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Well-partial-orderings and the big Veblen number

Jeroen Van der Meeren · **Michael Rathjen** ·
Andreas Weiermann

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Abstract In this article we characterize a countable ordinal known as the big Veblen number in terms of natural well-partially ordered tree-like structures. To this end, we consider generalized trees where the immediate subtrees are grouped in pairs with address-like objects.
Motivated by natural ordering properties, extracted from the standard notations for the big Veblen number, we investigate different choices for embeddability relations on the generalized trees. We observe that for addresses using one finite sequence only, the embeddability coincides with the classical tree embeddability, but in this article we are interested in more general situations (transfinite addresses and well-partially ordered addresses). We prove that the maximal order type of some of these new embeddability relations hit precisely the big Veblen ordinal $\vartheta\Omega^{\Omega}$. Somewhat surprisingly, changing a little bit the well-partially ordered addresses (going from multisets to finite sequences), the maximal order type hits an ordinal which exceeds the big Veblen number by far, namely $\vartheta\Omega^{\Omega^{\Omega}}$. Our results contribute to the research program (originally initiated by Diana Schmidt) on classifying properties of natural well-orderings in terms of order-theoretic properties of the functions generating the orderings.

Keywords Well-partial-orderings · Kruskal's theorem · Big Veblen number · Ordinal notation systems · Natural well-orderings · Maximal order type · Collapsing function · Recursively defined trees · Tree-embeddabilities

Jeroen Van der Meeren
Department of Mathematics, Ghent University, Krijgslaan 281, 9000 Gent, Belgium
E-mail: jvdm@cage.ugent.be

Michael Rathjen
Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England
E-mail: rathjen@maths.leeds.ac.uk

Andreas Weiermann
Department of Mathematics, Ghent University, Krijgslaan 281, 9000 Gent, Belgium
E-mail: andreas.weiermann@ugent.be

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1 Introduction

Well-quasi-orders are common (and sometimes reinvented [6]) ordering structures which play a prominent role e.g. in computer algebra, formal language theory, transition systems, graph theory and mathematical logic. Well-partial-orders are well-quasi-orders which are in addition antisymmetric. Hence, they are partial orders which are well-founded and do not admit infinite antichains. For the purpose of this article the difference between these notions will not play any role. In fact, any well-quasi-order can be considered as a well-partial-order after the obvious factorization.

In the late seventies Diana Schmidt (using results of De Jongh and Parikh) started a research program to classify the closure ordinals of ordinal functions in terms of underlying monotonicity properties [9]. She calculated the *maximal order type*, i.e. the lengths of the maximal possible linear (thence well-ordered) extension, of several tree embeddability relations for various classes of trees and she classified closure ordinals of several classes of monotonic increasing functions. At that time these results occurred as mere results in the theory of orderings, but later the proof-theoretic significance of her results have been clarified independently by Friedman [11] and by Rathjen and Weiermann [7]. In essence the well-foundedness of the maximal order types of the embeddability relations in question turned out to be equivalent with the corresponding well-partial-orderedness. Therefore, the maximal order type is in some sense the maximum of proof-theoretical ordinals of (natural) theories which do not prove the well-partial-orderedness. These results indicate a general and intrinsic significance of the invariant provided by the maximal order type.

Another very intriguing facet of maximal order types is their relationship with Feferman's natural well-ordering problem. It is well known in the proof-theoretic community that this is a very deep conceptual problem which is now unsolved for decades. So at a more pragmatic level it seems interesting to collect interesting properties of existing examples of natural well-orderings. With regard to this idea the research initiated by Diana Schmidt (and previously by de Jongh and Parikh) fits very well.

Typically an ordinal notation system T is a term representation of the least set T of ordinals such that $0 \in T$ and such that $f(t_1,\ldots,t_n) \in T$ provided that $t_1,\ldots,t_n$ were already in T where f is a constructor symbol (from a given signature). For example the constructor symbols could be functions symbols but more general operations can be allowed. Of course in general not much can be said about the order type of T. The situation changes somewhat surprisingly if we require conditions like increasingness, i.e. $t_i \leq f(t_1,\ldots,t_n)$ and monotonicity, i.e. $f(t_1,\ldots,t_n) \leq f(t'_1,\ldots,t'_n)$ provided that $t_i \leq t'_i$ for all $i \leq n$. Order-theoretic properties like these can impose a priori upper bounds on the order type of T. Diana Schmidt calculated bounds for closure ordinals for such monotonic increasing functions [8].

In her Habilitationsschrift [9] she reproved these results via calculating maximal order types of underlying well-partial-orderings and commented that by going over to well-partial-orderings she has been able to prove stronger results (with sometimes even simpler proofs). The basic idea is to take the set T in question and to restrict the ordering between terms to those cases which are justified by the monotonicity and increasingness conditions (subterm property). The new ordering becomes a well-partial-order and the maximal linear extension provides an upper bound for the order type of the original set T. It is interesting that in case of several examples of natural well-orderings the order type of T usually coincides with the maximal order type of the underlying well-partial-order. So in some sense, natural well-orderings produce the maximal possible order type out of the syntactical material given for defining T.

This line of research has been taken up in [14] where the last author extended Schmidt's approach to transfinite arities. In more detail, motivated by order-theoretic properties of the functions considered by Veblen and Schütte (see, for example, [10, 12] for further details), a well-partial-ordering (which we would denote in this article by $\mathcal{T}(M^\diamond(\tau\times\cdot)))$ has been considered, which corresponds to the ordinal $\vartheta(\Omega^\tau)$ using the ordinal notation system of [7]. The underlying set of this well-partial-ordering was introduced as follows: let $\underline{0}$ and $\underline{\psi}$ be two distinct symbols. For a countable ordinal τ let $\mathcal{T}(M^\diamond(\tau\times\cdot))$ be the least set T such that $\underline{0}\in T$ and such that if $\xi_1<\cdots<\xi_n<\tau$ and $t_1,\ldots,t_n\in T$, then $\langle\underline{\psi},\langle\xi_1,t_1\rangle,\ldots,\langle\xi_n,t_n\rangle\rangle\in T$. Let the underlying ordering $\leq_\tau$ be the least binary reflexive and transitive relation on $\mathcal{T}(M^\diamond(\tau\times\cdot))$ such that

1. $t_i\leq_\tau\langle\underline{\psi},\langle\xi_1,t_1\rangle,\ldots,\langle\xi_n,t_n\rangle\rangle$ $(1\leq i\leq n)$,
2. if $h:\{1,\ldots,m\}\to\{1,\ldots,n\}$ is a one-to-one mapping and if $\xi_i\leq\xi'_{h(i)}$ and $t_i\leq_\tau t'_{h(i)}$ for all $i=1,\ldots,m$, then

$$\langle\underline{\psi},\langle\xi_1,t_1\rangle,\ldots,\langle\xi_m,t_m\rangle\rangle\leq_\tau\langle\underline{\psi},\langle\xi'_1,t'_1\rangle,\ldots,\langle\xi'_n,t'_n\rangle\rangle.$$

Note that in the last condition the comparison is based on comparing multisets of pairs consisting of ordinals (the ordinal addresses) and previously defined terms. In [14] it is shown that the maximal order type of $\mathcal{T}(M^\diamond(\tau\times\cdot))$ is bounded by $\vartheta(\Omega^\tau)$ so that it can give rise to an ordinal notation system for $\vartheta(\Omega^\tau)$. Furthermore (by allowing the case $\tau=\Omega$), it has been indicated in [14] that the order type $\mathcal{T}(M^\diamond(\Omega\times\cdot))$ is bounded by the big Veblen number $\vartheta(\Omega^\Omega)$.

In some sense, these results are not fully satisfying since they refer (what the ordinal valued addresses in the terms concerns) to an underlying structure of ordinals and not to terms of the corresponding ordinal notation system! Therefore, the representation of $\vartheta(\Omega^\tau)$ using $\mathcal{T}(M^\diamond(\tau\times\cdot))$ provides an ordinal notation system which can be only be developed if we have an a priori effective term description for the segment τ. And in the case of $\mathcal{T}(M^\diamond(\Omega\times\cdot))$ it is even more difficult to use this set to built up a constructive notation system.

In this article we take a fresh look at the situation and we succeed in replacing the ordinal addresses by addresses consisting of previously defined terms (or of elements from a given well-partial-order). More specifically, we define two well-partial-orderings $\mathcal{T}(M^\diamond(\cdot\times\cdot))$ and $\mathcal{T}(M(\cdot\times\cdot))$ using multisets of pairs for which their maximal order types are equal to the big Veblen number $\vartheta(\Omega^\Omega)$. As a corollary, we obtain a new and intrinsic characterization of the ordinal notation systems for the big Veblen number. Furthermore, we study what would happen if we replace the multisets by sequences. To this end, we investigate the well-partial-order $\mathcal{T}((\cdot \times \cdot)^*)$, which is based on finite sequences of pairs of previously defined terms. The third author wondered in the ninetees if any ordinal notation system which respects the construction of finite *sequences* of pairs of terms is bounded in order type by $\vartheta(\Omega^\Omega)$. Somewhat surprisingly we show in this article that the relevant order type is equal to $\vartheta\left(\Omega^{\Omega^\Omega}\right)$, which is considerably bigger than the big Veblen number.

In subsequent work the authors intend to characterize the Howard-Bachmann ordinal in terms of tree-like well-partial-orders and to give proof-theoretic characterizations of the relevant systems of second order arithmetic. Moreover, we intend to determine finally the maximal order types of the Friedman-embeddability relation with the so called gap condition [11] or even the maximal order types of the embeddability relations studied by Kriz and Gordeev [5,3]. The present paper will be the first step in attacking these goals.

Technically, this article is organized as follows. In section two we start with some preliminaries and we include a slight correction of a previous proof of the third author regarding the maximal order type of the set of finite multisets over X with respect to a so called term ordering. (We include this side calculation because the term ordering is used in section three.) In section three we prove that the maximal order types of two well-partial-orderings, induced by comparisons of multisets of pairs, are equal to the big Veblen number, which is the limiting number for the Schütte-Veblen hierarchy. In section four we show that the maximal order type of a natural well-partial-order, which is based on comparing finite *sequences* of pairs, hits the bigger ordinal number $\vartheta\left(\Omega^{\Omega^\Omega}\right)$, which is far beyond the Schütte-Veblen hierarchy.

2 Preliminaries

2.1 Well-partial-orderings

In this section we recall some basic facts from the theory of well-partial-orderings. These orderings are defined as follows.

Definition 1 A **well-partial-ordering** (hereafter $\mathbb{wpo}$) is a partial ordering $(X, \leq_X)$ such that for every infinite sequence $(x_i)_{i=1}^{+\infty}$ of elements in X, there exists two indices i and j such that $i < j$ and $x_i \leq_X x_j$. We denote the $\mathbb{wpo}$ $(X, \leq_X)$ by X if the ordering is obvious from the context. We call a sequence $(x_i)_{i<\alpha}$ of elements in X (where $\alpha \leq \omega$) bad if there do not exist $i < j < \alpha$ such that $x_i \leq_X x_j$. If a sequence is not bad we call it good.

In the literature one frequently encounters the similar notion of a well-quasi-ordering (which lacks antisymmetry). For the purpose of this article the difference does not play any role since after an obvious factorization well-quasi-orderings can be considered as well-partial-orderings. Some standard facts about well-partial-orders are gathered in the following lemma.

Lemma 1 *1. A well-partial-ordering does not contain infinite bad sequences.*

2. *A well-partial-ordering is well-founded and does not admit infinite antichains.*
3. *Every extension of a well-partial-ordering to a linear ordering on the same domain is a well-ordering.*
4. *Every partially ordered extension of a well-partial-ordering on the same domain is a well-founded ordering.*
5. *Every infinite sequence of elements in a domain of a well-partial-ordering contains a weakly increasing subsequence.*

Well-partial-orderings therefore play an important role in termination proofs. In a groundbreaking paper, de Jongh and Parikh [4] have been able to isolate a mathematical invariant of well-partial-orderings which is crucial in determining the proof-theoretic strength of well-partial-orderings.

Definition 2 The **maximal order type** of the wpo $(X, \leq_X)$ is equal to

$$\sup\{\alpha \colon \leq_X \subseteq \preceq, \preceq \text{ is a well-ordering on } X \text{ and } otype(X, \preceq) = \alpha\}.$$

We denote this ordinal as $o(X, \leq_X)$ or as $o(X)$ if the ordering is obvious from the context.

The following theorem by de Jongh and Parikh [4] shows that this supremum is actually a maximum.

Theorem 1 (de Jongh and Parikh [4]) *Assume that $(X, \leq_X)$ is a wpo. Then there exists a well-ordering $\preceq$ on X which is an extension of $\leq_X$ such that $otype(X, \preceq) = o(X, \leq_X)$.*

The maximal order type is given by a set-theoretic definition. In case of concretely given well-partial-orderings, it is quite a few times possible to calculate these ordinal more explicitly. To do so, it turns out to be useful to approximate well-partial-orderings by suitable cofinal subsets, the so called 'left sets' of elements.

Definition 3 Let $(X, \leq_X)$ be a wpo and $x \in X$. Define $L(x)$ as the set $\{y \in X : x \not\leq_X y\}$ and $l(x) := o(L(x), \leq \restriction L(x))$.

The role of these sets become clear by the following structural theorem.

Theorem 2 (de Jongh and Parikh [4]) *Assume that X is a partial ordering. If $L(x)$ is a wpo for every $x \in X$, then X is a wpo. (The converse is trivially true.) In this case, $o(X) = \sup\{l(x) + 1 : x \in X\}$.*

Therefore the maximal order type is equal to the height of the tree of finite bad sequences and so in nice cases, the maximal order type can be calculated in a recursive way. Moreover, in many natural cases the maximal order type provides a bound for the proof-theoretic ordinal of the system of analysis needed to prove the well-partial-orderedness of the given well-partial-order. To obtain bounds on maximal order types, it turns out to be useful to consider mappings which preserve well-partial-orderedness. We call these mappings quasi-embeddings.

Definition 4 Let X and Y two posets. A map $e : X \to Y$ is called a **quasi-embedding** if for all $x, x' \in X$ with $e(x) \leq_Y e(x')$ we have $x \leq_X x'$.

This definition looks artificial at first sight but it turns out to be the appropriate notion to work with, as is indicated by the next lemma.

Lemma 2 *If X and Y are posets and $e : X \to Y$ is a quasi-embedding and Y is a* $\mathbb{wpo}$*, then X is a* $\mathbb{wpo}$ *and $o(X) \leq o(Y)$.*

2.2 Bounds for the maximal order types of multisets and finite sequences

In this section we recall some elementary theory for maximal order types. The material is basically known, but we use the opportunity to correct a minor error from a previous calculation of the last author. The study of maximal order types of multisets and sequences is relevant for considering tree-based well-partial-orderings later, since normal trees consist of a root and a sequence (or a multiset) of immediate subtrees.

Definition 5 Let X_0 and X_1 be two $\mathbb{wpo}$'s. Define the **disjoint union** $X_0 + X_1$ as the set $\{(x,0) : x \in X_0\} \cup \{(y,1) : y \in X_1\}$ with the following ordering:

$$(x,i) \leq (y,j) \Leftrightarrow i = j \text{ and } x \leq_{X_i} y.$$

For an arbitrary element (x,i) in $X_0 + X_1$, we omit the second coordinate i if it is clear from the context to which set the element x belongs to.
Define the **cartesian product** $X_0 \times X_1$ as the set $\{(x,y) : x \in X_0, y \in X_1\}$ with the following ordering:

$$(x,y) \leq (x',y') \Leftrightarrow x \leq_{X_0} x' \text{ and } y \leq_{X_1} y'.$$

Definition 6 Let X^* be the set of **finite sequences** over X ordered by

$$(x_1,\dots,x_n) \leq_X^* (y_1,\dots,y_m) \Longleftrightarrow (\exists 1 \leq i_1 < \dots < i_n \leq m)(\forall j \in \{1,\dots,n\})(x_j \leq_X y_{i_j}).$$

Remark that the meaning of $\leq_X^*$ is $(\leq_X)^*$. If the underlying ordering on X is clear from the context, we write $\leq^*$ instead of $\leq_X^*$.

De Jongh, Parikh and Schmidt provided precise bounds for the maximal order types of these well-partial-orderings.

Theorem 3 (de Jongh and Parikh[4], Schmidt[9]) *If X_0, X_1 and X are* $\mathbb{wpo}$*'s, then $X_0 + X_1$, $X_0 \times X_1$ and X^* are still* $\mathbb{wpo}$*'s, and*

$$\begin{aligned} o(X_0 + X_1) &= o(X_0) \oplus o(X_1), \\ o(X_0 \times X_1) &= o(X_0) \otimes o(X_1), \end{aligned}$$

where $\oplus$ and $\otimes$ are the natural sum and product between ordinals, and

$$o(X^*) = \begin{cases} \omega^{\omega^{o(X)-1}} & \text{if } o(X) \text{ is finite,} \\ \omega^{\omega^{o(X)+1}} & \text{if } o(X) = \varepsilon + n, \text{ with } \varepsilon \text{ an epsilon number and } n < \omega, \\ \omega^{\omega^{o(X)}} & \text{otherwise.} \end{cases}$$

We now consider two different embeddings on multisets. The first one is called the *term ordering* by Aschenbrenner and Pong [1]. The second one is called the *multiset ordering* in the term rewriting community.

