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RIEMANNIAN COMPARISON AND LENGTH OF EXISTENCE OF OPTIMAL CONTROLS

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ABSTRACT. In Riemannian geometry, there are various comparison theorems which estimate the distance to conjugate points on manifolds and hence the maximum length of an energy minimising extremal. We apply this to optimal control problems to estimate the maximum length of existence of an optimal trajectory where energy is measured by the cost function. We show this can be done for control systems where the control part of the cost function can be interpreted as a Riemann metric and the unforced dynamics satisfy an integrability condition.

1. INTRODUCTION

A continuing problem in optimal control is the generalisation of the classical notions from calculus of variations of conjugate points, Riccati equations and Jacobi equations and the interpretation of the corresponding necessary and sufficient conditions for optimality - see for instance [4, 12, 13].

In this paper we show that for a certain class of nonlinear optimal control problems it is possible to estimate the distance, in terms of the cost function, to the first conjugate point and hence to estimate how far an optimal control exists. This is done by showing that the Hamiltonian arising from the maximum principle can be associated with a geodesic problem on a Riemann manifold whose metric is determined by the control problem. It is then possible to apply a comparison theorem which estimates distance between conjugate points given bounds on the sectional curvature of the manifold.

This approach is similar to [8, 9] where linear quadratic optimal control was considered. There the control problem was interpreted as a Jacobi equation for an underlying geodesic problem. Here, however, the nonlinear control problem is interpreted directly as a geodesic problem.

2. SOME DIFFERENTIAL GEOMETRY

In this section we briefly describe those elements of Riemannian geometry required to state the comparison theorem. We will also describe the constant curvature models used for comparison. More details can be found in [6, 10]. The summation convention on repeated indices will be used.

Let M be an n-dimensional manifold with local coordinates $x^1, \dots, x^n$ in an open set $U \subseteq M$. Then a Riemann metric g is a positive definite symmetric bilinear form on the tangent space $T_p(M)$ at $p \in M$ which varies smoothly with p. In other words, g is an inner product on $T_p(M)$. Taking the coordinate vector fields $\partial/\partial x^1, \dots, \partial/\partial x^n$ as a basis for $T_p(M)$, then $dx^1, \dots, dx^n$ form a basis for the cotangent space $T_p^*(M)$ and

$$g = g_{ij} dx^i dx^j$$

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where g_{ij} is a symmetric matrix of smooth functions on U. We aim to map the control problem onto the open set U.

The energy of the Riemann metric is the Lagrangian function

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j .$$

The corresponding Hamiltonian is the Legendre transformation of L

$$y_i = \partial L / \partial \dot{x}^i \qquad\qquad H = y_i \dot{x}^i - L = \frac{1}{2} g^{ji} y_j y_i$$

where g^{ji} is the inverse matrix to g_{ij}. We will sometimes denote this induced inner product on $T_p^*(M)$ as g^{-1}. We will also use $< . | . >$ to denote the natural bilinear pairing $y_i \dot{x}^i$ between $T_p^*(M)$ and $T_p(M)$.

Geodesics are extremals of the energy L. They satisfy the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

which reduce to

$$\text{(1)} \qquad \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$$

where

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right) g^{lk} .$$

The arc length of a curve $\gamma(t)$ is the integral of $\sqrt{g(\dot{\gamma}, \dot{\gamma})}$ along γ. If the tangent vector $\dot{\gamma}$ has fixed energy then γ is parameterised proportional to arc length. The principle of least action of Maupertuis says that a geodesic of energy e extremises arc length amongst curves parameterised to have the same fixed energy and, conversely, that an extremal of arc length is, up to reparameterisation proportional to arc length, a geodesic.

The quantities Γ^k_{ij} are the Christoffel symbols defining the Levi-Civita connection associated with g

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = \Gamma^k_{ij} \partial/\partial x^k .$$

If $X = X^i \partial/\partial x^i$ and $Y = Y^i \partial/\partial x^i$ are vector fields then the covariant derivative of Y with respect to X is

$$\nabla_X(Y) = \left(\partial_X(Y^k) + \Gamma^k_{ij} X^i Y^j\right) \partial/\partial x^k$$

where ∂_X is the Lie derivative with respect to X. If Z is a third vector field, f is a function and $[.,.]$ denotes the Lie bracket on vector fields then the connection satisfies

$$\nabla_{fX+Z} Y = f\nabla_X Y + \nabla_Z Y$$
$$\nabla_X(fY + Z) = \partial_X(f) Y + f\nabla_X Y + \nabla_X Z$$
$$\nabla_X Y = \nabla_Y X + [X, Y].$$

This last equation says that the connection is torsion free. The Riemann metric is parallel with respect to the connection in the sense that

$$\partial_Z g(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Written covariantly, the geodesic equation (1) becomes

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

Linearising this equation along a geodesic $\gamma(t)$ gives Jacobi's equation

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} J + R(\dot{\gamma}, J)\dot{\gamma} = 0$$

where R is the curvature tensor associated with the Riemann metric. A solution J along γ is called a Jacobi field. Let $J' = \nabla_{\dot\gamma} J$. Being solutions of a second order linear differential equation, the space of Jacobi fields along γ is $2n$-dimensional and is completely determined by its initial conditions $J(0)$ and $J'(0)$.

