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Box-Particle Probability Hypothesis Density Filtering

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***Abstract*—This paper develops a novel approach for multi-target tracking, called box-particle probability hypothesis density filter (box-PHD filter). The approach is able to track multiple targets and estimates the unknown number of targets. Furthermore, it is capable of dealing with three sources of uncertainty: stochastic, set-theoretic and data association uncertainty. The box-PHD filter reduces the number of particles significantly, which improves the runtime considerably. The small number of box particles makes this approach attractive for distributed inference, especially when particles have to be shared over networks. A box-particle is a random sample that occupies a small and controllable rectangular region of non-zero volume. Manipulation of boxes utilizes methods from the field of interval analysis. The theoretical derivation of the box-PHD filter is presented followed by a comparative analysis with a standard sequential Monte Carlo (SMC) version of the PHD filter. To measure the performance objectively three measures are used: inclusion, volume and the optimum subpattern assignment metric. Our studies suggest that the box-PHD filter reaches similar accuracy results, like a SMC-PHD filter but with considerably less computational costs. Furthermore, we can show that in the presence of strongly biased measurement the box-PHD filter even outperforms the classical SMC-PHD filter.**

***Index Terms*—Multi-Target Tracking, Box-Particle Filters, Random Finite Sets, PHD Filter, Interval Measurements**

I. Introduction

MULTI-TARGET tracking is a common problem with many applications. In most of these the expected target number is not known a priori, so that it has to be estimated from the measured data. In general, multi-target tracking involves the joint estimation of states and the number of targets from a sequence of observations in the presence of detection uncertainty, association uncertainty and clutter [1]. Classical approaches such as the Joint Probabilistic Data Association filter (JPDAF) [2] and multi-hypothesis tracking (MHT) [3] need in general the knowledge of the expected number of targets. The *finite set statistics* (FISST) approach proposed by Mahler [4] is a systematic treatment for multi-target tracking with an unknown and variable number of objects. To reduce the complexity Mahler proposed an approximation of the original Bayes multi-target filter, the *Probability Hypothesis Density* filter (PHD). One of the main advantages of the PHD filter is that it avoids the data association problem and resolves the measurement origin uncertainty in an elegant way. In [5], [6] it was shown that the PHD filter outperforms the classical approaches such as the Kalman Filter, standard particle filters and the Multiple Hypothesis Tracking. Algorithms based on the JPDAF [7] tend to merge tracking results produced by closely spaced objects. This drawback cannot be observed, when using the PHD filter. Many implementations of the PHD filter have been proposed, either using sequential Monte Carlo methods [8]–[10], or with Gaussian mixtures [11]. An improved implementation of SMC-PHD filter was published in [12].

The traditional measurement noise expresses uncertainty due to randomness, often referred to as statistical uncertainty. In many practical applications, however, the standard measurement model is not adequate. Complex distributed surveillance systems, for example, are often operating under unknown synchronization biases and/or unknown system delays. The resulting measurements are affected by bounded errors of typically unknown distribution and biases, and can be expressed rather by intervals than by point values. An interval measurement expresses a type of uncertainty which is referred as the set-theoretic uncertainty [13], [14], vagueness [15] or imprecision [16]. Some of the first works about representing densities as a mixture of box-particles can be traced back to the early seventies, see [17] for a review. The concept of box-particle filtering in the context of tracking was introduced in [18]. In [19] it was shown that box-particles can be seen as supports of uniform probability density functions (PDFs), leading to Bayesian understanding of box-particle filters. In [20] a single target box-particle Bernoulli filter with box measurements is presented.

The main contribution of this work is a general derivation of box-particle methods in the context of multi-target tracking with an unknown number of targets, clutter and false alarms. We present here a box-particle version of the multi-target PHD filter. In addition, a comparison of the Box-PHD filter with a standard sequential Monte Carlo PHD Filter is performed. The optimum subpattern assignment (OSPA) metric [21] is used as performance measure, together with the criteria for measuring the *inclusion* of the true state and the *volume* of the posterior PDF [20].

The remaining part of this article is structured as follows. A brief introduction to Finite Set Statistics is given in Section II. The necessary interval methodology is explained in Section

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III. Section IV contains a general description of the PHD filter with a basic SMC implementation. The following Section V-A describes the steps needed to get from point particles to box-particles. The Box-PHD filter is derived and described in Section V. A numerical study is presented in Section VI. Conclusions are drawn in the final Section VII.

II. Finite Set Statistics

In a single-object system, the state and measurement at time k are represented as two random vectors of possibly different dimensions. These vectors evolve in time, but maintain their initial dimension. However, this is not the case in a multi-object system. Here the multi-object state and multi-object measurement are two collections of individual objects and measurements. The number of these may change over time and lead to another dimensions of the multi-object state and multi-object measurement. Furthermore, there exist no ordering for the elements of the multi-object state and measurement. Using the theory proposed in [22], the multi-object state and measurement are naturally represented as finite subsets $\mathbf{X}_k$ and $\mathbf{Z}_k$ defined as follows:

Let $N(k)$ be a random number of objects, which are located at $\mathbf{x}_{k,1}, ..., \mathbf{x}_{k,N(k)}$ in the single-object state space E_S, e.g. R^d then,

$$\mathbf{X}_k = \left\{ \mathbf{x}_{k,1}, ..., \mathbf{x}_{k,N(k)} \right\} \in \mathcal{F}(E_S) \tag{1}$$

is the multi-object state, where $\mathcal{F}(E_S)$ denotes the collection of all finite subsets of the space E_S. Analogous to this, we define the multi-object measurement

$$\mathbf{Z}_k = \left\{ \mathbf{z}_{k,1}, ..., \mathbf{z}_{k,M(k)} \right\} \in \mathcal{F}(E_O), \tag{2}$$

assuming that at the time step k we have $M(k)$ measurements $\mathbf{z}_{k,1}, ..., \mathbf{z}_{k,M(k)}$ in the single-object space E_O, which correspond to real targets and clutter. The sets $\mathbf{X}_k$ and $\mathbf{Z}_k$ are also called random finite sets. In analogy to the expectation for a random vector, a first-order moment of the posterior distribution for a random set is of interest here, which is the so called probability hypothesis density. The integral value of the PHD over a given region in state space leads to the expected number of objects within this region. Denote $f_{k|k}(\mathbf{x}_k)$ as the PHD associated with the multi-object posterior $p(\mathbf{X}_k|\mathbf{Z}^k)$ at a time step k, with $\mathbf{Z}^k$ denoting the accumulated measurements from the time steps 1 to k. The PHD filter consists of two steps: prediction and update [4].

The prediction can be realized through the following equation[1]:

$$f_{k|k-1}(\mathbf{x}_k) = b(\mathbf{x}_k) + \int p_s(\mathbf{x}_{k-1}) p(\mathbf{x}_k|\mathbf{x}_{k-1}) f_{k-1|k-1}(\mathbf{x}_{k-1}) \mathrm{d}\mathbf{x}_{k-1}, \tag{3}$$

where $b(\mathbf{x}_k)$ denotes the intensity function of spontaneous birth of new objects, $\mathbf{x}_{k-1}$. $p_s(\mathbf{x}_{k-1})$ is the probability that the object still exists at the time step k given its previous state $\mathbf{x}_{k-1}$, and $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ is the transition probability density of the individual objects. The update equation can be written as

$$f_{k|k}(\mathbf{x}_k) \cong F(\mathbf{Z}_k|\mathbf{x}_k) f_{k|k-1}(\mathbf{x}_k), \tag{4}$$

$$F(\mathbf{Z}_k|\mathbf{x}_k) = 1 - p_D(\mathbf{x}_k) + \sum_{\mathbf{z} \in \mathbf{Z}_k} \frac{p_D(\mathbf{x}_k) p(\mathbf{z}|\mathbf{x}_k)}{\lambda c(\mathbf{z}) + \int p_D(\mathbf{x}_k) p(\mathbf{z}|\mathbf{x}_k) f_{k|k-1}(\mathbf{x}_k) \mathrm{d}\mathbf{x}_k}, \tag{5}$$

with $p_D(\mathbf{x}_k)$ denotes the probability that an object in state $\mathbf{x}_k$ will be detected at time step k. Furthermore, $p(\mathbf{z}|\mathbf{x}_k)$ is the measurement likelihood, $c(\mathbf{z})$ the probability distribution for every clutter point and λ is the average number of clutter points per scan.