Definition 7 Let $M^{\diamond}(X, \leq_X)$ be the set of **finite multisets** over X ordered by

$$m \leq_X^{\diamond} m' \iff (\exists f : m \hookrightarrow m')(\forall x \in m)[x \leq_X f(x)].$$

Remark that f is an injective function. We also notate $\leq_X^{\diamond}$ as $\leq^{\diamond}$ and $M^{\diamond}(X, \leq_X)$ as $M^{\diamond}(X)$ if the underlying ordering on X is clear from the context.

Definition 8 Let $M(X, \leq_X)$ be the set of **finite multisets** over X ordered by

$$m \underline{<_X<_X} m' \iff m = m' \text{ or } (\forall x \in m \backslash (m \cap m'))(\exists y \in m' \backslash (m \cap m'))(x \leq_X y),$$

where $\backslash$ and $\cap$ refer to multiset operations. We sometimes notate $\underline{<_X<_X}$ as $\underline{<<}$ and $M(X, \leq_X)$ as $M(X)$ if the underlying ordering on X is clear from the context.

These two multiset-constructors on well-partial-orderings produce again $\mathbb{wpo}$'s (since there is a quasi-embedding to X^*) and their maximal order types in terms of $o(X)$ are known. The easier case concerns the multiset ordering.

Theorem 4 *Let $(X, \leq_X)$ be a $\mathbb{wpo}$. Then $M(X)$ is also a $\mathbb{wpo}$ and $o(M(X)) = \omega^{o(X)}$.*

Proof In [13], the third author proved that it is a $\mathbb{wpo}$ and $o(M(X)) \leq \omega^{o(X)}$. For the other direction, it suffices to show $o(M(X, \preceq)) \geq \omega^{o(X)}$ where $\preceq$ is a linear extension of $\leq_X$ on X having order type $o(X)$. This is because there exists a quasi-embedding from $M(X, \preceq)$ into $M(X, \leq_X)$. Hence it sufficient to prove that for every ordinal α, there exists a quasi-embedding from ω^{α} into $M(\alpha)$. Let e be defined in the following way

$$\begin{array}{rccc} e: & \omega^{\alpha} & \to & M(\alpha), \\ & 0 & \mapsto & [], \\ & \beta =_{CNF} \omega^{\beta_1} + \cdots + \omega^{\beta_n} & \mapsto & [\beta_1, \ldots, \beta_n], \end{array}$$

where *CNF* stands for Cantor Normal Form. One can prove easily that e is a quasi-embedding. $\square$

Definition 9 Let α be an ordinal. Define α' by

$$\alpha' := \begin{cases} \alpha + 1 & \text{if } \alpha = \varepsilon + n, \text{ with } \varepsilon \text{ an epsilon number and } n \text{ a natural number}, \\ \alpha & \text{otherwise.} \end{cases}$$

Notation 1 *Let $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ be an ordinal (CNF stands for Cantor Normal Form). We use the notation $\widehat{\alpha}$ for the ordinal $\omega^{\alpha_1'} + \cdots + \omega^{\alpha_n'}$. Note that $\widehat{\alpha \oplus \beta} = \widehat{\alpha} \oplus \widehat{\beta}$ and that $\alpha < \beta$ implies $\widehat{\alpha} < \widehat{\beta}$. Also $\widehat{0} = 0$.*

Lemma 3 *Assume X is a $\mathbb{wpo}$ with $o(X) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$. Then there exist finitely many elements $x_1, \ldots, x_m$ in X such that*

$$o(L_X(x_1) \cap \cdots \cap L_X(x_m)) = \omega^{\alpha_n}$$

and

$$o(U_X(x_1) \cup \cdots \cup U_X(x_m)) = \omega^{\alpha_1} + \cdots + \omega^{\alpha_{n-1}},$$

where $U_X(x_i) = \{y \in X : x_i \leq_X y\}$, the complement of $L_X(x_i)$.

Proof Corollary 2.17 in [4]. $\square$

Theorem 5 *Let $(X,\leq_X)$ be a well-partial-ordering. Then $M^\diamond(X)$ is also a* $\mathbb{wpo}$ *and* $o(M^\diamond(X)) = \omega^{\widehat{o(X)}}$.

A proof for Theorem 5 can be found in [16]. However, the proof contains a small error for some exceptional cases. For the sake of convenience, we give here a full correct proof by adapting the old proof.

Proof (of Theorem 5) **(i)** $M^\diamond(X)$ **is a** $\mathbb{wpo}$ **and** $o(M^\diamond(X)) \leq \omega^{\widehat{o(X)}}$.
We prove this by induction on $o(X)$. If $o(X)=0$, then $M^\diamond(X)=\{[]\}$. Hence, $M^\diamond(X)$ is a $\mathbb{wpo}$ and the inequality holds. Assume $o(X)>0$ and $o(X) =_{CNF} \omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$ with $n\geq 2$. Using Lemma 3, we obtain elements $x_1,\dots,x_m$ in X such that $o(L_X(x_1)\cap\cdots\cap L_X(x_m)) = \omega^{\alpha_n}$ and $o(U_X(x_1)\cup\cdots\cup U_X(x_m)) = \omega^{\alpha_1}+\cdots+\omega^{\alpha_{n-1}}$. Define X_1 as $U_X(x_1)\cup\cdots\cup U_X(x_m)$ and X_2 as $L_X(x_1)\cap\cdots\cap L_X(x_m)$. From the induction hypothesis, we gain that X_1 and X_2 are $\mathbb{wpo}$'s and $o(M^\diamond(X_i)) \leq \omega^{\widehat{o(X_i)}}$ for every i. Because X is the disjoint union (as a set, not as a partial ordering) of X_1 and X_2, one can define a natural quasi-embedding from $M^\diamond(X)$ in $M^\diamond(X_1)\times M^\diamond(X_2)$. Hence, by Lemma 2, we gain that $M^\diamond(X)$ is a $\mathbb{wpo}$ and

$$\begin{aligned} o(M^\diamond(X)) &\leq o(M^\diamond(X_1)\times M^\diamond(X_2)) \overset{IH}{\leq} \omega^{\widehat{o(X_1)}} \otimes \omega^{\widehat{o(X_2)}} \\ &= \omega^{\widehat{o(X_1)}\oplus\widehat{o(X_2)}} = \omega^{\widehat{o(X_1)\oplus o(X_2)}} = \omega^{\widehat{o(X)}}. \end{aligned}$$

Assume now $o(X)=\omega^{\alpha_1}$. If $\alpha_1=0$, then the claim trivially follows. So, suppose $\alpha_1>0$. In this case, we show that $L(w)$ is a $\mathbb{wpo}$ and $o(L(w)) < \omega^{\widehat{o(X)}}$ for all $w\in M^\diamond(X)$ by induction on the length of w. Hence, the claim follows by Theorem 2. If w is a multiset of length zero, then $L(w)$ is a $\mathbb{wpo}$ and $o(L(w)) = 0 < \omega^{\widehat{o(X)}}$. Assume $w=[w_1,\dots,w_n]$ with $n\geq 1$ and define w' as $[w_1,\dots,w_{n-1}]$. Then $v=[v_1,\dots,v_m]\in L(w)$ iff one of the following holds:

1. $w_n \not\leq_X v_i$ for all i,
2. $w_n \leq_X v_i$ and $[v_1,\dots,v_{i-1},v_{i+1},\dots,v_m]\in L(w')$ for a certain i. Choose i minimal with this condition.

If the first case holds, then v can be seen as an element of $M^\diamond(L_X(w_n))$. If the second case holds, then v can be seen as an element of $X\times L(w')$ by the identification $v\equiv(v_i,[v_1,\dots,v_{i-1},v_{i+1},\dots,v_m])$. Hence we can define a map

$$\begin{array}{rcll} e: L(w) & \to & M^\diamond(L_X(w_n)) + (X\times L(w')), & \\ v & \mapsto & v & \text{if the first case holds,} \\ v & \mapsto & (v_i,[v_1,\dots,v_{i-1},v_{i+1},\dots,v_m]) & \text{if the second case holds.} \end{array}$$

It is easy to check that this is a quasi-embedding. From the fact that $o(L_X(w_n)) < o(X)$ and the main and side induction hypothesis, we obtain that $M^\diamond(L_X(w_n)) + (X\times L(w'))$ is a $\mathbb{wpo}$. Hence, by Lemma 2, $L(w)$ is a $\mathbb{wpo}$ and

$$o(L(w)) \leq o(M^\diamond(L_X(w_n))) \oplus (o(X)\otimes o(L(w'))).$$

Because of the main induction hypothesis, we know that

$$o(M^{\diamond}(L_X(w_n))) \leq \omega^{\widehat{o(L_X(w_n))}} < \omega^{\widehat{o(X)}}.$$

Furthermore, from the side induction hypothesis, it follows that $o(L(w')) < \omega^{\widehat{o(X)}}$. We claim that $o(X) \otimes o(L(w')) < \omega^{\widehat{o(X)}}$. If the claim is valid, then $o(L(w)) < \omega^{\widehat{o(X)}}$ because $\omega^{\widehat{o(X)}}$ is an additive closed ordinal number. We know that $o(X) = \omega^{\alpha_1}$, so $\omega^{\widehat{o(X)}}$ is a multiplicative closed ordinal. So the claim follows if we can prove that $o(X) < \omega^{\widehat{o(X)}}$. If $\alpha_1' = \alpha_1$, then α_1 is not an epsilon number, hence $\alpha_1 < \omega^{\alpha_1} = \omega^{\alpha_1'} = \widehat{o(X)}$. So $o(X) = \omega^{\alpha_1} < \omega^{\widehat{o(X)}}$. If $\alpha_1' = \alpha_1 + 1$, then $\alpha_1 < \omega^{\alpha_1'} = \widehat{o(X)}$. So again $o(X) = \omega^{\alpha_1} < \omega^{\widehat{o(X)}}$.

(ii) $o(M^{\diamond}(X, \leq_X)) \geq \omega^{\widehat{o(X)}}$.
For this it suffices to show $o(M^{\diamond}(X, \preceq)) \geq \omega^{\widehat{o(X)}}$ where $\preceq$ is a linear extension of $\leq_X$ on X having order type $o(X)$. This is because there exists a quasi-embedding from $M^{\diamond}(X, \preceq)$ into $M^{\diamond}(X, \leq_X)$. Hence it sufficient to prove that for every ordinal α, there exists a quasi-embedding from $\omega^{\widehat{\alpha}}$ into $M^{\diamond}(\alpha)$. We do this by induction on α. If $\alpha = 0$, then the assertion is obvious, hence we can assume that $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > 0$. Applying the induction hypothesis on the fact that $\omega^{\alpha_2} + \cdots + \omega^{\alpha_n} < \alpha$, we obtain a quasi-embedding

$$f : \omega^{\omega^{\alpha_2} \widehat{+ \cdots +} \omega^{\alpha_n}} \to M^{\diamond}(\omega^{\alpha_2} + \cdots + \omega^{\alpha_n}).$$

If $n = 1$, then $\omega^{\omega^{\alpha_2} \widehat{+ \cdots +} \omega^{\alpha_n}} = \omega^{\widehat{0}} = \omega^0 = 1$.

a) Suppose $\alpha_1' = \alpha_1$.
Assume $\beta < \omega^{\widehat{\alpha}} = \omega^{\omega^{\alpha_1} + \omega^{\alpha_2'} + \cdots + \omega^{\alpha_n'}}$. Then

$$\beta = \omega^{\omega^{\alpha_1}} \gamma + \omega^{\beta_1} + \cdots + \omega^{\beta_r},$$

with $\beta_r \leq \cdots \leq \beta_1 < \omega^{\alpha_1}$. The inequality $\beta < \omega^{\widehat{\alpha}}$ yields

$$\gamma < \omega^{\omega^{\alpha_2'} + \cdots + \omega^{\alpha_n'}} = \omega^{\omega^{\alpha_2} \widehat{+ \cdots +} \omega^{\alpha_n}}.$$

Define $e : \omega^{\widehat{\alpha}} \to M^{\diamond}(\alpha)$ in the following way. If $f(\gamma) = [\delta_1, \ldots, \delta_k]$, let $e(\beta)$ be

$$[\omega^{\alpha_1} + \delta_1, \ldots, \omega^{\alpha_1} + \delta_k, \beta_1, \ldots, \beta_r].$$

Note that $e(\beta)$ is a multiset over α, because $\delta_i < \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}$ and $\beta_i < \omega^{\alpha_1} \leq \alpha$. If $n = 1$, then $f(\gamma) = f(0) = []$, hence $e(\beta) = [\beta_1, \ldots, \beta_r]$. We claim that $e(\beta) \leq^{\diamond} e(\beta')$ implies $\beta \leq \beta'$. Assume that $\beta' = \omega^{\omega^{\alpha_1}} \gamma' + \omega^{\beta_1'} + \cdots + \omega^{\beta_{r'}'}$, $\beta_1' \geq \cdots \geq \beta_{r'}'$ and $f(\gamma') = [\delta_1', \ldots, \delta_{k'}']$. Then

$$[\omega^{\alpha_1} + \delta_1, \ldots, \omega^{\alpha_1} + \delta_k, \beta_1, \ldots, \beta_r] \leq^{\diamond} [\omega^{\alpha_1} + \delta_1', \ldots, \omega^{\alpha_1} + \delta_{k'}', \beta_1', \ldots, \beta_{r'}'].$$

We know that $\beta'_1,\dots,\beta'_{r'} < \omega^{\alpha_1}$, hence

$$[\omega^{\alpha_1}+\delta_1,\dots,\omega^{\alpha_1}+\delta_k] \leq^{\diamond} [\omega^{\alpha_1}+\delta'_1,\dots,\omega^{\alpha_1}+\delta'_{k'}],$$

so

$$f(\gamma) = [\delta_1,\dots,\delta_k] \leq^{\diamond} [\delta'_1,\dots,\delta'_{k'}] = f(\gamma').$$

Because f is a quasi-embedding, we obtain $\gamma \leq \gamma'$. If $\gamma < \gamma'$, then $\beta < \beta'$. If $\gamma = \gamma'$, then

$$f(\gamma) = [\delta_1,\dots,\delta_k] = [\delta'_1,\dots,\delta'_{k'}] = f(\gamma')$$

and

$$[\omega^{\alpha_1}+\delta_1,\dots,\omega^{\alpha_1}+\delta_k,\beta_1,\dots,\beta_r] \leq^{\diamond} [\omega^{\alpha_1}+\delta'_1,\dots,\omega^{\alpha_1}+\delta'_{k'},\beta'_1,\dots,\beta'_{r'}]$$

implies

$$[\beta_1,\dots,\beta_r] \leq^{\diamond} [\beta'_1,\dots,\beta'_{r'}].$$

This yields $\omega^{\beta_1}+\cdots+\omega^{\beta_r} \leq \omega^{\beta'_1}+\cdots+\omega^{\beta'_{r'}}$, hence $\beta \leq \beta'$.

b) Suppose $\alpha'_1 = \alpha_1+1$.
In this case, α_1 equal to $\varepsilon+m$ with ε an epsilon number and m a natural number. Suppose $\beta < \omega^{\widehat{\alpha}} = \omega^{\varepsilon\omega^{m+1}+\omega^{\alpha'_2}+\cdots+\omega^{\alpha'_n}}$. Then

$$\beta = \omega^{\varepsilon\omega^{m+1}}\gamma+\delta,$$

for certain $\gamma < \omega^{\omega^{\alpha'_2}+\cdots+\omega^{\alpha'_n}} = \omega^{\widehat{\omega^{\alpha_2}+\cdots+\omega^{\alpha_n}}}$ and $\delta < \omega^{\varepsilon\omega^{m+1}} = \varepsilon^{\left(\omega^{m+1}\right)} = \omega^{\omega^{\varepsilon+m+1}}$. In this case, we also have the function f and assume $f(\gamma) = [\gamma_1,\dots,\gamma_k]$. Now $\delta < \varepsilon^{\left(\omega^{m+1}\right)} = \varepsilon^{(\omega\omega^m)}$, hence

$$\begin{aligned}\delta = {} & \varepsilon^{\omega\beta_1+k_1}a_{1,k_1}+\varepsilon^{\omega\beta_1+k_1-1}a_{1,k_1-1}+\cdots+\varepsilon^{\omega\beta_1}a_{1,0}\\ & +\dots\\ & +\varepsilon^{\omega\beta_l+k_l}a_{l,k_l}+\varepsilon^{\omega\beta_l+k_l-1}a_{l,k_l-1}+\cdots+\varepsilon^{\omega\beta_l}a_{l,0},\end{aligned}$$

with $a_{i,j} < \varepsilon$ for every i and j, $a_{i,k_i} \neq 0$ for $i = 1,\dots,l$, $k_i < \omega$ and $\omega^m > \beta_1 > \cdots > \beta_l \geq 0$. Note that if $m = 0$, then $l = 1$ and $\beta_1 = 0$. Define now

$$\begin{aligned} e : \omega^{\widehat{\alpha}} & \to M^{\diamond}(\alpha),\\ \beta & \mapsto [\varepsilon\omega^m+\gamma_1,\dots,\varepsilon\omega^m+\gamma_k,\\ & \qquad \varepsilon\beta_1+a_{1,k_1},\varepsilon\beta_1+a_{1,k_1}+a_{1,k_1-1},\dots,\varepsilon\beta_1+a_{1,k_1}+a_{1,k_1-1}+\cdots+a_{1,0},\\ & \qquad \dots,\\ & \qquad \varepsilon\beta_l+a_{l,k_l},\varepsilon\beta_l+a_{l,k_l}+a_{l,k_l-1},\dots,\varepsilon\beta_l+a_{l,k_l}+a_{l,k_l-1}+\cdots+a_{l,0}].\end{aligned}$$