If J is orthogonal to γ, then so is J'. It follows that the space of Jacobi fields along γ orthogonal to γ which vanish at $t = 0$ is $(n-1)$-dimensional. A Jacobi field in this space describes an infinitesimal variation of γ through neighbouring geodesics starting from the same point p. Conjugate points to p along γ correspond to zeroes of non-trivial Jacobi fields in this family or, equivalently, to singularities of the exponential map from $T_p(M) \to M$.

Note that it follows from $\nabla_{\dot\gamma}\dot\gamma = 0$ and the skew-symmetry of the curvature tensor $R(\dot\gamma, \dot\gamma)\dot\gamma = 0$ that a Jacobi field parallel to γ and vanishing at $t = 0$ is just a linear field $at\dot\gamma$ for some constant a. Thus to determine the position of conjugate points we can restrict attention to Jacobi fields orthogonal to the geodesic as the parallel component vanishes nowhere apart from the initial point.

The components of the curvature tensor are given by

$$R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\frac{\partial}{\partial x^l} = R^i_{ljk}\frac{\partial}{\partial x^i}$$

where

$$R^i_{ljk} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{kl}}{\partial x^j} + \sum_{r=1}^{n}\left(\Gamma^i_{kr}\Gamma^r_{jl} - \Gamma^i_{jr}\Gamma^r_{kl}\right).$$

If e_1 and e_2 are unit vectors in the directions $\dot\gamma(t)$ and $J(t)$ at some point $\gamma(t)$ along γ then the sectional curvature of the plane spanned by $\dot\gamma$ and J is $g(R(e_1, e_2)e_1, e_2)$.

Example 2.1. Consider the sphere $x^2 + y^2 + z^2 = r^2$ in $\mathbf{R}^3$. Take as coordinates x^1 measuring the angle of latitude from the North pole and x^2 measuring longitude. The Riemann metric pulled back from $\mathbf{R}^3$ is

$$g = r^2(dx^1)^2 + r^2\sin^2(x^1)(dx^2)^2.$$

From the above formulae the only non-zero Christoffel symbols are

$$\Gamma^2_{12} = \Gamma^2_{21} = \cos(x^1)/\sin(x^1)$$

and

$$\Gamma^1_{22} = -\sin(x^1)\cos(x^1).$$

The non-zero components of the curvature tensor are

$$R^2_{112} = -R^2_{121} = 1 \qquad R^1_{221} = -R^1_{212} = \sin^2(x^1).$$

Consider a geodesic $\gamma(t)$ starting from the North pole. It is a great circle given by $x^1 = t$ and $x^2 = \text{const}$. The space of Jacobi fields orthogonal to $\gamma(t)$ is generated over $\mathbf{R}$ by the vector field $\partial/\partial x^2$ along $\gamma(t)$. It can be verified by direct calculation that Jacobi's equation holds

$$\nabla_{\partial/\partial x^1}\nabla_{\partial/\partial x^1}\partial/\partial x^2 = -\partial/\partial x^2 = -R(\partial/\partial x^1, \partial/\partial x^2)\partial/\partial x^1.$$

The first conjugate point to the North pole occurs at the South pole when $x^1 = \pi$, a distance of $r\pi$ in the Riemann metric. The sectional curvature of the plane tangent to γ at $\gamma(t)$ spanned by the unit vectors $e_1 = (1/r)\partial/\partial x^1$ and $e_2 = (1/r\sin(x^1))\partial/\partial x^2$ is

$$g(R(e_1, e_2)e_1, e_2) = g\left(\frac{1}{r^3\sin(x^1)}\frac{\partial}{\partial x^2}, \frac{1}{r\sin(x^1)}\frac{\partial}{\partial x^2}\right) = \frac{1}{r^2}.$$

This calculation can be re-done in the orthonormal frame field e_1, e_2. Writing the orthogonal Jacobi field as $J = f(t)e_2$, Jacobi's equation becomes

$$\frac{d^2}{dt^2}f(t) + \frac{1}{r^2}f(t) = 0$$

with solution $\sin(t/r)e_2$. In this frame it is easy to see that the Jacobi field vanishes at $t = \pm kr\pi$, i.e. at the North and South poles, and that the sectional curvature of a unit plane tangent to γ is $1/r^2$.

Remark. For more complicated examples, the Christoffel symbols and the curvature can be calculated using the tensor package in Maple. The actual quantity calculated in Maple is the covariant curvature tensor

$$R_{jklh} = R_{lhjk} = g\left(R\left(\partial/\partial x^j, \partial/\partial x^k\right)\partial/\partial x^l, \partial/\partial x^h\right).$$

The curvature tensor defined above can then be obtained as $R^i_{ljk} = g^{ih}R_{lhjk}$.

Example 2.2. The above generalises to the following constant curvature manifolds which will be used for comparison.

