Evaluation of a Bound on Nonlinear Output Frequency Responses Using a Genetic Algorithm

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Abstract - In this paper a Genetic Algorithm (GA) based approach is proposed to evaluate a bound on output frequency responses of nonlinear systems. In this approach a GA is applied to solve an optimisation problem associated with generalised frequency response functions of nonlinear systems. The result is then used, together with other techniques developed by the authors, to evaluate a bound on the system output frequency response. The new bound is shown to be more accurate and less conservative compared to the result obtained using previous methods and a simulation study is included to demonstrate the effectiveness and performance of the new approach. The proposed approach is of practical importance because the bound can be used as part of engineering designs when it is necessary to assess the response of nonlinear mechanical or civil engineering structures to maximum loads.

1. Introduction

System theories and methods are usually developed either in the time or in the frequency domain. The time domain approach is useful when the problem is stated in terms of a differential or difference equation where time is the primary variable. However, although the differential and difference equations are basic system descriptions, many specifications and criteria for systems in mechanical, electrical and electronic, and control engineering are expressed in terms of steady-state frequency domain concepts such as the magnitude and phase responses. This results in the extensive use of the Fourier and/or Laplace transforms for analysis and design of systems, leading to frequency as the independent variable rather than time. Historically, frequency domain methods of linear systems tended to dominate the early theory and practice of system studies and were the foundation of classical control and signal processing techniques. This is due largely to the mathematical simplicity that linear frequency domain methods afford and the physical insight that the methods produce in linear system analysis. The extension of frequency domain analysis methods to nonlinear systems began in the late 1950s where the concept of Generalised Frequency Response Functions (GFRFs) was introduced by George in 1959 [1]. GFRFs represent the nonlinear frequency response behaviours in the form of multidimensional transfer functions and extend the concept of the linear system transfer function to the nonlinear case. The estimation and analysis of the GFRFs have been studied by several workers [2]-[12] and important properties and characteristics of practical nonlinear systems have been investigated on the basis of a graphical interpretation of these functions [13][14].

An important issue concerning system analysis in the frequency domain involves investigating how a system output frequency response is determined as a function of the frequency characteristics of the input and the frequency domain properties of the system. Solutions to this rely on the analysis of the relationship between the frequency characteristics of the system input and output. In the linear system case, this frequency domain input and output relationship is well known, the spectra of the system input and output are linearly related by the system frequency response function. However, extension of the linear result to the nonlinear case is far from straightforward. The analytical expression of output frequency responses of nonlinear systems is associated with the frequency domain Volterra series expansion which involves the summation of the association of variables [15] up to possibly an infinite order of system nonlinearities. In order to simplify the analysis for nonlinear system output frequency responses, in our previous work [16][17], a concept known as the bound on output frequency responses of nonlinear systems was introduced and an expression for the bound was derived. The result reveals how the effects of the GFRFs and the input characteristics can be analysed as separate influences on system output frequency responses.

The bound is in fact the worse case output frequency response of nonlinear systems which is believed to be of considerable engineering significance because practical engineering designs in mechanical and civil engineering, for example, often consider situations where associated structures are subject to possibly maximum loads. In the previous work [16][17], practical computation of this bound involved a recursive evaluation of gain bounds on the GFRFs of nonlinear systems. This may sometimes result in a relatively conservative result due to the amplifying effects of the recursive evaluation algorithm which deals with an optimisation problem associated with GFRFs in an indirect way. In order to overcome this problem, in the present study, a Genetic Algorithm (GA) based approach is proposed to evaluate the bound. GAs are optima searching algorithms based on the mechanics of natural selection and natural genetics, which only use information of objective functions and require no derivatives or other auxiliary knowledge. This advantage implies that GAs are very appropriate tools when dealing directly with optimisation problems associated with GFRFs because the analytical expressions of GFRFs are normally so complicated that derivatives or other auxiliary knowledge are difficult to use. The new approach employs a GA to evaluate the gain bound on the nth-order GFRF over the n-dimensional frequency space hyper-plane $\omega_1+\cdots+\omega_n=\omega$ so as to produce a more accurate and less conservative result regarding output frequency responses of nonlinear systems. The result can be used to analyse the behaviours of nonlinear systems in the frequency domain and can also be taken as the basis for engineering designs when the nonlinear systems involved represent structures in mechanical or civil engineering. A simulation study is included to illustrate the implementation and demonstrate the effectiveness of the new approach.

