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Abstract

A new algorithm, for identifying linear and nonlinear differential equation models, is introduced. The algorithm avoids the computation of derivatives by using the generalised frequency response functions to reconstruct the model. It is shown that the model can be constructed sequentially by building in first the linear terms, then the quadratic terms and so on in a manner that determines the significant model terms at each step and provides unbiased estimates in the presence of noise. Simulated examples and the identification of a model relating the action of waveforces to an offshore structure are included to demonstrate the procedure.

1. Introduction

Parameter estimation of discrete-time systems has attracted much more attention than that of continuous-time systems even though most physical systems are naturally continuous. This is because the former is better suited for digital computer implementation. However, in many situations, a continuous time model is desired, for example in control applications or to provide a more direct link to the physical properties and operation of the underlying system.(Young, 1981; Gawthrop, 1982; Unbehanen and Rao, 1987).

Parameter estimation of the linear transfer function in the frequency domain can be traced back to Levi(1959), who first considered the synthesis of s-transfer function models from complex frequency response data. Levi's method, however, gave biased estimates. To overcome this drawback, Sanathanan and Koerner(1963) introduced an iterative weighted least square approach to asymptotically eliminate the bias introduced in Levi's scheme. Lawrence and Roger(1979) derived a recursive form of the Sanathanan and Koerner algorithm in order to obtain a faster convergence. Whitfield(1986) introduced a unified structure of the existing least-square based algorithms and an alternative formulation in the time domain. Recently, Pintelon etc.(1994) surveyed estimators of the linear transfer function in the frequency domain.

Parameter estimation of nonlinear systems in the frequency domain has received increasing attention. The use of modulating functions to convert differential equations into an algebraic form in the frequency domain has been considered by numerous authors (Pearson and Lee. 1985a, 1985b; Patra and Unbehauen, 1994). But this approach is limited to specified classes of nonlinear systems, namely, linear, integrable and convolvable terms and does not work well in the presence of noise(Patra and Unbehauen, 1994). These disadvantages limit the practical application of these methods.

Alternatively, the generalised frequency response functions(GFRF) can be used to identify continuous time nonlinear systems in the frequency domain. The generalised frequency response functions represent extensions of the linear frequency response function to nonlinear systems and are defined as the multidimensional Fourier transforms of the Volterra kernels. In practical applications, the GFRF's can be estimated either by extending the traditional FFT spectral estimation methods to multidimensions(Kim and Powers, 1988), or by fitting a parametric NARMAX model and then mapping this into the frequency domain(Peyton Jones and Billings, 1989). The first approach can involve the use of special input signals and large data sets. The second approach was used by Tsang and Billings(1992) and Swain and Billings(1995) to estimate nonlinear differential equations models. However, the basic idea of using the generalised frequency response functions as a basis to reconstruct a nonlinear model of the system can be employed irrespectively of the computation method for the GFRF's. The great advantage of this approach is that the direct calculation of the derivatives of the input-output signals is avoided. However only the noise-free case was considered in previous studies and as a consequence the results were heavily dependent on the frequency range that could be analysed.

In the present study the reconstruction of both linear and nonlinear differential equation models based on the GFRF's is studied. A new algorithm is introduced which produces unbiased estimates in the presence of noise. One of the properties of the algorithm is that the nonlinear model can be constructed sequentially by building in the linear model terms, followed by the quadratic terms and so on. At each stage the significance of each candidate model term is assessed and only relevant model terms are included. This enables the nonlinear differential equation model to be constructed componentwise to produce a parsimonious system description.

2. Generalised Frequency Response Functions

The traditional description of nonlinear systems is the Volterra functional series(Schetzen, 1980)

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{1}$$

where $y_n(t)$ is the 'n-th order output' of the system

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \qquad n > 0 \tag{2}$$

$h_n(\tau_1, \cdots, \tau_n)$ is called 'nth-order Kernel' or 'nth-order impulse response function'. If n=1, this reduces to the familiar linear convolution integral.

The multi-dimensional Fourier transform of the nth-order impulse response function yields the 'nth-order frequency response function' or the Generalised Frequency Response Function (GFRF)

$$H_n(j\omega_1, \cdots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \exp(-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)) d\tau_1 \cdots d\tau_n \tag{3}$$

Taking the inverse Fourier transform of (3), substituting into (2) and carrying out the multiple integration on $\tau_1 \cdots \tau_n$ gives

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) \times \exp(j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)) d\omega \tag{4}$$

The frequency domain representation of (1) using (4) is given by (Liu and Vinh, 1991)

$$Y(j\omega) = H_1(j\omega)U(j\omega) + \sum_{n=2}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j(\omega - \omega_2 - \cdots - \omega_n), j\omega_2, \cdots, j\omega_n) \\ U(j(\omega - \omega_2 - \cdots - \omega_n), j\omega_2, \cdots, j\omega_n) \prod_{i=2}^{n} U(j\omega_i) d\omega \tag{5}$$

where $Y(j\omega), U(j\omega)$ are the Fourier Transforms of y(t) and u(t) and $H_n(j\omega_1, j\omega_2, \cdots, j\omega_n)$ is referred to as the n'th order Generalised Frequency Response Function(GFRF).

Bendat and Piersol(1986), Chua and Liao(1989), Kim and Powers(1988), and Nam and Powers(1994) discussed nonparametric estimation of the $H_n(\cdot)$ based on FFT type procedures. The method of Nam and Powers will be used to illustrate the basic principle of these approaches. By assuming that only terms up to third order are present in (1) and replacing integral operations by summations, a simplified discrete version of (1) and (4) is obtained:

$$Y(j\omega)=H_1(j\omega)U(j\omega)+\sum_{\substack{p,q \\ (p+q=\omega)}}H_2(jp,jq)U(jp)U(jq)$$
$$+\sum_{\substack{k,l,m \\ (k+l+m=\omega)}}H_3(jk,jl,jm)U(jk)U(jl)U(jm)+\varepsilon(j\omega) \tag{6}$$

Although (6) is nonlinear in terms of the input, it is linear in terms of the unknown transfer functions. Therefore, by considering all permutations of the frequency distribution within the interested frequency region, a set of overdetermined linear equations can be derived, and the standard linear least squares estimator can be applied to determine H_1, H_2 and H_3.

Peyton Jones and Billings(1989) proposed an alternative approach to get the H_n from sampled input-output data. They first fit a nonlinear autoregressive moving average with exogenous input(NARMAX) model to the system and then map this model into the frequency domain. The advantage of this method is that it significantly reduces the computational requirements, a much smaller data set is required, and there is no a prior assumption that the system only contains terms up to H_3, all the H_n's can be readily determined.

3. Mapping Nonlinear Differential Equations Into the GFRF's

Given an ordinary nonlinear differential equation(NDE) several methods are available which can be used to map this into the frequency domain. Here we briefly introduce the method proposed by Billings and Peyton Jones(1990).