Note that $e(\beta) \in M^{\diamond}(\alpha)$, because $\varepsilon\omega^m+\gamma_i < \varepsilon\omega^n+\omega^{\alpha_2}+\cdots+\omega^{\alpha_n} = \alpha$ and $\varepsilon\beta_i+a_{i,k_i}+a_{i,k_i-1}+\cdots+a_{i,j} < \varepsilon\beta_i+\varepsilon = \varepsilon(\beta_i+1) \leq \varepsilon\omega^m \leq \alpha$. If $n = 0$, then $f(\gamma) = f(0) = []$, hence $e(\beta) = [\varepsilon\beta_1+a_{1,k_1},\varepsilon\beta_1+a_{1,k_1}+a_{1,k_1-1},\dots,\varepsilon\beta_1+a_{1,k_1}+a_{1,k_1-1}+$

$\cdots + a_{1,0}, \ldots, \varepsilon\beta_l + a_{l,k_l}, \varepsilon\beta_l + a_{l,k_l} + a_{l,k_l-1}, \ldots, \varepsilon\beta_l + a_{l,k_l} + a_{l,k_l-1} + \cdots + a_{l,0}]$.
We claim that $e(\beta) \leq^\diamond e(\beta')$ implies $\beta \leq \beta'$. Assume that $e(\beta) \leq^\diamond e(\beta')$ with $\beta' = \omega^{\varepsilon\omega^{m+1}}\gamma' + \delta'$, $\delta' < \omega^{\varepsilon\omega^{m+1}}$, $f(\gamma') = [\gamma'_1, \ldots, \gamma'_{k'}]$ and

$$\begin{aligned}\delta' = {} & \varepsilon^{\omega\beta'_1 + k'_1} a'_{1,k'_1} + \varepsilon^{\omega\beta'_1 + k'_1 - 1} a'_{1,k'_1-1} + \cdots + \varepsilon^{\omega\beta'_1} a'_{1,0} \\ & + \ldots \\ & + \varepsilon^{\omega\beta'_{l'} + k'_{l'}} a'_{l',k'_{l'}} + \varepsilon^{\omega\beta'_{l'} + k'_{l'} - 1} a'_{l',k'_{l'}-1} + \cdots + \varepsilon^{\omega\beta'_{l'}} a'_{l',0},\end{aligned}$$

with $a'_{i,j} < \varepsilon$ for every i and j, $a'_{i,k'_i} \neq 0$ for $i = 1, \ldots, l'$, $k'_i < \omega$ and $\omega^m > \beta'_1 > \cdots > \beta'_{l'} \geq 0$. Then

$$\begin{aligned} & [\varepsilon\omega^m + \gamma_1, \ldots, \varepsilon\omega^m + \gamma_k, \varepsilon\beta_1 + a_{1,k_1}, \ldots, \varepsilon\beta_1 + a_{1,k_1} + \cdots + a_{1,0}, \ldots, \\ & \varepsilon\beta_l + a_{l,k_l}, \ldots, \varepsilon\beta_l + a_{l,k_l} + \cdots + a_{l,0}] \\ \leq^\diamond & [\varepsilon\omega^m + \gamma'_1, \ldots, \varepsilon\omega^m + \gamma'_{k'}, \varepsilon\beta'_1 + a'_{1,k'_1}, \ldots, \varepsilon\beta'_1 + a'_{1,k'_1} + \cdots + a'_{1,0}, \ldots, \\ & \varepsilon\beta'_{l'} + a'_{l',k'_{l'}}, \ldots, \varepsilon\beta'_{l'} + a'_{l',k'_{l'}} + \cdots + a'_{l',0}]\end{aligned}$$

implies

$$\begin{aligned} & [\varepsilon\omega^m + \gamma_1, \ldots, \varepsilon\omega^m + \gamma_k] \\ \leq^\diamond & [\varepsilon\omega^m + \gamma'_1, \ldots, \varepsilon\omega^m + \gamma'_{k'}],\end{aligned}$$

because $\varepsilon\beta'_i + a'_{i,k'_i} + a'_{i,k'_i-1} + \cdots + a'_{i,j} < \varepsilon\omega^m$. Hence

$$f(\gamma) = [\gamma_1, \ldots, \gamma_k] \leq^\diamond [\gamma'_1, \ldots, \gamma'_{k'}] = f(\gamma').$$

Because f is a quasi-embedding, we obtain that $\gamma \leq \gamma'$. If $\gamma < \gamma'$, then $\beta < \beta'$. If $\gamma = \gamma'$, then

$$f(\gamma) = [\gamma_1, \ldots, \gamma_k] = [\gamma'_1, \ldots, \gamma'_{k'}] = f(\gamma').$$

This yields

$$\begin{aligned} & [\varepsilon\beta_1 + a_{1,k_1}, \ldots, \varepsilon\beta_1 + a_{1,k_1} + \cdots + a_{1,0}, \ldots, \\ & \varepsilon\beta_l + a_{l,k_l}, \ldots, \varepsilon\beta_l + a_{l,k_l} + \cdots + a_{l,0}] \\ \leq^\diamond & [\varepsilon\beta'_1 + a'_{1,k'_1}, \ldots, \varepsilon\beta'_1 + a'_{1,k'_1} + \cdots + a'_{1,0}, \ldots, \\ & \varepsilon\beta'_{l'} + a'_{l',k'_{l'}}, \ldots, \varepsilon\beta'_{l'} + a'_{l',k'_{l'}} + \cdots + a'_{l',0}].\end{aligned}$$

The case $n = 0$ implies directly this inequality because $\gamma = \gamma' = 0$.
Suppose that $\varepsilon\beta_1 + a_{1,k_1}$ is mapped, according to the $\leq^\diamond$-relation, to $\varepsilon\beta'_i + a'_{i,k'_i} + \cdots + a'_{i,j}$. Then $\varepsilon\beta_1 + a_{1,k_1} \leq \varepsilon\beta'_i + a'_{i,k'_i} + \cdots + a'_{i,j}$, so $\beta_1 \leq \beta'_i$ because otherwise $\varepsilon\beta'_i + a'_{i,k'_i} + \cdots + a'_{i,j} < \varepsilon(\beta'_i + 1) \leq \varepsilon\beta_1 \leq \varepsilon\beta_1 + a_{1,k_1}$, a contradiction. Hence $\beta_1 \leq \beta'_i \leq \beta'_1$. If $\beta_1 < \beta'_1$, then $\delta < \delta'$, hence $\beta < \beta'$. If $\beta_1 = \beta'_1$, then i must be 1 and every $\varepsilon\beta_1 + a_{1,k_1} + \cdots + a_{1,j}$ must be mapped on a $\varepsilon\beta'_1 + a'_{1,k'_1} + \cdots + a'_{1,p}$, because otherwise

$\varepsilon\beta_1 + a_{1,k_1} + \cdots + a_{1,j} \leq \varepsilon\beta_i' + a'_{i,k_i'} + \cdots + a_{i,p}$ with $i \neq 1$, hence $\beta_1 \leq \beta_i' < \beta_1'$ like before, a contradiction. So

$$\begin{aligned}&[\varepsilon\beta_1 + a_{1,k_1}, \ldots, \varepsilon\beta_1 + a_{1,k_1} + \cdots + a_{1,0}]\\ \leq^{\diamond} &[\varepsilon\beta_1' + a'_{1,k_1'}, \ldots, \varepsilon\beta_1' + a'_{1,k_1'} + \cdots + a'_{1,0}],\end{aligned}$$

hence

$$[a_{1,k_1}, \ldots, a_{1,k_1} + \cdots + a_{1,0}] \leq^{\diamond} [a'_{1,k_1'}, \ldots, a'_{1,k_1'} + \cdots + a'_{1,0}].$$

From this it follows that $k_1 \leq k_1'$. If $k_1 < k_1'$, then $\delta < \delta'$, hence $\beta < \beta'$. Assume now $k_1 = k_1'$. Then the multisets have the same size. If the multisets are equal, then it is easy to see that $a_{1,j} = a'_{1,j}$ for every j. Otherwise, you can prove by induction on k_1, that this implies the existence of an index j such that $a_{1,k_1} = a'_{1,k_1'}$, ..., $a_{1,j+1} = a'_{1,j+1}$ and $a_{1,j} < a'_{1,j}$. In the former case, we gain $[\varepsilon\beta_2 + a_{2,k_2}, \ldots, \varepsilon\beta_2 + a_{2,k_2} + \cdots + a_{2,0}, \ldots, \varepsilon\beta_l + a_{l,k_l}, \ldots, \varepsilon\beta_l + a_{l,k_l} + \cdots + a_{l,0}] \leq^{\diamond} [\varepsilon\beta_2' + a'_{2,k_2'}, \ldots, \varepsilon\beta_2' + a'_{2,k_2'} + \cdots + a'_{2,0}, \ldots, \varepsilon\beta'_{l'} + a'_{l',k'_{l'}}, \ldots, \varepsilon\beta'_{l'} + a'_{l',k'_{l'}} + \cdots + a'_{l',0}]$, from which we can conclude $\beta \leq \beta'$ by induction on $l + l'$. In the latter case, we can also conclude that $\beta \leq \beta'$. □

2.3 Recursively defined trees

We are interested in well-partial-orderings based on trees for which the maximal order type is equal to the Schütte-Veblen ordinal number $\vartheta(\Omega^\Omega)$. For the sake of convenience, we first recall the definition of the ϑ-function. Afterwards, we introduce the new well-partially ordered tree-like structures. More information about the theta-function and its connections with the Ψ-function (developed by Buchholz [2]) can be found in [7].

Definition 10 Let Ω denote the first uncountable ordinal. Every ordinal $0 < \alpha < \varepsilon_{\Omega+1}$ can be written as $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ with $\beta_i < \Omega$ and $\alpha > \alpha_1 > \cdots > \alpha_n$. Define the set of coefficients recursively as $K(\alpha) = \{\beta_1, \ldots, \beta_n\} \cup K(\alpha_1) \cup \cdots \cup K(\alpha_n)$. Define $k(\alpha)$ as the ordinal $\max(K(\alpha))$. Let $k(0)$ be 0.

Definition 11 Let P denote the set of the additive closed ordinal numbers $\{\omega^\alpha : \alpha \in ON\}$. For every ordinal $\alpha < \varepsilon_{\Omega+1}$, define $\vartheta(\alpha)$ as

$$\min\{\zeta \in P : k(\alpha) < \zeta \text{ and } \forall\beta < \alpha\,(k(\beta) < \zeta \rightarrow \vartheta(\beta) < \zeta)\}.$$

In the sequel we restrict ourselves to ordinals below $\varepsilon_{\Omega+1}$. It can be shown by an easy cardinality argument that $\vartheta\alpha < \Omega$. The definition of ϑ yields easily that the ordering between ϑ-terms can be described as follows.

Lemma 4 *We have* $\vartheta\alpha < \vartheta\beta \iff \begin{cases}\alpha < \beta \text{ and } k(\alpha) < \vartheta\beta\\ \beta < \alpha \text{ and } \vartheta\alpha \leq k(\beta).\end{cases}$

One can find a proof of the previous lemma in [7] (Lemma 1.2.7), but note that they approached the theta-function differently than in this article. We need the following additional lemma's. Both of them can be proved by direct calculations.

Lemma 5 *Suppose α and β are ordinals below $\varepsilon_{\Omega+1}$. Then*

$$k(\alpha \oplus \beta) \leq k(\alpha) \oplus k(\beta),$$
$$k(\alpha \otimes \beta) \leq \max\{k(\alpha) \oplus k(\beta), k(\alpha) \otimes k(\beta) \otimes \omega\}.$$

Furthermore, $k(\alpha), k(\beta) \leq k(\alpha \oplus \beta)$ and $k(\alpha) \leq k(\alpha \otimes \beta)$ if $\beta > 0$.

Lemma 6 *Suppose α and β are countable ordinal numbers with $\alpha, \beta < \gamma$ for an epsilon number γ. $k(o((\Omega\alpha + \beta)^*)) < \gamma$.*

From now on we only consider $\mathbb{wpo}$'s which are countable. Our exposition would also apply after suitable modifications to uncountable $\mathbb{wpo}$'s when one replaces the first uncountable ordinal Ω by the next regular cardinal above the cardinality of the $\mathbb{wpo}$'s in question.

For introducing specific classes of trees later, we need the following definition of a class *Map* of operators which map given well-partial-orderings to new well-partial-orderings in a constructive way.

Definition 12 Define *Map* as the least set satisfying the following:

1. $\cdot \in Map$, ($\cdot$ plays the role of a place holder).
2. If $\mathbb{X}$ is a (as we agreed earlier on) countable $\mathbb{wpo}$, then $\mathbb{X} \in Map$,
3. If $W_1, W_2 \in Map$, then $W_1 + W_2$, $W_1 \times W_2$, W_1^* , $M(W_1)$ and $M^\diamond(W_1)$ are also elements of Map.

Every element W of *Map* can be seen as a mapping from the set of partial orderings to the set of partial orderings: one gains the partial ordering $W(X)$ by putting the partial order X into the $\cdot$. For example, if $W = (\cdot^* \times \mathbb{X})$, then $W(X)$ is the partial ordering $(X^* \times \mathbb{X})$. Furthermore, if X is a $\mathbb{wpo}$, then $W(X)$ is also a $\mathbb{wpo}$.
A crucial fact is that every element of $W(X)$ can be described using a term in finitely many elements in X. In our example, the element $((x_1, \ldots, x_n), \mathbb{x}) \in W(X) = (X^* \times \mathbb{X})$ can be described by a concrete term using the elements $x_1, \ldots, x_n$. Therefore, an arbitrary element of $W(X)$ can be represented as $w(x_1, \ldots, x_n)$, with a term w and $x_i \in X$. By abstracting elements away, 'an element of the mapping W' can be represented as $w(\cdot, \ldots, \cdot)$. E.g. for our element, this is equal to $((\cdot, \ldots, \cdot), \mathbb{x})$. This element of the mapping W maps elements of the partial ordering X to an element of the partial ordering $W(X)$.

Note that a map W is effectively given in the finitely many components which enter via the second step of the definition into its construction. Later in this article, we need the so-called Lifting Lemma.

Lemma 7 (Lifting Lemma) *Assume that $W \in Map$ and let q be a quasi-embedding from the partial ordering Y to the partial ordering Z. Then for every $y_1, \ldots, y_n$, $y'_1, \ldots, y'_m \in Y$ and all elements $w(\cdot, \ldots, \cdot), v(\cdot, \ldots, \cdot)$ in W, $w(q(y_1), \ldots, q(y_n)) \leq_{W(Z)} v(q(y'_1), \ldots, q(y'_m))$ implies $w(y_1, \ldots, y_n) \leq_{W(Y)} v(y'_1, \ldots, y'_m)$.*

Proof This can be proved by a routine induction on the length of construction of W. For example, let us assume that the Lifting Lemma is valid for W_1 and W_2. To prove that it is also valid for $W = W_1 \times W_2$ pick $y_1,\dots,y_{n_1},y_{n_1+1},\dots,y_{n_1+n_2}$ and $y'_1,\dots,y'_{m_1},y'_{m_1+1},\dots,y'_{m_1+m_2}$ arbitrarily from Y and choose $w_1(\cdot,\dots,\cdot)$, $v_1(\cdot,\dots,\cdot)$ in W_1 and $w_2(\cdot,\dots,\cdot)$, $v_2(\cdot,\dots,\cdot)$ in W_2. Assume

$$\begin{aligned}&(w_1(q(y_1),\dots,q(y_{n_1})),w_2(q(y_{n_1+1}),\dots,q(y_{n_1+n_2})))\\ \leq_{W(Z)}&(v_1(q(y'_1),\dots,q(y'_{m_1})),v_2(q(y'_{m_1+1}),\dots,q(y'_{m_1+m_2}))).\end{aligned}$$

This inequality yields

$$\begin{aligned}w_1(q(y_1),\dots,q(y_{n_1})) &\leq_{W_1(Z)} v_1(q(y'_1),\dots,q(y'_{m_1})),\\ w_2(q(y_{n_1+1}),\dots,q(y_{n_1+n_2})) &\leq_{W_2(Z)} v_2(q(y'_{m_1+1}),\dots,q(y'_{m_1+m_2})).\end{aligned}$$

By the induction hypothesis, we obtain

$$\begin{aligned}w_1(y_1,\dots,y_{n_1}) &\leq_{W_1(Y)} v_1(y'_1,\dots,y'_{m_1}),\\ w_2(y_{n_1+1},\dots,y_{n_1+n_2}) &\leq_{W_2(Y)} v_2(y'_{m_1+1},\dots,y'_{m_1+m_2}),\end{aligned}$$

and therefore,

$$\begin{aligned}&(w_1(y_1,\dots,y_{n_1}),w_2(y_{n_1+1},\dots,y_{n_1+n_2}))\\ \leq_{W(Y)}&(v_1(y'_1,\dots,y'_{m_1}),v_2(y'_{m_1+1},\dots,y'_{m_1+m_2})).\end{aligned}$$

Another example is the following. Assume that the Lifting Lemma is valid for W_1. We want to prove that it is also valid for $M(W_1)$. Pick $y^1_1,\dots,y^1_{n_1},\dots,y^k_1,\dots y^k_{n_k}$ and $y'^1_1,\dots,y'^1_{m_1},\dots,y'^l_1,\dots,y'^l_{m_l}$ arbitrarily from Y and choose $w_1,\dots,w_k,v_1,\dots v_l$ in W_1. Assume

$$\begin{aligned}m &= [w_1(q(y^1_1),\dots,q(y^1_{n_1})),\dots,w_k(q(y^k_1),\dots,q(y^k_{n_k}))]\\ \leq_{M(W_1(Z))}m' &= [v_1(q(y'^1_1),\dots,q(y'^1_{m_1})),\dots,v_l(q(y'^l_1),\dots,q(y'^l_{m_l}))].\end{aligned}$$