2. Bound on the output frequency responses of nonlinear systems

In the linear system case, it is well known that the frequency domain relationship between the system input and output can be simply written as

$$Y(j\omega)=H(j\omega)U(j\omega) \qquad (2.1)$$

where $Y(j\omega)$ and $U(j\omega)$ represent the Fourier transforms of the system output and input and $H(j\omega)$ is the system frequency response function. For nonlinear systems which are stable at the zero equilibrium point and can be described in the neighbourhood of the point by the Volterra series [18], the extension of the linear frequency domain description (2.1) can be obtained as [19]

$$Y(j\omega) = \sum_{n=1}^{N} Y_n(j\omega) \tag{2.2}$$

where

$$Y_n(j\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\cdots\omega_n=\omega} H_n(j\omega_1 \cdots j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_\omega \tag{2.3}$$

and N is the maximum order of the system nonlinearities.

In (2.2) and (2.3), $Y_n(j\omega)$ represents the nth-order system output frequency characteristic, $H_n(j\omega_1 \cdots j\omega_n)$ is known as the nth-order GFRF of the system, and $\int_{\omega_1+\cdots+\omega_n=\omega} (.)\, d\sigma_\omega$ denotes the integration of (.) over the n-dimensional hyperplane $\omega = \omega_1 + \cdots + \omega_n$.

In order to simplify the analysis of output frequency responses of nonlinear systems, a concept known as the bound on the system output frequency responses was proposed and an expression for the bound was derived from (2.2) and (2.3) [16][17] to yield a simplified nonlinear system input and output frequency domain relationship such that

$$Y^B(\omega) = \sum_{n=1}^{N} Y_n^B(\omega) \tag{2.4}$$

where

$$Y_n^B(\omega) = \frac{1}{(2\pi)^{n-1}} \left| H_n\left(j\omega_{n1}^{\#} \cdots j\omega_{nn}^{\#}\right) \right| \underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{2.5}$$

In (2.4) and (2.5), $Y^B(\omega)$ and $Y_n^B(\omega)$ represent the bound on the system magnitude frequency response $|Y(j\omega)|$ and the bound on the system nth-order magnitude frequency response $|Y_n(j\omega)|$ respectively, $\{\omega_{n1}^{\#}, \omega_{n2}^{\#}, \ldots, \omega_{nn}^{\#}\}$ denotes the co-ordinates of a point on the n-dimensional hyperplane $\omega_1 +, \ldots, +\omega_n = \omega$, and $\underbrace{|U|*\ldots*|U(j\omega)|}_{n}$ represents the n-dimensional convolution integration of the input magnitude frequency characteristic defined by

$$\underbrace{|U|*\ldots*|U(j\omega)|}_{n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |U(j\omega_1)|, \ldots, |U(j\omega_{n-1})| \; |U[j(\omega - \omega_1, \ldots, -\omega_{n-1})]| d\omega_1\, d\omega_2, \ldots d\omega_{n-1} \tag{2.6}$$

Concerning the relationship with $|Y(j\omega)|$, the new concept $Y^B(\omega)$ possesses the following properties.