Consider a continuous system characterised by the differential equation

$$f(y,\frac{dy}{dt},\cdots,\frac{d^p y}{dt^p};u,\frac{du}{dt},\cdots,\frac{d^q u}{dt^q})=0 \tag{7}$$

A polynomial form of (7) for a wide class of nonlinear system can be expressed as

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=0}^{L}C_{p,q}(l_1,\cdots,l_{p+q})\prod_{i=1}^{p}D^{l_i}y(t)\prod_{i=p+1}^{p+q}D^{l_i}u(t)=0 \tag{8}$$

where M is the order of the nonlinearity, L is the order of the derivatives in terms of the input and output, $p+q=m$, and the operator D is defined by

$$D^l x(t)=\frac{d^l x(t)}{dt^l} \qquad l\geq 0$$

Assume $C_{1,0}\neq 0$,and arrange (8) to give

$$-C_{1,0}(0)y(t)=\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{\substack{l_1,l_{p+q}=0 \\ \text{except } C_{1,0}(0)}}^{L}C_{p,q}(l_1,\cdots,l_{p+q})\prod_{i=1}^{p}D^{l_i}y(t)\prod_{i=p+1}^{p+q}D^{l_i}u(t) \tag{9}$$

For example, the NDE model

$$y+a_1Dy+a_2D^2y+a_3y^2+a_4Dy\cdot y^2=b_0u+b_1Du+b_2u^2$$

can be expressed in the form of model (8) as

$C_{1,0}(0)=1, C_{1,0}(1)=a_1, C_{1,0}(2)=a_2, C_{2,0}(0,0)=a_3, C_{3,0}(1,0,0)=a_4,$
$C_{0,1}(0)=b_0, C_{0,1}(1)=b_1, C_{0,2}(0,0)=b_2$

By adopting the method of Billings and Peyton Jones(1990), (8) can be mapped into the frequency domain as

$$
\begin{aligned}
&-[\sum_{l_1=0}^{L} C_{1,0}(l_1)(j\omega_1+\cdots+j\omega_n)^{l_1}]H_n^{asym}(j\omega_1,\cdots,j\omega_n) \\
&= \sum_{l_1,l_n=1}^{L} C_{0,n}(l_1,\cdots,l_n)(j\omega_1)^{l_1}\cdots(j\omega_n)^{l_n} \\
&+ \sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,l_n=0}^{L} C_{p,q}(l_1,\cdots,l_{p+q})(j\omega_{n-q-1})^{l_{p-1}}\cdots(j\omega_n)^{l_{p-q}} H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q}) \\
&+ \sum_{p=2}^{n}\sum_{l_1,l_p=0}^{L} C_{p,0}(l_1,\cdots,l_p) H_{n,p}(j\omega_1,\cdots,j\omega_{n-q})
\end{aligned}
\tag{10}
$$

where the recursive relation is given by

$$
H_{n,p}^{asym}(\cdot)=\sum_{i=1}^{n-p+1} H_i(\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(\omega_{i+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_i)^{l_p} \tag{11}
$$

The recursion finishes with p=1 and $H_{n,1}(j\omega_1,\cdots,j\omega_n)$ has the property

$$
H_{n,1}(j\omega_1,\cdots,j\omega_n)=H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{l_1} \tag{12}
$$

The n-th order transfer function of eqn (10) is not necessarily unique in that changing the order of any two arguments generates a new function without changing the value of $y_n(t)$ in eqn (4). The symmetric version of $H_n(\cdot)$ is normally used because it is unique and has values that are independent of the order of the arguments. This is given as

$$
H_n^{sym}(j\omega_1,\ldots,j\omega_n)=\frac{1}{n!}\sum_{\substack{all\ permutations\\ of\ \omega_1,\ldots,\omega_n}} H_n^{asym}(j\omega_1,\ldots,j\omega_n) \tag{13}
$$

Without loss of generality, we assume $C_{1,0}(0)=-1$;

4. Parameter Estimation

Inspection of (10)–(12) reveals that H_1 contains only the terms and coefficients which relate to the linear terms and coefficients in the model of equation (8). Similarly H_2 only involves the terms and coefficients of the linear and quadratic nonlinearity of (8), and so on. This implies that parameters corresponding to each order of nonlinearity starting with the linear terms can be estimated sequentially and independently.

4.1 Synthesis of the Linear Transfer Functions

Setting n=1 in (10) yields

$$
H_1(j\omega)=\frac{\sum_{l_1=0}^{L} C_{0,1}(l_1)(j\omega)^{l_1}}{-\sum_{l_1=1}^{L} C_{1,0}(l_1)(j\omega)^{l_1}}=\frac{B(j\omega)}{1+A(j\omega)} \tag{14}
$$

With the assumption $C_{1,0}(0)=-1$ then

$$
B(j\omega)=\sum_{l_1=0}^{L} C_{0,1}(l_1)(j\omega)^{l_1},\quad A(j\omega)=-\sum_{l_1=1}^{L} C_{1,0}(l_1)(j\omega)^{l_1}
$$

In practice the estimates of $H_1(j\omega)$ will usually involve some noise so that

$$\begin{aligned}\bar{H}_1(j\omega) &= H_1(j\omega) + N_1(j\omega)\\ &= \frac{B(j\omega)}{1+A(j\omega)} + N_1(j\omega)\end{aligned} \tag{15}$$

where $\bar{H}_1(j\omega)$ is the estimate of $H_1(j\omega)$, and $N_1(j\omega)$ is assumed to be independent, zero mean complex white noise. (See Appendix A for the definition of complex noise).
If H_1 is to be modelled. the error criterion to be minimized will be

$$J = \sum_{r=1}^{M} \left|N_1(j\omega_r)\right|^2 = \sum_{r=1}^{M} \left|\bar{H}_1(j\omega_r) - H_1(j\omega_r)\right|^2 \tag{16}$$

Because the parameters in $B(j\omega)$ occur nonlinearly within the modulus. this problem belongs to the nonlinear least squares family, and optimization techniques would be required to yield a solution .

Levi(1959) suggested a modified error criterion by multiplying both sides of (15) with $[1+A(j\omega)]$ and re-arranging as

$$[1+A(j\omega)]\bar{H}_1(j\omega) = B(j\omega) + [1+A(j\omega)]N_1(j\omega) \tag{17}$$

Let

$$N_1'(j\omega) = [1+A(j\omega)]N_1(j\omega) \tag{18}$$

For estimation purpose, (17) can be represented as

$$z(j\omega) = \sum_{i=1}^{2L+1} \theta_i P_i(j\omega) + N_1'(j\omega) \tag{19}$$

where

$z(j\omega) = \bar{H}_1(j\omega)$

$\theta_1 = C_{0,1}(0)$, $\quad P_1 = 1$

$\theta_2 = C_{0,1}(1)$, $\quad P_2 = j\omega$

$\vdots \qquad\qquad \vdots$

$\theta_{L+1} = C_{0,1}(L)$. $\quad P_{L+1} = (j\omega)^L$

$\theta_{L+2} = C_{1,0}(1)$. $\quad P_{L+2} = (j\omega)^1 \bar{H}_1(j\omega)$

$\vdots \qquad\qquad \vdots$

$\theta_{2L+1} = C_{1,0}(L)$, $\quad P_{2L+1} = (j\omega)^L \bar{H}_1(j\omega)$

If 'N' measurements of $z(j\omega)$ and $P_i(j\omega)$ are available, at $\omega(i)$, $i = 1,\ldots,N$, then (19) can be expressed in form as

$$\mathrm{Z} = \mathrm{P}\Phi + \xi \tag{20}$$

$$\mathrm{Z} = \begin{bmatrix} z(j\omega(1)) \\ z(j\omega(1)) \\ \vdots \\ z(j\omega(N)) \end{bmatrix}_{N\times 1}, \quad \Phi = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{2L+1} \end{bmatrix}_{(2L+1)\times 1}, \quad \xi = \begin{bmatrix} N_1'(j\omega(1)) \\ N_1'(j\omega(1)) \\ \vdots \\ N_1'(j\omega(N)) \end{bmatrix}_{N\times 1}$$

where

$$\mathrm{P} = \begin{bmatrix} P_1(j\omega(1)) & P_2(j\omega(1)) \cdots & P_{2L+1}(j\omega(1)) \\ P_1(j\omega(1)) & P_2(j\omega(1)) \cdots & P_{2L+1}(j\omega(1)) \\ \vdots & \vdots & \cdots \vdots \\ P_1(j\omega(N)) & P_2(j\omega(N)) \cdots & P_{2L+1}(j\omega(N)) \end{bmatrix}_{N\times(2L+1)}$$

