequency responses to multiple inputs

It has been shown in Section 3 that, based on the expression for the output frequency responses of nonlinear systems to multiple inputs given by equation (25), output components of nonlinear systems under multiple inputs at any frequencies of interest can be readily evaluated. Equations (39), (40), and (48) derived in Section 4 provide an effective algorithm for determining possible output frequencies of nonlinear systems in the multiple input situation. Based on these two theoretical results, a new method is proposed below to evaluate nonlinear output responses to multiple inputs.

The basic idea of this new method is to determine all possible system output frequencies and the frequencies contributed by each order of system nonlinearities using the algorithm derived in Section 4. Thus, if the frequencies of interest are beyond the range of possible output frequencies, it is known immediately that the output responses at these frequencies are zero. If the frequencies of interest are within the range of possible output frequencies then the frequencies contributed by each order of system nonlinearities provide important information concerning which order of system nonlinearities could have contribution to these frequencies of interest. Moreover, system output responses at the frequencies of interest are evaluated using equation (25) and the computation is implemented by first calculating the responses at these frequencies contributed by the nonlinear orders which really have contributions to these frequency components and then simply making a summation of the results obtained for corresponding nonlinear orders.

The procedure of the new method is summarised in the following, which requires the frequency domain model of the considered nonlinear system, i.e. the GFRFs, $H_n(j\omega_1,\cdots,j\omega_n)$, $n=1,\ldots,N$, are known a priori,

(i) Calculate all possible output frequencies using equations (39), (40) and (48) to yield the set Ω.

(ii) For n=1,2,...,N, calculate Ω_n to determine a set S_Ω which is composed of the numbers of the nonlinearity orders which have contributions to the output frequency $\omega_A \in \Omega$ at which the output component is to be evaluated.

(iii) Compute $\overline{\mathbf{Y}}_n(j\omega_A)$ as below

$$\overline{\mathbf{Y}}(j\omega_A)=\sum_{n\in S_\Omega}\overline{\mathbf{Y}}_n(j\omega_A) \tag{49}$$

where

$$\begin{aligned}\overline{\mathbf{Y}}_n[j\omega_A]&=\frac{1}{2^{n_1}}\sum_{\omega_{i_1}+,\cdots,+\omega_{i_n}=\omega_A}A(\omega_{i_1})\ldots A(\omega_{i_n})H_n(j\omega_{i_1},\ldots,j\omega_{i_n})\\&=\frac{1}{2^{n_1}}\sum_{i_1=-R,i_1\neq 0}^{R}\cdots\sum_{i_{n-1}=-R,i_{n-1}\neq 0}^{R}A(\omega_{i_1})\ldots A(\omega_{i_{n-1}})A(\omega_A-\omega_{i_1}-,\ldots,-\omega_{i_{n-1}})\\&\qquad H_n(j\omega_{i_1},\ldots,j\omega_{i_{n-1}},\omega_A-\omega_{i_1}-,\ldots,-\omega_{i_{n-1}})\end{aligned} \tag{50}$$

$$\text{and} \quad n_1 = \begin{cases} n & \text{if } \omega_A = 0 \\ n-1 & \text{otherwise} \end{cases}$$

(iv) Evaluate the output response at frequency ω_A as

$$y_{\omega_A}(t) = |\overline{Y}(j\omega_A)|\cos(\omega_A t + \angle\overline{Y}(j\omega_A)) \tag{51}$$

The example of the OTA-C circuit in Section 1 is considered again to illustrate the application of this method in the following.

Assume that $f[v_a(t)]$ can be approximated sufficiently well by a third order polynomial

$$f[v_a(t)] = \sum_{n=1}^{3} g_n v_a^n(t)$$

where $g_n \neq 0$, for $n = 1,2,3$, and the output response of the circuit to the multiple input

$$u(t) = \cos\omega_1 t + \cos\omega_2 t = \cos t + \cos 3t$$

at the frequency of interest $\omega_A = 5$ is to be evaluated.

The GFRFs of the circuit system are given by equation (8). They are all zero in this case for $n > 3$ but not zero for $n = 1,2,3$ because $f[v_a(t)]$ can be approximated well by a third order polynomial and $g_n \neq 0$, for $n = 1,2,3$. Obviously the maximum order of system nonlinearities in this case is $N = 3$.