(i) $Y^B(\omega) \geq |Y(j\omega)|$

(ii) $Y^B(\omega) = |Y(j\omega)|$ for N = 1

(iii) $Y^B(\omega) = |Y(j\omega)| = 0$ when ω is beyond the frequency range [20] produced by the system nonlinearities up to the Nth order

The practical evaluation of the bound $Y^B(\omega)$ involves the calculation of the n-dimensional convolution integration (2.6) and evaluation of a bound on the GFRF denoted by $|H_n(j\omega_1 \cdots j\omega_n)|_\omega^B$ where $\omega_1, \cdots, \omega_n$ are subject to the constraint

$$\omega_1 +, \ldots, +\omega_n = \omega$$

because the exact position of the point $\{\omega_{n1}^\#, \omega_{n2}^\#, \ldots, \omega_{nn}^\#\}$ is hard to determine. The result obtained is

$$\overline{Y}^B(\omega) = \sum_{n=1}^{N} \overline{Y}_n^B(\omega) \tag{2.7}$$

where

$$\overline{Y}_n^B(\omega) = \frac{1}{(2\pi)^{(n-1)}} |H_n(j\omega_1, \ldots, j\omega_n)|_\omega^B \underbrace{|U|*, \ldots, *|U(j\omega)|}_{n} \tag{2.8}$$

which possesses the same properties as $Y^B(\omega)$ in terms of the relationship with $|Y(j\omega)|$.

In the previous work [16][17], an effective algorithm has been developed to calculate the n-dimensional convolution integration above but $|H_n(j\omega_1 \cdots j\omega_n)|_\omega^B$ was evaluated indirectly using a recursive algorithm for the gain bound of the GFRF $H_n(j\omega_1 \cdots j\omega_n)$ which, due to the amplifying effects of the recursive calculation, may sometimes result in a relatively conservative result.

The problem of evaluating $|H_n(j\omega_1 \cdots j\omega_n)|_\omega^B$ is in fact an optimisation problem associated with the magnitude characteristic of the GFRF $H_n(j\omega_1 \cdots j\omega_n)$ such that

$$\max_{\substack{\omega \cdots \omega_{n1} \\ \text{s.t. } \omega_1 + \cdots + \omega_n = \omega}} |H_n(j\omega_1 \cdots j\omega_n)| \tag{2.9}$$

Therefore, applying an effective optimisation method to solve this problem directly is the key point if a more accurate and less conservative $\overline{Y}^B(\omega)$ is to be obtained

3. Evaluation of $|H_n(j\omega_1 \cdots j\omega_n)|_\omega^B$ using a genetic algorithm

The GA is a stochastic global search method that mimics the metaphor of natural biological evolution. GAs operate on a population of potential solutions applying the principle of survival of the fittest to produce hopefully better and better approximations to a solution.

When a simple GA is applied, the individuals in a population of potential solutions which are initially given randomly are first encoded as strings composed over some alphabet(s) such as, for example, the binary alphabet $\{0,1\}$. Then, three operators known as reproduction, crossover, and mutation which are borrowed from natural genetics are applied to the strings respectively to produce the next generation of potential solutions which are hopefully better solutions due to the effects of these operations.

The specific operations of using a GA for the optimisation problem (2.9) are described below to illustrate how a GA can be applied to deal with the optimisation problem associated with GFRFs of nonlinear systems.

3.1 Encoding

For this specific application of a genetic algorithm, the individuals in each generation represent potential solutions to problem (2.9). Therefore the n variables $\omega_1,\cdots,\omega_n$ in each individual are subject to the constraint $\omega_1+,\ldots,+\omega_n=\omega$ and because of this only n-1 variables are independent and need to be encoded before GA operators are applied. Thus the encoding for each individual is implemented as follows

binary string 1	binary string 2	... ...	binary string n-1

where the binary strings are strings composed of 0 and 1 and are all of the same length.