Finally (20) should be partitioned into real and imaginary parts as

$$\begin{bmatrix}\mathrm{Re}(Z)\\ \mathrm{Im}(Z)\end{bmatrix}=\begin{bmatrix}\mathrm{Re}(\mathrm{P})\\ \mathrm{Im}(\mathrm{P})\end{bmatrix}\Phi+\begin{bmatrix}\mathrm{Re}(\xi)\\ \mathrm{Im}(\xi)\end{bmatrix} \tag{21}$$

Linear least squares can now be applied to estimate the parameters. However, the estimates will be biased because $N_1'(j\omega)$ is no longer white.

Alternatively, postulate a filter $F(j\omega)$ onto (17) so that

$$[1+A(j\omega)]F(j\omega)\tilde{H}_1(j\omega)=F(j\omega)B(j\omega)+\{F(j\omega)[1+A(j\omega)]\}N_1(j\omega) \tag{22}$$

If $F(j\omega)$ selected as $F(j\omega)=\dfrac{1}{1+A(j\omega)}$ and

$$\tilde{H}_1^F(j\omega)=F(j\omega)\tilde{H}_1(j\omega)\ ,\quad B^F(j\omega)\triangleq F(j\omega)B(j\omega)$$

then (22) becomes

$$[1+A(j\omega)]\tilde{H}_1^F(j\omega)=B^F(j\omega)+N_1(j\omega) \tag{23}$$

The least squares estimate will therefore be unbiased if this operates on the filtered frequency response data $\tilde{H}_1^F(j\omega)$ and $B^F(j\omega)$.

This is essentially a complex generalised least square (GLS) framework(Clarke,1967). The steps are summarized below:

(1). Form Z and P of (20), then find the least square estimates of $\hat{\Phi}$, i.e. $\hat{A}(j\omega)$ and $\hat{B}(j\omega)$.

(2). Form $\hat{F}(j\omega)=\dfrac{1}{1+\hat{A}(j\omega)}$.

(3). Compute the filtered data

$$\tilde{H}_1^F(j\omega)=\hat{F}(j\omega)\tilde{H}_1(j\omega)$$

$$B^F(j\omega)=\hat{F}(j\omega)B(j\omega)$$

(4). Form Z and P of (20) using the filtered data $\tilde{H}_1^F(j\omega)$ and $B^F(j\omega)$, then find the least square estimates of $\hat{\Phi}$.

(5). Go to (2) and repeat until convergence.

4.2 Estimation of the Second Order Nonlinearities

Setting n=2 in (10) yields

$$\begin{aligned}-[\sum_{l_1=0}^{L}C_{1,0}(l_1)(j\omega_1+j\omega_2)^{l_1}]H_2^{asym}(j\omega_1,j\omega_2)=&\sum_{l_1,l_2=0}^{L}C_{0,2}(l_1,l_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2}\\&+\sum_{l_1,l_2=0}^{L}C_{1,1}(l_1,l_2)(j\omega_2)^{l_2}H_{1,1}(j\omega_1)\\&+\sum_{l_1,l_2=0}^{L}C_{2,0}(l_1,l_2)H_{2,2}^{asym}(j\omega_1,j\omega_2)\end{aligned} \tag{24}$$

With the recursive relation

$$H_{1,1}(j\omega)=H_1(j\omega_2)(j\omega_1)^{l_1}$$

$$H_{2,2}^{asym}(j\omega_1,j\omega_2)=H_1(j\omega_1)H_{1,1}(j\omega_2)(j\omega_1)^{l_2}=H_1(j\omega_1)H_1(j\omega_2)(j\omega_2)^{l_1}(j\omega_1)^{l_2}$$

In which $H_1(\cdot)$ is the noise-free part in eqn (15). In practice $H_1(\cdot)$ would be formed by the coefficients $C_{1,0}(\cdot)$ and $C_{0,1}(\cdot)$ estimated in the linear term identification described in 4.1.

The corresponding noisy case is

$$\tilde{H}_2^{asym}(j\omega_1,j\omega_2)=H_2^{asym}(j\omega_1,j\omega_2)+N_2(j\omega_1,j\omega_2) \tag{25}$$

where $\bar{H}_2^{asym}(j\omega_1, j\omega_2)$ is obtained from section 3 and $N_2(j\omega_1, j\omega_2)$ is independent, zero mean white noise over two frequency axes.

The coefficients $C_{1,0}(l_1)$ $(l_1 = 1,\ldots,L)$ have been estimated at the previous stage. Therefore setting

$$-\sum_{l_1=0}^{L} C_{1,0}(l_1)(j\omega_1 + j\omega_2)^{l_1} = A_2(j\omega_1, j\omega_2) \tag{26}$$

allows eqn (24) to be rewritten as

$$\begin{aligned} H_2^{asym}(j\omega_1, j\omega_2) = \frac{1}{A_2(j\omega_1, j\omega_2)} [& \sum_{l_1,l_2=0}^{L} C_{0,2}(l_1,l_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2} \\ & + \sum_{l_1,l_2=0}^{L} C_{1,1}(l_1,l_2)(j\omega_2)^{l_2} H_{1,1}(j\omega_1) \\ & + \sum_{l_1,l_2=0}^{L} C_{2,0}(l_1,l_2) H_{2,2}^{asym}(j\omega_1, j\omega_2)] \end{aligned} \tag{27}$$

Substituting (27) into (25) yields

$$\begin{aligned} \bar{H}_2^{asym}(j\omega_1, j\omega_2) = \frac{1}{A_2(j\omega_1, j\omega_2)} [& \sum_{l_1,l_2=0}^{L} C_{0,2}(l_1,l_2)(j\omega_1)^{l_1}(j\omega_2)^{l_2} \\ & + \sum_{l_1,l_2=0}^{L} C_{1,1}(l_1,l_2)(j\omega_2)^{l_2} H_{1,1}(j\omega_1) \\ & + \sum_{l_1,l_2=0}^{L} C_{2,0}(l_1,l_2) H_{2,2}^{asym}(j\omega_1, j\omega_2)] \\ + N_2(j\omega_1, j\omega_2) & \end{aligned} \tag{28}$$

Let

$z(j\omega_1, j\omega_2) = \bar{H}_2^{asym}(j\omega_1, j\omega_2)$

$\theta_1 = C_{0,2}(0,0)$, $\quad P_1 = 1/A_2(j\omega_1, j\omega_2)$

$\theta_2 = C_{0,2}(0,1)$, $\quad P_2 = (j\omega_2)/A_2(j\omega_1, j\omega_2)$

$\vdots \qquad\qquad \vdots$

$\theta_{6L+6} = C_{2,0}(L,L)$, $\quad P_{6L+6} = \left[H_1(j\omega_1)H_1(j\omega_2)(j\omega_1)^L(j\omega_2)^L\right]/A_2(j\omega_1, j\omega_2)$