Because $H_2(.) \neq 0$, $p^* = 1$. Thus, using (39), (40) and (48) with

$$R = 2, \quad \overline{W_1} = [1,3]', \quad I = [1,1,1,1]', \quad \text{and} \quad W = [-3,-1,1,3]'$$

yields

$$\Omega = \Omega_3 \cup \Omega_2 = \{0,1,2,3,4,5,6,7,9\}$$

indicating that $\omega_A = 5$ belongs to the frequencies which possibly appear in the system output.

Ω_1, Ω_2, Ω_3 obtained using (39) in this case are

$$\Omega_1 = \{1,3\}, \quad \Omega_2 = \{0,2,4,6\}, \quad \Omega_3 = \{1,3,5,7,9\}$$

So $S_\Omega = \{3\}$ for the frequency of interest $\omega_A = 5$.

Using equation (50) and considering that in this specific case,

$$R = 2,\ A_1 = A_2 = 1,\ \omega_1 = 1,\ \omega_2 = 3,$$

$$A(\omega) = \begin{cases} A_i = 1 & if\ \omega \in \{\omega_i,\ i = \pm 1, \pm 2\} = \{-3, -1, 1, 3\} \\ 0 & \text{otherwise} \end{cases}$$

$$H_3(j\omega_1, j\omega_2, j\omega_3) = \frac{g_3 R_o}{j(\omega_1 + \omega_2 + \omega_3)R_o C + 1}$$

it follows that

$$\overline{\mathbf{Y}}(j\omega_A) = \sum_{n \in S_\Omega} \overline{\mathbf{Y}}_n(j\omega_A) = \overline{\mathbf{Y}}_3(j5)$$

$$= \frac{1}{4} \sum_{i_1=-2, i_1 \neq 0}^{2} \sum_{i_2=-2, i_2 \neq 0}^{2} A(\omega_{i_1})A(\omega_{i_2})A(5-\omega_{i_1}-\omega_{i_2})H_3[j\omega_{i_1}, j\omega_{i_2}, j(5-\omega_{i_1}-\omega_{i_2})]$$

$$= \frac{1}{4}\frac{g_3 R_o}{j5R_o C + 1}\begin{Bmatrix} A(5-\omega_{-2}-\omega_{-2}) + A(5-\omega_{-2}-\omega_{-1}) + A(5-\omega_{-2}-\omega_1) + A(5-\omega_{-2}-\omega_2) \\ A(5-\omega_{-1}-\omega_{-2}) + A(5-\omega_{-1}-\omega_{-1}) + A(5-\omega_{-1}-\omega_1) + A(5-\omega_{-1}-\omega_2) \\ A(5-\omega_1-\omega_{-2}) + A(5-\omega_1-\omega_{-1}) + A(5-\omega_1-\omega_1) + A(5-\omega_1-\omega_2) \\ A(5-\omega_2-\omega_{-2}) + A(5-\omega_2-\omega_{-1}) + A(5-\omega_2-\omega_1) + A(5-\omega_2-\omega_2) \end{Bmatrix}$$

$$= \frac{1}{4}\frac{g_3 R_o}{j5R_o C + 1}\begin{Bmatrix} A(5+3+3) + A(5+3+1) + A(5+3-1) + A(5+3-3) \\ A(5+1+3) + A(5+1+1) + A(5+1-1) + A(5+1-3) \\ A(5-1+3) + A(5-1+1) + A(5-1-1) + A(5-1-3) \\ A(5-3+3) + A(5-3+1) + A(5-3-1) + A(5-3-3) \end{Bmatrix}$$

$$= \frac{1}{4}\frac{g_3 R_o}{j5R_o C + 1}\begin{Bmatrix} A(3) \\ + A(3) + A(1) \\ + A(3) + A(1) + A(-1) \end{Bmatrix} = \frac{6}{4}\frac{g_3 R_o}{j5R_o C + 1}$$

Thus, in this case, the output response component of the circuit at $\omega_A = 5$ is

$$y_{\omega_A=5}(t) = \frac{1.5|g_3|R_o}{\sqrt{(5R_0 C)^2 + 1}}\cos\left(5\mathrm{t} - \angle\tan^{-1}(5\mathrm{R_oC})\right)$$

The method developed and illustrated above provides an effective means for evaluating the output frequency responses of nonlinear systems under multiple inputs based on the system frequency domain descriptions. Exact evaluation of system output frequency responses can only be achieved using both system models and exact knowledge of the corresponding input spectra. Multiple input signals can easily be generated with all parameters of the signals under control. Methods are currently available for estimating the GFRFs of nonlinear systems [9] [10] [11] [12] and for many systems such as electronic circuits the GFRFs can even be derived directly from the system structure and parameters. Therefore, this method can, hopefully, be widely applied to analyse nonlinear behaviours of practical systems including electronic circuits at the system/circuit design and simulation stages.