1. The n-sphere of radius r in $\mathbf{R}^{n+1}$ has sectional curvature $1/r^2$ and conjugate points $r\pi$ apart.
2. The n-plane in $\mathbf{R}^{n+1}$ has sectional curvature zero and no points conjugate to a given point.
3. The n-hyperboloid in $\mathbf{R}^{n+1}$ has sectional curvature $-1/r^2$ and no points conjugate to a given point.

We now consider geodesics starting orthogonal to a submanifold K of dimension $0 < k < n$ rather than from a point. Note that the orthogonality is part of the Euler-Lagrange necessary conditions. Suppose to begin with that $k = n - 1$, i.e. K is a hypersurface. Let ξ be the positively oriented field of unit vectors orthogonal to K and X be a vector field tangent to K. Then differentiating $g(\xi, \xi) = 1$ gives $g(\nabla_X\xi, \xi) = 0$. So $\nabla_X\xi$ is tangent to K. The linear mapping on vector fields tangent to K defined by $\nabla_X\xi$ is called the second fundamental form of K.

Strictly speaking, the second fundamental form acting on vector fields X and Y tangent to K is defined as $\alpha(X, Y) = g(-\nabla_X\xi, Y)$ (see [6]). We take $\nabla_X\xi$ to be the second fundamental form in this paper because with the given sign it corresponds to the Riccati matrix of the hypersurface K.

To see this, consider the family of geodesics starting orthogonal to K in the direction of ξ and parameterised by arc length. Then ξ is the field of tangent vectors to these geodesics at K and can be extended along the geodesics to a vector field in a neighbourhood of K. Pick one of the geodesics and denote it by γ. Suppose J is an orthogonal Jacobi field along γ which is tangent to K at $t = 0$ and which varies γ through neighbouring geodesics in the same family. Then J can be extended to Jacobi fields on the neighbouring geodesics. This construction results in fields ξ normal to and J tangent to the family of hypersurfaces equidistant to K along the normal geodesics. It follows that the flow generated by J and the geodesic flow generated by ξ commute and so $[J, \xi] = 0$. (This can be proved more formally by taking an exponential or normal coordinate system in a neighbourhood of K.) Then, since the Levi-Civita connection has no torsion,

$$\nabla_J\xi = \nabla_\xi J + [J, \xi] = \nabla_\xi J \tag{2}$$

i.e. the covariant derivative of J along γ is given by the second fundamental form of the relevant equidistant hypersurface. Let $J' = \nabla_\xi J$ and let P be the matrix representing the second fundamental form $\nabla\xi$. Then (2) gives $J' = PJ$. Thus P is, by definition, the Riccati matrix. A further covariant differentiation of $J' = PJ$ gives the Riccati equation for P (see [5]).

3. On the n-hyperboloid of radius r, if P has all eigenvalues either
   (a) equal to $(1/r)\cosh(c)/\sinh(c)$ for $c \in (-\infty, 0) \cup (0, \infty)$ then there is no focal point along $\operatorname{sign}(c)\gamma$ and one focal point at distance $r|c|$ along $-\operatorname{sign}(c)\gamma$, or
   (b) equal to $-1/r$, $(1/r)\sinh(c)/\cosh(c)$ for $c \in (-\infty, \infty)$ or $1/r$ then there is no focal point along γ or $-\gamma$. The limiting cases $\pm 1/r$ correspond to asymptotic stability/instability of J and J'.

We now state the comparison theorem. There are various comparison theorems in Riemannian geometry. Corollary 2.6 below was essentially first stated by Bonnet for geodesics on a surface and can be viewed as a generalisation of Sturm's theorem on zeroes of scalar equations of the form $\ddot{x} + f(t)x = 0$. Rauch's comparison theorem (see [6]) gives a comparison of the lengths of Jacobi fields along geodesics starting from points on two different manifolds under suitable curvature and nonconjugacy hypotheses. This is generalised to geodesics starting from submanifolds by Warner in [11]. We only require comparison with manifolds of constant curvature and so we state the following which is Theorem 4.1 of [11], although part 1 follows immediately from Rauch's theorem. This avoids many of the technical details of the statement of the most general comparison theorem, but the reader should be aware that a more general theorem exists.

Theorem 2.5. *Let M and N be Riemannian manifolds of the same dimension n. Let K and L be submanifolds of M and N of dimension $0 \leq k < n$. Let $g = g(t)$, $0 \leq t \leq b$ be a geodesic of M parameterised by its arc length with $g(0) \in K$ and $\dot{g}(0) \in T_{g(0)}(K)^{\perp}$. Let $h = h(t)$, $0 \leq t \leq b$ be a geodesic of N parameterised by its arc length with $h(0) \in L$ and $\dot{h}(0) \in T_{h(0)}(L)^{\perp}$. Assume that for each $t \in [0, b]$ and for arbitrary planes S tangent to g at $g(t)$ and T tangent to h at $h(t)$, the sectional curvatures $K_M(S)$ and $K_N(T)$ satisfy*

$$K_M(S) \leq K_N(T).$$

Also assume either

1. *$k = 0$ (i.e. K and L are points). Then if there are no conjugate points on h for $t \in (0, b]$, there are no conjugate points on g for $t \in (0, b]$.*
2. *$k > 0$ and*

$$\text{minimum eigenvalue of } P \geq \text{maximum eigenvalue of } Q$$

   where P is the second fundamental form of K in the direction $\dot{g}(0)$ and Q is the second fundamental form of L in the direction $\dot{h}(0)$. Then if there are no focal points on h for $t \in (0, b]$, there are no focal points on g for $t \in (0, b]$.