So that (25) can be expressed as

$$z(j\omega_1, j\omega_2) = \sum_{i=1}^{6L+6} \theta_i P_i(j\omega_1, j\omega_2) + N_2(j\omega_1, j\omega_2) \tag{29}$$

If '2N' measurements of $z(j\omega_1, j\omega_2)$ and $P_i(j\omega_1, j\omega_2)$ over two frequency dimensions are available, at $\omega_1(i), \omega_2(i)$, $i = 1,\ldots,N$, then (29) can be expressed in form as

$$Z = P\Phi + \xi \tag{30}$$

where

$$Z = \begin{bmatrix} z(j\omega_1(1), j\omega_2(1)) \\ z(j\omega_1(1), j\omega_2(2)) \\ \vdots \\ z(j\omega_1(N), j\omega_2(N)) \end{bmatrix}_{2N\times 1}, \quad \Phi = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{6L+6} \end{bmatrix}_{(6L+6)\times 1}, \xi = \begin{bmatrix} N_2(j\omega_1(1), j\omega_2(1)) \\ N_2(j\omega_1(1), j\omega_2(2)) \\ \vdots \\ N_2(j\omega_1(N), j\omega_2(N)) \end{bmatrix}_{2N\times 1}$$

$$P = \begin{bmatrix} P_1(j\omega_1(1), j\omega_2(1)) & P_2(j\omega_1(1), j\omega_2(1)) & \cdots & P_{6L+6}(j\omega_1(1), j\omega_2(1)) \\ P_1(j\omega_1(1), j\omega_2(2)) & P_2(j\omega_1(1), j\omega_2(2)) & \cdots & P_{6L+6}(j\omega_1(1), j\omega_2(2)) \\ \vdots & \vdots & \cdots & \vdots \\ P_1(j\omega_1(N), j\omega_2(N)) & P_2(j\omega_1(N), j\omega_2(N)) & \cdots & P_{6L+6}(j\omega_1(N), j\omega_2(N)) \end{bmatrix}_{2N\times(6L+6)}$$

Standard linear least squares s can now be applied to estimate Φ in eqn (30).

In fact, eqn (27) only holds when $\bar{H}_2$ and H_2 are both symmetric. It is therefore essential to symmetrisise $\bar{H}_2^{asym}$ and H_2^{asym} by applying eqn (13).

4.3 Estimation of the Third Order Nonlinearities

Setting n=3 in (10) yields

$$\begin{aligned} &-[\sum_{l_1=0}^{L} C_{1,0}(l_1)(j\omega_1 + j\omega_2 + j\omega_3)^{l_1}]H_3^{asym}(j\omega_1, j\omega_2, j\omega_3) \\ &= [\sum_{l_1,l_2=0}^{L} C_{1,1}(l_1,l_2)(j\omega_3)^{l_2} H_{2,1}^{asym}(j\omega_1, j\omega_2) + \sum_{l_1,l_2=0}^{L} C_{2,0}(l_1,l_2) H_{3,2}^{asym}(j\omega_1, j\omega_2, j\omega_3)] \\ &\quad + [\sum_{l_1,l_2,l_3=0}^{L} C_{0,3}(l_1,l_2,l_3)(j\omega_1)^{l_1}(j\omega_2)^{l_2}(j\omega_3)^{l_3} + \sum_{l_1,l_2,l_3=0}^{L} C_{2,1}(l_1,l_2,l_3)(j\omega_3)^{l_3} H_{2,2}^{asym}(j\omega_1, j\omega_2) \\ &\quad + \sum_{l_1,l_2,l_3=0}^{L} C_{1,2}(l_1,l_2,l_3)(j\omega_2)^{l_2}(j\omega_3)^{l_3} H_{1,1}(j\omega_1) \\ &\quad + \sum_{l_1,l_2,l_3=0}^{L} C_{3,0}(l_1,l_2,l_3) H_{3,3}^{asym}(j\omega_1, j\omega_2, j\omega_3)] \end{aligned} \tag{31}$$

With the recursive relation

$$H_{n,p}^{asym}(\cdot) = \sum_{k=1}^{n-p+1} H_k(j\omega_1,\cdots,j\omega_k) H_{n-k,p-1}(j\omega_{k+1},\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_k)^{l_p}, \; n,p=2,3$$

and the property

$$H_{n,1}(j\omega_1,\cdots,j\omega_n) = H_n(j\omega_1,\cdots,j\omega_n)(j\omega_1+\cdots+j\omega_n)^{l_1}, \quad n=1,2,3$$

Using the parameters estimated in previous steps allows specifies the terms

$$-\sum_{l_1=0}^{L} C_{1,0}(l_1)(j\omega_1 + j\omega_2 + j\omega_3)^{l_1} = A_3(j\omega_1, j\omega_2, j\omega_3) \tag{32}$$

So that from (31)

$$H_3^{asym}(j\omega_1,j\omega_2,j\omega_3)$$

$$=\frac{1}{A_3(\cdot)}[\sum_{l_1,l_2=0}^{L} C_{2,1}(l_1,l_2)(j\omega_3)^{l_2} H_{2,1}^{asym}(j\omega_1,j\omega_2) + \sum_{l_1,l_2=0}^{L} C_{2,0}(l_1,l_2) H_{3,2}^{asym}(j\omega_1,j\omega_2,j\omega_3)]$$

$$+\frac{1}{A_3(\cdot)}[\sum_{l_1,l_2,l_3=0}^{L} C_{0,3}(l_1,l_2,l_3)(j\omega_1)^{l_1}(j\omega_2)^{l_2}(j\omega_3)^{l_3} + \sum_{l_1,l_2,l_3=0}^{L} C_{2,1}(l_1,l_2,l_3)(j\omega_3)^{l_3} H_{2,2}^{asym}(j\omega_1,j\omega_2)$$

$$+\sum_{l_1,l_2,l_3=0}^{L} C_{1,2}(l_1,l_2,l_3)(j\omega_2)^{l_2}(j\omega_3)^{l_3} H_{1,1}(j\omega_1)$$

$$+\sum_{l_1,l_2,l_3=0}^{L} C_{3,0}(l_1,l_2,l_3) H_{3,3}^{asym}(j\omega_1,j\omega_2,j\omega_3)] \quad (33)$$

Analogously the noisy case is

$$\tilde{H}_3^{asym}(j\omega_1,j\omega_2,j\omega_3) = H_3^{asym}(j\omega_1,j\omega_2,j\omega_3) + N_3(j\omega_1,j\omega_2,j\omega_3) \quad (34)$$

where $\tilde{H}_3^{asym}(j\omega_1,j\omega_2,j\omega_3)$ is obtained from section 3 and $N_3(j\omega_1,j\omega_2,j\omega_3)$ is independent, zero mean white noise over three frequency axes.

Note that the parameters in the first pair of square bracketson the RHS of (34) have been estimated in previous stages, so this part can be moved to the LHS.

Analogously to the quadratic case, the $\tilde{H}_3^{asym}$ and H_3^{asym} should be symmetrised to obtain a unique representation by applying eqn (13).

Extending the approach in the quadratic case, if '3N' frequency response data can be obtained symmetrically over $\omega_1,\omega_2,\omega_3$, unbiased estimates of the unknown coefficients in (33) can be expected.

This procedure can be continued to higher order nonlinearities.