Grey coding is used for the conversion between binary strings and real values. For example, in the case of n=3 and when values taken by ω_1,ω_2, and ω_3 are all in the range of [-b.b], the code

1101 0010

represents an individual with

$$\omega_1=\frac{2b}{\left(2^3+2^2+2+1\right)}\left(1\times 2^3+1\times 2^2+0\times 2+1\times 2^0\right)-b=\frac{11}{15}b$$

$$\omega_2=\frac{2b}{\left(2^3+2^2+2+1\right)}\left(0\times 2^3+0\times 2^2+1\times 2+0\times 2^0\right)-b=-\frac{11}{15}b$$

$$\omega_3=\omega-\omega_1-\omega_2=\omega$$

3.2 Reproduction

Reproduction is a process in which individual strings are copied according to their fitness function values. The fitness function is normally used to transform an objective function value of the involved optimisation problem into a measure of relative fitness, thus

$$F(x)=g(f(x)) \tag{3.1}$$

where f is the objective function, g transforms the values of the objective function to non-negative numbers and F is the resulting relative fitness. Copying strings according to their fitness values implies that strings with a higher value have a higher probability of contributing one or more offspring in the next generation. This is an artificial version of natural selection, a Darwinian survival of the fittest among string creatures.

For the present application, the transformation from the objective function value to the relative fitness is implemented in a way such that individuals are assigned a fitness according to equation

$$F(x_i)=2-MAX+2(MAX-1)\frac{x_i-1}{Nind-1} \tag{3.2}$$

where MAX is a number typically chosen in the interval [1.1,2]. $Nind$ is the total number of individuals in one generation, and x_i is the position of i-th individual in the generation ordered in terms of the objective function values of individuals.

The reproduction operator is implemented using a biased roulette wheel where each individual in the generation has a roulette wheel slot size in proportion to its fitness. A random number in the range of 0 to 2π is generated. A copy of the string of an individual is selected for mating for the offspring in the next generation if the random number falls in the slot corresponding to the individual. This process is repeated until the desired number of individuals have been selected.

For example, consider an discrete time nonlinear system with the difference equation description

$$y(t) = 0.5u(t-1) + 0.8y(t-1) - 0.64y(t-2) - 0.4u(t-1)u(t-3) + 0.2y(t-1)u(t-1) \tag{3.3}$$

and evaluate the maximum value of the third order GFRF $H_3(j\omega_1, j\omega_2, j\omega_3)$ of the system under the constraint $\omega_1 + \omega_2 + \omega_3 = 0$.

Suppose that in this case $\omega_1, \omega_2, \omega_3$ are all in the range of [-b,b] with b=1 and the initial generation of individuals is given randomly as follows

0010 1100
0111 0111
1101 1011
1001 0011
1001 1101

which represent a generation of individuals in real values as follows

-0.6000	0.0667	0.5333
-0.3333	-0.3333	0.6667
0.2000	0.7333	-0.9333
0.8667	-0.7333	-0.1333
0.8667	0.2000	-1.0667

The objective function values of individuals in this generation can be obtained by calculating the corresponding $|H_3(j\omega_1, j\omega_2, j\omega_3)|$ using the recursive algorithm for computing GFRFs developed in [8] to yield the result

codes of individuals	objective function values
0010 1100	0.0259
0111 0111	0.0375
1101 1011	0.0760
1001 0011	0.0546
1001 1101	0.0551

which, after sorting in ascending order, can be written as

codes of individuals	objective function values
0010 1100	0.0259
0111 0111	0.0375
1001 0011	0.0546
1001 1101	0.0551
1101 1011	0.0760

The corresponding fitness values can then be obtained from (3.2) with $MAX = 2$ as

codes of individuals	fitness values
0010 1100	0
0111 0111	0.5
1101 1011	2
1001 0011	1
1001 1101	1.5

Thus , by implementation of the reproduction operation using a biased roulette wheel based on the above fitness values, four (equal to 5×0.8 where 0.8 is known as the generation gap representing the desired number of new individuals to be produced from the present generation) individuals are selected as

codes of selected individuals	objective function values
0111 0111	0.0375
1001 0011	0.0546
1001 1101	0.0551
1101 1011	0.0760

Clearly, after the reproduction operation, the individuals with relatively greater objective function values are selected. The results will be further processed using the next GA operator, crossover.