Our application of this theorem is described in the following corollary which can be deduced by applying the above theorem to the manifold M, on which the control problem will be defined, and either the n-sphere, plane or hyperboloid. The corollary is stated for initial points/submanifolds but applies equally to terminal points/submanifolds by applying the theorem to $-g$. This changes the sign of the second fundamental form and so the relevant inequalities below will be reversed. Note also that g is parameterised by arc length and so the distance in the Riemann metric to a conjugate/focal point is given by the parameter value of the conjugate/focal point.

Corollary 2.6. *Let M, K, g, S and P be as in Theorem 2.5. Further, let λ and μ denote the minimum and maximum eigenvalues of P. In parts 1 to 3 of the following, (a) describes the case $k = 0$ and (b) the case $k > 0$. Parts 4 and 5 describe the case $k > 0$.*

1. *Let $K_M(S) \geq 1/r^2$ for some $r > 0$. Then*

(a) *g has a conjugate point within distance $r\pi$.*
(b) *if $\mu \leq (1/r)\cos(c)/\sin(c)$ for some $c \in (0,\pi)$ then g has a focal point within distance $r(\pi - c)$.*

2. *Let $K_M(S) \leq 1/R^2$ for some $R > 0$. Then*
   (a) *g has no conjugate point within distance $R\pi$.*
   (b) *if $\lambda \geq (1/R)\cos(d)/\sin(d)$ for some $d \in (0,\pi)$ then g has no focal point within distance $R(\pi - d)$.*
3. *Let $K_M(S) \leq 0$. Then*
   (a) *g has no conjugate points.*
   (b) *if $\lambda \geq 1/c$ for some $c \in (-\infty, 0)$ then g has no focal point within distance $-c$. If $\lambda \geq 0$ then g has no focal point.*
4. *Let $K_M(S) \leq -1/r^2$ for some $r > 0$. Then 3.(b) can be improved as follows:*
   (a) *if $\lambda \geq (1/r)\cosh(c)/\sinh(c)$ for some $c \in (-\infty, 0)$ then g has no focal point within distance $-rc$.*
   (b) *if $\lambda \geq -1/r$ then g has no focal point.*
5. *Let $K_M(S) \geq -1/R^2$ for some $R > 0$ and $\mu \leq (1/R)\cosh(d)/\sinh(d)$ for some $d \in (-\infty, 0)$ then g has a focal point within distance $-Rd$.*

3. Applications to Optimal Control

Consider an optimal control problem of the form: minimise

$$S = \int_{t_0}^{t_f} l(x, u)\,dt \tag{3}$$

subject to $\dot{x} = f(x, u)$ where $x, u \in \mathbf{R}^n$. In this section we deal with extremals starting from a point p or orthogonal, in a scalar product to be defined below, to a k-dimensional submanifold K $(0 < k < n)$. The results will also apply to extremals ending at p or orthogonal to K. Note that the orthogonality is one of the necessary conditions for an extremal of S. More general initial or final conditions involving a term of the form $\theta(x(t_0), t_0)$ or $\theta(x(t_f), t_f)$ in S, and hence non-orthogonal extremals, will be considered in the next section.

In this section it will be shown that a certain class of control problems with the above initial or final conditions can, after applying Pontryagin's maximum principle, be considered as geodesic problems on manifolds where the Riemann metric is determined by l and f.

The positive definiteness of the Riemann metric guarantees that the resulting extremals starting or ending on p or K are locally optimal i.e. that an optimal control exists locally. In cases where the resulting curvature is suitably bounded Corollary 2.6 can be applied to determine distance bounds on the occurence of the first conjugate or focal point. This gives distance bounds on the existence of the optimal control or implies that it exists indefinitely. By indefinitely is meant as $t_f \to \infty$ or $t_0 \to -\infty$ depending on which end of the optimisation is free.

The existence of optimal solutions satisfying both initial and final conditions is, in general, a difficult question. In this context it would involve determining firstly whether the final point or manifold was equidistant along all extremals from the initial point or manifold and then whether it was within the distance bounds for optimality determined by the curvature.

We begin with our basic class of geodesic control problems.