Hence, the two multisets are equal or $(\forall x\in m\backslash(m\cap m'))(\exists y\in m'\backslash(m\cap m'))(x\leq_{W_1(Z)} y)$. In the former case, the proof easily follows because $w_i(q(y^i_1),\dots,q(y^i_{n_i}))=_{W_1(Z)} v_j(q(y'^j_1),\dots,q(y'^j_{m_j}))$ implies $w_i(y^i_1,\dots,y^i_{n_i})=_{W_1(Y)} v_j(y'^j_1,\dots,y'^j_{m_j})$ by using the induction hypothesis twice.
In the case of $(\forall x\in m\backslash(m\cap m'))(\exists y\in m'\backslash(m\cap m'))(x\leq_{W_1(Z)} y)$, the same observation implies that $w_i(q(y^i_1),\dots,q(y^i_{n_i}))$ is an element of $m\cap m'$ iff $w_i(y^i_1,\dots,y^i_{n_i})$ is an element of the intersection of $\overline{m}$ and $\overline{m'}$, where

$$\begin{aligned}\overline{m} &= [w_1(y^1_1,\dots,y^1_{n_1}),\dots,w_k(y^k_1,\dots,y^k_{n_k})],\\ \overline{m'} &= [v_1(y'^1_1,\dots,y'^1_{m_1}),\dots,v_l(y'^l_1,\dots,y'^l_{m_l})].\end{aligned}$$

Additionally, $v_j(q(y'^j_1),\dots,q(y'^j_{m_j}))$ is an element of $m\cap m'$ iff $v_j(y'^j_1,\dots,y'^j_{m_j})$ is an element of $\overline{m}\cap\overline{m'}$. Take $w_i(y^i_1,\dots,y^i_{n_i})\in\overline{m}\backslash(\overline{m}\cap\overline{m'})$. Then $w_i(q(y^i_1),\dots,q(y^i_{n_i}))\in m\backslash(m\cap m')$, hence there exists $v_j(q(y'^j_1),\dots,q(y'^j_{m_j}))\in m'\backslash(m\cap m')$ such that

$$w_i(q(y^i_1),\dots,q(y^i_{n_i}))\leq_{W_1(Z)} v_j(q(y'^j_1),\dots,q(y'^j_{m_j})).$$

The induction hypothesis implies that $w_i(y^i_1,\dots,y^i_{n_i}) \leq_{W_1(Y)} v_j(y'^j_1,\dots,y'^j_{m_j})$. Furthermore,

$$v_j(y'^j_1,\dots,y'^j_{m_j}) \in \overline{m'}\backslash(\overline{m}\cap\overline{m'}),$$

because $v_j(q(y'^j_1),\dots,q(y'^j_{m_j})) \in m'\backslash(m\cap m')$. We can conclude that $\overline{m} \leq_{M(W_1(Y))} \overline{m'}$. □

In the following definition we introduce the tree-structures we are interested in. We formulate the definition with regard to a general $W \in Map$ and we will later restrict ourselves to specific choices of W. The results which we will prove will hold for general W but carrying out a general proof will be rather messy.

Definition 13 $\mathcal{T}(W)$ is recursively defined as follows:

1. $\circ$ is an element of $\mathcal{T}(W)$,
2. if $w(\cdot,\dots,\cdot)$ is an element of W and $t_1,\dots,t_n$ are elements of $\mathcal{T}(W)$, then the term $\circ[w(t_1,\dots,t_n)]$ is an element of $\mathcal{T}(W)$. Most of the time, we notate this element as $\circ w(t_1,\dots,t_n)$: the element $w(\cdot,\dots,\cdot)$ of W has quite often enough brackets in its description.

Note that $w(t_1,\dots,t_n)$ is an element of $W(\mathcal{T}(W))$. We say that t has a bigger complexity than $t_1,\dots,t_n$.

Definition 14 We define $\leq_{\mathcal{T}(W)}$ on $\mathcal{T}(W)$ as follows:

1. $\circ \leq_{\mathcal{T}(W)} t$ for every t in $\mathcal{T}(W)$,
2. if $s \leq_{\mathcal{T}(W)} t_j$ for a certain j, then $s \leq_{\mathcal{T}(W)} \circ[w(t_1,\dots,t_n)]$,
3. if $w(t_1,\dots,t_n) \leq_{W(\mathcal{T}(W))} w'(t'_1,\dots,t'_m)$,
   then $\circ[w(t_1,\dots,t_n)] \leq_{\mathcal{T}(W)} \circ[w'(t'_1,\dots,t'_m)]$.

Example 1 The partial ordering $\mathcal{T}(M^\diamond(\cdot\times\cdot))$ can be seen a couple $(T,\leq_T)$ such that T and $\leq_T$ is chosen in the least possible way satisfying

- $\circ \in T$,
- if $t_1,\dots,t_{2n} \in T$, then $\circ[(t_1,t_2),\dots,(t_{2n-1},t_{2n})] \in T$,

and

- $\circ \leq_T t$ for every $t \in T$,
- if $s \leq_T t_j$, then $s \leq_T \circ[(t_1,t_2),\dots,(t_{2n-1},t_{2n})]$,
- if $[(t_1,t_2),\dots,(t_{2n-1},t_{2n})] \leq_{M^\diamond(T\times T)} [(t'_1,t'_2),\dots,(t'_{2m-1},t'_{2m})]$,
  then $\circ[(t_1,t_2),\dots,(t_{2n-1},t_{2n})] \leq_T \circ[(t'_1,t'_2),\dots,(t'_{2m-1},t'_{2m})]$.

The general conjecture is that for every W in Map (with the exclusion of some obvious exceptions) the structure $\mathcal{T}(W)$ is actually a wpo and its maximal order type is equal to $\vartheta(o(W(\Omega)))$. In this article, we prove that this conjecture is true for $W = M^\diamond(\cdot\times\cdot)$, $W = M(\cdot\times\cdot)$ and $W = (\cdot\times\cdot)^*$. Furthermore, we also prove for related subcases that $\mathcal{T}(W)$ is a wpo and $\vartheta(o(W(\Omega)))$ is an upperbound of its maximal order type.
Also note that the trees with the ‘normal’ embeddability relation (like in Kruskal’s theorem) can also be captured with this structure: it is equal to $\mathcal{T}(W)$ with $W(X) =$

$X^*\backslash\{()\}$, the set of finite sequences over X without the empty sequence. This is because a tree, as in Kruskal's theorem, can be seen as root with a finite sequence of immediate subtrees. We have to cancel out the empty sequence, because otherwise we have two trees with no immediate subtrees, namely $\circ$ and $\circ[()]$.

Notation 2 *Suppose $t = \circ[w(t_1,\dots,t_n)]$ is an element of $\mathcal{T}(W)$. By $\times t$, we mean the element $w(t_1,\dots,t_n)$ of $W(\mathcal{T}(W))$. (This comes down to deleting the root of the tree in question.)*

3 An order-theoretic characterization of tree-like $\mathbb{wpo}$'s based on multisets of pairs

It this section, we show that the partial orderings $\mathcal{T}(M^\diamond(\cdot\times\cdot))$ and $\mathcal{T}(M(\cdot\times\cdot))$ corresponds to the Schütte-Veblen ordinal, i.e. we prove that $\mathcal{T}(M^\diamond(\cdot\times\cdot))$ and $\mathcal{T}(M(\cdot\times\cdot))$ are $\mathbb{wpo}$'s with corresponding maximal order type $\vartheta(\Omega^\Omega)$.

Definition 15 Let α be an ordinal. Define $\breve{\alpha}$ by

$$\breve{\alpha} := \begin{cases} \alpha+1 & \text{if } \alpha<\omega, \\ \alpha & \text{otherwise.} \end{cases}$$

Notation 3 *Let $\alpha =_{CNF} \omega^{\alpha_1}+\dots+\omega^{\alpha_n}$ be an ordinal. We use the notation $\widetilde{\alpha}$ for the ordinal $\omega^{\breve{\alpha}_1}+\dots+\omega^{\breve{\alpha}_n}$.*

Some easy consequences:

Lemma 8 *1. $\widetilde{\alpha}$ is always a limit ordinal,*
2. $\alpha<\beta$ implies $\widetilde{\alpha}<\widetilde{\beta}$,
3. $\omega^{\widehat{\Omega\alpha}} = \Omega^{\widetilde{\alpha}}$ for every countable ordinal α.

Notation 4 *Let γ be an ordinal number. Define $\gamma^{\overline{1}}$ as γ and $\gamma^{\overline{n+1}}$ as $\gamma^{\overline{n}}\otimes\gamma$.*

The following theorem is needed for proving the main Theorem 7.

Theorem 6 *Assume that $\mathbb{X}_{k,l}$ and $\mathbb{Y}_{k,l}$ are countable $\mathbb{wpo}$'s. Let*

$$W(X) = \sum_{k=0}^{K}\sum_{l=0}^{L} M^\diamond(X\times\mathbb{X}_{k,l})\times X^l\times\mathbb{Y}_{k,l},$$

where X^l denotes the product $X\times\dots\times X$ with l X's. Then $\mathcal{T}(W)$ is a $\mathbb{wpo}$ and $o(\mathcal{T}(W))\leq\vartheta(o(W(\Omega)))$.

Proof We will prove the theorem by main induction on the ordinal $o(W(\Omega))$. Without loss of generality, we can assume that $\mathbb{Y}_{k,l}$ are nonempty $\mathbb{wpo}$'s. If $o(W(\Omega))<\Omega$, then $W(X)$ does not contain X (or W does not contain $\cdot$) and it is equal to a countable $\mathbb{wpo}$ $\mathbb{Z}$: in this case, $L=0$ and $\mathbb{X}_{k,0}=\emptyset$. Therefore, $W(X)\cong\sum_{k=0}^{K}\mathbb{Y}_{k,0}$ which we call $\mathbb{Z}$. Hence $\mathcal{T}(W)\cong\mathbb{Z}\cup\{0\}$, with 0 smaller than every element in $\mathbb{Z}$. So $o(\mathcal{T}(W)) = o(\mathbb{Z})+1\leq\vartheta(o(\mathbb{Z})) = \vartheta(o(W(\Omega)))$.

If $o(W(\Omega)) \geq \Omega$, in other words X really occurs in $W(X)$, then $\vartheta(o(W(\Omega)))$ is an epsilon number. We want to prove that $L(t)$ is a $\mathbb{wpo}$ and $l(t) < \vartheta(o(W(\Omega)))$ for every t in $\mathcal{T}(W)$. We do this by induction on the complexity of t. The theorem then follows from Theorem 2. If $t = \circ$, then $L(t)$ is the empty $\mathbb{wpo}$ and $l(t) = 0 < \vartheta(o(W(\Omega)))$. Assume that $t = \circ([(t^1_1,x_1),\ldots,(t^1_n,x_n)],(t^2_1,\ldots,t^2_b),y)$ with $L(t^j_i)$ $\mathbb{wpo}$'s and $l(t^j_i) < \vartheta(o(W(\Omega)))$, $x_i \in \mathbb{X}_{a,b}$ and $y \in \mathbb{Y}_{a,b}$. We will show that $L(t)$ is a $\mathbb{wpo}$ and $l(t) < \vartheta(o(W(\Omega)))$.

Suppose that $s = \circ([(s^1_1,\overline{x_1}),\ldots,(s^1_m,\overline{x_m})],(s^2_1,\ldots,s^2_{\overline{b}}),\overline{y})$, with $\overline{x_i} \in \mathbb{X}_{\overline{a},\overline{b}}$ and $\overline{y} \in \mathbb{Y}_{\overline{a},\overline{b}}$. $t \leq_{\mathcal{T}(W)} s$ is valid iff $t \leq_{\mathcal{T}(W)} s^j_i$ for a certain i and j or $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, and

$$
\begin{gathered}
(t^2_1,\ldots,t^2_b) \leq (s^2_1,\ldots,s^2_{\overline{b}}),\\
[(t^1_1,x_1),\ldots,(t^1_n,x_n)] \leq^{\diamond} [(s^1_1,\overline{x_1}),\ldots,(s^1_m,\overline{x_m})].
\end{gathered}
$$

Therefore, $s \in L(t)$ iff $s^j_i \in L(t)$ for every i and j and one of the following holds

1. $a \neq \overline{a}$,
2. $a = \overline{a}$, $b \neq \overline{b}$,
3. $a = \overline{a}$, $b = \overline{b}$, $y \not\leq \overline{y}$,
4. $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, $(t^2_1,\ldots,t^2_b) \not\leq (s^2_1,\ldots,s^2_{\overline{b}})$,
5. $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, $(t^2_1,\ldots,t^2_b) \leq (s^2_1,\ldots,s^2_{\overline{b}})$, $[(t^1_1,x_1),\ldots,(t^1_n,x_n)] \not\leq^{\diamond} [(s^1_1,\overline{x_1}),\ldots,(s^1_m,\overline{x_m})]$.

If (4.) holds, there must be a minimal index $l(s)$ such that

$$t^2_1 \leq s^2_1,\ldots,t^2_{l(s)-1} \leq s^2_{l(s)-1}, t^2_{l(s)} \not\leq s^2_{l(s)}.$$

If (5.) holds, we must be in one of the following groups

1. $(t^1_1,x_1) \not\leq (s^1_i,\overline{x_i})$ for every i,
2. there exists i_1 such that $(t^1_1,x_1) \leq (s^1_{i_1},\overline{x_{i_1}})$ and $(t^1_2,x_2) \not\leq (s^1_i,\overline{x_i})$ for every $i \neq i_1$ (choose i_1 minimal),
3. there exist i_1 and i_2 such that $(t^1_1,x_1) \leq (s^1_{i_1},\overline{x_{i_1}})$, $(t^1_2,x_2) \leq (s^1_{i_2},\overline{x_{i_2}})$ and $(t^1_3,x_3) \not\leq (s^1_i,\overline{x_i})$ for every $i \neq i_1,i_2$ (choose i_1,i_2 minimal with respect to the lexicographic ordering on the couples (i_1,i_2) for which this holds),

   ...

n. there exist indices $i_1,\ldots,i_{n-1}$ such that $(t^1_1,x_1) \leq (s^1_{i_1},\overline{x_{i_1}})$, $(t^1_2,x_2) \leq (s^1_{i_2},\overline{x_{i_2}})$, ..., $(t^1_{n-1},x_{n-1}) \leq (s^1_{i_{n-1}},\overline{x_{i_{n-1}}})$ and $(t^1_n,x_n) \not\leq (s^1_i,\overline{x_i})$ for every $i \neq i_1,\ldots,i_{n-1}$ (choose $i_1,\ldots,i_{n-1}$ minimal with respect to the lexicographic ordering on the $(n-1)$-tuples $(i_1,i_2,\ldots,i_{n-1})$ for which this holds).

It is easy to see that $(t_i^1, x_i) \not\leq (s_j^1, \overline{x_j})$ is equivalent with saying $s_j^1 \in L(t_i^1)$ or ($t_i^1 \leq s_j^1$ and $\overline{x_j} \in L_{\mathbb{X}_{a,b}}(x_i)$). Define $W'(X)$ as

$$\begin{aligned}
&\sum_{k=0,k\neq a}^{K} \sum_{l=0}^{L} M^\diamond(X \times \mathbb{X}_{k,l}) \times X^l \times \mathbb{Y}_{k,l} \\
&+ \sum_{l=0,l\neq b}^{L} M^\diamond(X \times \mathbb{X}_{a,l}) \times X^l \times \mathbb{Y}_{a,l} \\
&+ M^\diamond(X \times \mathbb{X}_{a,b}) \times X^b \times L_{\mathbb{Y}_{a,b}}(y) \\
&+ \sum_{l=1}^{b} M^\diamond(X \times \mathbb{X}_{a,b}) \times X^{b-1} \times L_{\mathcal{T}(W)}(t_l^2) \times \mathbb{Y}_{a,b} \\
&+ \sum_{k=1}^{n} (X \times \mathbb{X}_{a,b})^{k-1} \times M^\diamond(L_{\mathcal{T}(W)}(t_k^1) \times \mathbb{X}_{a,b}) \times M^\diamond(X \times L_{\mathbb{X}_{a,b}}(x_k)) \times X^b \times \mathbb{Y}_{a,b}
\end{aligned}$$

The five terms separated by $+$ correspond to the five groups (1. - 5.) in which s can lie in. The index l in the fourth line corresponds to $l(s)$. The index k in the fifth line corresponds to which case (1. - n.) we are at that moment.
So recall that $s \in L(t)$ iff $s_i^j \in L(t)$ for every i and j and $\times t \not\leq_{W(\mathcal{T}(W))} \times s$, where we characterized $\times t \not\leq_{W(\mathcal{T}(W))} \times s$ by the five cases 1. - 5., hence it is characterized by W'. Therefore, $\times s$ (with $s \in L(t)$) can be interpreted as an element $w'(s_1, \dots, s_r)$ of $W'(L(t))$ with every s_k equal to a certain $s_i^j \in L(t)$ and $w'(\cdot, \dots, \cdot) \in W'$. Let $w'(s_1, \dots, s_r)$ be this interpretation of $\times s$ and $w''(s'_1, \dots, s'_{r'})$ the interpretation of $\times s'$ for an arbitrary $s' \in L(t)$. It can be proved in a straightforward way that the inequality $w'(s_1, \dots, s_r) \leq_{W'(\mathcal{T}(W))} w''(s'_1, \dots s'_{r'})$ implies $\times s \leq_{W(\mathcal{T}(W))} \times s'$, hence $s \leq_{\mathcal{T}(W)} s'$. Because a similar argument (completely written out) can be found in the proof of Theorem 7, we will skip the detailed verification of this fact.