3.3 Crossover

Crossover is the most important GA operator. This operator first randomly selects two strings of individuals from the results of the reproduction operation and then exchanges the right parts of the two strings from a randomly selected crossover point. Thus the resulting offspring strings possess the characteristics of both parent strings and compared with the parents the offspring can hopefully represent similar or even better solutions to the corresponding optimisation problem according to the principles in natural genetics.

For the example above, the crossover operator produces the following result

codes of individuals after crossover	objective function values
0111 0011	0.0829
1001 0111	0.0748
1001 1011	0.0787
1101 1101	0.0606

It can be observed that the first and second strings of selected individuals exchange their right parts from the fifth position to yield the first and second individuals after crossover, and the third and fourth strings of the selected individuals exchange their right parts from the sixth position to yield the third and fourth individuals after crossover.

3.4 Mutation

Mutation is a local operator which implements an occasional (with small probability) random alteration of the value of a string position. In the case of binary coding, this simply means changing a 1 to a 0 and vice visa. Mutation is needed because even though reproduction and crossover effectively search and recombine extant notions, occasionally they may become premature and lose some potentially useful genetic material (1's or 0's at particular locations). In the above example, the mutation operator is implemented with a probability $0.7/[(n-1)\times 4] = 0.7/8$ and the results obtained from the above codes of the individuals after crossover are

codes of individuals after mutation	objective function values
0111 1001	0.0748
1001 0110	0.0774
1101 1011	0.0760
1101 1101	0.0606

where it can be observed that mutation operators take effect in the strings of the 1st, 2nd and 3rd individuals.

3.5 Reinsertion

In many applications of GAs such as in the example above, fewer individuals are produced by reproduction operation than the size of the original generation. This was determined by the generation gap factor which represents the desired number of new individuals. To maintain the size of the original generation, after reproduction, crossover, and mutation, the obtained new individuals have to be reinserted into the old generation to yield the next generation. When selecting which members of the old generation should be replaced, the most apparent strategy is to replace the least fit members determinstically. In the above example, reinserting the individuals after mutation into the original generation based on this strategy yields

codes of individuals in the new generation	objective function values
0111 1001	0.0748
1001 0110	0.0774
1101 1011	0.0760
1101 1011	0.0760
1101 1101	0.0606

It can be found that except for the 4th individual the individuals in the old generation are all replaced by the offspring obtained after the three GA operations. What is worth pointing out here is that in this example although the new generation of individuals seem not much better than the old one in terms of objective function values, the trend of pursuing the better solutions can still be observed from the above described process which is only the operations of a GA for one generation.

After repeating the above process for 500 times, that is, after 500 generations of GA operations, the objective function values of the obtained individuals are

0.1480
0.0296
0.1480
0.1480
0.1480

where 0.1480 is just the maximum value of GFRF $|H_3(j\omega_1, j\omega_2, j\omega_3)|$ when $\omega_1, \omega_2, \omega_3$ are subject to the constraints $\omega_1 + \omega_2 + \omega_3 = 0$ and $\omega_i \in [-1,1]$, $i = 1,2,3$.

4. Procedures of the GA based approach and a simulation example

4.1 The procedures

The procedures of a GA based approach for evaluating $\overline{Y}^B(\omega)$ defined by (2.7) and (2.8) are summarised in the following which can easily be implemented in practical applications.