Theorem 3.1. *Suppose $l(x, u) = \frac{1}{2}u^T g^{-1}(x)u$ and $f(x, u) = g^{-1}(x)u$ where $g^{-1} = [g^{ij}(x)]$ is an $n \times n$ matrix of differentiable functions on $\mathbf{R}^n$ which is symmetric and positive definite for each $x \in \mathbf{R}^n$. Then extremals of (3) are geodesics on a manifold with local coordinates $x = (x^1, \ldots, x^n) \in \mathbf{R}^n$ and Riemann metric $g = g_{ij}dx^i dx^j$.*

Proof. Take as Hamiltonian $H(x, y, u) = y^T f(x, u) - l(x, u)$ where $y \in \mathbf{R}^n$ is a vector of Lagrange multipliers. Pontryagin's maximum principle says that an optimal control u^* satisfies

$$H(x, y, u^*) = \sup_u H(x, y, u)$$

and the adjoint vector y satisfies

$$\dot{y} = -\partial H/\partial x = -y^T \partial f/\partial x + \partial l/\partial x.$$

Now $H(x, y, u) = y_i g^{ij} u_j - \frac{1}{2} u_i g^{ij} u_j$. So u^* satisfies $\partial H/\partial u = y_i g^{ij} - u_i g^{ij} = 0 \Leftrightarrow u_i = y_i$. Then $H = \frac{1}{2} y_i g^{ij} y_j$. This is just the Hamiltonian associated with the Riemann metric g as can be seen by applying the inverse Legendre transformation

$$\dot{x}^i = \partial H/\partial y_i = y_j g^{ji} \Rightarrow y_j = \dot{x}^i g_{ij}$$

and

$$L = y_i \dot{x}^i - H = \dot{x}^j g_{ji} \dot{x}^i - \frac{1}{2} \dot{x}^j g_{ji} \dot{x}^i = \frac{1}{2} \dot{x}^j g_{ji} \dot{x}^i$$

which is the Lagrangian given by the energy of the metric. Similarly the adjoint equation transforms into the Euler-Lagrange equation corresponding to L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{d}{dt} y_i = \frac{\partial}{\partial x^i}(-H) = \frac{\partial}{\partial x^i}(y_i \dot{x}^i - H) = \frac{\partial L}{\partial x^i}.$$

Hence the extremals are geodesics with respect to g. □

Note the reference above to extremals starting orthogonal to a submanifold K meant orthogonal in the Riemann metric g defined by l and f. In the Hamiltonian setting this means that the adjoint vector y satisfies $\pi(y) \in K$ and $< v \mid y >= 0$ for all $v \in T_{\pi(y)}(K)$ where π is the projection map $T^*(M) \to M$.

Example 3.2. Take

$$l = \frac{1}{2} u_1^2 + \frac{1}{2} \frac{1}{\sin^2(x^1)} u_2^2$$

and

$$\frac{d}{dt}\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} = \begin{pmatrix} u_1 \\ \frac{u_2}{\sin^2(x^1)} \end{pmatrix}.$$

Then

$$H = yf - l = y_1 u_1 + \frac{y_2 u_2}{\sin^2(x^1)} - \frac{1}{2} u_1^2 - \frac{1}{2} \frac{u_2^2}{\sin^2(x^1)}.$$

For an optimal u Pontryagin's maximum principle gives

$$H = \frac{1}{2} y_1^2 + \frac{1}{2} \frac{y_2^2}{\sin^2(x^1)}.$$

It can be seen that this is the Hamiltonian associated with the Riemann metric on a 2-sphere of radius 1 (c.f. Example 2.1). Hence u cannot be optimal beyond distance π which is the maximum distance realisable in this cost function. Basically, optimality is equivalent to x^1 and x^2 being the image of the so-called normal coordinates on the sphere under the exponential map associated with the Riemann metric. This map has a singularity at $x^1 = \pi$.

Example 3.3. In [3] Bloch and Crouch consider optimal control problems on the adjoint orbits of left invariant Lie groups and on Grassmannian manifolds. They show that the optimal trajectories correspond to the geodesic flow on these manifolds. Examples include geodesic flow on the n-sphere and on $SO(n)$ which is equivalent to the equations of motion of the generalised rigid body. Corollary 2.6

can therefore be applied to determine the length of existence of optimal controls. This involves translating the above coordinate based expressions into the invariant notation of [3] based on double Lie brackets and is related to the curvature and cut locus properties of distance spheres in complex projective space. This will be considered in a future paper.

To extend this to control problems with unforced dynamics we need the following two results from Hamiltonian mechanics.

Theorem 3.4. *(Jacobi) Let g be a Riemann metric and V be a function on $\mathbf{R}^n$ which is bounded above by a constant e on some region of $\mathbf{R}^n$. Consider a Lagrangian $L = \frac{1}{2}\dot{x}^T g\dot{x} - V(x)$ with corresponding Hamiltonian $H = \frac{1}{2}y^T g^{-1} y + V$ where $y = \partial L/\partial\dot{x}$. Then extremals of L with total energy $H = e$ are geodesics of the Riemann metric $\widehat{g} = (e - V)g$ up to reparameterisation with energy 1. i.e. they are extremals of $\widehat{L} = \frac{1}{2}(e - V)\dot{x}^T g\dot{x}$.*

Proof. Section 45A of [2] or Theorem 3.7.7 of [1]. Note that Arnold uses the term geodesic to mean extremal of arc length. $\widehat{g}$ is known as the Jacobi metric. □