There exists a quasi-embedding f from $L(t)$ in $\mathcal{T}(W')$: define $f(\circ)$ as $\circ$. Assume

$$s = \circ([(s_1^1, \overline{x_1}), \dots, (s_m^1, \overline{x_m})], (s_1^2, \dots, s_{\overline{b}}^2), \overline{y}) \in L(t)$$

and suppose that $f(s_i^j)$ is already defined. Let $w'(s_1, \dots, s_r)$ be the interpretation of $\times s$ in $W'(L(t))$. Then $\{s_1, \dots, s_r\} \subseteq \{s_1^1, \dots, s_m^1, s_1^2, \dots, s_{\overline{b}}^2\}$ and define $f(s)$ as the element $\circ(w'(f(s_1), \dots, f(s_r)))$ in $\mathcal{T}(W')$.

We show that f is a quasi-embedding. We will prove, by induction on the sum of the complexities of s and s', that $f(s) \leq_{\mathcal{T}(W')} f(s')$ implies $s \leq_{\mathcal{T}(W)} s'$. If either s or s' is equal to $\circ$, this is trivial. Suppose $f(s) \leq_{\mathcal{T}(W')} f(s')$ with $f(s) = \circ[w'(f(s_1), \dots, f(s_r))]$ and $f(s') = \circ[w''(f(s'_1), \dots, f(s'_{r'}))]$. Then $f(s) \leq_{\mathcal{T}(W')} f(s'_i)$ for a certain i or $w'(f(s_1), \dots, f(s_r)) \leq_{W'(\mathcal{T}(W'))} w''(f(s'_1), \dots, f(s'_{r'}))$. In the former case, we obtain by the induction hypothesis, that $s \leq_{\mathcal{T}(W)} s'_i \leq_{\mathcal{T}(W)} s'$. In the latter case, f is a quasi-embedding on the set $S = \{s_1^1, \dots, s_{\overline{b}}^2\} \cup \{s_1'^1, \dots, s_{\overline{b'}}'^2\} \subseteq \mathcal{T}(W)$ to the set $f(S) \subseteq \mathcal{T}(W')$ by the induction hypothesis. Therefore, by the Lifting Lemma

$$w'(s_1, \dots, s_r) \leq_{W'(\mathcal{T}(W))} w''(s'_1, \dots s'_{r'}).$$

Hence, $s \leq_{\mathcal{T}(W)} s'$.

Because of Lemma 2, we obtain $o(L(t)) \leq o(\mathcal{T}(W'))$. If we can prove the inequalities $o(W'(\Omega)) < o(W(\Omega))$ and $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$, we gain by the main induction hypothesis

$$\begin{aligned} o(L(t)) &\leq o(\mathcal{T}(W')) \\ &\leq \vartheta(o(W'(\Omega))) \\ &< \vartheta(o(W(\Omega))), \end{aligned}$$

and that $L(t)$ is a $\mathbb{wpo}$ by Lemma 2. Hence, we are done.

a) $o(W'(\Omega)) < o(W(\Omega))$.
For notational convenience, we write sometimes $\mathbb{Y}$ instead of $o(\mathbb{Y})$ for $\mathbb{wpo}$'s $\mathbb{Y}$. $o(W'(\Omega)) < o(W(\Omega))$ is equivalent with saying (using Theorem 3 and 5)

$$\begin{aligned} &\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes l_{\mathbb{Y}_{a,b}}(y) \\ &\oplus \bigoplus_{l=1}^{b} \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^{b-1} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b} \\ &\oplus \bigoplus_{k=1}^{n} (\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} \\ < \, &\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b}. \end{aligned}$$

It is easy to see that there exists a finite N such that for every k,

$$(\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} < \Omega^N.$$

Note that all occurring $\mathbb{wpo}$'s are countable. Furthermore,

$$\begin{aligned} \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} &= \Omega^{\widetilde{l_{\mathbb{X}_{a,b}}(x_k)}}, \\ \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} &= \Omega^{\widetilde{\mathbb{X}_{a,b}}}, \end{aligned}$$

using Notation 1 and 3. Because $\widetilde{X_{a,b}}$ is a limit ordinal and $\widetilde{l_{\mathbb{X}_{a,b}}(x_k)} < \widetilde{\mathbb{X}_{a,b}}$, we obtain

$$\begin{aligned} &(\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} \\ < \, &\Omega^{\widetilde{l_{\mathbb{X}_{a,b}}(x_k)}+N} \\ < \, &\Omega^{\widetilde{\mathbb{X}_{a,b}}} \\ = \, &\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}}. \end{aligned}$$

The last ordinal number is additive closed, hence

$$\bigoplus_{k=1}^{n} (\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} < \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}}.$$

Similarly,

$$\left(\bigoplus_{l=1}^{b} \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^{b-1} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b}\right) \oplus \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \leq \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b,$$

from which we can conclude

$$\begin{aligned}
&\left(\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes l_{\mathbb{Y}_{a,b}}(y)\right) \oplus \left(\bigoplus_{l=1}^{b} \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^{b-1} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b}\right) \\
&\oplus \left(\bigoplus_{k=1}^{n} (\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b}\right) \\
< &\left(\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes l_{\mathbb{Y}_{a,b}}(y)\right) \oplus \left(\bigoplus_{l=1}^{b} \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^{b-1} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b}\right) \\
&\oplus \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \\
\leq &\left(\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes l_{\mathbb{Y}_{a,b}}(y)\right) \oplus \left(\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b\right) \\
\leq &\ \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b}.
\end{aligned}$$

This strict inequality also holds in the exceptional cases $b = 0$ and $n = 0$.

b) $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.

$$o(W(\Omega)) = \bigoplus_{k=0}^{K} \bigoplus_{l=0}^{L} \omega^{\widehat{\Omega \otimes \mathbb{X}_{k,l}}} \otimes \Omega^l \otimes \mathbb{Y}_{k,l} = \bigoplus_{k=0}^{K} \bigoplus_{l=0}^{L} \Omega^{\widetilde{\mathbb{X}_{k,l}} \oplus l} \otimes \mathbb{Y}_{k,l},$$

from which we obtain

$$\mathbb{Y}_{k,l}, \widetilde{\mathbb{X}_{k,l}} \oplus l \leq k(o(W(\Omega))) < \vartheta(o(W(\Omega))). \tag{1}$$

Furthermore, $\mathbb{X}_{k,l} \leq \widetilde{\mathbb{X}_{k,l}}$. Now, $o(W'(\Omega))$ is equal to

$$\begin{aligned}
&\bigoplus_{k=0, k\neq a}^{K} \bigoplus_{l=0}^{L} \Omega^{\widetilde{\mathbb{X}_{k,l}} \oplus l} \otimes \mathbb{Y}_{k,l} \\
&\oplus \bigoplus_{l=0, l\neq b}^{L} \Omega^{\widetilde{\mathbb{X}_{a,l}} \oplus l} \otimes \mathbb{Y}_{a,l} \\
&\oplus \Omega^{\widetilde{\mathbb{X}_{a,b}} \oplus b} \otimes l_{\mathbb{Y}_{a,b}}(y) \\
&\oplus \bigoplus_{l=1}^{b} \Omega^{\widetilde{\mathbb{X}_{a,b}} \oplus (b-1)} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b} \\
&\oplus \bigoplus_{k=1}^{n} \Omega^{(k-1) \oplus \widetilde{l_{\mathbb{X}_{a,b}}(x_k)} \oplus b} \otimes \mathbb{X}_{a,b}^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1) \otimes \mathbb{X}_{a,b}}} \otimes \mathbb{Y}_{a,b}.
\end{aligned}$$

Hence, $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$ by Lemma 5, inequality (1), $\widehat{o(\mathbb{X})} \leq o(\mathbb{X}) \otimes \omega$, $\widetilde{o(\mathbb{X})} \leq o(\mathbb{X}) \otimes \omega$, $l_{\mathcal{T}(W)}(t^i_j) < \vartheta(o(W(\Omega)))$ and the fact that $\vartheta(o(W(\Omega)))$ is an epsilon number. □

Theorem 7 *The partial orderings $\mathcal{T}(M(\cdot \times \cdot))$ and $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ are* wpo*'s and $o(\mathcal{T}(M(\cdot \times \cdot))) \leq o(\mathcal{T}(M^\diamond(\cdot \times \cdot))) \leq \vartheta(\Omega^\Omega)$.*

Proof By induction on the complexity of the elements in $\mathcal{T}(M(\cdot \times \cdot))$ and $\mathcal{T}(M^\diamond(\cdot \times \cdot))$, one can define easily a quasi-embedding from $\mathcal{T}(M(\cdot \times \cdot))$ in $\mathcal{T}(M^\diamond(\cdot \times \cdot))$. Hence if $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ is a wpo, then $\mathcal{T}(M(\cdot \times \cdot))$ is a wpo and $o(\mathcal{T}(M(\cdot \times \cdot))) \leq o(\mathcal{T}(M^\diamond(\cdot \times \cdot)))$ by Lemma 2. To prove the inequality $o(\mathcal{T}(M^\diamond(\cdot \times \cdot))) \leq \vartheta(\Omega^\Omega)$ and the well-partial-orderedness of $\mathcal{T}(M^\diamond(\cdot \times \cdot))$, we show that $L(t)$ is a wpo and $l(t) < \vartheta(\Omega^\Omega)$ for every t in $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ by induction on the complexity of t. The theorem then follows from Theorem 2. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(\Omega^\Omega)$. Assume $t = \circ[(s_1,t_1)\dots,(s_n,t_n)]$ with $L(t_i)$, $L(s_i)$ wpo's and $l(t_i), l(s_i) < \vartheta(\Omega^\Omega)$.

Take an arbitrary $v = \circ[(u_1,v_1),\dots,(u_m,v_m)]$ in $\mathcal{T}(M^\diamond(\cdot \times \cdot))$. Then

$$\begin{aligned} t \leq v \Leftrightarrow\ & t \leq u_i \text{ or } t \leq v_i \text{ for a certain } i \\ & \text{or } (\exists f : \{1,\dots,n\} \hookrightarrow \{1,\dots,m\})(\forall i \in \{1,\dots,n\})((s_i,t_i) \leq (u_{f(i)}, v_{f(i)})). \end{aligned}$$

Hence $v = \circ[(u_1,v_1),\dots,(u_m,v_m)]$ is an element of $L(t)$ iff $u_i, v_i \in L(t)$ for every i and

$$(\forall f : \{1,\dots,n\} \hookrightarrow \{1,\dots,m\})(\exists i \in \{1,\dots,n\})((s_i,t_i) \nleq (u_{f(i)}, v_{f(i)})).$$

Therefore, $v \in L(t)$ ($v \neq \circ$) iff $u_i, v_i \in L(t)$ for every i and one of the following holds:

1. $(s_1,t_1) \nleq (u_i,v_i)$ for every i,
2. there exists i_1 such that $(s_1,t_1) \leq (u_{i_1},t_{i_1})$ and $(s_2,t_2) \nleq (u_i,v_i)$ for every $i \neq i_1$ (i_1 minimal chosen),
3. there exist i_1 and i_2 such that $(s_1,t_1) \leq (u_{i_1},t_{i_1})$; $(s_2,t_2) \leq (u_{i_2},t_{i_2})$ and $(s_3,t_3) \nleq (u_i,v_i)$ for every $i \neq i_1, i_2$ (choose i_1, i_2 minimal with respect to the lexicographic ordering on the couples (i_1,i_2) for which this holds),

   ...

n. there exist indices $i_1,\dots,i_{n-1}$ such that $(s_1,t_1) \leq (u_{i_1},t_{i_1})$; $(s_2,t_2) \leq (u_{i_2},t_{i_2})$; $\dots (s_{n-1},t_{n-1}) \leq (u_{i_{n-1}},t_{i_{n-1}})$ and $(s_n,t_n) \nleq (u_i,v_i)$ for every $i \neq i_1,\dots,i_{n-1}$ (pick the indices $i_1,\dots,i_{n-1}$ minimal with respect to the lexicographic ordering on the $(n-1)$-tuples $(i_1,i_2,\dots,i_{n-1})$ for which this holds).

Also note that $\circ \in L(t)$. Define

$$W'(X) := \sum_{k=1}^{n} (X \times X)^{k-1} \times M^\diamond\left((L(s_k) \times X) + (X \times L(t_k))\right).$$

k represents which case (1.-n.) holds. Define the map $f : L(t) \to \mathcal{T}(W')$ recursively as follows. First, let $f(\circ)$ be $\circ$. Secondly, suppose $v = \circ[(u_1,v_1),\dots,(u_m,v_m)] \in L(t)$ and that $f(u_i)$, $f(v_i)$ are already defined. Assume that v lies in group k (hence we have indices $i_1,\dots,i_{k-1}$) and take $\{j_1,\dots,j_l\}$ as the subset of $\{1,\dots,m\} \backslash \{i_1,\dots,i_{k-1}\}$ such that $u_{j_p} \in L(s_k)$ for every p. Define $\{r_1, \ldots, r_t\}$ as $\{1, \ldots, m\} \backslash \{i_1, \ldots, i_{k-1}, j_1, \ldots, j_l\}$. Note that $v_{r_p} \in L(t_k)$ for every p. Let $f(v)$ be the following element in $\mathcal{T}(W')$:

$$\circ\Big((f(u_{i_1}), f(v_{i_1})), \ldots, (f(u_{i_{k-1}}), f(v_{i_{k-1}})),$$
$$[(u_{j_1}, f(v_{j_1})), \ldots, (u_{j_l}, f(v_{j_l})), (f(u_{r_1}), v_{r_1}) \ldots, (f(u_{r_t}), v_{r_t})] \Big). \quad (2)$$

Assuming that f is a quasi-embedding and using Lemma 2 and Theorem 6, we have that $L(t)$ is a $\mathbb{wpo}$ and

$$o(L(t)) \leq o(\mathcal{T}(W')) \leq \vartheta \left(\bigoplus_{k=1}^{n} \Omega^{2k-2} \omega^{\Omega \otimes (\widehat{l(s_k) \oplus l(t_k)})} \right).$$

Seeing that

$$l(s_k) \oplus l(t_k) < \vartheta(\Omega^{\Omega}),$$

it can be shown in a similar way as in the proof of Theorem 6 that

$$\vartheta \left(\bigoplus_{k=1}^{n} \Omega^{2k-2} \omega^{\Omega \otimes (\widehat{l(s_k) \oplus l(t_k)})} \right) < \vartheta(\Omega^{\Omega}),$$

hence $o(L(t)) < \vartheta(\Omega^{\Omega})$.

We still have to prove that f is a quasi-embedding. We show that $f(v) \leq f(v')$ implies $v \leq v'$ by induction on the complexity of v'. If $f(v) \leq f(\circ) = \circ$, then $v = \circ \leq v'$. Assume $v' = \circ[(u'_1, v'_1), \ldots, (u'_{m'}, v'_{m'})] \in L(t)$ with $f(v')$ defined as

$$\circ\Big((f(u'_{i'_1}), f(v'_{i'_1})), \ldots, (f(u'_{i'_{k'-1}}), f(v'_{i'_{k'-1}})),$$
$$[(u'_{j'_1}, f(v'_{j'_1})), \ldots, (u'_{j'_{l'}}, f(v'_{j'_{l'}})), (f(u'_{r'_1}), v'_{r'_1}) \ldots, (f(u'_{r'_{t'}}), v'_{r'_{t'}})] \Big)$$

and suppose $f(v) \leq f(v')$. We show that $v \leq v'$ holds. If $v = \circ$, this is trivial. Say that $f(v)$ is defined as in (2). Because $f(v) \leq f(v')$, we obtain $f(v) \leq f(u'_p)$ or $f(v) \leq f(v'_p)$ for a certain p or $k = k'$ and

$$(f(u_{i_1}), f(v_{i_1})) \leq (f(u'_{i'_1}), f(v'_{i'_1})),$$
$$\ldots$$
$$(f(u_{i_{k-1}}), f(v_{i_{k-1}})) \leq (f(u'_{i'_{k'-1}}), f(v'_{i'_{k'-1}})),$$

and

$$[(u_{j_1}, f(v_{j_1})), \ldots, (u_{j_l}, f(v_{j_l})), (f(u_{r_1}), v_{r_1}) \ldots, (f(u_{r_t}), v_{r_t})]$$
$$\leq^{\diamond} [(u'_{j'_1}, f(v'_{j'_1})), \ldots, (u'_{j'_{l'}}, f(v'_{j'_{l'}})), (f(u'_{r'_1}), v'_{r'_1}) \ldots, (f(u'_{r'_{t'}}), v'_{r'_{t'}})].$$

In the two former cases, we obtain by the induction hypothesis that $v \leq u'_p$ or $v \leq v'_p$, hence $v \leq v'$. In the latter case, from the induction hypothesis follows $(u_{i_p}, v_{i_p}) \leq$

$(u'_{i'_p}, v'_{i'_p})$ for every $p = 1, \ldots, k-1$. Furthermore, there exists an injective function $g : \{j_1, \ldots, j_l, r_1, \ldots, r_t\} \to \{j'_1, \ldots, j'_{l'}, r'_1, \ldots, r'_{t'}\}$ such that $g(j_p) = j'_l$ for a certain l and $g(r_p) = r'_l$ for a certain l with[1]

$$(u_{j_p}, f(v_{j_p})) \le (u'_{g(j_p)}, f(v'_{g(j_p)}))$$

and

$$(f(u_{r_p}), v_{r_p}) \le (f(u'_{g(r_p)}), v'_{g(r_p)})$$

for every p. Using the induction hypothesis, we gain

$$(u_{j_p}, v_{j_p}) \le (u'_{g(j_p)}, v'_{g(j_p)})$$

and

$$(u_{r_p}, v_{r_p}) \le (u'_{g(r_p)}, v'_{g(r_p)}).$$

Therefore

$$\begin{aligned} &[(u_{j_1}, v_{j_1}), \ldots, (u_{j_l}, v_{j_l}), (u_{r_1}, v_{r_1}) \ldots, (u_{r_t}, v_{r_t})] \\ \le^{\diamond} &[(u'_{j'_1}, v'_{j'_1}), \ldots, (u'_{j'_{l'}}, v'_{j'_{l'}}), (u'_{r'_1}, v'_{r'_1}) \ldots, (u'_{r'_{t'}}, v'_{r'_{t'}})]. \end{aligned}$$

Together with $(u_{i_p}, v_{i_p}) \le (u'_{i'_p}, v'_{i'_p})$ for every $p = 1, \ldots, k = k'$, we can conclude that

$$[(u_1, v_1), \ldots, (u_m, v_m)] \le^{\diamond} [(u'_1, v'_1), \ldots, (u'_{m'}, v'_{m'})],$$

hence $v \le v'$. □

Now, we will show that $\vartheta(\Omega^{\Omega})$ is also a lower bound of the maximal order types of these $\mathbb{wpo}$'s. First an additional lemma.