5. Model Structure Determination

The sequential construction of the model starting with the linear terms, followed by the quadratic terms, and so on as described in the previous sections forms the basis of the solution. But in practice only a few of the numerous possible candidate linear, quadratic, cubic etc. terms will be relevant. It is therefore important , when no a prior information is available regarding the continuous time model, to be able to select significant model terms at each stage in the model reconstruction. This can be achieved using a modification of the orthogonal least squares method (OLS) (Billings, et al. 1988).

Consider a system expressed by

$$z=\sum_{i=1}^{M}\theta_i p_i+\varepsilon \quad (35)$$

where $\theta_i, i=1,\ldots,M$ are unknown parameters.

Reformulating eqn (35) in the form of an auxiliary model yields

$$z=\sum_{i=1}^{M} g_i w_i+\varepsilon \quad (36)$$

where $g_i, i=1,\ldots,M$ are auxiliary parameters and $w_i, i=1,\ldots,M$ are constructed to be orthogonal over the data record such that

$$\sum_{t=1}^{N} w_j(t) w_{k-1}=0, \qquad j=0,1,\ldots,k \quad (37)$$

where N is the length of the data record.

Multiplying the auxiliary model (36) by itself, using the orthogonal property (37) and taking the time average gives

$$\frac{1}{N}\sum_{t=1}^{N} z^2(t) = \frac{1}{N}\sum_{t=1}^{N}\left\{\sum_{i=0}^{M} g_i^2 w_i^2(t)\right\} + \frac{1}{N}\sum_{t=1}^{N}\varepsilon^2(t) \tag{38}$$

Define

$$ERR_i = \frac{\sum_{t=1}^{N} g_i^2 w_i^2(t)}{\sum_{t=1}^{N} z^2(t) - \frac{1}{N}\left\{\sum_{t=1}^{N} z(t)\right\}^2} \times 100 \tag{39}$$

for $i = 1,2,\ldots,M$. The quantity ERR_i is called the Error Reduction Ratio and provides an indication of which terms should be included in the model in accordance with their contribution to the energy of the dependent variable. Terms whose ERR values are less than a pre-defined threshold value (e.g., 0.01) can be considered to be insignificant and negligible. However, this idea cannot be applied directly in the iterative linear transfer function identification. Since Levi's approach places too much emphasis on high frequency response data, in which the SNR is relatively small, generally the result of the first iteration would be biased and this will not give the correct indication of the significance of each term. Some modification therefore must be made when implementing OLS, mainly in the linear term reconstruction. Simulations suggest that the best solution to this problem is to begin with an overparameterized linear model structure. When the parameters of this model converge, eliminate those terms where the ERR values are below the threshold. Finally re-estimate the parameters for this reduced linear model structure and hence obtain the final coefficients.

The reconstruction algorithm including the OLS term-selection criterion is summarised below:

I). Linear term identification

1. Construct the Bode plot of the linear transfer function using the frequency response data, and determine an overestimate of the possible model order.
2. Assume an overparameterized model structure, and apply the orthogonal least squares(OLS) estimator using the GLS criterion described in 4.1 until the parameters converge.
3. Check each ERR_i and eliminate terms with an ERR value below the threshold.
4. Re-pass the frequency response data through the iterative algorithm using the structure obtained in step 3 to obtain estimates of the final linear model.

II).Quadratic nonlinear term identification

5. Build a new $H_1(j\omega)$ using the results obtained in step 4.
6. Assume an overparameterized quadratic model structure and apply the OLS estimator to eqn (25)-(28).
7. Select the terms according to a pre-defined threshold value and re-apply the OLS estimator to eqn (25)-(28) to obtain the final quadratic model.

III). n-th order nonlinear term identification (n>2)

7. Build the new $H_1(\cdot), H_2(\cdot), \ldots, H_{n-1}(\cdot)$ using the results obtained in previous identification steps.
8. Assume an overparameterized n-th order nonlinear model structure and apply the OLS estimator to eqn (10).
9. Select the terms according to a pre-defined threshold value and re-apply OLS estimator to eqn (10) to obtain the final model.

6. Simulation and Real Examples

6.1 A Linear system

Consider a 4-th order linear system defined by

$$H(s) = \frac{\omega_{n_1}^2 . \omega_{n_2}^2}{(s^2 + 2\xi_1\omega_{n_1}s + \omega_{n_1}^2)(s^2 + 2\xi_2\omega_{n_2}s + \omega_{n_2}^2)} \tag{40}$$

where

$$\xi_1 = 0.04, \quad \omega_{n_1} = 2*\pi*13$$

$$\xi_2 = 0.08, \quad \omega_{n_2} = 2*\pi*30$$

and the two resonant peaks appear at 13HZ and 30HZ respectively.

Two methods of obtaining the frequency response data which is the input to the identification procedure will be investigated. Initially the ideal case of generating the data directly from eqn (40) will be used. Then the more realistic approach of generating data from a simulation of eqn (40), identifying a discrete time domain model and using this to generate the frequency response data will be considered.

6.1.1. Generating the frequency response data directly from H_1

A total of 380 frequency data points were generated using eqn (40) over 0~50 HZ and independent, zero mean complex white noise was added with a standard deviation of 0.20.

6.1.1.1. Structure detection

The OLS estimator failed to detect the correct structure based on Levi's method according to the *ERR* values when an overparameterized structure was initialized with 6 linear input terms and 3 linear output terms. The results are given in Table 1(a). Inspection of Table 1(a) shows that the *ERR* values of the terms du/dt, d^2u/dt^2 and d^5y/dt^5 which should not be in the model are slightly greater than the *ERR* value for the correct model terms d^3y/dt^3 and dy/dt. This is a direct result of the bias associated with Levi's method noted in the introduction.

After 5 iterations using the new GLS approach of section 4.1, all the parameters converged, and the *ERR* values reflected the correct model terms (Table 1(a)). The *ERR* values of extra terms du/dt, d^2u/dt^2 and d^5y/dt^5 were now so small that they are clearly negligible (Table 1(a)).

6.1.1.2. Parameter estimation

Eliminating terms with *ERR* values below the threshold, according to the GLS results, and re-estimating the models provided the results in Table 1(b). The H_1 error in Table 1(b) is the sum of the absolute difference in the amplitude of the frequency response data between the identified differential equation and the true system over the range 0~50 HZ .

Normally, without loss of generality, we always assume $C_{1,0}(0) = -1$ in eqn (14), that is the coefficient of the linear y term is 1. However, in this example in order to easily compare the form of the final estimated model with the true model eqn (40), the parameters were re-arranged so that the coefficient of d^4y/dt^4 was 1. The results in Table 1(b) clearly show that the GLS based results are excellent but, as expected, the estimates based on Levi's method are biased.

6.1.2. Generating the frequency response data via an identified ARMAX model

6.1.2.1. Identification of the ARMAX model

The system defined by eqn (40) was excited by a uniformly distributed white noise sequence and the input-output data were then sampled at 500HZ to produce 1000 data points for identification. Independent white noise was then added to the output data to give a SNR of 40 dB. After passing all the model validations, a 35 term ARX model with a 25 term MA noise model was estimated using the input-output data. Figure 1 illustrates the Bode plot of the true continuous system and the identified ARMAX model and shows that they overlap almost perfectly.