Theorem 3.5. *Let $(x, y) \to (X, Y)$ be a change of variables in phase space $\mathbf{R}^{2n}$ satisfying $y^T dx - hdt = Y^T dX - Hdt + dS$ where $h(x, y)$ is a Hamiltonian function. $H(X, Y) = h(x, y)$ is the same Hamiltonian in the new coordinates and S is some function of X, Y and t. Then the trajectories of the phase flow:*

$$\dot{x} = \frac{\partial h}{\partial y} \qquad \dot{y} = -\frac{\partial h}{\partial x}$$

are represented in the (X, Y) coordinates by the integral curves of the canonical equations

$$\dot{X} = \frac{\partial H}{\partial Y} \qquad \dot{Y} = -\frac{\partial H}{\partial X}.$$

Proof. Section 45A of [2]. □

Theorem 3.6. *Suppose in (3) that $l(x, u) = U(x) + \frac{1}{2}u^T g^{-1}(x)u$ for some differentiable function U and $n \times n$ matrix of differentiable functions $g^{-1} = [g^{ij}(x)]$ which is symmetric and positive definite for each $x \in \mathbf{R}^n$. Suppose also that $\dot{x} = f(x, u) = a(x) + g^{-1}(x)u$ for some vector valued function a satisfying the integrability condition $[g(x)a(x)]^T dx = dS(x)$ for some function S. Define a potential function $V(x) = -\frac{1}{2}a(x)^T g(x)a(x) - U(x)$ and a Hamiltonian $h(x, y, u) = y^T f - l$ where $y \in \mathbf{R}^n$ is an adjoint vector of Lagrange multipliers. Then extremals of (3) with total energy $h = e$ are geodesics of the Jacobi metric $\widehat{g} = (e - V)g$ up to reparameterisation with energy 1.*

Proof. We have

$$h(x, y, u) = y_i a^i + y_i g^{ij} u_j - U - \frac{1}{2}u_i g^{ij} u_j.$$

Maximising this with respect to u gives $u_i = y_i$ and so for an optimal control

$$h(x, y) = \frac{1}{2}y^T g^{-1} y + y^T a - U.$$

Completing the square gives

$$\begin{aligned} h(x, y) &= \frac{1}{2}(y^T + a^T g)g^{-1}(y + ga) - \frac{1}{2}a^T ga - U \\ &= \frac{1}{2}Y^T g^{-1} Y + V(X) \end{aligned}$$

under the change of variables $X = x, Y = y + ga$. Extremals of (3) satisfy the dynamic constraint equation $\dot{x} = f = \partial h/\partial y$ and the adjoint equation $\dot{y} = -\partial h/\partial x$ i.e. they are trajectories of the phase flow for h. Now

$$Y^T dX = (y + ga)^T dx = y^T dx + dS$$

by hypothesis. Hence by Theorem 3.5 the extremals of (3) in X, Y coordinates are integral curves of the phase flow corresponding to $H(X, Y) = \frac{1}{2}Y^T g^{-1} Y + V(X)$. Then by Jacobi's theorem those extremals with total energy $h = H = e$ are geodesics in the metric $\hat{g} = (e - V)g$ up to reparameterisation. □

Example 3.7. Systems such as

$$\dot{x}_1 = x_2 + u_1$$
$$\dot{x}_2 = x_1 + u_2$$

and

$$\dot{x}_1 = x_1^2 + x_2^2 + u_1$$
$$\dot{x}_2 = 2x_1 x_2 + u_2$$

satisfy the integrability condition for $a(x)$ in Theorem 3.6. For the second system, for example, $(x_1^2 + x_2^2)dx_1 + 2x_1x_2dx_2 = d(\frac{1}{3}x_1^3 + x_1x_2^2)$.

Remark. In [7] we studied dynamical systems which can be lifted to Clifford algebras. The two systems in the example above have unforced dynamics which are liftable to the universal Clifford algebra $\mathbf{R} \oplus \mathbf{R}$ of dimension 2 over $\mathbf{R}$ with quadratic form $Q(x) = -x^2$. Theorem 3.6 can be applied to systems liftable to the other Clifford algebras, but then it is necessary to consider control cost matrices $g^{-1}(x)$ which are indefinite or negative definite to take account of the quadratic form defining the Clifford algebra. For example, consider the system

$$\dot{x}_1 = x_1^2 - x_2^2 + u_1$$
$$\dot{x}_2 = 2x_1x_2 - u_2$$

with cost function $l(x, u) = U(x) + \frac{1}{2}u_1^2 - \frac{1}{2}u_2^2$. This system has unforced dynamics liftable to the universal Clifford algebra $\mathbf{C}$ of dimension 2 over $\mathbf{R}$ with quadratic form $Q(x) = x^2$. Forming the Hamiltonian and applying the maximum principle as in the proof of Theorem 3.6 gives

$$\begin{aligned} h &= y_1(x_1^2 - x_2^2) + \frac{1}{2}y_1^2 + 2y_2x_1x_2 - \frac{1}{2}y_2^2 - U(x) \\ &= \frac{1}{2}(y_1 + x_1^2 - x_2^2)^2 - \frac{1}{2}(y_2 - 2x_1x_2)^2 + V(x) \\ &= \frac{1}{2}Y^T g^{-1} Y + V(X). \end{aligned}$$