III. Interval Analysis

This section gives a short introduction to the field of interval analysis, which will be used in this article. For more informations see [23]. The original idea of interval analysis was to deal with intervals instead of real numbers for exact computation in the presence of rounding errors. However, this field has strongly increased its potential applications. We will use the main concepts to represent particles not as delta-peaks but as boxes in the state space. An interval $[x] = [\underline{x}, \overline{x}] \in \mathbb{IR}$ is a closed and connected subset of the real numbers $\mathbb{R}$, with $\underline{x} \in \mathbb{R}$ representing its lower bound and $\overline{x} \in \mathbb{R}$ its upper bound. In multiple dimensions d this interval becomes a box $[\mathbf{x}] \in \mathbb{IR}^d$ defined as a Cartesian product of d intervals: $[\mathbf{x}] = [x_1] \times ... \times [x_d]$. Here the operator $|[.]|$ denotes the volume of a box $[\mathbf{x}]$. The function $\mathrm{mid}([\mathbf{x}])$ returns the center of a box. Elementary arithmetic operations, basic functions and operations between sets have been naturally extended to the interval analysis context.

For general functions the concept of *inclusion functions* has been developed. An *inclusion function* $[\mathbf{g}]$ for a given function $\mathbf{g}$ is defined such that the image of any box $[\mathbf{x}]$ by $[\mathbf{g}]$ is a box $[\mathbf{g}]([\mathbf{x}])$ containing $\mathbf{g}([\mathbf{x}])$. An inclusion function which leads to the smallest box area is needed. Hence, the size of the box $[\mathbf{g}]([\mathbf{x}])$ should be minimal but at the same time has to cover the whole image of a box $[\mathbf{x}]$. An important class in the context of tracking are the *natural inclusion functions*.

Definition 1. *Assume* $\mathbf{g}\colon \mathbb{R}^d \to \mathbb{R}$, $(x_1, ..., x_d) \mapsto g(x_1, ..., x_d)$ *is a function expressed as a finite composition of the operators* $+, -, *, /$ *and standard mathematical functions (sin, cos, exp, ...). A* ***natural inclusion function*** *is obtained by replacing each real variable and each operator or function by its interval counterpart.*

In general, natural inclusion functions are not minimal, but many functions can be modified in order to satisfy the conditions in the following theorem and then their natural inclusion functions are minimal. Proofs and examples can be found in [23].

Definition 2. *An inclusion function* $[\mathbf{g}]$ *for* $\mathbf{g}$ *is convergent if, for any sequence of boxes* $[\mathbf{x}](k)$,

$$\lim_{k \to \infty} |[\mathbf{x}](k)| = 0 \Rightarrow \lim_{k \to \infty} [\mathbf{g}]([\mathbf{x}](k)) = 0, \tag{6}$$

[1]Target spawning is not considered in this paper.

with $|[\mathbf{x}](k)|$ *being the volume of the box* $[\mathbf{x}](k)$.

Theorem 1. *If* $\mathbf{g}$ *involves only continuous operators and continuous elementary functions then* $[\mathbf{g}]$ *is convergent. If, furthermore, each of the variables* $x_1, ..., x_2$ *occurs at most once in the formal expression of* $\mathbf{g}$, *then* $[\mathbf{g}]$ *is minimal.*

The next important concept is *contraction*, which is needed for the definition of likelihood functions and the update step of the proposed filters. The contraction operation actually represents an optimization procedure which finds the smallest box which satisfies certain constraints. One elegant way of performing this optimization is by formulating it as a Constraint Satisfaction Problem. The Constraint Satisfaction Problem (CSP) [23], often denoted by $\mathcal{H}$, can be written as:

$$\mathcal{H} : (\mathbf{g}(\mathbf{x}) = \mathbf{0},\ \mathbf{x} \in [\mathbf{x}]) . \tag{7}$$

A common interpretation of (7) is: find the box enclosure of the set of vectors $\mathbf{x}$ belonging to a given prior domain $[\mathbf{x}]$ satisfying a set of m contraints $\mathbf{g} = (g_1, ..., g_m)^T$, with g_i a real valued function. The solution consists of all $\mathbf{x}$, that satisfy $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ or written as a set:

$$\mathbb{S} = \{\mathbf{x} \in [\mathbf{x}] \mid \mathbf{g}(\mathbf{x}) = \mathbf{0}\}. \tag{8}$$

A *contraction* of $\mathcal{H}$ means replacing $[\mathbf{x}]$ by a smaller box $[\mathbf{x}]'$ under the constraint $\mathbb{S} \subseteq [\mathbf{x}]' \subseteq [\mathbf{x}]$. There are several methods to build a contractor for $\mathcal{H}$, e.g. by the Gauss elimination, Gauss-Seidel algorithm and linear programming. In this work, however, we will use *Constraint Propagation* (CP), or sometimes referred as forward-backward propagation, for its suitability in the context of tracking problems. An example of a CP algorithm is given in Appendix A.

IV. THE SMC–PHD FILTER

Inspired by the works of Vo et al. [10] and Ristic et al. [12] on efficient sequential Monte Carlo methods for the PHD filter, an improved SMC-PHD filter [12] is briefly presented in this paper to make it self-contained. The main improvement is a measurement steered particle placement for target birth. In addition, a target state and covariance matrix estimation without the need of clustering is introduced. The state of an individual object is represented by $\mathbf{x}_k \in \mathbb{R}^{n_x}$ and each measurement as $\mathbf{z}_k \in \mathbb{R}^{n_z}$. Assume that the transitional density $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ is known through an evolution model $\mathbf{f}_k$, nonlinear in general, that is

$$\mathbf{x}_k = \mathbf{f}_k(\mathbf{x}_{k-1}) + \mathbf{w}_k, \tag{9}$$

with $\mathbf{w}_k$ a zero mean Gaussian white process noise .

The SMC-PHD filter consists in 6 steps, which are summarized in what follows. Here the particle set represents the target intensity $f_{k|k}(\mathbf{x})$ of the PHD filter, which corresponds to the multi-target state. Given from the previous time step we have the particle set:

$$\{(\mathbf{x}_i, w_i)\}_{i=1}^{N_k}, \tag{10}$$

with $\mathbf{x}_i \in \mathbb{R}^{n_x}$, w_i the corresponding weight and N_k denoting the number of particles, estimated at time step t_{k-1}. Recall that the integral over this intensity (or sum, if using particles) is the estimated expected number of targets and it is not necessary equal to one. The implementation details using a particle PHD representation is presented below.

1) **Predict target intensity**
The resampled particle set gained from the previous step is denoted by $\{\mathbf{x}_i, w_i\}_{i=1}^{N_k}$. These particles represent the intensity over the state space. Another interpretation is, that every particle represents a possible target state (called microstates in the language of thermodynamics), so that the prediction of the whole set can be modeled by applying a transition model to every particle and adding some noise to it. The weights remain unchanged at this step. In practical implementations this has the same effect as predicting the intensity distribution over the state space with a closed formula.
In order to avoid sampling a high number $N_{k,new}$ of newborn particles, the authors in [12] propose to sample new born particles according to the measurement set $\boldsymbol{Z}_{k-1} = \{\mathbf{z}_{k,1}, ..., \mathbf{z}_{k,M_{k-1}}\}$ from the previous time step t_{k-1}. For each measurement $\mathbf{z}_{k-1,j}, j = 1, ..., M_{k-1}$, $N^j_{k,new} = N_{k,new}/M_{k-1}$ new particles $\tilde{\mathbf{x}}_i$ are drawn from a distribution $\beta_k(\mathbf{x}|\mathbf{z}_{k-1,j})$. In [12] , $\beta_k(.|\mathbf{z}_{k-1,j})$ is constructed with the assumption that the state vector can be separated into directly measured component vector and unmeasured component vector. The measured component of the newborn particles can be sampled by inverting the measurement function while the unmeasured component are sampled uniformly (see [12] for more details).
The weights of the new born particles are set to
$$w_i = \frac{\nu_k}{N_{k,new}}, \quad i = N_k + 1, ..., N_k + N_{k,new}, \tag{11}$$
with ν_k, as in [12] is a prior expected number of target births at time k. The predicted particle set contains the new born and persistent particles and is $\{\tilde{\mathbf{x}}_i, w_i\}_{i=1}^{N_k+N_{k,new}}$.