Terms	Parameter estimates using Levi	*ERR* of Levi	Parameter estimates using GLS	*ERR* of GLS
d^5y/dt^5	5.2538e-4	0.2387%	-1.1171e-05	4.13e-08%
d^4y/dt^4	1	25.1192 %	1	1.2813%
d^3y/dt^3	68.953	0.3706%	32.289	0.0365%
d^2y/dt^2	70543	58.0506%	42709	98.2920%
dy/dt	8.740e-5	0.2113%	4.057e-5	0.1768%
y	5.087e-8	----------	2.394e+8	-----------
u	-5.38e+8	3.4611%	-2.424e+8	0.2039%
du/dt	8.8259e+5	0.9247%	4.4376e+4	7.58e-05%
d^2u/dt^2	-6479.59	1.3761%	-544.43	3.76e-05 %

Table 1(a). Initial identification results based on noise corrupted H_1 for the system in eqn (40)

Terms	True Parameters	Parameter estimates using Levi	Parameter estimates using GLS
d^4y/dt^4	1	1	1
d^3y/dt^3	36.69	22.78	34.74
d^2y/dt^2	4.2400e+4	65792	4.2473e+4
dy/dt	4.334e+5	308679	4.195e+5
y	2.3705e+8	4.789e+8	2.369e+8
u	-2.3705e+8	-2.885e+8	-2.364e+8
H_1 error	-------------	588.63	8.23

Table 1(b). Final parameter estimates based on noise corrupted H_1 for the system in eqn (40)

6.1.2.2. Reconstruction of the continuous time model

The OLS estimator again failed to detect the correct structure according to the *ERR* values based on Levi's method when an initial overparameterized structure was used (6 linear input terms and 3 linear output terms as in subsection 6.1.1). The results are listed in Table 2(a). After 5 iterations of the GLS approach all the parameters converged and as above the *ERR* values of the incorrect model terms du/dt, d^2u/dt^2 and d^5y/dt^5 became very small. Deleting these terms and re-estimating the model coefficients produced the final estimates in Table 2(b).

Figure 2 shows a comparison of the Bode plot of the reconstructed continuous time system model and the true continuous time model.

Terms	Parameter estimates using Levi	ERR of Levi	Parameter estimates using GLS	ERR of GLS
d^5y/dt^5	-6.473e-04	1.0323%	-4.642e-04	2.31e-05%
d^4y/dt^4	1	25.491%	1	1.0117%
d^3y/dt^3	60.583	0.103%	17.277	0.0301%
d^2y/dt^2	4.205e+4	66.389%	4.277e+04	98.5789%
dy/dt	5.660e+05	0.3484%	3.282e+05	0.1825%
y	2.349e+8	----------	2.399e+8	-----------
u	-2.487e+8	6.4182%	-2.416e+8	0.1968%
du/dt	1.318e+05	0.0072%	9.265e+04	1.784e-07%
d^2u/dt^2	-1108	0.128%	-156.32	5.72e-06%

Table 2(a). Initial identification results based on H_1 generated from estimated ARMAX model

Terms	True Parameters	Parameter estimates using Levi	Parameter estimates using GLS
d^4y/dt^4	1	1	1
d^3y/dt^3	36.69	33.978	36.421
d^2y/dt^2	4.2400e+4	4.1957e+4	4.2351e+4
dy/dt	4.334e+5	4.010e+5	4.319e+5
y	2.3705e+8	2.343e+8	2.366e+8
u	-2.3705e+8	-2.151e+8	-2.3706e+8
H_1 error	-------------	40.2879	3.1347

Table 2(b). Final parameter estimates based on H_1 generated from estimated ARMAX model

A comparison of the results in Tables 1(b) and 2(b) shows that while the GLS estimates are excellent in both cases, the parameter estimates from Levi's method are much better based on the identified ARX model. This is to be expected because the noise which was added to H_1 in 6.1.1 will induce bias in Levi's method whereas the ARX approach accommodates the noise in a noise model which is then discarded to compute the frequency response estimates.

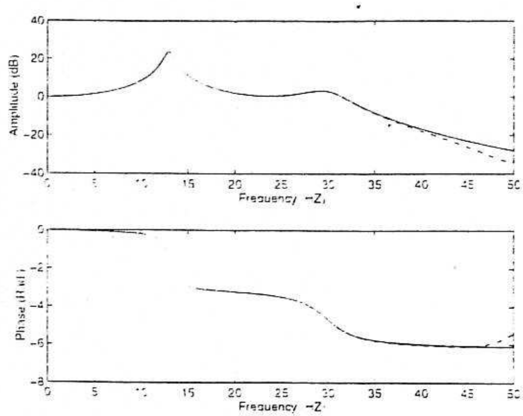

Fig. 1 Comparison of the frequency response gain & phase from the true (—) and identified ARMAX model(---)

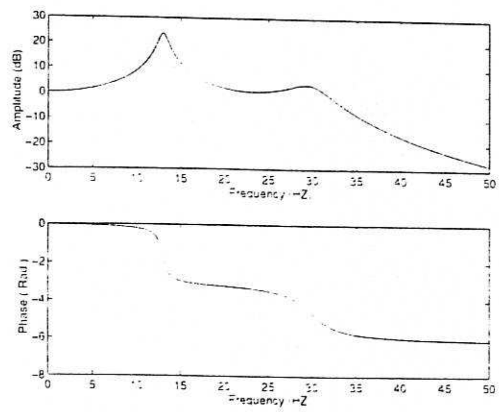

Fig. 2 Comparison of the frequency response gain & phase from the true model (—) and reconstructed model (---)

6.2. A Nonlinear system

Consider the quadratic nonlinear system

$$0 = 1\,y + 0.04\,dy/dt + 0.01\,d^2y/dt^2 - 1\,u + 0.18\,y^2 \qquad (41)$$

As in the linear case two methods of generating the frequency response data which is the input to the identification procedure will be investigated. Initially data will be generated from the exact frequency response functions $H_1(\cdot)$ and $H_2(\cdot)$ corrupted by additive white noise. Then the more realistic approach of fitting a NARMAX model to the noisy output data, using this model to generate $H_1(j\omega_1)$ and $H_2(j\omega_1, j\omega_2)$ and then reconstructing the nonlinear differential equation will be studied.

6.2.1. Generating the frequency response data directly from H_1 and H_2

Analogously, mapping (41) into the frequency domain to produce the exact frequency response functions and then adding independent zero-mean complex white noise with standard deviation 0.1 to both H_1 and H_2 respectively over the uniformly spaced frequency range $f_1 \in [0,20]$ HZ for $H_1(j\omega_1)$ and f_1 and $f_2 \in [0,15]$ HZ for $H_2(j\omega_1, j\omega_2)$ produced the data set for the direct identification of continuous linear and nonlinear terms.

6.2.1.1 Linear term identification

The identification was based on 200 noise-corrupted frequency response data points selected over the uniformly spaced frequency range $f_1 \in [0,20]$ HZ of $H_1(j\omega_1)$. Analogously to subsection 6.1.1, an initial overparameterized linear model structure was used with 4 input terms and 2 output terms. Inspection of the results in Table 3(a) shows that the OLS estimator fails to detect the correct model terms using ERR values based on Levi's method, and the sum of the ERR values is far less than 100% due to the bias induced by the noise that was added to the exact values of $H_1(j\omega_1)$ and $H_2(j\omega_1, j\omega_2)$. Although the Bode plot based on the estimates from the GLS approach is a good fit to the true system, the ERR values in Table 3(a) suggest that the terms d^3y/dt^3 and du/dt should be deleted. Deleting these incorrect terms and re-estimating produced the final results in Table 3(b).