6. Conclusions

The behaviour of practical systems, including electronic circuits, usually exhibits nonlinear characteristics although measures are often taken to try to compensate for undesirable nonlinear effects. It is therefore important to evaluate system output responses so as to estimate how the nonlinearities affect the system performance. Multiple inputs are typical signals which are used to excite systems when the system performance in the frequency domain is to be investigated. The existing methods for analysis of the responses of nonlinear systems under multiple inputs are almost all based on the concept of 'frequency mix vector'. This concept is useful for explaining how the output frequencies of nonlinear systems are generated but it is difficult to use to evaluate the output response at frequencies of interest. In order to overcome this problem, a new method is developed in the present study to evaluate the frequency domain responses of nonlinear systems under multiple inputs. This is based on two theoretical results concerning the output frequency response of nonlinear systems to multiple inputs and the determination of the output frequencies of nonlinear systems subject to multiple input excitations. This new method provides an effective means for evaluating nonlinear output frequency responses to multiple inputs from system frequency domain models and nonlinearities and will hopefully find wide applications at the system design and simulation stages.

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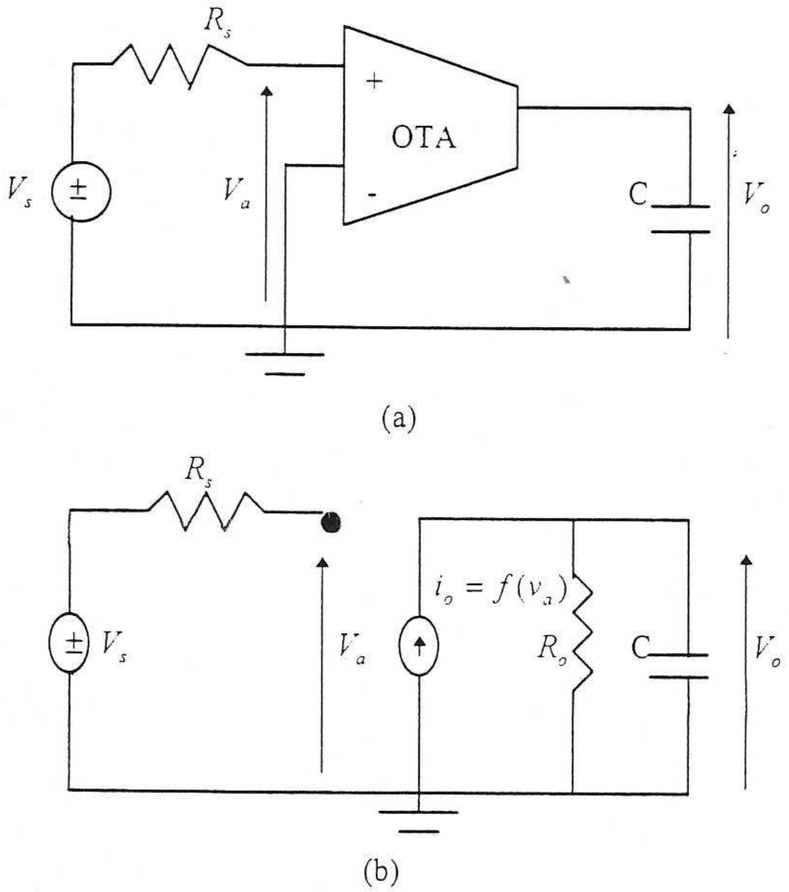

Figure 1. (a) OTA-C integrator (b) Nonlinear equivalent circuit

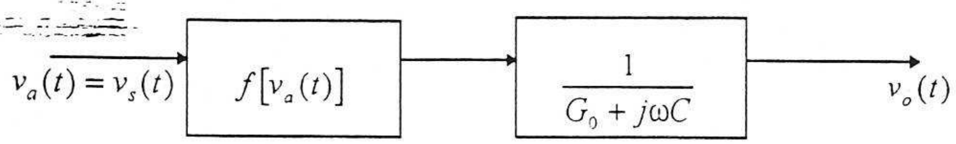

Figure 2. Block diagram of the OTA-C integrator