Lemma 9 *Suppose $1 < \alpha < \vartheta(\Omega^{\Omega})$ and $\alpha \in P$, the set of additive closed ordinal numbers. Then there exists unique $0 < \beta_i < \Omega$ and $\alpha_i < \Omega$ such that $\alpha = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n)$, $\alpha_1 > \cdots > \alpha_n$.*

Proof In [15], the third author proved this for $\vartheta(\Omega^{\omega})$. By standard properties of the ϑ-function, it holds that for every ordinal $\alpha < \vartheta(\varepsilon_{\Omega+1}) \cap P$, there exists a unique $\xi < \varepsilon_{\Omega+1}$ such that $\alpha = \vartheta(\xi)$. (A proof of this fact can be found in an unpublished article of Buchholz.) Denote ξ as $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ with $0 < \beta_i < \Omega$ and $\alpha_1 > \cdots > \alpha_n$. We only need to show that $\alpha_1 < \Omega$. If $\alpha_1 \ge \Omega$, then $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \ge \Omega^{\Omega}$. So, $\alpha < \vartheta(\Omega^{\Omega})$ can only holds if $\alpha \le k(\Omega^{\Omega})$. But $k(\Omega^{\Omega}) = 1 < \alpha$, hence α_1 has to be smaller than Ω. □

Theorem 8 $o(\mathcal{T}(M(\cdot \times \cdot))) \ge \vartheta(\Omega^{\Omega})$.

[1] Because $(u_{j_p}, f(v_{j_p}))$ and $(f(u_{r_l}), v_{r_l})$ are incomparable.

Proof Define

$$\begin{aligned}
g : \vartheta(\Omega^\Omega) &\to \mathcal{T}(M(\cdot\times\cdot)),\\
0 &\mapsto \circ,\\
1 &\mapsto \circ[(\circ,\circ)],\\
\alpha =_{CNF} \omega^{\alpha_1}+\cdots+\omega^{\alpha_n}, n\geq 2 &\mapsto \circ[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)],\\
\alpha = \omega^\beta = \vartheta(\Omega^{\alpha_1}\beta_1+\cdots+\Omega^{\alpha_n}\beta_n) > 1 &\mapsto \circ[(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))].
\end{aligned}$$

In this definition we assume that $\beta_i > 0$ as in Lemma 9. Obviously we see that $\beta = 0$ iff $g(\beta) = \circ$ and $\beta = 1$ iff $g(\beta) = \circ[(\circ,\circ)]$. If we prove that g is a quasi-embedding, we can conclude the theorem by Lemma 2. We show that $g(\alpha) \leq g(\alpha')$ implies $\alpha \leq \alpha'$ by induction on $\alpha \oplus \alpha'$. The cases α and/or α' equal to 0 or 1 are trivial, so we may assume that $\alpha, \alpha' > 1$.

a) $\alpha' =_{CNF} \omega^{\alpha'_1}+\cdots+\omega^{\alpha'_m}$, $m \geq 2$.

i) $\alpha =_{CNF} \omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$, $n\geq 2$.

If

$$g(\alpha) = \circ[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \leq \circ[(g(\alpha'_1),\circ),\ldots,(g(\alpha'_m),\circ)] = g(\alpha').$$

then $g(\alpha) \leq g(\alpha'_i)$ for a certain i or

$$[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \leq [(g(\alpha'_1),\circ),\ldots,(g(\alpha'_m),\circ)]. \tag{3}$$

In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha'_i < \alpha'$. In the latter case, let p denote $[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \cap [(g(\alpha'_1),\circ),\ldots,(g(\alpha'_m),\circ)]$.
If the intersection $p = [(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)]$, then there exists a set of different indices $\{i_1,\ldots,i_n\}$ such that $(g(\alpha_j),\circ) = (g(\alpha'_{i_j}),\circ)$ for every $j = 1,\ldots,n$. By the induction hypothesis we obtain $\alpha_j = \alpha'_{i_j}$. So, $\alpha \leq \alpha'$.
Suppose now $p \subsetneq [(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)]$ and say that i is the minimum index such that $(g(\alpha_i),\circ) \notin p$. By inequality (3), there exists a j such that $g(\alpha_i) < g(\alpha'_j)$. Hence, by the induction hypothesis, we obtain $\alpha_i < \alpha'_j$. Because $(\alpha_i)_i$ is a descending sequence, we gain

$$\omega^{\alpha_i}+\cdots+\omega^{\alpha_n} < \omega^{\alpha'_j}.$$

Furthermore, for every $k < i$, there exists a $l_k \neq j$ such that $g(\alpha_k) = g(\alpha'_{l_k})$ with $l_{k_1} \neq l_{k_2}$ if $k_1 \neq k_2$. From the induction hypothesis, it follows that $\alpha_k = \alpha'_{l_k}$. Hence

$$\alpha = \omega^{\alpha_1}+\cdots+\omega^{\alpha_n} < \omega^{\alpha'_{l_1}}+\cdots+\omega^{\alpha'_{l_{i-1}}}+\omega^{\alpha'_j} \leq \omega^{\alpha'_1}+\cdots+\omega^{\alpha'_m} = \alpha'.$$

ii) $1 < \alpha = \omega^\beta = \vartheta(\Omega^{\alpha_1}\beta_1+\cdots+\Omega^{\alpha_n}\beta_n)$.

$\beta_i > 0$, hence $g(\beta_i) \neq \circ$. Assume $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\alpha'_i)$ for a certain i or $[(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))] \leq [(g(\alpha'_1),\circ),\ldots,(g(\alpha'_m),\circ)]$. In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha'_i < \alpha'$. The latter case is impossible because $g(\beta_i) \not\leq \circ$.

b) $1 < \alpha' = \omega^{\beta'} = \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m)$.
We know that $g(\beta'_i) \neq \circ$.

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, $n \geq 2$.
Suppose $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\alpha'_i)$ for a certain i, or $g(\alpha) \leq g(\beta'_i)$ for a certain i, or

$$[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \leq [(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))].$$

In the two former cases, we obtain by the induction hypothesis that $\alpha \leq \alpha'_i$ or $\alpha \leq \beta'_i$. In both cases, $\alpha \leq k(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m) < \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m) = \alpha'$. If

$$[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \leq [(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))]$$

holds, we see that

$$[(g(\alpha_1),\circ),\ldots,(g(\alpha_n),\circ)] \cap [(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))] = \emptyset,$$

because $g(\beta'_i) \neq \circ$. Hence, for every i there exists a j such that $g(\alpha_i) \leq g(\alpha'_j)$. By the induction hypothesis, we attain $\alpha_i \leq \alpha'_j < \alpha'$. If $\alpha'_1 > 0$, then α' is an epsilon number, so $\alpha < \alpha'$. Suppose $\alpha' = \vartheta(\Omega^0\beta_1)$ with $\beta_1 > 0$. Then $g(\alpha') = \circ[(\circ, g(\beta'_1))]$, hence $g(\alpha_i) = \circ$ for every i. We obtain $\alpha < \omega \leq \alpha'$.

ii) $1 < \alpha = \omega^{\beta} = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n)$.
If

$$\begin{aligned} g(\alpha) &= \circ[(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))] \\ &\leq g(\alpha') = \circ[(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))], \end{aligned}$$

then either $g(\alpha) \leq g(\alpha'_i)$ or $g(\alpha) \leq g(\beta'_i)$ for a certain i or

$$\begin{aligned} &[(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))] \\ &\leq [(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))]. \end{aligned} \tag{4}$$

In the former cases, $\alpha \leq \alpha'_i < \alpha'$ or $\alpha \leq \beta'_i < \alpha'$ by the induction hypothesis. In the latter case, let p denote the intersection $[(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))] \cap [(g(\alpha'_1),g(\beta'_1)),\ldots,(g(\alpha'_m),g(\beta'_m))]$.
If $p = [(g(\alpha_1),g(\beta_1)),\ldots,(g(\alpha_n),g(\beta_n))]$, then there exists a set of different indices $\{i_1,\ldots,i_n\}$ such that $(g(\alpha_j),g(\beta_j)) = (g(\alpha'_{i_j}),g(\beta'_{i_j}))$ for every $j = 1,\ldots,n$. By the induction hypothesis we obtain $\alpha_j = \alpha'_{i_j}$ and $\beta_j = \beta'_{i_j}$. Thus

$$\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \leq \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m.$$

Furthermore,

$$\begin{aligned} k(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n) &= \max\{\alpha_i,\beta_i \mid i = 1,\ldots,n\} \\ \leq \max\{\alpha'_i,\beta'_i \mid i = 1,\ldots,m\} &< \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m), \end{aligned}$$

from which we can conclude that

$$\alpha = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n) \leq \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m) = \alpha'.$$

Suppose now $p \subsetneq [(g(\alpha_1), g(\beta_1)), \ldots, (g(\alpha_n), g(\beta_n))]$ and say that i is the minimum index such that $(g(\alpha_i), g(\beta_i)) \notin p$. By inequality (4), there exists a j such that $g(\alpha_i) \leq g(\alpha'_j)$ and $g(\beta_i) \leq g(\beta'_j)$, but $(g(\alpha_i), g(\beta_i)) \neq (g(\alpha'_j), g(\beta'_j)) \notin p$. Hence, by the induction hypothesis, we obtain that $\alpha_i \leq \alpha'_j$ and $\beta_i \leq \beta'_j$, but $(\alpha_i, \beta_i) \neq (\alpha'_j, \beta'_j)$. Because $(\alpha_i)_i$ is a strictly descending sequence, we gain

$$\Omega^{\alpha_{i+1}}\beta_{i+1} + \cdots + \Omega^{\alpha_n}\beta_n < \Omega^{\alpha_i},$$

hence

$$\Omega^{\alpha_i}\beta_i + \cdots + \Omega^{\alpha_n}\beta_n < \Omega^{\alpha'_j}\beta'_j.$$

Also, for every $k < i$, there exists a $l_k \neq j$ such that $g(\alpha_k) = g(\alpha'_{l_k})$ and $g(\beta_k) = g(\beta'_{l_k})$ with $l_{k_1} \neq l_{k_2}$ if $k_1 \neq k_2$. From the induction hypothesis, it follows that $\alpha_k = \alpha'_{l_k}$ and $\beta_k = \beta'_{l_k}$. Hence

$$\begin{aligned}
&\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \\
&< \Omega^{\alpha'_{l_1}}\beta'_{l_1} + \cdots + \Omega^{\alpha'_{l_{i-1}}}\beta'_{l_{i-1}} + \Omega^{\alpha'_j}\beta'_j \\
&\leq \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m.
\end{aligned}$$

Furthermore, by inequality (4) and the induction hypothesis, we know that for every i, there exists a j such that $\alpha_i \leq \alpha'_j$ and $\beta_i \leq \beta'_j$, hence

$$\max\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \leq \max\{\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_m\}.$$

So

$$\begin{aligned}
&k\,(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n) \\
&\leq k\left(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m\right) \\
&< \vartheta\left(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m\right).
\end{aligned}$$

We conclude

$$\alpha = \vartheta\,(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n) \leq \vartheta\left(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m\right) = \alpha'.$$

□

Corollary 1 $o(\mathcal{T}(M(\cdot \times \cdot))) = o(\mathcal{T}(M^{\diamond}(\cdot \times \cdot))) = \vartheta(\Omega^{\Omega})$.

Proof Follows from Theorems 7 and 8. □

4 An order-theoretic characterization of tree-like $\mathbb{wpo}$'s which are based on finite sequences of pairs

In this section, we show that using finite sequences instead of finite multisets implies a $\mathbb{wpo}$ that has a bigger maximal order type! The following theorem is needed for proving the main Theorem 10.

Theorem 9 *Let $\mathbb{Y}_i^j$ and $\mathbb{Z}_i$ be countable $\mathbb{wpo}$'s and n_i and m_i be natural numbers. If*

$$W(X) = \sum_{i=0}^{N} \left(\left(\mathbb{Y}_1^i \times X + \mathbb{Z}_1^i\right)^* \times \cdots \times \left(\mathbb{Y}_{n_i}^i \times X + \mathbb{Z}_{n_i}^i\right)^* \times X^{m_i} \times \mathbb{Z}_i \right),$$

then $\mathcal{T}(W)$ is a $\mathbb{wpo}$ and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.

Proof We will prove the theorem by main induction on the ordinal $o(W(\Omega))$. Without loss of generality, we may assume that $\mathbb{Y}_j^i$ and $\mathbb{Z}_i$ are non-empty $\mathbb{wpo}$'s (unless $W(X) \cong \emptyset$). If $o(W(\Omega)) < \Omega$, then $W(X)$ does not contain X (or W does not contain $\cdot$) and it is equal to a countable $\mathbb{wpo}$ $\mathbb{Z}$: in this case, $n_i = m_i = 0$ for all i. Therefore, $W(X) \cong \sum_{i=0}^{N} \mathbb{Z}_i$, which we call $\mathbb{Z}$. Hence $\mathcal{T}(W) \cong \mathbb{Z} \cup \{0\}$, with 0 smaller than every element in $\mathbb{Z}$. Then $o(\mathcal{T}(W)) = o(\mathbb{Z}) + 1 \leq \vartheta(o(\mathbb{Z})) = \vartheta(o(W(\Omega)))$.
If $o(W(\Omega)) \geq \Omega$, in other words X really occurs in $W(X)$, then $\vartheta(o(W(\Omega)))$ is an epsilon number. We want to prove that $L(t)$ is a $\mathbb{wpo}$ and $l(t) < \vartheta(o(W(\Omega)))$ for every t in $\mathcal{T}(W)$, by induction on the complexity of t. If $t = \circ$, then $L(t)$ is the empty $\mathbb{wpo}$ and $l(t) = 0 < \vartheta(o(W(\Omega)))$. Assume

$$t = \circ((\overline{t_1}, \ldots, \overline{t_{n_k}}), (t_1, \ldots, t_{m_k}), z)$$

with $\overline{t_j} = ((\overline{t_j})_1, \ldots, (\overline{t_j})_{p_j})$ and either $(\overline{t_j})_i = z_i^j$ or $(\overline{t_j})_i = (y_i^j, t_i^j)$ with $L(t_i)$, $L(t_i^j)$ $\mathbb{wpo}$'s and $l(t_i), l(t_i^j) < \vartheta(o(W(\Omega)))$, $y_i^j \in \mathbb{Y}_j^k$, $z_i^j \in \mathbb{Z}_j^k$ and $z \in \mathbb{Z}_k$. Suppose s is an arbitrary element in $\mathcal{T}(W)$, different from $\circ$. Then

$$\begin{aligned} s &= \circ((\overline{s_1}, \ldots, \overline{s_{n_l}}), (s_1, \ldots, s_{m_l}), z'), \\ \overline{s_j} &= ((\overline{s_j})_1, \ldots, (\overline{s_j})_{q_j}), \\ (\overline{s_j})_i &= z_i'^j \text{ or } (y_i'^j, s_i^j) \end{aligned} \tag{5}$$

with $z' \in \mathbb{Z}_l$, $y_i'^j \in \mathbb{Y}_j^l$ and $z_i'^j \in \mathbb{Y}_j^l$. $s \in L(t)$ holds iff $s_i \in L(t)$, $s_i^j \in L(t)$ and one of the following holds:

1. $k \neq l$,
2. $k = l$, $z' \in L_{\mathbb{Z}_k}(z)$,
3. $k = l$, $z \leq_{\mathbb{Z}_k} z'$, $(t_1, \ldots, t_{m_k}) \not\leq (s_1, \ldots, s_{m_k})$,
4. $k = l$, $z \leq_{\mathbb{Z}_k} z'$, $(t_1, \ldots, t_{m_k}) \leq (s_1, \ldots, s_{m_k})$, $(\overline{t_1}, \ldots, \overline{t_{n_k}}) \not\leq (\overline{s_1}, \ldots, \overline{s_{n_k}})$.