(i) Establish from the input and output data using the NARMAX methodology [21]-[25] or derive from a differential equation description, a NARX (nonlinear autoregresive with exogenous input) model of the nonlinear system under study such that

$$y(t) = \sum_{m=1}^{\overline{M}} y_m(t) \qquad (4.1)$$

where $y_m(t)$, the 'NARX mth-order output' of the system, is given by

$$y_m(t) = \sum_{p=0}^{m} \sum_{k_1, k_{p+q}=1}^{K} c_{pq}(k_1, \ldots, k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \qquad (4.2)$$

with

$$p+q=m, \quad k_i = 1,\ldots,K, \; i = 1,\ldots,p+q, \text{ and } \sum_{k_1, k_{p+q}=1}^{K} \equiv \sum_{k_1=1}^{K} \cdots \sum_{k_{p+q}=1}^{K}$$

This is a discrete time model description of the original system. A simple example of a NARX model has been given in equation (3.3) which can be expressed in the general form of equations (4.1) and (4.2) by

$c_{01}(1) = 0.5$; $c_{10}(1) = 0.8$; $c_{01}(2) = -0.64$; $c_{02}(1,3) = -0.4$; $c_{11}(1,1) = 0.2$; else $c_{pq}(.) = 0$

(ii) Map the NARX model which is the time domain description of the nonlinear system into the frequency domain to obtain the GFRFs of the system using the recursive algorithm developed in [8] as below

$$\{1 - \sum_{k_1=1}^{K} c_{10}(k_1) \exp[-j(\omega_{d1} +, \cdots, +\omega_{dn}) k_1]\} H^d{}_n(j\omega_{d1}, \ldots, j\omega_{dn})$$

$$= \sum_{k_1, k_n=1}^{K} c_{0n}(k_1, \ldots, k_n) \exp[-j(\omega_{d1} k_1 +, \cdots, +\omega_{dn} k_n)]$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1, k_{p+q}=1}^{K} c_{pq}(k_1, \ldots, k_{p+q}) \exp[-j(\omega_{d(n-q+1)} k_{p+1} + \cdots + \omega_{dn} k_{p+q})] H^d{}_{n-q,p}(j\omega_{d1}, \ldots, j\omega_{d(n-q)})$$

$$+ \sum_{p=2}^{n} \sum_{k_1, k_p=1}^{K} c_{p0}(k_1, \ldots, k_p) H^d{}_{n,p}(j\omega_{d1}, \ldots, j\omega_{dn}) \qquad (4.3)$$

where

$$H^d{}_{np}(j\omega_{d1},...,j\omega_{dn}) = \sum_{i=1}^{n-p+1} H^d{}_i(j\omega_{d1},...,j\omega_{di}) H^d{}_{n-i,p-1}(j\omega_{d(i+1)},...,j\omega_{dn}) \exp[-j(\omega_{d1}+\cdot\cdot+\omega_{di})k_p] \tag{4.4}$$

with

$$H^d{}_{n1}(j\omega_{d1},...,j\omega_{dn}) = H^d{}_n(j\omega_{d1},...,j\omega_{dn}) \exp[-j(\omega_{d1}+\cdot\cdot+\omega_{dn})k_1] \tag{4.5}$$

In (4.3), (4.4) and (4.5), superscript d represents the results for the system discrete time model and ω_{di}, $i = 1,...,n$ denote the discrete angular frequencies which are valid only within the interval $[-\pi,\pi]$.

The relationship between ω_{di}, $i = 1,...,n$ and the angular frequencies ω_i, $i = 1,...,n$, is

$$\omega_{di} = T_s \omega_i \quad i = 1,\ldots,n \tag{4.6}$$

where T_s is the system sampling period and the GFRFs of the corresponding continuous time system can be obtained from the results for the system discrete time model as

$$H_n(j\omega_1 \cdots j\omega_n) = H^d{}_n(jT_s\omega_1 \cdots jT_s\omega_n) \tag{4.7}$$

(iii) Evaluate $\left|H_n(j\omega_1 \cdots j\omega_n)\right|_{\omega}^{B}$ for n=1,...,N using a GA as illustrated in Section 3.