6.2.1.2 Nonlinear term identification

The identification was based on 26×26 uniformly spaced points from the noise corrupted $H_2(j\omega_1, j\omega_2)$ for f_1 and $f_2 \in [0,15]$ HZ. Initially an overparamerized model structure was used. The results from the OLS estimator are listed in Table 3(c), which shows that the ERR values of the incorrect terms are very small and are negligible. Deleting these incorrect terms and re-estimating produced the results in Table 3(d), together with the results from the Swain and Billings(1995) procedure.

Terms	True Parameters	Parameter estimates using Levi	*ERR* of Levi	Parameter estimates using GLS	*ERR* of GLS
d^3y/dt^3	--------	4.6836e-08	0.0006%	-6.6064e-09	5.268e-10%
d^2y/dt^2	0.01	0.0003644	4.6306%	0.01	85.8606%
dy/dt	0.04	0.0017875	0.0235%	0.039967	6.9035%
y	1	1	------------	1	-----------
u	-1	-0.046388	0.5680%	-1.0028	6.9860%
du/dt	-------	0.0001381	0.0314%	0.001042	0.0014%

Table 3(a). Initial linear identification results based on noise corrupted $H_1(j\omega_1)$ for the nonlinear system eqn (41)

Terms	True Parameters	Parameter estimates using Levi	Parameter estimates using GLS
d^2y/dt^2	0.01	0.0003624	0.009946
dy/dt	0.04	0.001367	0.03981
y	1	1	1
u	-1	-0.04776	-0.9987

Table 3(b). Final linear parameter estimates based on noise corrupted $H_1(j\omega_1)$ for the nonlinear system eqn (41)

Terms	estimates	*ERR*
y^2	0.1823	96.3084%
$(du/dt)^2$	0.00002	0.0031%
$(dy/dt)^2$	-0.00004	0.0059%
$y\,u$	-0.0035	0.0015%
$y*(dy/dt)$	0.0002	0.0004%

Table 3(c). Initial nonlinear terms identification results based on noise corrupted $H_2(j\omega_1, j\omega_2)$ for the nonlinear system eqn (41)

Terms	True Parameter	By Swain and Billings's algorithm		By present formulation	
y^2	0.18	0.1822	*ERR* 1.414%	0.1811	*ERR* 96.308%

Table 3(d). Final nonlinear identification results based on noise corrupted $H_2(j\omega_1, j\omega_2)$ for the nonlinear system eqn (41)

Although the estimate from Swain and Billings(1995) is good. the associated *ERR* value is very small (1.414%). That is because Swain and Billings assumed that there are no noise terms in (26). Even if the noise is white as in this example. when the model is multiplied out the term $A_2(j\omega_1, j\omega_2)$ results in coloured noise corruption and this will induce bias.

6.2.2. Generating the frequency response data from an identified NARMAX model

The model of eqn (41) was excited by a uniformly distributed white noise sequence and 1.000 input-output data were obtained by sampling the input and output at 400HZ. The output data were then corrupted by independent white noise to give a SNR of 40 dB. The input-output data were used to identify a NARMAX model using the methods described in Billings and Chen (1989) to produce the final model.

$$\begin{aligned} y(t) = {} & 0.1479\, y(t-1) - 0.0631\, y(t-8) - 0.3318\, y(t-2) - 0.03072\, u(t-2) \\ & - 0.04197\, u(t-4) - 0.03406\, u(t-6) - 0.3848\, y(t-4) - 0.02821\, u(t-3) \\ & + 0.2161\, y(t-3) - 0.05358\, y(t-2)y(t-2) - 0.08622\, y(t-9) - 0.02748\, u(t-8) \\ & - 0.2515\, y(t-12) + 0.01469\, u(t-10) + 0.02267\, u(t-5) + 0.01309\, u(t-1) \\ & - 0.002974\, u(t-5)u(t-6) + 0.03092\, y(t-1)y(t-1) - 0.02547\, y(t-13) \\ & - 0.1730\, y(t-5) + 0.01613\, u(t-7) + 0.01030\, u(t-9) - 0.01915\, y(t-6)y(t-6) \\ & + 0.1184\, y(t-7) + 0.03260\, y(t-6) + \Theta_\xi + \xi(t) \end{aligned} \tag{42}$$

where Θ_ξ represents the noise model terms.

6.2.2.1 Linear term reconstruction

Following the same procedure as in subsection 6.2.1.1 produced the results in Tables 4(a) and (b). The Levi parameter estimates in Table 4(b) are much closer to the true values compared to the results of Table 3(b) because of the improved noise suppression which occurs with the parameter estimation approach.

Terms	True Parameters	Parameter estimates using Levi	*ERR* of Levi	Parameter estimates using GLS	*ERR* of GLS
d^3y/dt^3	--------	1.4355e-06	0.0081%	2.6742e-06	7.347e-07%
d^2y/dt^2	0.01	0.007236	20.2696%	0.01009	86.0066%
dy/dt	0.04	0.03196	7.2366%	0.040375	6.9265%
y	1	1	------------	1	-----------
u	-1	-0.7639	44.7336%	-0.994	7.0653%
du/dt	-------	0.0005151	0.6901%	0.0004416	3.94e-05%

Table 4(a). Initial linear identification results based on $H_1(j\omega_1)$ generated from the identified NARMAX model for the nonlinear system eqn (41)

In fact the Swain and Billings method could only detect the correct y^2 term only when the frequency range was restricted to a narrow region where the SNR was high.

6.3. A Real Application

An accurate and precise prediction of wave forces on offshore structures that are subjected to random ocean waves is an essential prerequisite for design. Wave forces on structures composed of slender members are traditionally calculated on the basis of the Morison equation which was introduced by Morison et al (1950) as a semi-intuitive expression for predicting the force exerted on a body in a viscous fluid under unsteady flow conditions. The Morison equation generally predicts the main trends in measured data quite well: however some characteristics of the flow are not well represented. For example in sinusoidal oscillatory flows the force variation at the fundamental frequency may be well predicted while that at higher harmonics is not. This implies that peak forces may be poorly predicted. A poor representation of the high frequency content of the forces is a serious limitation for the determination of the fatigue life of a structural element. Hence Morison's equation needs to be extended.

A sophisticated system identification technique based on the NARMAX model was used by Worden et al (1994) to model the wave force dynamics of U-tube, De-Voorst and Christchurch Bay data. Although the discrete NARMAX models obtained could adequately represent the dynamics of the input-output data and performed well compared to the Morison equation so far as the predictive performance was concerned, the parameters of the discrete models could not be easily related to the physical parameters of the inherently continuous time system. In addition, the discrete representation of a system varies with the sampling rate but the continuous time representation is unique. Hence, it is often desirable to fit continuous time models to the input-output data to obtain physically interpretable parameters. In the present fluid loading system for example the continuous time model allows the model terms to be related back to fundamental hydrodynamic effects.

In this application, 1000 pairs of the velocity(input) and force(output) time histories with a fixed cylinder which were obtained from the University of Salford (Baker,1994) were used in the NARMAX model identification. The force was measured on a small cylindrical element and the input velocity was the ambient horizontal water particle velocity at the mid point of the element. These data were sampled at 25HZ. The NAMAX model identified using the data set is given by

$$\begin{aligned} y(t) = {} & 1.4282\, y(t-1) - 0.21091\, y(t-2) - 0.26914 y(t-2) \\ & + 1.4318\, u(t-1) - 1.3002\, u(t-3) + 11.090\, u(t-3)u(t-3)u(t-3) \\ & - 7.2738\, u(t-2)u(t-3)u(t-3) \\ & + \Theta_\xi + \xi(t) \end{aligned} \tag{43}$$

Since the NARMAX model of (43) does not contain any quadratic term $H_2(\cdot)$ will be zero. The linear gain and phase plot and the third order gain plot are shown in figure 3 and 4 respectively . The peak magnitude of $H_3(\cdot)$ is found to be 41.26dB. This compares with a maximum linear gain of 20.56dB and shows that the system possesses a very dominant nonlinear characteristic.