If (3.) holds, there must be a minimal index $l(s)$ such that

$$t_1 \leq s_1, \ldots, t_{l(s)-1} \leq s_{l(s)-1}, t_{l(s)} \not\leq s_{l(s)}.$$

If (4.) holds, there must be a minimal index $k(s)$ such that

$$\overline{t_1} \leq \overline{s_1}, \dots, \overline{t_{k(s)-1}} \leq \overline{s_{k(s)-1}}, \overline{t_{k(s)}} \not\leq \overline{s_{k(s)}}.$$

In this case

$$\overline{t_{k(s)}} = ((\overline{t_{k(s)}})_1, \dots, (\overline{t_{k(s)}})_{p_{k(s)}}) \not\leq ((\overline{s_{k(s)}})_1, \dots, (\overline{s_{k(s)}})_{q_{k(s)}}) = \overline{s_{k(s)}}$$

is valid iff one of the following cases holds

1. $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every j,
2. there exists an index j_1 such that $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every $j < j_1$, $(\overline{t_{k(s)}})_1 \leq (\overline{s_{k(s)}})_{j_1}$ and $(\overline{t_{k(s)}})_2 \not\leq (\overline{s_{k(s)}})_j$ for every $j > j_1$,

   ...

$p_{k(s)}$. there exist $p_{k(s)} - 1$ indices $j_1 < \cdots < j_{p_{k(s)}-1}$ such that $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every $j < j_1$, $(\overline{t_{k(s)}})_1 \leq (\overline{s_{k(s)}})_{j_1}$, $(\overline{t_{k(s)}})_2 \not\leq (\overline{s_{k(s)}})_j$ for every $j_2 > j > j_1$, ..., $(\overline{t_{k(s)}})_{p_{k(s)}-1} \leq (\overline{s_{k(s)}})_{j_{p_{k(s)}-1}}$ and $(\overline{t_{k(s)}})_{p_{k(s)}} \not\leq (\overline{s_{k(s)}})_j$ for every $j > j_{p_{k(s)}-1}$.

If $(\overline{t_j})_i = z_i^j$, define L_i^j as $\mathbb{Y}_j^k \times X + L_{\mathbb{Z}_j^k}(z_i^j)$. If $(\overline{t_j})_i = (y_i^j, t_i^j)$, define L_i^j as $(L_{\mathbb{Y}_j^k}(y_i^j) \times X) + (\mathbb{Y}_j^k \times L(t_i^j)) + \mathbb{Z}_j^k$. Define $W'(X)$ as follows

$$\begin{aligned}
&\sum_{i=0, i\neq k}^{N} \left(\left(\mathbb{Y}_1^i \times X + \mathbb{Z}_1^i\right)^* \times \cdots \times \left(\mathbb{Y}_{n_i}^i \times X + \mathbb{Z}_{n_i}^i\right)^* \times X^{m_i} \times \mathbb{Z}_i \right) \\
&+ \left(\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k} \times L_{\mathbb{Z}_k}(z) \right) \\
&+ \sum_{i=1}^{m_k} \left(\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k-1} \times L(t_i) \times \mathbb{Z}_k \right) \\
&+ \sum_{j=1}^{n_k} \sum_{i=1}^{p_j} \Big[\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{j-1}^k \times X + \mathbb{Z}_{j-1}^k\right)^* \\
&\quad \times (\mathbb{Y}_j^k \times X + \mathbb{Z}_j^k)^{i-1} \times (L_1^j)^* \times \cdots \times (L_i^j)^* \\
&\quad \times \left(\mathbb{Y}_{j+1}^k \times X + \mathbb{Z}_{j+1}^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k} \times \mathbb{Z}_k \Big].
\end{aligned}$$

The four cases separated by a $+$ represents the four groups in which s can lie in. The index i in the third term represents $l(s)$. The index j, respectively i, in the fourth term represents $k(s)$, respectively case 1. - $p_{k(s)}$. We can interpret $\times s$ with $s \in L(t)$ as an element of $W'(L(t))$ like in Theorem 6 and 7. With this in mind, we can define a map $f : L(t) \to \mathcal{T}(W')$ as follows. First define $f(\circ)$ as $\circ$. Then, assuming s as in (5) and assuming that $f(s_i)$ and $f(s_i^j)$ are already defined, let $f(s)$ be $\circ[w(f(s_1'), \dots, f(s_r'))]$, where $w(s_1', \dots, s_r')$ is the interpretation of $\times s$ as an element in $W'(L(t))$ and $\{s_1', \dots, s_r'\} \subseteq \{s_1, \dots, s_{m_l}, s_1^1, \dots, s_{q_{n_l}}^{n_l}\}$.
It can be proved in a similar way as in Theorem 7 that f is a quasi-embedding.

By Lemma 2 we obtain

$$o(L(t)) \leq o(\mathcal{T}(W')).$$

If $o(W'(\Omega)) < o(W(\Omega))$, we gain that $\mathcal{T}(W')$ is a wpo (hence $L(t)$ is a wpo) and $o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega)))$ by the main induction hypothesis. If additionally the inequality $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$ holds, then

$$o(L(t)) \leq o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega))) < \vartheta(o(W(\Omega))).$$

So the only two remaining things we have to prove are $o(W'(\Omega)) < o(W(\Omega))$ and $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.

a) $o(W'(\Omega)) < o(W(\Omega))$.
For notational convenience, we will write sometimes $\mathbb{Y}$, respectively $\mathbb{Z}^*$ instead of $o(\mathbb{Y})$, respectively $o(\mathbb{Z}^*)$ for wpo's $\mathbb{Y}$ and $\mathbb{Z}$. It is easy to see that $\mathbb{Y}^k_j \otimes \Omega \oplus l_{\mathbb{Z}^k_j}(z^j_i) < \mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j$ and $(l_{\mathbb{Y}^k_j}(y^j_i) \otimes \Omega) \oplus (\mathbb{Y}^k_j \otimes l(t^j_i)) \oplus \mathbb{Z}^k_j < \mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j$ and $\Omega^{m_k} \otimes \mathbb{Z}_k < \Omega^\omega \leq \left(\mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j\right)^*$, hence

$$\left(\mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j\right)^{\overline{i-1}} \otimes o((L^j_1)^*) \otimes \cdots \otimes o((L^j_i)^*) \otimes \Omega^{m_k} \otimes \mathbb{Z}_k < \left(\mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j\right)^*.$$

We attain

$$\begin{aligned}
&\bigoplus_{j=1}^{n_k} \bigoplus_{i=1}^{p_j} \Big[\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{j-1} \otimes \Omega \oplus \mathbb{Z}^k_{j-1}\right)^* \otimes \left(\mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j\right)^{\overline{i-1}} \\
&\otimes o((L^j_1)^*) \otimes \cdots \otimes o((L^j_i)^*) \otimes \left(\mathbb{Y}^k_{j+1} \otimes \Omega \oplus \mathbb{Z}^k_{j+1}\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \\
&\otimes \Omega^{m_k} \otimes \mathbb{Z}_k \Big] \\
< \ &\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^*,
\end{aligned}$$

hence

$$\begin{aligned}
&\left(\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k} \otimes L_{\mathbb{Z}_k}(z)\right) \\
&\oplus \bigoplus_{i=1}^{m_k} \left(\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k-1} \otimes L(t_i) \otimes \mathbb{Z}_k\right) \\
&\oplus \bigoplus_{j=1}^{n_k} \bigoplus_{i=1}^{p_j} \Big[\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{j-1} \otimes \Omega \oplus \mathbb{Z}^k_{j-1}\right)^* \\
&\quad \otimes \left(\mathbb{Y}^k_j \otimes \Omega \oplus \mathbb{Z}^k_j\right)^{\overline{i-1}} \otimes o((L^j_1)^*) \otimes \cdots \otimes o((L^j_i)^*) \\
&\quad \otimes \left(\mathbb{Y}^k_{j+1} \otimes \Omega \oplus \mathbb{Z}^k_{j+1}\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k} \otimes \mathbb{Z}_k\Big] \\
< \ &\left(\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k} \otimes L_{\mathbb{Z}_k}(z)\right) \\
&\oplus \ \left(\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k}\right) \\
\leq \ &\left(\mathbb{Y}^k_1 \otimes \Omega \oplus \mathbb{Z}^k_1\right)^* \otimes \cdots \otimes \left(\mathbb{Y}^k_{n_k} \otimes \Omega \oplus \mathbb{Z}^k_{n_k}\right)^* \otimes \Omega^{m_k} \otimes \mathbb{Z}_k.
\end{aligned}$$

This inequality yields $o(W'(\Omega)) < o(W(\Omega))$.

b) $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.
This can be proved similarly as in Theorem 6. □

Theorem 10 *$\mathcal{T}((\cdot\times\cdot)^*)$ is a wpo and $o(\mathcal{T}((\cdot\times\cdot)^*)) \leq \vartheta(\Omega^{\Omega^\Omega})$.*

Proof We show that $L(t)$ is a wpo and $l(t) < \vartheta(\Omega^{\Omega^\Omega})$ holds for every t in $\mathcal{T}((\cdot\times\cdot)^*)$ by induction on the complexity of t. The theorem then follows from Theorem 2. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(\Omega^{\Omega^\Omega})$. Assume $t = \circ((t_1^1,t_2^1),\dots,(t_1^k,t_2^k))$ with $L(t_i^j)$ wpo's and $l(t_i^j) < \vartheta(\Omega^{\Omega^\Omega})$ and suppose that $s = \circ((s_1^1,s_2^1),\dots,(s_1^l,s_2^l))$. Then $t \leq s$ iff $t \leq s_i^j$ for certain i and j or

$$((t_1^1,t_2^1),\dots,(t_1^k,t_2^k)) \leq ((s_1^1,s_2^1),\dots,(s_1^l,s_2^l)).$$

Hence, $s \in L(t)$ if $s_i^j \in L(t)$ for every i and j and one of the following holds

1. $(t_1^1,t_2^1) \not\leq (s_1^i,s_2^i)$ for every i,
2. there exists an index l_1 such that $(t_1^1,t_2^1) \not\leq (s_1^i,s_2^i)$ for every $i < l_1$, $(t_1^1,t_2^1) \leq (s_1^{l_1},s_2^{l_1})$ and $(t_1^2,t_2^2) \not\leq (s_1^i,s_2^i)$ for every $l_1 < i$,

   ...

k. there exist indices $l_1 < \cdots < l_{k-1}$ such that $(t_1^1,t_2^1) \not\leq (s_1^i,s_2^i)$ for every $i < l_1$, $(t_1^1,t_2^1) \leq (s_1^{l_1},s_2^{l_1})$, $(t_1^2,t_2^2) \not\leq (s_1^i,s_2^i)$ for every $l_1 < i < l_2$, $(t_1^2,t_2^2) \leq (s_1^{l_2},s_2^{l_2})$, ..., $(t_1^{k-1},t_2^{k-1}) \leq (s_1^{l_{k-1}},s_2^{l_{k-1}})$ and $(t_1^k,t_2^k) \not\leq (s_1^i,s_2^i)$ for every $l_{k-1} < i$.

Also $(t_1^i,t_2^i) \not\leq (s_1^j,s_2^j)$ is valid if one of the following holds

a. $s_1^j \in L(t_1^i)$,
b. $t_1^i \leq s_1^j$ and $s_2^j \in L(t_2^i)$.

Let $W'(X)$ be

$$\sum_{j=1}^{k} \left(\prod_{p=1}^{j-1} \left(Y_p \times X^2\right) \right) \times Y_j$$

for

$$Y_p = \left((L(t_1^p)\times X) + (X \times L(t_2^p))\right)^*.$$

Define the mapping $f : L(t) \to \mathcal{T}(W')$ recursively as follows. First let $f(\circ)$ be $\circ$. Assume $s = \circ((s_1^1,s_2^1),\dots,(s_1^l,s_2^l)) \in L(t)$ and $f(s_j^i)$ is already defined for every i and j. We only consider that 2. and always *b.* holds. We will use the same indices as there. The other cases can be treated in a similar way. Define $f(s)$ then as

$$\circ\Big(((s_1^1,f(s_2^1)),\dots,(s_1^{l_1-1},f(s_2^{l_1-1}))),(f(s_1^{l_1}),f(s_2^{l_1})),$$
$$((s_1^{l_1+1},f(s_2^{l_1+1})),\dots,(s_1^l,f(s_2^l)))\Big).$$

One can prove that f is a quasi-embedding in the same manner as Theorem 7. By Lemma 2 and Theorem 9 we obtain that $L(t)$ is a wpo and

$$o(L(t)) \leq o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega))).$$

The only remaining thing that needs a proof is $\vartheta(o(W'(\Omega))) < \vartheta(\Omega^{\Omega^\Omega})$. It is known that $o(L(t_i^j)) < \vartheta(\Omega^{\Omega^\Omega}) < \Omega$, hence $(o(L(t_1^j)) \otimes \Omega) \oplus (\Omega \otimes o(L(t_2^j))) + 1 < \Omega^2$. We gain

$$\Big((o(L(t_1^j)) \otimes \Omega) \oplus (\Omega \otimes o(L(t_2^j)))\Big)^* < \omega^{\omega^{\Omega^2}} = \Omega^{\Omega^\Omega},$$

hence $o(W'(\Omega)) < \Omega^{\Omega^\Omega}$. Furthermore, $o(W'(\Omega))$ is equal to

$$\begin{aligned}
&\bigoplus_{j=1}^{k} \Big(\big((l(t_1^1) \otimes \Omega) \oplus (\Omega \otimes l(t_2^1))\big)^* \otimes \Omega^2 \otimes \dots \\
&\quad \otimes \Omega^2 \otimes \Big((l(t_1^j) \otimes \Omega) \oplus (\Omega \otimes l(t_2^j))\Big)^* \Big) \\
= &\bigoplus_{j=1}^{k} \Big(\Omega^{2(j-1)} \otimes \big(\Omega \otimes (l(t_1^1) \oplus l(t_2^1))\big)^* \otimes \dots \otimes \Big(\Omega \otimes (l(t_1^j) \oplus l(t_2^j))\Big)^* \Big).
\end{aligned}$$

Because $l(t_i^j) < \vartheta(\Omega^{\Omega^\Omega})$ and $\vartheta(\Omega^{\Omega^\Omega})$ is an epsilon number, we have $l(t_1^j) \oplus l(t_2^j) < \vartheta(\Omega^{\Omega^\Omega})$. Hence, using Lemma 6, we see that $k\Big(\Omega \otimes (l(t_1^j) \oplus l(t_2^j))\Big)^*$ is strictly smaller than $\vartheta(\Omega^{\Omega^\Omega})$. Furthermore, from Lemma 5 it follows that the coefficients of $o(W'(\Omega))$ are strictly smaller than $\vartheta(\Omega^{\Omega^\Omega})$. $\square$

We proved that $\vartheta(\Omega^{\Omega^\Omega})$ is an upper bound for the maximal order type of the wpo $\mathcal{T}((\cdot \times \cdot)^*)$. The next theorem claims that this ordinal is also a lower bound.

Theorem 11 *If $W(X) = (X \times X)^*$, then $o(\mathcal{T}(W)) \geq \vartheta(\Omega^{\Omega^\Omega})$.*

Proof We define a quasi-embedding g from $\vartheta(\Omega^{\Omega^\Omega})$ to $\mathcal{T}((\cdot \times \cdot)^*)$ in the following recursive way: let $g(0)$ be $\circ$. If $\alpha =_{CNF} \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ with $n \geq 2$, define $g(\alpha)$ as $\circ((g(\alpha_1), \circ), \dots, (g(\alpha_n), \circ))$. Now let $\alpha = \vartheta(\beta) < \vartheta(\Omega^{\Omega^\Omega})$. Then $\beta < \Omega^{\Omega^\Omega} = \omega^{\omega^{\Omega^2}}$. If $\beta < \Omega$, define $g(\alpha)$ as $\circ((g(\beta), g(\beta)))$. Hence $g(1) = \circ((\circ, \circ))$. Assume $\beta \geq \Omega$.