2) **Compute Correction Term**
For all new measurements $\mathbf{z}_j$, with $j = 1, ..., m_k$ compute:
$$\lambda_{k|k-1}(\mathbf{z}_j) = \lambda c(\mathbf{z}_j) + \sum_{i=1}^{N_k+N_{k,new}} p_k(\mathbf{z}_j \mid \tilde{\mathbf{x}}_i) p_k^D(\tilde{\mathbf{x}}_i) w_i \tag{12}$$

3) **Update**
Given m_k new measurements the update of the state intensity is realized through a correction of the individual particle weights. For every particle $(\mathbf{x}_i, w_i)$, with $i = 1, .., N_k + N_{k,new}$ set:
$$\hat{w}_i = \left[(1 - p_k^D(\tilde{\mathbf{x}}_i)) + \sum_{j=1}^{m_k} \frac{p_k(\mathbf{z}_j \mid \tilde{\mathbf{x}}_i) p_k^D(\tilde{\mathbf{x}}_i)}{\lambda_{k|k-1}(\mathbf{z}_j)} \right] \cdot w_i \tag{13}$$

4) **Estimate target states**
To avoid a clustering step we use the methodology presented in [24]. First, compute the following weights

for all new measurements $\mathbf{z}_j, j = 1, ..., m_k$ and all persistent particles, i.e. not the new born particles $\tilde{\mathbf{x}}_i, i = 1, ..., N_k$.

$$w_{j,i} = \frac{p_k(\mathbf{z}_j \mid \tilde{\mathbf{x}}_i) p_k^D(\tilde{\mathbf{x}}_i)}{\lambda_{k|k-1}(\mathbf{z}_j)} \cdot w_i \tag{14}$$

Then compute the following sum

$$W_j = \sum_{i=1}^{N_k} w_{j,i}, \tag{15}$$

which can be seen as a probability of existence for target j, similarly to the multi-target multi-Bernoulli filter [25]. For further analysis, only those j for which W_j is above a specified threshold τ are considered, i.e.

$$\mathcal{J} = \{j | W_j > \tau, j = 1, ..., m_k\} \tag{16}$$

For all $j \in \mathcal{J}$ the estimated point states are then:

$$\hat{\mathbf{y}}_j = \sum_{i=1}^{N_k} \tilde{\mathbf{x}}_i \cdot w_{j,i}. \tag{17}$$

Note that only targets that have been detected at time step t_k can be reported as present. This follows the lack of "memory" of a PHD filter. The full characteristics are discussed in [26], [27]. In practice τ is usually set as $\tau = 0.75$.

5) **Estimate covariance matrices**
For each estimated state $\hat{\mathbf{y}}_j$ compute its covariance matrix:

$$\mathbf{P}_j = \sum_{i=1}^{N_k} w_{j,i} \left[(\tilde{\mathbf{x}}_i - \hat{\mathbf{y}}_j)(\tilde{\mathbf{x}}_i - \hat{\mathbf{y}}_j)^T \right], \tag{18}$$

The matrix $\mathbf{P}_j$ is not an error covariance matrix in the sense of single target Bayes filtering, but it characterizes the particle distribution of state $\hat{\mathbf{y}}_j$.

6) **Resampling**
Compute first the estimated expected number of targets

$$\eta_k = \sum_{i=1}^{N_k + N_{k,new}} \hat{w}_i. \tag{19}$$

Let N_{k+1} be the number of resampled particles, then any standard resampling technique for particle filtering can be used. Rescale the weights by η_k to get a new particle set $\{\mathbf{x}_i, \eta_k / N_{k+1}\}_{i=1}^{N_{k+1}}$.

V. Derivation of the Box particle PHD filter

A. From Particle to Box

Recall that applying particle filters to the PHD filter leads to a particle approximation of the intensity $f_{k|k}(\mathbf{x})$ by a set of N_k weighted random samples $\{(\mathbf{x}_i, w_i)\}_{i=1}^{N_k}$. The approximation can be written as:

$$f_{k|k}(\mathbf{x}) \approx \sum_{i=1}^{N_k} w_i \delta_{\mathbf{x}_i}(\mathbf{x}), \tag{20}$$

with $\delta_{\mathbf{x}_i}(\mathbf{x})$ the Dirac delta function concentrated at $\mathbf{x}_i$. The sum (20) converges to $f_{k|k}(\mathbf{x})$, with $N_k \to \infty$ [28]. The number of particles used is a key issue to the overall filter performance. In general, the higher the number of particles, the better the approximation and with it the performance. However, a high number of particles leads often to a computationally demanding scenario. In [18] the authors presented a natural way to deal with the decrease of N_k by using boxes instead of point particles and combining particle filter techniques with interval analysis methods. Moreover, in [19] the authors propose to interpret box-particles as supports of uniform PDFs, so that (20) changes to:

$$f_{k|k}(\mathbf{x}) \approx \sum_{i=1}^{N_k} w_i U_{[\mathbf{x}_i]}(\mathbf{x}), \tag{21}$$

with $U_{[\mathbf{x}_i]}(\mathbf{x})$ denoting the uniform PDF over the box $[\mathbf{x}_i]$.

Similarly to the scheme of the SMC-PHD filter the box-PHD filter can be summarized in 7 steps that are derived and presented in the following sections. Step 1 corresponds to the time update, steps 2-5 to the measurement update and steps 6 and 7 to the resampling. A brief summary is also provided later in Algorithm 1.

B. Time update step

Assume that from the previous time step we have the weighted box particle set[2], $\{([\mathbf{x}_i], w_i)\}_{i=1}^{N_k}$ approximating the intensity (21). With $[\mathbf{x}_i] \in \mathbb{IR}^{n_x}$, w_i the corresponding weight and N_k denoting the number of particles. The Box-PF approximation of the PHD prediction equation (3) requires to approximate two terms: the birth intensity $b(\mathbf{x}_k)$ and the persistent intensity.

1) **Predict target intensity**

As for the SMC-PHD filter, the approach in [12] is used here to approximate the newborn particles. Denote by $N_{k,new}$ the number of newborn particles to be sampled. For each measurement $\mathbf{z}_{k-1,j}, j = 1, ..., M_{k-1}$, $N_{k,new}^j = N_{k,new}/M_{k-1}$ new box particles $[\tilde{\mathbf{x}}_i]$ are drawn from a distribution $\beta_k(\mathbf{x}|\mathbf{z}_{k-1,j})$ that is,

$$b(\mathbf{x}) \approx \sum_{j=1}^{M_k} \beta_k(\mathbf{x}|\mathbf{z}_{k-1,j}) \quad \text{with} \tag{22}$$

$$\beta_k(\mathbf{x}|\mathbf{z}_{k-1,j}) \approx \frac{1}{N_{k,new}^j} \sum_{i=1}^{N_{k,new}^j} U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}) \tag{23}$$

As described previously for the PF in section IV, $\beta_k(.|\mathbf{z}_{k-1,j})$ is constructed by separating the state into directly measured component and unmeasured component. The measured components of the newborn box $[\tilde{\mathbf{x}}_i]$ in (23) are chosen by inverting each measurement box $[\mathbf{z}_{k-1,j}]$ while the unmeasured component are chosen according to a prior support. The weights of the new born box particles are set to

$$w_i = \frac{\nu_k}{N_{k,new}}, \quad i = N_k + 1, ..., N_k + N_{k,new}, \tag{24}$$

[2] For simplicity of notation, we skip the time index k for the particle in the rest of the paper when it is not needed.

with ν_k, as in [12] is a prior expected number of target births at time k.

Next, it remains to propagate the persistent box particles, and hence to approximate the integral in (3) $\int p_s\ p(\mathbf{x}_k|\mathbf{x}_{k-1})f_{k-1|k-1}(\mathbf{x}_{k-1})\mathrm{d}\mathbf{x}_{k-1}$ can be approximated. Recall that the transitional density $p(\mathbf{x}_k|\mathbf{x}_{k-1})$ is known through an evolution model $\mathbf{f}_k$ (cf. equation (9)). It is assumed furthermore that $\mathbf{w}_k$ is a bounded noise[3] in a box $[\mathbf{w}_k]$. According to [19] the following approximations are made with uniform PDFs (similarly to what is commonly used in the SMC-PHD time update step with dirac functions):

$$\int p(\mathbf{x}_k|\mathbf{x}_{k-1})f_{k-1|k-1}(\mathbf{x}_{k-1})\mathrm{d}\mathbf{x}_{k-1} \approx w_i \sum_{i=1}^{N_k} U_{[\mathbf{f}_k]([\mathbf{x}_i])+[\mathbf{w}_k]}(\mathbf{x}_k) \qquad (25)$$

Equation (25) means that the persistent box particles are propagated using a transition function's inclusion function $[\mathbf{f}]$. Since the image of a box particle $\mathbf{f}_k([\mathbf{x}_i])$ is not necessarily a box, an inclusion function has to be used.