This is a canonical change of variables because

$$Y^T dX = y_1 dx_1 + (x_1^2 - x_2^2)dx_1 + y_2 dx_2 - 2x_1x_2dx_2 = y^T dx + d(\frac{1}{3}x_1^3 - x_1x_2^2).$$

Allowing indefinite Riemann metrics would require distinguishing between time-like and space-like trajectories in a pseudo-Riemannian manifold. We will consider this in a future paper.

Having shown that the extremals of (3) with the given l and f are geodesics in the Jacobi metric, the next step is to apply Corollary 2.6 in cases where the corresponding curvature is suitably bounded. However there are some points to be addressed before this can be done.

The first thing to note is that the change of variables in the cotangent space in the proof of Theorem 3.6 does not change the integral curves involved and so has no effect on the position of conjugate points. However any estimates of the distance to conjugate points using Corollary 2.6 will be with respect to the Jacobi metric. This will have to be rescaled to give the distance with respect to the original cost function. This is illustrated in Example 3.11 below.

The second point concerns the following question. Let c be an extremal of $H = \frac{1}{2}Y^T g^{-1} Y + V(X)$ with energy $H = e$ starting orthogonal with respect to g to a submanifold K of M of dimension k, $0 \leq k < n$. Let $\dot{c}$ denote the field of tangent vectors to c and P be the second fundamental form of K with respect to g in the direction $\dot{c}(0)$ if $k > 0$. If J is a vector field along c, let $J' = \nabla_{\dot{c}} J$ where ∇ is covariant differentiation with respect to the connection for g. Note that orthogonality in g implies orthogonality in $\hat{g} = (e - V)g$. Note also that we have included extremals starting from a point as case $k = 0$ for brevity in the following discussion. In this case, when we say $J(0) \in T_{c(0)}(K)$ we will mean $J(0) = 0$, the term focal point will mean conjugate point and the term involving P will vanish from the initial condition below. Then the question is, can valid conclusions about focal points on c be drawn from applying Corollary 2.6 to c as a geodesic of the Jacobi metric $\hat{g} = (e - V)g$?

The occurence of focal points to K on c is determined, as in the geodesic case, by the vanishing of non-trivial Jacobi fields satisfying the relevant initial conditions. The Jacobi equation for H is the linearisation along c of the canonical equations for H (see [10], Chapter IV). As in the geodesic case, this is a second order n-dimensional system of linear differential equations and so the space of Jacobi fields along c is $2n$-dimensional. However, unlike the geodesic case, it cannot be assumed that the space of Jacobi fields can be decomposed into those parallel to and those orthogonal to c and that the parallel fields only vanish at one point. Hence the initial conditions for K are the same as in the geodesic case without the condition that $J'(0)$ be orthogonal to c, namely $J(0) \in T_{c(0)}(K)$ and $PJ(0) - \|\dot{c}(0)\|^{-1} J'(0) \in T_{c(0)}(K)^{\perp}$. The space of Jacobi fields determining focal points to K is therefore n-dimensional. Denote this space by L.

Suppose the level surfaces of V are $(n-1)$-dimensional and grad$V$ is parallel to $\dot{c}$. Then under these hypotheses, L does decompose into fields parallel and orthogonal to c and we will show the parallel fields do not vanish before $V = e$.

Consider first the $(n-1)$-dimensional subspace of L containing Jacobi fields orthogonal to c which generate variations of c through extremals on the same energy level surface $H = e$. By Jacobi's theorem these extremals are all geodesics in the same Jacobi metric $\hat{g} = (e - V)g$ after reparameterisation. Thus the distance along c to a focal point corresponding to this subspace is estimated by Corollary 2.6.

It should be noted that the Jacobi metric has singularities on the boundary of the region $V < e$. So, while the extremals may still be optimal for H, the estimates from Corollary 2.6 for the $(n-1)$-dimensional subspace above are only valid up to the boundary of this region. In general, however, for a control problem $U(x) \geq 0$ and, by hypothesis, $a^T g a \geq 0$ for all x. Hence $e - V = e + \frac{1}{2}a^T g a + U > 0$ for $e > 0$ and so such singularities do not occur.

What about the remaining 1-dimensional subspace of solutions to Jacobi's equation? Denote this subspace by L_1. It is shown below that with the above hypotheses on V, there are no focal points on c corresponding to L_1 up to the boundary of the region $V < e$. Thus the estimates from Corollary 2.6 cover the full n-dimensional space L of Jacobi fields satisfying the initial conditions for K up to the boundary of the region $V < e$.