For every ordinal $\delta < \omega^{\omega^{\Omega^2}}$ with $\delta \geq \omega$, there exists unique ordinals $k < \omega$, $\overline{\delta} < \Omega^2$, $\delta_0, \dots, \delta_k < \omega^{\omega^{\overline{\delta}}}$ with $\delta_k > 0$ such that

$$\delta = \omega^{\omega^{\overline{\delta}} \cdot k} \delta_k + \dots + \omega^{\omega^{\overline{\delta}} \cdot 1} \delta_1 + \delta_0. \tag{6}$$

Note that $k > 0$, because otherwise $\delta < \omega^{\omega^0} = \omega$, a contradiction. From $\overline{\delta} < \Omega^2$, we obtain two unique countable ordinals $\overline{\delta_1}$ and $\overline{\delta_2}$ such that $\overline{\delta} = \Omega \overline{\delta_1} + \overline{\delta_2}$. Now, define

$f(\delta) \in W(\Omega) = (\Omega \times \Omega)^*$ for every $\delta < \omega^{\omega^{\Omega^2}}$ recursively as follows. If $\delta = n < \omega$, let $f(\delta)$ be $((\circ,\circ),\dots,(\circ,\circ))$, where $(\circ,\circ)$ occurs $n+1$ times. If $\delta \geq \omega$, notate δ as in (6) and let $f(\delta)$ be

$$f(\delta_k)^\frown((1+\overline{\delta_1},1+\overline{\delta_2}))^\frown f(\delta_{k-1})\dots((1+\overline{\delta_1},1+\overline{\delta_2}))^\frown f(\delta_0),$$

where $^\frown$ represents the concatenation of the strings. Remark that the length of the finite sequence $f(\delta)$ with $\delta > 0$ is strictly bigger than 1. Before we give the definition of $g(\alpha) = g(\vartheta(\beta))$, we first want to prove that for the largest countable ordinal occurring in $f(\delta) \in (\Omega \times \Omega)^*$, call it $Max(f(\delta))$, is less than or equal to $k(\delta)+\omega$. Furthermore, we want to prove that $k(\delta) < \omega^{\omega^{Max(f(\delta))+1}}$. We prove both inequalities by induction on δ. If $\delta < \omega$, they are trivial. Assume $\delta \geq \omega$. Then, as in (6),

$$\begin{aligned}\delta &= \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot k}\delta_k + \dots + \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0\\ &= \Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k}\delta_k + \dots + \Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0.\end{aligned}$$

From the induction hypothesis, we can conclude that

$$Max(f(\delta)) \leq \max\left\{1+\overline{\delta_1},1+\overline{\delta_2},k(\delta_0)+\omega,\dots,k(\delta_k)+\omega\right\}$$

and $k(\delta_i) < \omega^{\omega^{Max(f(\delta_i))+1}}$ for all i. Using the second part of Lemma 5, we see that $k(\delta_0),\dots,k(\delta_k) \leq k(\delta)$ and $k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k}) \leq k(\delta)$. The latter implies that

$$k(\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k) = \max\{-1+\overline{\delta_1},\omega^{\overline{\delta_2}}\cdot k\} \leq k(\delta).$$

Hence, $1+\overline{\delta_1} \leq k(\delta)+\omega$ and $1+\overline{\delta_2} \leq k(\delta)+\omega$. We conclude that $Max(f(\delta)) \leq k(\delta)+\omega$. Using the first part of Lemma 5, we obtain

$$\begin{aligned}&k(\delta)\\ &\leq k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k}\delta_k)\oplus\dots\oplus k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot 1}\delta_1)\oplus k(\delta_0)\\ &\leq \max\{k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k})\oplus k(\delta_k), k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k})\otimes k(\delta_k)\otimes\omega\}\\ &\quad\oplus\dots\\ &\quad\oplus\max\{k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot 1})\oplus k(\delta_1), k(\Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot 1})\otimes k(\delta_1)\otimes\omega\}\\ &\quad\oplus k(\delta_0)\\ &\leq \max\{k(-1+\overline{\delta_1})\oplus k(\delta_k), k(\omega^{\overline{\delta_2}}\cdot k)\oplus k(\delta_k),\\ &\qquad\quad k(-1+\overline{\delta_1})\otimes k(\delta_k)\otimes\omega, k(\omega^{\overline{\delta_2}}\cdot k)\otimes k(\delta_k)\otimes\omega\}\\ &\quad\oplus\dots\\ &\quad\oplus\max\{k(-1+\overline{\delta_1})\oplus k(\delta_1), k(\omega^{\overline{\delta_2}}\cdot 1)\oplus k(\delta_1),\\ &\qquad\quad k(-1+\overline{\delta_1})\otimes k(\delta_1)\otimes\omega, k(\omega^{\overline{\delta_2}}\cdot 1)\otimes k(\delta_1)\otimes\omega\}\\ &\quad\oplus k(\delta_0).\end{aligned}$$

Because $k(\delta_i) < \omega^{\omega^{Max(f(\delta_i))+1}} \leq \omega^{\omega^{Max(f(\delta))+1}}$ and $k(1+\overline{\delta_1}) = 1+\overline{\delta_i} \leq Max(f(\delta)) < \omega^{\omega^{Max(f(\delta))+1}}$ and $k(\omega^{\overline{\delta_2}} \cdot i) = \omega^{\overline{\delta_2}} \cdot i \leq \omega^{Max(f(\delta))} \cdot i < \omega^{\omega^{Max(f(\delta))+1}}$ and $\omega^{\omega^{Max(f(\delta))+1}}$ is an additive and multiplicative closed ordinal number, we can conclude that $k(\delta) < \omega^{\omega^{Max(f(\delta))+1}}$.

We still want to prove one more thing, before we give the definition of $g(\alpha)$: if $\delta < \omega^{\omega^{\Omega\zeta+\eta}}$ for certain countable ordinals ζ and η, then for all pairs (δ_1^i, δ_2^i) occurring in $f(\delta)$ we have $(\delta_1^i, \delta_2^i) <_{lex} (1+\zeta, 1+\eta)$, where $<_{lex}$ is the lexicographical ordering between pairs. We prove this by induction on δ. If $\delta < \omega$, then this is trivial. Assume that $\omega \leq \delta < \omega^{\omega^{\Omega\zeta+\eta}}$. Then $\delta_0, \dots, \delta_k < \omega^{\omega^{\Omega\zeta+\eta}}$, hence for all pairs (δ_1^i, δ_2^i) occurring in $f(\delta_0), \dots, f(\delta_k)$, we have $(\delta_1^i, \delta_2^i) <_{lex} (1+\zeta, 1+\eta)$. Furthermore, from $\delta < \omega^{\omega^{\Omega\zeta+\eta}}$ we see that $(\overline{\delta_1}, \overline{\delta_2}) <_{lex} (\zeta, \eta)$. Hence, $(1+\overline{\delta_1}, 1+\overline{\delta_2}) <_{lex} (1+\zeta, 1+\eta)$. Therefore, for all pairs (δ_1^i, δ_2^i) occurring in $f(\delta)$ we have $(\delta_1^i, \delta_2^i) <_{lex} (1+\zeta, 1+\eta)$.

Now we are ready to define $g(\alpha)$ for $\alpha = \vartheta(\beta)$ with $\beta < \omega^{\omega^{\Omega^2}}$ and $\beta \geq \Omega$: assume that $f(\beta) = ((\beta_1^1, \beta_2^1), \dots, (\beta_1^n, \beta_2^n)) \in W(\Omega)$. Then define $g(\alpha)$ as

$$\circ\left((g(\beta_1^1), g(\beta_2^1)), \dots, (g(\beta_1^n), g(\beta_2^n))\right)$$

$g(\alpha)$ is well-defined, because for every i and j, $\beta_j^i \leq k(\beta) + \omega < \vartheta(\beta) = \alpha$. Obviously, we see that $\alpha = 0$ iff $g(\alpha) = \circ$ and $\alpha = n < \omega$ iff $g(\alpha) = \circ((\circ,\circ), \dots, (\circ,\circ))$, where $(\circ,\circ)$ occurs n times.

The last part of this theorem consists of proving that g is a quasi-embedding: from Lemma 2 we can then conclude this theorem. We show that $g(\alpha) \leq g(\alpha')$ implies $\alpha \leq \alpha'$ for all $\alpha, \alpha' < \vartheta(\Omega^{\Omega^\Omega})$ by induction on $\alpha \oplus \alpha'$. If α or α' is equal to 0, this is trivial. So we may assume that $\alpha, \alpha' > 0$.

a) $\alpha' =_{CNF} \omega^{\alpha_1'} + \cdots + \omega^{\alpha_m'}, m \geq 2$.

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, $n \geq 2$.

If

$$g(\alpha) = \circ((g(\alpha_1), \circ), \dots, (g(\alpha_n), \circ)) \leq \circ\left((g(\alpha_1'), \circ), \dots, (g(\alpha_m'), \circ)\right) = g(\alpha').$$

then $g(\alpha) \leq g(\alpha_i')$ for a certain i or

$$((g(\alpha_1), \circ), \dots, (g(\alpha_n), \circ)) \leq \left((g(\alpha_1'), \circ), \dots, (g(\alpha_m'), \circ)\right).$$

In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha_i' < \alpha'$. In the latter case, there exist indices $1 \leq i_1 < \cdots < i_n \leq m$ such that $g(\alpha_j) \leq g(\alpha_{i_j}')$. By the induction hypothesis, we gain that $\alpha_j \leq \alpha_{i_j}'$ for every j. Hence $\alpha \leq \alpha'$.

ii) $\alpha = \vartheta(\beta)$.

If $\beta = 0$, then $\alpha = 1 \leq \alpha'$. Assume that $0 < \beta < \Omega$, then $g(\alpha) = \circ((g(\beta), g(\beta)))$.

Hence, $g(\alpha) \leq g(\alpha') = ((g(\alpha'_1), \circ), \ldots, (g(\alpha'_m), \circ))$ implies $g(\alpha) \leq g(\alpha'_i)$ for a certain i because $g(\beta) \not\leq \circ$. The induction hypothesis implies $\alpha \leq \alpha'_i < \alpha'$. Now suppose that $\beta \geq \Omega$ and $f(\beta) = ((\beta_1^1, \beta_2^1), \ldots, (\beta_1^n, \beta_2^n))$. At least one β_2^i is strictly bigger than 0, so $g(\alpha) \leq g(\alpha')$ implies $g(\alpha) \leq g(\alpha'_i)$ for a certain i. Therefore, $\alpha \leq \alpha'_i < \alpha'$ like before.

b) $\alpha' = \vartheta(\beta')$**.**

If $\beta' < \Omega$, then $g(\alpha) \leq g(\alpha') = \circ((g(\beta'), g(\beta')))$ implies $g(\alpha) \leq g(\beta')$ or that $\alpha = \vartheta(\beta)$ with $\beta < \Omega$ and $g(\beta) \leq g(\beta')$. The other cases are simply not possible because in these cases, the length of the corresponding finite sequence of $g(\alpha)$ is always strictly bigger than 1. We can conclude that $\alpha \leq \alpha'$. Assume from now on that $\beta' \geq \Omega$ and $f(\beta') = ((\beta_1'^1, \beta_2'^1), \ldots, (\beta_1'^m, \beta_2'^m))$.

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, $n \geq 2$.

Suppose $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\beta_j'^i)$ for certain i and j or

$$\begin{aligned} &((g(\alpha_1), \circ), \ldots (g(\alpha_n), \circ)) \\ &\leq ((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))). \end{aligned}$$

The induction hypothesis and the fact that every ordinal in $f(\beta')$ is less than or equal to $k(\beta') + \omega < \alpha'$ implies in the first case $\alpha \leq \beta_j'^i < \alpha'$, what we want, and in the latter case

$$\begin{aligned} &((\alpha_1, \circ), \ldots (\alpha_n, \circ)) \\ &\leq ((\beta_1'^1, \beta_2'^1), \ldots, (\beta_1'^m, \beta_2'^m)). \end{aligned}$$

Hence, for every i there exists an index j such that $\alpha_i \leq \beta_1'^j \leq k(\beta') + \omega < \alpha'$. We know that α' is an epsilon number, because $\beta' > \Omega$. So, $\alpha < \alpha'$.

ii) $\alpha = \vartheta(\beta)$.

If $\beta < \Omega$, then

$$\begin{aligned} &g(\alpha) = \circ((g(\beta), g(\beta))) \\ &\leq g(\alpha') = \circ\left((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))\right) \end{aligned}$$

implies either $g(\alpha) \leq g(\beta_j'^i)$ for certain i and j or

$$((g(\beta), g(\beta))) \leq \left((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))\right).$$

The induction hypothesis in the former case implies $\alpha \leq \beta_j'^i < \alpha'$ and in the latter case, it implies $\beta \leq \beta_s'^r < \alpha' = \vartheta(\beta')$ for certain r and s. Hence, in the latter case $\vartheta(\beta) \leq \vartheta(\beta')$ because $\beta < \Omega \leq \beta'$ and $k(\beta) = \beta < \vartheta(\beta')$.

Assume now that $\beta \geq \Omega$ and $f(\beta) = ((\beta_1^1, \beta_2^1), \ldots, (\beta_1^n, \beta_2^n))$. $g(\alpha) \leq g(\alpha')$ then either implies $g(\alpha) \leq g(\beta_j'^i)$ for certain i and j or

$$\left((g(\beta_1^1), g(\beta_2^1)), \ldots, (g(\beta_1^n), g(\beta_2^n))\right) \leq \left((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))\right).$$

In the former case, the induction hypothesis implies $\alpha \leq \beta_j'^i < \alpha'$. In the latter case, it implies

$$f(\beta) = \big((\beta_1^1,\beta_2^1),\ldots,(\beta_1^n,\beta_2^n)\big) \leq \big((\beta_1'^1,\beta_2'^1),\ldots,(\beta_1'^m,\beta_2'^m)\big) = f(\beta'). \quad (7)$$

Therefore, for every i and j, there exist r and s such that $\beta_j^i \leq \beta_s'^r < \vartheta(\beta') = \alpha'$. Hence $k(\beta) \leq \omega^{\omega^{Max(f(\beta))+1}} = \omega^{\omega^{Max_{i,j}\{\beta_j^i\}+1}} < \alpha'$, because α' is an epsilon number. If we now could prove that $f(\delta) \leq f(\delta')$ implies $\delta \leq \delta'$ for all $\delta,\delta' < \omega^{\omega^{\Omega^2}}$, we are done because (7) then implies $\beta \leq \beta'$. Hence, $\alpha = \vartheta(\beta) \leq \vartheta(\beta') = \alpha'$.

So assume that $f(\delta) \leq f(\delta')$. We will prove by induction on $\delta \oplus \delta'$ that $\delta \leq \delta'$. Assume that $\delta' < \omega$. Then $f(\delta) \leq f(\delta') = ((\circ,\circ),\ldots,(\circ,\circ))$, where $(\circ,\circ)$ occurs $\delta'+1$ many times. Hence $f(\delta)$ is also of the form $((\circ,\circ),\ldots,(\circ,\circ))$, so $\delta < \omega$ and $\delta \leq \delta'$. Assume that $\delta' \geq \omega$. If $\delta < \omega$, then $\delta \leq \delta'$ trivially holds. Assume that $\delta \geq \omega$. Like in (6), there exist unique ordinals $k,l < \omega$, $\overline{\delta_1},\overline{\delta_2},\overline{\delta_1'},\overline{\delta_2'} < \Omega$, $\delta_0,\ldots,\delta_k < \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}}$ with $\delta_k > 0$, $\delta_0',\ldots,\delta_l' < \omega^{\omega^{\Omega\overline{\delta_1'}+\overline{\delta_2'}}}$ with $\delta_l' > 0$ such that

$$\delta = \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot k}\delta_k + \cdots + \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0 \quad (8)$$

$$\delta' = \omega^{\omega^{\Omega\overline{\delta_1'}+\overline{\delta_2'}}\cdot l}\delta_l' + \cdots + \omega^{\omega^{\Omega\overline{\delta_1'}+\overline{\delta_2'}}\cdot 1}\delta_1' + \delta_0'. \quad (9)$$

$f(\delta) \leq f(\delta')$ then implies

$$\begin{aligned} & f(\delta_k)^\frown((1+\overline{\delta_1},1+\overline{\delta_2}))^\frown f(\delta_{k-1})\ldots((1+\overline{\delta_1},1+\overline{\delta_2}))^\frown f(\delta_0) \\ \leq\, & f(\delta_l')^\frown((1+\overline{\delta_1'},1+\overline{\delta_2'}))^\frown f(\delta_{l-1}')\ldots((1+\overline{\delta_1'},1+\overline{\delta_2'}))^\frown f(\delta_0'). \end{aligned} \quad (10)$$

Because $f(\delta_i') < \omega^{\omega^{\Omega\overline{\delta_1'}+\overline{\delta_2'}}}$, all pairs occurring in $f(\delta_i')$ is lexicographically strictly smaller than $(1+\overline{\delta_1'},1+\overline{\delta_2'})$. So if a certain $((1+\overline{\delta_1},1+\overline{\delta_2}))$ occurring in $f(\delta)$ would not be mapped onto $((1+\overline{\delta_1'},1+\overline{\delta_2'}))$ in the inequality (10), then $((1+\overline{\delta_1},1+\overline{\delta_2}))$ is lexicographically smaller than a pair in $f(\delta_i')$ for a certain i, hence $((1+\overline{\delta_1},1+\overline{\delta_2})) < (1+\overline{\delta_1'},1+\overline{\delta_2'})$. Therefore, $\delta < \delta'$. Assume now that every $((1+\overline{\delta_1},1+\overline{\delta_2}))$ occurring in $f(\delta)$ is mapped onto a $((1+\overline{\delta_1'},1+\overline{\delta_2'}))$ in $f(\delta')$ following the inequality (10). Hence $((1+\overline{\delta_1},1+\overline{\delta_2})) \leq_{lex} ((1+\overline{\delta_1'},1+\overline{\delta_2'}))$. If $((1+\overline{\delta_1},1+\overline{\delta_2})) <_{lex} ((1+\overline{\delta_1'},1+\overline{\delta_2'}))$, then $\delta < \delta'$. Assume $((1+\overline{\delta_1},1+\overline{\delta_2})) = ((1+\overline{\delta_1'},1+\overline{\delta_2'}))$. If $k < l$, then $\delta < \delta'$, so assume from now on that $k = l$. Therefore, inequality (10) implies $f(\delta_i) \leq f(\delta_i')$ for all $i = 1,\ldots,k$. From the induction hypothesis, this implies $\delta_i \leq \delta_i'$. We can conclude that $\delta \leq \delta'$. This ends this proof. □

Corollary 2 $o(\mathcal{T}((\cdot\times\cdot)^*)) = \vartheta(\Omega^{\Omega^\Omega})$.

Proof Follows from Theorems 10 and 11. □

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