The new set of predicted box particles is the union of the newborn box particles and the predicted persistent particle, that we denote $\{[\tilde{\mathbf{x}}_i], w_i\}_{i=1}^{N_k+N_{k,new}}$. The predicted PHD has the expression:

$$f_{k|k-1}(\mathbf{x}_k) \approx \sum_{i=1}^{N_k+N_{k,new}} w_i U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k). \qquad (26)$$

C. Generalized likelihood

In the measurement update step, an important challenge is how to implement the likelihood for the set of box-particles representing the PHD. For the M_k new measurements $\mathbf{z}_{k,j}$, in the context of this article, box measurements $[\mathbf{z}_{k,j}]$ are associated to them to model the noise. The sensor noise statistic is not modelled using a density (that in practice is often unknown). Instead, the only assumption that is made is that the sensor error range is known (in practice this information is known a priori). The likelihood terms $p([\mathbf{z}] \mid \mathbf{x})$, we are interested in, are called *generalized likelihood* . In [29], the generalized likelihood expression is derived and can be written:

$$p([\mathbf{z}] \mid \mathbf{x}) = Pr\big\{\mathbf{h}(\mathbf{x}) + \mathbf{v} \in [\mathbf{z}]\big\}, \qquad (27)$$

with $\mathbf{h}$ denoting the measurement model and $\mathbf{v}$ the stochastic noise associated to it (note that, without loss of generality, here we consider an additive noise). If we assume that the measurement model is deterministic and we neglect the effect of $\mathbf{v}$ (in [30] the expression of the generalized likelihood with the stochastic noise can be found), $p([\mathbf{z}] \mid \mathbf{x})$ has the form:

$$p([\mathbf{z}] \mid \mathbf{x}) = Pr\big\{\mathbf{h}(\mathbf{x}) \in [\mathbf{z}]\big\} = U_{[\mathbf{z}]}(\mathbf{h}(\mathbf{x})), \qquad (28)$$

Note that, in equation (28), for a more general problem, each measurement can be characterized using a weighted mixture of boxes (see [19]) to account for measurement noises with known statistics (e.g. Gaussian noise for instance). In that case, the generalized likelihood can be also written as a weighted mixture of uniform PDFs.

D. Measurement update Step

Using the set of box particles $\{[\tilde{\mathbf{x}}_i], w_i\}_{i=1}^{N_k+N_{k,new}}$ approximating the predicted intensity $f_{k|k-1}(\mathbf{x}_k)$ and using the expression of the generalized likelihood (28), the terms in the correction step (5) are to be calculated.

2) **Compute Correction Term**

First, the denominator terms in the right-hand side of equation (5), denoted here $\lambda_{k|k-1}([\mathbf{z}_j])$ have the form:

$$\lambda_{k|k-1}([\mathbf{z}_j]) = \lambda c([\mathbf{z}_j]) + \int p_D\, p([\mathbf{z}_j]|\mathbf{x}_k)f_{k|k-1}(\mathbf{x}_k)\mathrm{d}\mathbf{x}_k. \qquad (29)$$

Here, p_D is assumed constant. Using (26) and (28), the term $p([\mathbf{z}_j]|\mathbf{x}_k)f_{k|k-1}(\mathbf{x}_k)$ in (29) can be written as:

$$p([\mathbf{z}_j]|\mathbf{x}_k)f_{k|k-1}(\mathbf{x}_k) \approx \textstyle\sum_{i=1}^{N_k+N_{k,new}} w_i U_{[\mathbf{z}_j]}(\mathbf{h}(\mathbf{x}_k))U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k). \qquad (30)$$

The term $U_{[\mathbf{z}_j]}(\mathbf{h}(\mathbf{x}_k))U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k)$ in (30) is also a constant function with a support being the following set $S_i \subset E_S$, where

$$S_i = \{\tilde{\mathbf{x}} \in [\tilde{\mathbf{x}}_i] \mid \mathbf{h}(\tilde{\mathbf{x}}_i) \in [\mathbf{z}_j]\}. \qquad (31)$$

Equation (31) defines the solution set of a CSP and from its expression, we can deduce that predicted box particles $[\tilde{\mathbf{x}}_i]$, have to be contracted with respect to the measurement $[\mathbf{z}_j]$. Let us define the function $[h_{\mathrm{CP}}]([\mathbf{x}], [\mathbf{z}])$ that returns the contracted version of $[\mathbf{x}]$ under the constraints given by the measurement function h. In this paper, $[h_{\mathrm{CP}}]$ is obtained via the CP algorithm (see [23]). An example of this contraction step is also given in the appendix (c.f. Appendix A). Following this notation:

$$U_{[\mathbf{z}_j]}(\mathbf{h}(\mathbf{x}_k))U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k) \approx \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|} U_{[\hat{\mathbf{x}}_{i,j}]}(\mathbf{x}_k), \qquad (32)$$

where we denote $[\hat{\mathbf{x}}_{i,j}] = [h_{\mathrm{CP}}]([\tilde{\mathbf{x}}_i], [\mathbf{z}_j])$.

Consequently, equation (30) can be further developed into:

$$p([\mathbf{z}_j]|\mathbf{x}_k)f_{k|k-1}(\mathbf{x}_k) \approx \textstyle\sum_{i=1}^{N_k+N_{k,new}} w_i \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|} U_{[\hat{\mathbf{x}}_{i,j}]}(\mathbf{x}_k). \qquad (33)$$

Note that this result (33) is always true for box particle filter implementations and can be interpreted as: the likelihood calculation requires i) contraction for the box

[3]Without loss of generality, for simplicity noise $\mathbf{w}_k$ is restricted to be additive and bounded. In [19], the general case is considered with noise $\mathbf{w}_k$ approximated using a mixture of uniform PDFs.

particles and ii) a likelihood value proportional to the ratio between the volume of the newly contracted box particle and the original one.

Furthermore, using the expression (33), equation (29) can now be written in the form

$$\begin{aligned}\lambda_{k|k-1}([\mathbf{z}_j]) \approx\ & \lambda c([\mathbf{z}_j]) + \\ & p_D \sum_{i=1}^{N_k+N_{k,new}} w_i \int \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|} U_{[\hat{\mathbf{x}}_{i,j}]}(\mathbf{x}_k) \mathrm{d}\mathbf{x}_k \\ & = \lambda c([\mathbf{z}_j]) + p_D \sum_{i=1}^{N_k+N_{k,new}} w_i \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|}\end{aligned} \quad (34)$$

3) **Update**

By inserting the expression (30) inside the PHD update equations (4) and (5) the updated intensity can be approximated with box particles according to

$$\begin{aligned} f_{k|k}(\mathbf{x}_k) \approx\ & (1-p_D) \sum_{i=1}^{N_k+N_{k,new}} w_i U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k) + \\ & \sum_{j=1}^{M(k)} \sum_{i=1}^{N_k+N_{k,new}} w_i \frac{p_D\, U_{[\mathbf{z}_j]}(\mathbf{h}(\mathbf{x}_k)) U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k)}{\lambda_{k|k-1}([\mathbf{z}_j])} \\ \approx\ & (1-p_D) \sum_{i=1}^{N_k+N_{k,new}} w_i U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k) + \\ & p_D \sum_{j=1}^{M(k)} \sum_{i=1}^{N_k+N_{k,new}} w_i \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|\lambda_{k|k-1}([\mathbf{z}_j])} U_{[\hat{\mathbf{x}}_{i,j}]}(\mathbf{x}_k)\end{aligned} \quad (35)$$

Equation (35), means that, given M_k new measurements the update of the state intensity is realized through contraction step of the box particles and $(N_k + N_{k,new}).(M(k)+1)$ new box particles approximate the updated intensity. The box particle weights are updated according to two groups that reflect the two terms summed in equation (35) :

$$\hat{w}_i = [(1-p_D)] \cdot w_i, \quad (36)$$

$$\hat{w}_i = \left[p_D \sum_{j=1}^{M(k)} \frac{|[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|\lambda_{k|k-1}([\mathbf{z}_j])} \right] \cdot w_i. \quad (37)$$

To avoid this approximation with a potentially huge quantity of box particles, a strategy scoring each measurement is introduced later in this paper in step 6.