After computing the GFRF's from (43), 200 equally spaced frequency response function data were generated in the frequency range of 0-8HZ. A linear model structure with 6 input terms and 4 output terms was used initially. The results are given in Table 5(a).

Note that although the Bode plots of both results in Table 5(a) fit $H_1(\cdot)$ well, the negligible terms indicated by the *ERR* values shows that the models are significantly different. Levi's result shows that d^2u/dt^2 and d^3u/dt^3 are negligible terms while the GLS result suggests that d^3u/dt^3, d^5y/dt^5 and d^4y/dt^4 are negligible which also implies that the initial model structure is overparameterized.

Deleting the three negligible terms and re-estimating the data yields the final GLS results in Table 5(b).

Terms	Parameter estimates using Levi	ERR of Levi (sum=99.92%)	Parameter estimates using GLS	ERR of GLS (sum=99.99%)
d^5y/dt^5	1.8761e-07	10.4695%	1.690e-07	0.0013%
d^4y/dt^4	2.259e-05	1.6997%	2.1881e-05	0.1645%
d^3y/dt^3	1.489e-03	10.6780%	1.457e-03	2.2145%
d^2y/dt^2	0.04494	20.1051%	0.04475	17.7140%
dy/dt	0.2409	25.6696%	0.239	2.5992%
y	1	----------	1	-----------
u	-2.6107	17.5938%	-2.5368	64.1513%
du/dt	-2.1396	13.4546%	-2.1218	7.1437%
d^2u/dt^2	0.01434	0.0324%	0.01293	6.0113%
d^3u/dt^3	1.414e-04	0.2139%	1.6652e-04	0.0001%

Table 5(a). Initial linear identification results for the real system

Term	y	dy/dt	d^2y/dt^2	d^3y/dt^3	u	du/dt	d^2u/dt^2
Estimate	1	0.2253	0.04022	6.58e-4	-2.4504	-2.101	0.0339

Table 5(b). Final linear identification result for the real system

For the reconstruction of the nonlinear third order part frequency response data from 25 equally spaced points for each frequency axis were generated over the frequency range $f_1 = -2-2HZ,\ f_2 = 0-2HZ,\ f_3 = 0-2HZ$. The first two most significant terms u^3 and u^2Du were detected by the OLS estimator with a sum of ERR values at 99.3124% which suggests that these two terms are adequate to capture almost all of the nonlinear dynamics of the system. The final model was given by

$$0 = 6.58\mathrm{e}\text{-}4\frac{d^3y}{dt^3} + 0.04022\frac{d^2y}{dt^2} + 0.2253\frac{dy}{dt} + y - 2.4504u - 2.101\frac{du}{dt} + 0.0339\frac{d^2u}{dt^2}$$
$$- 76.1733u^3 + 17.9824u^2\frac{du}{dt} \qquad (44)$$

A comparison of the reconstructed GFRF are shown in figures 3,4,5. The comparison of the output response of the estimated continuous time model (44) and the original output data at **double** the original sampling rate (50HZ) is illustrated in figure 6. This further validates the reconstructed continuous time model which performs well in predicting the output force.

7. Conclusions

A new algorithm for reconstructing linear and nonlinear differential equation models from frequency response data has been introduced. It has been shown that by combining the procedure of Generalised Least Squares, developed for purely linear systems, with the orthogonal estimator and the error reduction ratio that nonlinear differential equation models can be identified without the need to compute higher order derivatives of noisy data which can lead to numerical problems.

One advantage of the new method is that the continuous time model can be constructed in stages. Initially the linear model terms are determined followed by the quadratic terms and so on. A second advantage is that no a prior knowledge is required regarding the model structure or which terms should be included in the model. The significant model terms are selected as part of the model estimation and this produces a powerful new procedure for identifying nonlinear differential equation models of unknown systems and often leads to concise model structures that can be related back to the physical components of the underlying system.

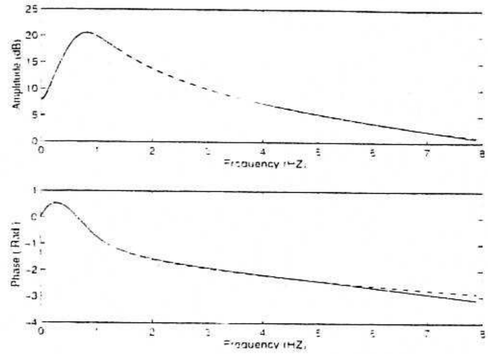

Fig 3. Comparison of H_1 for the real system from eqn (43)(—) and (44) (---)

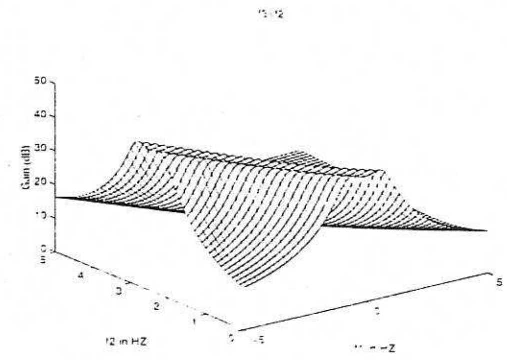

Fig 4. Gain plot of H_3 for the real system from eqn (43)

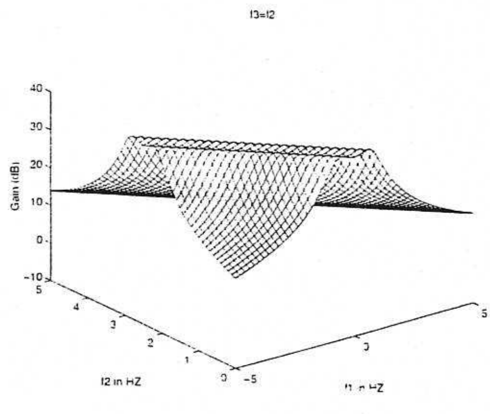

Fig 5. Gain plot of H_3 for the real system from eqn (44)

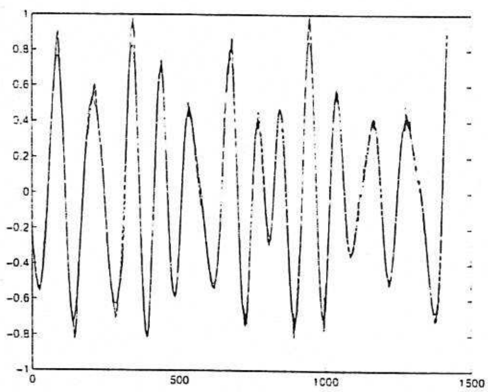

Fig 6. Comparison of output response for the real system, actual output (—) and estimated continuous time model(---)

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Appendix A

Definition of Complex noise(Pintelon. R. etc.1994)

$z = x + jy$ is a zero-mean complex random variable if

$$E\{z\} = 0,\ E\{x^2\} = E\{y^2\} \text{ and } E\{xy\} = 0$$

With the following properties

$$E\{|z|^2\} = 2E\{x^2\} = 2E\{y^2\} \text{ and } E\{z^2\} = 0$$