(iv) Calculate $\underbrace{|U|*...*|U(j\omega)|}_{n}$ for n=1,...,N using an algorithm developed in [16] as follows

(a) Choose a sufficiently large even integer M and perform the Fourier transform of M sampled values of the system input to yield

$$\tilde{U}(i) = \left|U_d[j\frac{2\pi}{M}(i - \frac{M}{2} + 1)]\right|, \qquad i = 0,1,\ldots, M-1 \tag{4.8}$$

where $U_d\left(j\frac{2\pi}{M}i\right)$, $i = -\left(\frac{M}{2} - 1\right),\cdots,\frac{M}{2}$ are the Fourier transform results.

(b) Determine an integer i_ω as

$$i_\omega = round\left[\frac{\omega T_s M}{2\pi}\right] \tag{4.9}$$

where round(.) denotes to take the integer near to (.).

(c) Evaluate $\underbrace{|U|*...*|U(j\omega)|}_{n}$ as follows

$$\underbrace{|U|*\cdots*|U(j\omega)|}_{n} = T_s\tilde{\tilde{U}}[i_\omega + (\frac{M}{2} - 1)n](\frac{2\pi}{M})^{n-1} \tag{4.10}$$

where $\tilde{\tilde{U}}[i_\omega + (\frac{M}{2} - 1)n]$ is obtained from the result of the following n-folds vector convolution

$$\{\tilde{\tilde{U}}[0]...,\tilde{\tilde{U}}[n(M-1)]\} = Conv\underbrace{\{[\tilde{U}(0),...,\tilde{U}(M-1)],...,[\tilde{U}(0),...,\tilde{U}(M-1)]\}}_{n} \tag{4.11}$$

(v) Evaluate $\overline{Y}^B(\omega)$ using equations (2.7) and (2.8).

In the above procedures the maximum order N of system nonlinearities is assumed to be known a priori. In practice, N could be chosen as N=3 or N=4. When combined with a GA, the techniques in [26] for truncation of nonlinear system expressions in the frequency domain can also be used to determine a more accurate result of N. The details of this will be discussed in another paper.

If the original system is a discrete time system, then T_s =1 in the above procedures and consequently

$$\omega_{di} = \omega_i \quad i = 1,\dots,n$$

$$H_n(j\omega_1 \cdots j\omega_n) = H^d{}_n(jT_s\omega_1 \cdots jT_s\omega_n)$$

and

$$\underbrace{|U| * \cdots * |U(j\omega)|}_{n} = \tilde{\tilde{U}}[i_\omega + (\frac{M}{2} - 1)n](\frac{2\pi}{M})^{n-1}$$

where $i_\omega = round\left[\frac{\omega M}{2\pi}\right]$.

4.2 A Simulation Example

Consider a discrete time nonlinear system with the NARX model given by equation (3.3). The bound on the output frequency response of the system to the time series input

$$u(t) = \frac{3}{\pi}\frac{\sin 0.8378t}{t} \quad t = 0, \pm 1, \pm 2, \dots \tag{4.12}$$

is to be evaluated over five frequencies

0, 0.5, 1, 1.5, 2

using the procedures described in Section 4.1 above.

(i) The NARX model of the nonlinear system has been given by equation (3.3).

(ii) Mapping the NARX model into the frequency domain up to the 3rd order can easily be performed using the recursive algorithm (4.3) -(4.5) to yield

$$H_1(j\omega_1) = \frac{0.5\exp(-j\omega_1)}{[1 - 0.8\exp(-j\omega_1) + 0.64\exp(-2j\omega_1)]} \tag{4.13}$$

$$H_2(j\omega_1, j\omega_2) = \frac{\{-0.4\exp[-j(\omega_1 + 3\omega_2)] + 0.2\exp[-j(\omega_2 + \omega_1)]H_1(j\omega_1)\}}{\{1 - 0.8\exp[-j(\omega_1 + \omega_2)] + 0.64\exp[-j2(\omega_1 + \omega_2)]\}} \tag{4.14}$$

$$H_3(j\omega_1,j\omega_2,j\omega_3)=\frac{0.2\exp[-j(\omega_2+\omega+\omega_3)]H_2(j\omega_1,j\omega_2)}{\{1-0.8\exp[-j(\omega_1+\omega_2+\omega_3)]+0.64\exp[-j2(\omega_1+\omega_2+\omega_3)]\}\}} \tag{4.15}$$