4) **Estimate target states**

To avoid a clustering step we use the methodology in [24] also presented in section IV for the SMC-PHD implementation. First, using equation (37) we compute the following weights for all the new measurements $[\mathbf{z}_j], j = 1, ..., m_k$ and all the persistent box particles $[\tilde{\mathbf{x}}_i]$ or uniform PDF $U_{[\tilde{\mathbf{x}}_i]}, i = 1, ..., N_k$ (the new born box particles are not used in this calculation).

$$w_{j,i} = \frac{p_D\, |[\hat{\mathbf{x}}_{i,j}]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}_j]|\lambda_{k|k-1}([\mathbf{z}_j])} \cdot w_i \quad (38)$$

Then compute the following sum

$$W_j = \sum_{i=1}^{N_k} w_{j,i}, \quad (39)$$

which can be seen as a probability of existence for target j, similarly to the multi-target multi-Bernoulli filter. For further analysis only those j are considered for which W_j is above a specified threshold τ, i.e.

$$\mathcal{J} = \{j | W_j > \tau, j = 1, ..., m_k\} \quad (40)$$

For all $j \in \mathcal{J}$ the estimated point states are then:

$$\hat{\mathbf{y}}_j = \frac{1}{W_j} \sum_{i=1}^{N_k} \mathrm{mid}([\tilde{\mathbf{x}}_i]) \cdot w_{j,i}. \quad (41)$$

For all $j \in \mathcal{J}$ the estimated box states are then:

$$[\hat{\mathbf{y}}_j] = \frac{1}{W_j} \sum_{i=1}^{N_k} [\tilde{\mathbf{x}}_i] \cdot w_{j,i}. \quad (42)$$

In Equations (41) and (42) we added, in contrast to [12], the normalization term $\frac{1}{W_j}$ to receive more accurate state estimates when W_j is not practically one.

5) **Estimate covariance matrices**

Using the interpretation of box-particles as a mixture of uniform PDFs, the covariance matrix for each state is computed as

$$\mathbf{P}_j = \sum_{i=1}^{N_k} \frac{w_{j,i}}{W_j} \left[(\mathrm{mid}([\tilde{\mathbf{x}}_i]) - \hat{\mathbf{y}}_j)(\mathrm{mid}([\tilde{\mathbf{x}}_i]) - \hat{\mathbf{y}}_j)^T + \mathbf{\Sigma}_{\mathbf{U_i}} \right], \quad (43)$$

with $\mathbf{\Sigma}_{\mathbf{U_i}}$ a diagonal matrix of the form

$$\mathbf{\Sigma}_{\mathbf{U_i}} = \begin{pmatrix} |([\mathbf{x}_i])_1|^2/12 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & |([\mathbf{x}_i])_{n_x}|^2/12 \end{pmatrix} \quad (44)$$

containing the standard derivations for the individual uniform PDFs. In Equation (43) we added, in contrast to [12], the normalization term $\frac{1}{W_j}$ to receive more accurate covariance matrix estimates when W_j is not practically one. The matrix $\mathbf{P}_j$ is not an error covariance matrix in the sense of single target Bayes filtering, but it characterizes the particle distribution of state $\hat{\mathbf{y}}_j$.

6) **Contract particles**

It has been shown in (35) that each box particle has to be duplicated and contracted by each measurement. To avoid this non-desirable number of paper we propose to contract each box particles $[\tilde{\mathbf{x}}_i], i = 1, ..., N_k + N_{k,new}$ with its corresponding measurement. The corresponding measurement is defined through:

$$[\mathbf{z}^i] = \arg\max_{w_{j,i}} \{[\mathbf{z}_j], w_{j,i} > 0\}. \quad (45)$$

If no $[\mathbf{z}^i]$ is found, the box particle $[\tilde{\mathbf{x}}_i]$ is not contracted, else $[\tilde{\mathbf{x}}_i]$ is set to

$$[\hat{\mathbf{x}}_i] = [\mathbf{h}_{\mathrm{CP}}]([\tilde{\mathbf{x}}_i], [\mathbf{z}^i]). \quad (46)$$

More formally, denote by S_1 the set of box particles $[\tilde{\mathbf{x}}_i], i = 1, ..., N_k + N_{k,new}$ for which $[\mathbf{z}^i]$ exists and denote by S_2 the remaining box particles. The posterior intensity $f_{k|k}(\mathbf{x}_k)$ given in equation (35), can be further approximated into the following mixture of $N_k + N_{k,new}$ PDFs:

$$\begin{aligned} f_{k|k}(\mathbf{x}_k) \approx\ & (1-p_D) \sum_{[\tilde{\mathbf{x}}_i] \in S_2} w_i U_{[\tilde{\mathbf{x}}_i]}(\mathbf{x}_k) + \\ & p_D \sum_{[\tilde{\mathbf{x}}_i] \in S_1} w_i \frac{|[\hat{\mathbf{x}}_i]|}{|[\tilde{\mathbf{x}}_i]||[\mathbf{z}^i]|\lambda_{k|k-1}([\mathbf{z}^i])} U_{[\hat{\mathbf{x}}_i]}(\mathbf{x}_k)\end{aligned} \quad (47)$$

7) **Resampling**
Compute first the estimated expected number of targets

$$\eta_k = \sum_{i=1}^{N_k+N_{k,new}} \hat{w}_i. \quad (48)$$

Let N_{k+1} be the number of resampled particles. As explained in [19], instead of replicating box-particles which have been selected more than once in the resampling step, we divide them into smaller box-particles as many times as they were selected. Several strategies of subdivision can be used (e.g. according to the largest box face). In this paper we randomly pick a dimension to be divided for the selected box-particle. Next, rescale the weights by η_k to get a new particle set $\{[\mathbf{x}_i], \eta_k/N_{k+1}\}_{i=1}^{N_{k+1}}$.

The box-PHD filter is summarized as Algorithm 1.

Algorithm 1 The box-PHD filter

In: $\{([\mathbf{x}_i], w_i)\}_{i=1}^{N_k}, \boldsymbol{Z}_k, \boldsymbol{Z}_{k-1}$
Out: $\{([\mathbf{x}_i], w_i)\}_{i=1}^{N_{k+1}}, \{[\hat{\mathbf{y}}_j], \hat{\mathbf{P}}_j\}$

1) Predict target intensity
   - For $i = 1, ..., N_k$ apply (50) to get $\tilde{\mathbf{x}}_i$.
   - Sample $N_{k,new}$ many new particles according to $\boldsymbol{Z}_{k-1}$
   - Weights for new particles are w_i (24)
2) Compute correction term
   - $\lambda_{k|k-1}([\mathbf{z}_j])$, according to (29)
3) Update target intensity
   - For every particle $([\tilde{\mathbf{x}}_i], w_i)$, with $i = 1, .., N_k + N_{k,new}$ set the new weight according to (35).
4) Compute target states
   - Compute the set $\mathcal{J}$ (40)
   - For all $j \in \mathcal{J}$: $[\hat{\mathbf{y}}_j] = \frac{1}{W_j} \sum_{i=1}^{N_k} w_{j,i} [\tilde{\mathbf{x}}_i]$ (42)
5) Compute covariance matrices
   - For all $j \in \mathcal{J}$ compute $\mathbf{P}_j$ according to (43).
6) Contract boxes
   - $[\hat{\mathbf{x}}_i] = [\mathbf{h}_{\text{CP}}]([\tilde{\mathbf{x}}_i], [\mathbf{z}])$ (46)
7) Resample
   - Use a resampling strategy with subdivision of boxes to get $\{([\mathbf{x}_i], w_i)\}_{i=1}^{N_k+1}$

VI. Numerical Studies

This section gives numerical studies for the proposed Box-particle PHD filter algorithm. For comparison with traditional particle filter techniques we use a point particle sequential Monte Carlo PHD (SMC-PHD) filter. As performance measure the optimum subpattern assignment (OSPA) metric [21] is used for performance measure, together with the criteria for measuring the *inclusion* of the true state and the *volume* of the posterior PDF. The later two were introduced in [20], [30]. Both filters have been implemented in C++ in a similar way. In addition the *Boost Interval Arithmetic Library* [31] was used to handle interval datatypes.

A. Testing Scenario

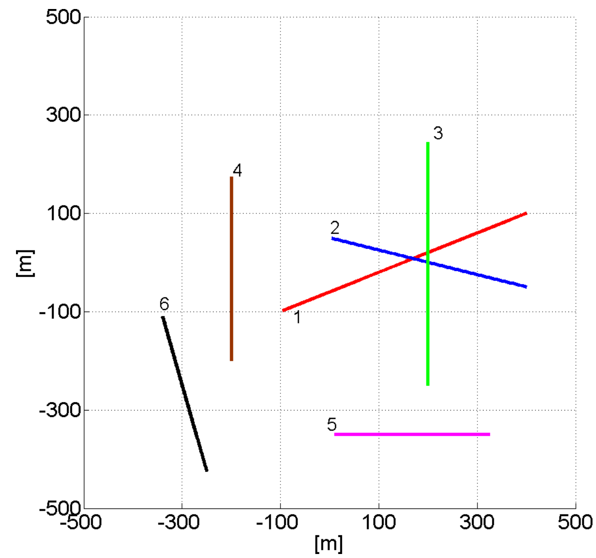

Fig. 1. Linear scenario used for performance evaluation. Six targets move inertially. The individual starting points of each target correspond to the denoted target ID number. Targets 1 - 3 are present for all time steps. Target 4 is presented between time step 15 and 90. Targets 5 and 6 are present between time step 30 and 75.

We analyze the behavior of both filters in a demanding linear scenario. Herein six inertial moved targets are placed in an area $A = [-500, 500]m \times [-500, 500]m$. The unit is assumed to be meters. The state space is $\mathcal{S} \subset \mathbb{R}^4$, where the first two components correspond to the x and y coordinates and the third and fourth their velocities. The measurement space consists of $[x]$ and $[y]$ measurements, so $\mathbf{Z} \subset \mathbb{IR}^2$. New measurements occur for the sake of simplicity every second. The measurement noise is white Gaussian noise with a standard deviation $\sigma_x = \sigma_y = 15m$. The probability of detection is set equal for all states to $p_k^D([\mathbf{x}]) = 0.95$. Target placement and direction of movement is visualized in Figure 1. Targets $1-3$ are present for all time steps. Target 4 is presented between time step 15 and 90. Targets 5 and 6 are present between time step 30 and 75. The whole scenario has a length of 100 time steps (seconds). The number of clutter measurements is estimated following a Poisson distribution with the mean value $|A| \cdot \rho_A$:

$$p(n_c) = \frac{1}{n_c!}(A \cdot \rho_A)^{n_c} \exp(-|A| \cdot \rho_A), \quad (49)$$

with $|A|$ denoting the volume of a observed area and ρ_A a parameter describing the clutter rate. For this scenario we used $\rho_A = 4 \cdot 10^{-6}$. Clutter measurements are generated by an i.i.d. process.

To initialize the particle cloud at time step $t_k = 0$, $N_0 \in \mathbb{N}^+$ particles are distributed uniformly across the state space $\mathcal{S}$, e.g. $N_0 = 1000$. The weights are set to $w_i = 1/N_0$.

Assuming a constant velocity model in two dimensions the prediction of the persistent particles can be modeled by:

$$[\tilde{\mathbf{x}}_i] = \begin{pmatrix} 1 & 0 & \Delta_t & 0 \\ 0 & 1 & 0 & \Delta_t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [\mathbf{x}_i] + [\boldsymbol{\nu}], \tag{50}$$

with $\Delta_t = t_k - t_{k-1}$ and $\boldsymbol{\nu}$ a 3σ interval of some white process noise, defined by a covariance matrix $\mathbf{\Sigma}$. Hidden in equation (50) are inclusion functions for the individual dimension of the state space. A close look reveals that every variable only appears once (for each dimension) and that all operations are continuous, so these natural inclusion functions are minimal and the propagated boxes have minimal size. This fact holds for constant velocity models with arbitrary dimensions.

B. Performance Measures

Let us define $d^{(c)}(\mathbf{x}, \mathbf{y}) := \min(c, d(\mathbf{x}, \mathbf{y}))$ as the distance between $\mathbf{x}, \mathbf{y}$ cut off at $c > 0$, and π_l the set of permutations on $\{1, 2, \ldots, l\}$ for any $l \in \mathbb{N} = \{1, 2, \ldots\}$. For $1 \leq p \leq \infty, c > 0$, and arbitrary finite subsets $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$ and $\mathbf{Y} = \{\mathbf{y}_1, \ldots \mathbf{y}_n\}$ of $\mathcal{S}$, with $m, n \in \mathbb{N}_0$, the OSPA metric [21] is defined as:

$$\overline{d}_p^{(c)}(\mathbf{X},\mathbf{Y}) := \left(\frac{1}{n}\left(\min_{\pi \in \Pi_n}\left(\sum_{i=1}^{m} d^{(c)}(\mathbf{x}_i, \mathbf{y}_{\pi(i)})^p\right) + c^p(n-m)\right)\right)^{\frac{1}{p}}. \tag{51}$$

For the OSPA metric (51) we use directly the state estimates if using the SMC-PHD filter. To apply the OSPA metric to the Box-PHD filter we use the point state estimates $\hat{\mathbf{y}}_j$ gained in Equation (41) of the proposed algorithm. Alternatively, one can use the center points of the box states $\mathrm{mid}([\hat{\mathbf{y}}_j])$, which have the same values as $\hat{\mathbf{y}}_j$.

The *inclusion* value ρ measures whether the state vector is contained in the support of the posterior PDF, or in the case of the PHD filter the posterior intensity. Given the ground truth for all targets $\mathbf{y}_l^*$, with l a index over the true number of targets, the inclusion for the SMC-PHD filter can be computed by evaluating:

$$\rho_l^{\text{SMC}} = \begin{cases} 1 & \exists j : (\hat{\mathbf{y}}_j - \mathbf{y}_l^*)\mathbf{P}_j^{-1}(\hat{\mathbf{y}}_j - \mathbf{y}_l^*)^T < \kappa \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

The condition in (52) checks if the ground truth is contained in the error ellipse defined by covariance matrix $\mathbf{P}_j$. The term κ defines the size of the error ellipse, e.g., use $\kappa = 11.8$ for a 3σ–ellipse in two dimensions [32]. The *inclusion* for the Box-PHD filter is much simpler to compute: Check if the ground truth $\mathbf{y}_l^*$ is contained in one of the state boxes $[\hat{\mathbf{y}}_j]$. If this is true the inclusion value is one, otherwise zero. Then ρ_l for the box-PHD filter is given by:

$$\rho_l^{\text{box}} = \begin{cases} 1 & \text{for } \mathbf{y}_l^* \in [\hat{\mathbf{y}}_j] \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

The *volume* criteria measures the spread of the particle distribution for a given state. To have a fair comparison between both filters we compute the volume for the SMC-PHD filter as:

$$\nu_j^{\text{SMC}} = \sqrt{6 \cdot \sqrt{\mathbf{P}_j(1,1)} + 6 \cdot \sqrt{\mathbf{P}_j(2,2)}}. \tag{54}$$

The volume in Equation (54) is the square root of the widths of a box containing the 3σ–ellipse of state j. Note that we only consider here the position information, since the entries of $\mathbf{P}_j$ have different units. For the Box-PHD filter the volume is computed as the square root of the widths of the box states, giving:

$$\nu_j^{\text{box}} = \sqrt{|[\hat{\mathbf{y}}_i](1)| + |[\hat{\mathbf{y}}_i](2)|} \tag{55}$$