(iii) Evaluate $\left|H_n(j\omega_1\cdots j\omega_n)\right|_\omega^B$ for n=1,2,3 and ω=0,0.5,1,1.5, and 2 using a GA as illustrated in Section 3. In this case, the values taken by $\omega_{,1}\cdots,\omega_n$ are all in the range of [-0.8378, 0.8378] due to the input excitation (4.12); the number of individuals in each generation is Nind=10; the length of binary strings for each individual is 10×(n-1); and the generation gap factor is 0.8. The results obtained after 200 generations of GA operations are given in Table 1.

Table 1

ω	0	0.5	1	1.5	2
$\left\|H_1(j\omega_1)\right\|_\omega^B$	0.5952	0.7551	1.5877	0.6472	0.3293
$\left\|H_2(j\omega_1,j\omega_2)\right\|_\omega^B$	0.3345	0.4340	1.0700	0.4637	0.2382
$\left\|H_{3n}(j\omega_1,j\omega_2,j\omega_3)\right\|_\omega^B$	0.1076	0.1130	0.3455	0.2511	0.0779

Table 2

ω	0	0.5	1	1.5	2
$\frac{1}{(2\pi)^0}\left\|U(j\omega)\right\|$	3.0005	3.0009	0.0014	0.0003	0.0002
$\frac{1}{(2\pi)^1}\left\|U\right\|*\left\|U(j\omega)\right\|$	2.3995	1.6858	0.9682	0.2527	0.0003
$\frac{1}{(2\pi)^2}\left\|U\right\|*\left\|U\right\|*\left\|U(j\omega)\right\|$	1.4414	1.2708	0.7840	0.3523	0.0910

(iv) Evaluate $\frac{1}{(2\pi)^{n-1}}\underbrace{|U|*...*|U(j\omega)|}_{n}$ for n=1,2,3 and ω=0,0.5,1,1.5, and 2 using the algorithm (4.8)-(4.11) with $T_s=1$ to yield the results given in Table 2.

(v) Evaluate $\overline{Y}^B(\omega)$ for ω=0,0.5,1,1.5, and 2 using equations (2.7) and (2.8) with N=3 and the results in Tables 1 and 2. The results are given in Table 3 together with the real system output magnitude frequency responses over these frequencies.

Table 3

ω	0	0.5	1	1.5	2
$\overline{Y}^B(\omega)$	2.7436	3.1412	1.3091	0.2058	0.0072
$\left\|Y(j\omega)\right\|$	1.4804	1.9452	0.8823	0.1333	0.0070

It can be observed by comparing $\overline{Y}^B(\omega)$ and $|Y(j\omega)|$ in Table 3 that $|Y(j\omega)|$ is not only bounded by $\overline{Y}^B(\omega)$ but the trend of variation of $|Y(j\omega)|$ is also correctly reflected by $\overline{Y}^B(\omega)$. The simulation study therefore demonstrates the effectiveness and good performance of the proposed GA based approach.

5. Conclusions

The bound on output frequency responses of nonlinear systems which was introduced by the authors in [16] reflects the system worse case output frequency response. Evaluation of the bound involves an optimisation problem associated with GFRFs of nonlinear systems. In the present study, a GA based approach was proposed to evaluate this bound in which a GA was applied to deal with the optimisation problem Compared to previous approaches, a more accurate and less conservative result can be obtained using the new approach because the optimisation issue is addressed directly using a GA. A simulation study demonstrated the effectiveness and good performance of the new approach. The proposed approach is believed to be of engineering significance because the bound on output frequency responses can be used in engineering designs of nonlinear mechanical or civil engineering structures, for example, to assess the response to maximum loads.

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