C. Simulations

1) Accuracy Test: In the first simulation we investigate the accuracy achieved with the Box-PHD filter in comparison with the SMC-PHD filter. To do so we will use the linear scenario described earlier. A visualization of the Box-PHD filter for the linear scenario can be seen in Figure 2. Figure 3 shows the mean OSPA values achieved with both filters on the given scenario. We can observe that the OSPA values are in general very low. This means that the SMC-PHD filter and the Box-PHD filter behave very good in this scenario. However, we can also observe that the Box-PHD filter has a little higher values than the SMC-PHD filter. The authors of [20] already noticed that point estimates gained from box-particles can have a slight bias. Therefore they introduced two new measurements criteria *inclusion* and *volume*. The mean results for 1000 Monte Carlo trials and all targets are shown in Figures 4 and 5, respectively. It can be easily seen that the inclusion and volume values react to target appearance and target disappearance. In general we can say that the Box-PHD filter has a higher volume then the SMC-PHD filter. This can be seen as a drawback of the box-particle technique. However, a closer look on the inclusion values reveals that the higher volume leads to better values for the inclusion criteria. So we can state that the SMC-PHD filter converges quickly to the solution and therefore it can happen sometimes that the true target state is not in the support of any covariance matrix $\mathbf{P}_j$. From an engineering point of view both filters reach similar results in this scenario. This fact can also be seen in Figure 6. Here, the estimated mean number of states is depicted. The curves of both filters are practically identical. Nevertheless, the number of particles needed for the Box-PHD filter is much smaller in comparison with the SMC-PHD filter, which yields in a better runtime shown in Table 1. The mean speedup factor for the Box-PHD filter is 10.9. The number of particles used in this scenario where 1875 for the SMC-PHD filter and only 63 for the Box-PHD filter. Both filters have been implemented in C++. The Box-PHD filter uses in addition the Boost Interval Analysis Library. Experiments were performed on a Intel Core 2 Duo (2.53GHz) PC with 4GB RAM. Additional performance measures on the complexity of the approach have been published in [30]. Nevertheless, Fig. 7 shows mean OSPA values for 1000 Monte Carlo trials on the above scenario, where the number of box-particles used is varied. It can be seen that as few as 10 box-particles are only needed in order to reach acceptable OSPA values. Worth mentioning is also that the maximum accuracy is already achieved by 50 box-particles for this scenario.

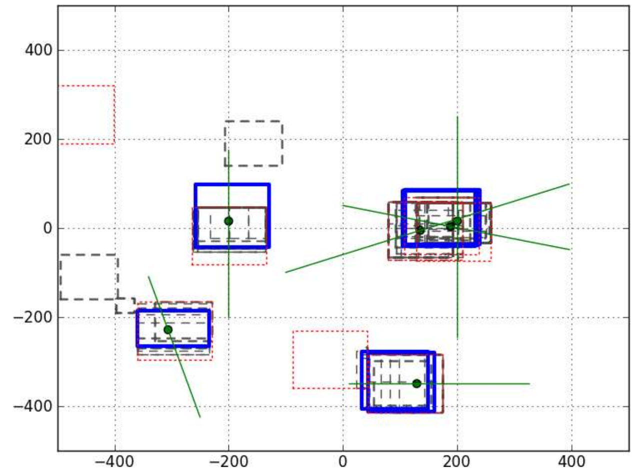

Fig. 2. Visualization of proposed Box-PHD filter. The green solid lines are the true target trajectories. The blue solid boxes correspond to a projection of the estimated box states into 2D. The box-particles are visualized as dashed black boxes, while red dotted boxes are the measurements.

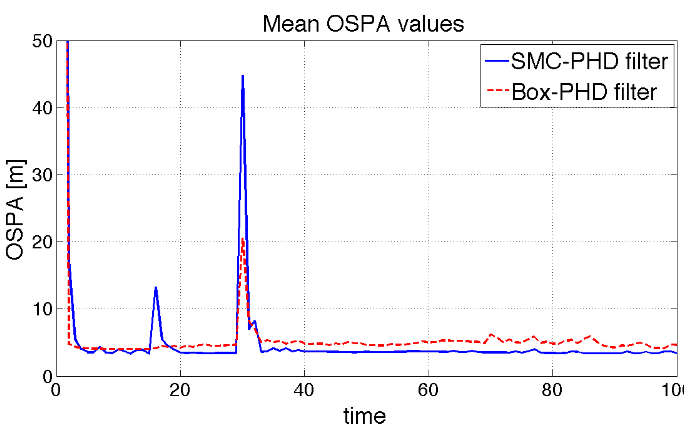

Fig. 3. Mean OSPA values for 1000 Monte Carlo trials on linear scenario for both filters.

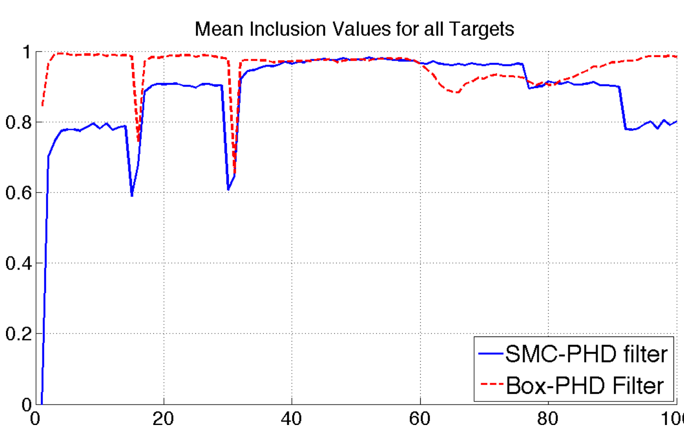

Fig. 4. Mean inclusion values for 1000 Monte Carlo trials and all targets on linear scenario without biased measurements for both filters.

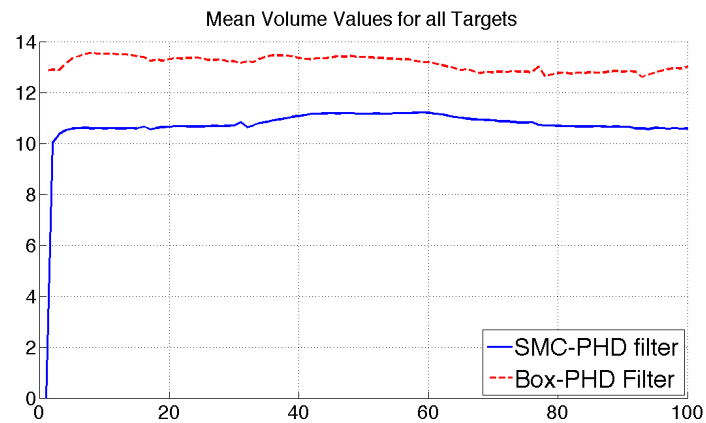

Fig. 5. Mean volume values for 1000 Monte Carlo trials and all targets on linear scenario without biased measurements for both filters.

	processing time (msec)	speedup
SMC-PHD filter	10.3428	1.0
Box-PHD filter	0.95167	10.9

TABLE I
MEAN RUNTIMES FOR PROCESSING ONE TIME STEP. VALUES COMPUTED OVER 1000 MONTE CARLO TRIALS AND FOR ALL TIME STEPS OF THE LINEAR SCENARIO.

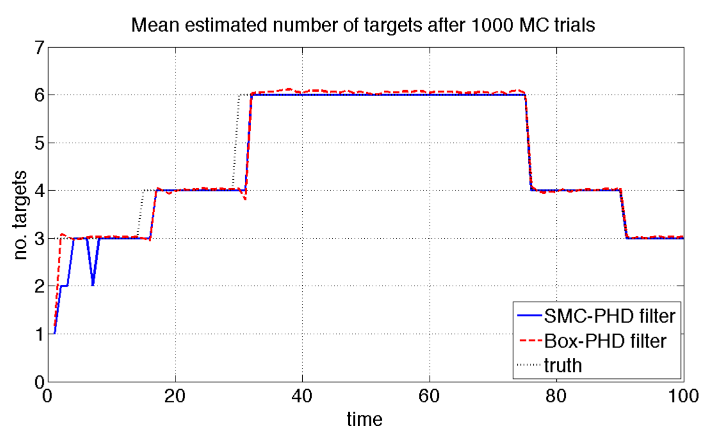

Fig. 6. Mean estimated number of states for 1000 Monte Carlo trials on linear scenario.

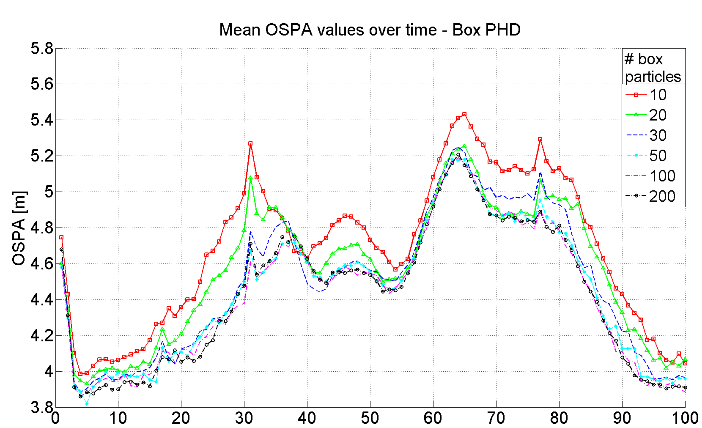

Fig. 7. Mean OSPA values for a varying number of box-particles over time.

2) Strong Bias: In the next simulation we investigate the behavior of both filter when the sensor measurements have a strong bias, i.e., the bias is bigger then the white process noise of the sensor. The examples are similar to those considered in [33] and in [30]. The linear scenario is used again and we added to every measurement a bias of 30[m] for the x measurement and a bias of 10 for the y measurement. The volume of both filters does not change, which can be seen in Figure 9. The inclusion criteria on the other hand changes dramatically for the SMC-PHD filter the value drops to values around 0.5[m], c.f. Figure 8. This means that approximately 50% of the time the true target state is not within the posterior intensity of the filter. This indicates filter divergence, which is considered a *catastrophic* event in target tracking. The Box-PHD filter, on the other hand, reaches values similar to the first simulation without bias. These results lead to the conclusion that the box-PHD filter outperforms the point SMC-PHD filter in scenarios with strongly biased measurements.

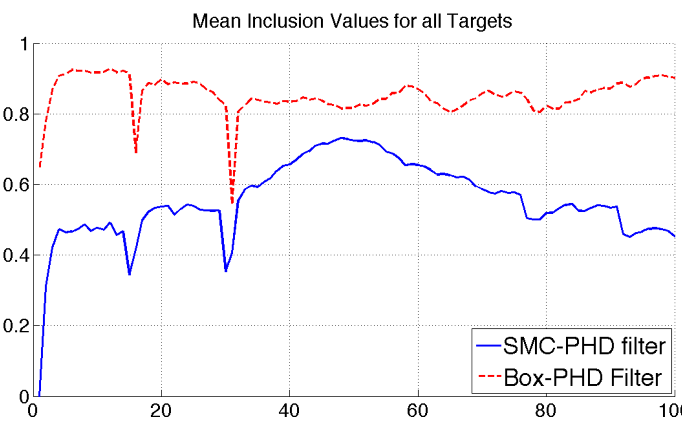

Fig. 8. Mean inclusion values for 1000 Monte Carlo trials and all targets on linear scenario with biased measurements for both filters.

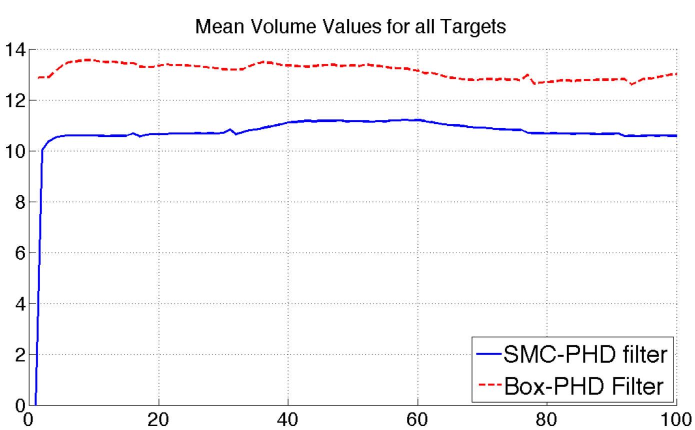

Fig. 9. Mean volume values for 1000 Monte Carlo trials and all targets on linear scenario with biased measurements for both filters.

VII. Conclusion

In this paper we presented a novel technique for non-linear multi-target tracking with a box-particle based filter, called the Box-PHD filter. The theoretical backbone of this is the random finite set theory, which can be used to derive the general intensity filter equations. For the implementation, however, methods from interval analysis are used additionally to get a box-particle representation of the PHD filter. This representation allows a decrement of the number of particles needed. In our simulations we could reduce the number of particles by a factor of approximately thirty and reduce the computation time by a factor of approximately eleven. On the other hand, the accuracy of the filter was not remarkably reduced. Especially in the presence of strong bias we could show that the Box-PHD filter can outperform the SMC-PHD filter with point particles.

Appendix A
Contraction Example

Assume the following scenario: A sensor measures azimuth α and range r in a local sensor coordinate system. The objective is to track a target in a global Cartesian coordinate system with these measurements. A measurement is then $\mathbf{z} = (\alpha, r)^T$, while the state is represented by $\mathbf{x} = (x, y)^T$. The point measurement function is defined as

$$\mathbf{z} = \mathbf{h}(\mathbf{x}) = \begin{pmatrix} \arctan\left(\frac{y-y_0}{x-x_0}\right) \\ \sqrt{(x-x_0)^2 + (y-y_0)^2} \end{pmatrix}, \tag{56}$$

where $(x_0, y_0)^T$ is the sensor position in a global coordinate system. Equation (56) defines two constraints that will be used to contract a state box $[\mathbf{x}]$. Assuming box measurements $[\mathbf{z}] = [\alpha] \times [r]$ and box states $[\mathbf{x}] = [x] \times [y]$ a contractor $[\mathbf{h}_{CP}]([\mathbf{x}] \mid [\mathbf{z}])$ based on constraint propagation [23] is given by the following algorithm:

0) Input: $[\mathbf{x}] = [x] \times [y],\ [\mathbf{z}] = [\alpha] \times [r]$
   Output: $[\mathbf{x}] = [x] \times [y]$
1) for contraint 1 do:

$$[x] := [x] \cap \sqrt{[r]^2 - ([y] - [y_0])^2} \tag{57}$$
$$[y] := [y] \cap \sqrt{[r]^2 - ([x] - [x_0])^2} \tag{58}$$
$$[r] := [r] \cap \sqrt{([x] - [x_0])^2 - ([y] - [y_0])^2} \tag{59}$$

2) for contraint 2 do:

$$[x] := [x] \cap \frac{[y] - [y_0]}{[\tan]([\alpha])} \tag{60}$$
$$[y] := [y] \cap [x] \cdot [\tan]([\alpha]) \tag{61}$$
$$[\alpha] := [\alpha] \cap [\arctan]\left(\frac{[y] - [y_0]}{[x] - [x_0]}\right) \tag{62}$$

3) if the boxes $[\mathbf{x}]$ and $[\mathbf{z}]$ are changed return to step 1.

The box $[x_0] \times [y_0]$ represents the sensor position as a singleton. In practice we found it useful to stop this iteration after a finite number of loops, e.g. three, without any lack of performance. The quotient of the contracted box volume and the original box volume is used is used to calculate the likelihood. Figure 10 visualizes the idea.

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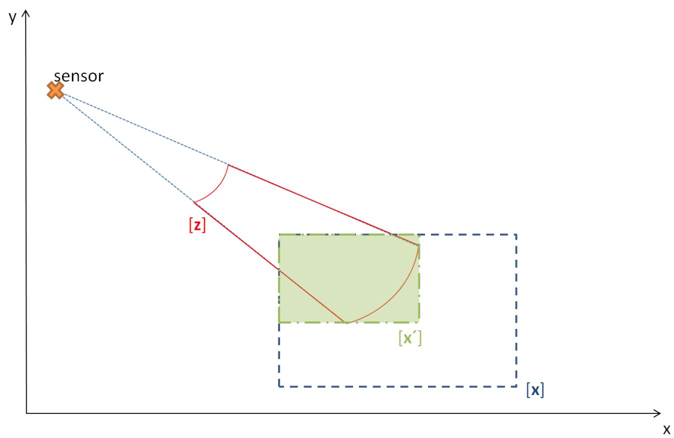

Fig. 10. Contraction example. The box $[\mathbf{x}]$ is contracted by the measurement box $[\mathbf{z}]$. The result is the green box $[\mathbf{x}']$.

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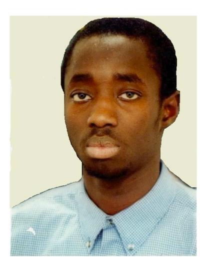

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