Recall from Section 2 that geodesics corresponding to $H = \frac{1}{2}y^T g^{-1} y$ are integral curves of $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ and the linearisation of this equation is the Jacobi equation $\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} J + R(\dot{\gamma}, J)\dot{\gamma} = 0$. Similarly it is shown in Theorem 3.7.4 of [1] that extremals of $H = \frac{1}{2}y^T g^{-1} y + V(x)$ are integral curves of

$$\nabla_{\dot{c}}\dot{c} = -\mathrm{grad}V(c(t)) \tag{4}$$

where ∇ is again with respect to the connection for g. Note that grad$V$ is the vector field satisfying $\partial_X V = g(\mathrm{grad}V, X)$ for any vector field X. By definition of ∇

$$\partial_Y g(\mathrm{grad}V, X) = g(\nabla_Y \mathrm{grad}V, X) + g(\mathrm{grad}V, \nabla_Y X).$$

Following [5], define the Hessian of V to be

$$g(\nabla_Y \mathrm{grad}V, X) = \partial_Y(\partial_X V) - \partial_{\nabla_Y X}V.$$

The following generalisation of Jacobi's equation then holds true with essentially the same proof in local coordinates. We indicate how it starts.

Lemma 3.8. *The linearisation of (4) along an extremal $c(t)$ is*

$$\nabla_{\dot{c}}\nabla_{\dot{c}}J + R(\dot{c}, J)\dot{c} = -\nabla_J \mathit{grad}V \tag{5}$$

where R is the curvature tensor corresponding to g.

Proof. In local coordinates (4) is

$$\frac{dx^i}{dt} = \dot{x}^i \qquad \frac{d\dot{x}^j}{dt} = -\Gamma^j_{kl}\dot{x}^k\dot{x}^l - g^{jk}\frac{\partial V}{\partial x^k}.$$

If dx^i/ds represents a vector field along c then the linearisation is

$$\frac{d}{dt}\left(\frac{dx^i}{ds}\right) = \frac{d\dot{x}^i}{ds}$$

$$\frac{d}{dt}\left(\frac{d\dot{x}^j}{ds}\right) = -\frac{\partial\Gamma^j_{kl}}{\partial x^i}\frac{dx^i}{ds}\dot{x}^k\dot{x}^l - \Gamma^j_{kl}\frac{d\dot{x}^k}{ds}\dot{x}^l - \Gamma^j_{kl}\dot{x}^k\frac{d\dot{x}^l}{ds} - g^{jk}\frac{\partial^2 V}{\partial x^i\partial x^k}\frac{dx^i}{ds} - \frac{\partial g^{jk}}{\partial x^i}\frac{\partial V}{\partial x^k}\frac{dx^i}{ds}.$$

Using the definitions of ∇ and R in local coordinates from Section 2, adding and subtracting the relevant terms and rearranging gives the result. □

Consider a vector field along c of the form $J = f(t)\dot{c}$ for some function f with initial conditions $f(0) = 0$ and $f'(0) = a$ for some constant a. Suppose J is a solution of the Jacobi equation (5). Then J will be an infinitesimal variation of c through extremals which have the same image as c as curves in $\mathbf{R}^n$ but which have different magnitude tangent vectors with respect to g at $t = 0$. So J will be a variation of c through extremals on energy level surfaces $H \neq e$. J will therefore span L_1. The next result gives conditions for J to be a solution of (5).

Proposition 3.9. *If $\mathit{grad}V$ is parallel to $\dot{c}$ and f satisfies the differential equation*

$$f'(t) = a\exp\left(\int_0^t \frac{g(\mathit{grad}V, \dot{c})}{(e - V(c(t)))}dt\right)$$
$$f(0) = 0$$

then J is a solution of Jacobi's equation (5).

Proof. By hypothesis

$$\mathrm{grad}V = \frac{g(\mathrm{grad}V, \dot{c})}{g(\dot{c}, \dot{c})}\dot{c} = \frac{g(\mathrm{grad}V, \dot{c})}{2(e - V)}\dot{c}.$$

Also by the skew symmetry of the curvature tensor, $R(\dot{c}, J)\dot{c} = fR(\dot{c}, \dot{c})\dot{c} = 0$. Then differentiating the expression for f' gives

$$\begin{aligned}
\frac{f''}{f'} &= \frac{g(\mathrm{grad}V, \dot{c})}{e - V} && \Leftrightarrow \\
f'' g(\dot{c}, \dot{c}) &= 2f' g(\mathrm{grad}V, \dot{c}) && \Leftrightarrow \\
f''\dot{c} - 2f'\mathrm{grad}V &= 0 && \Leftrightarrow \\
f''\dot{c} + 2f'\nabla_{\dot{c}}\dot{c} - f\nabla_{\dot{c}}\mathrm{grad}V &= -f\nabla_{\dot{c}}\mathrm{grad}V && \Leftrightarrow \\
\nabla_{\dot{c}}\nabla_{\dot{c}}(f\dot{c}) &= -f\nabla_{\dot{c}}\mathrm{grad}V && \Leftrightarrow \\
\nabla_{\dot{c}}\nabla_{\dot{c}}J + R(\dot{c}, J)\dot{c} &= -\nabla_J \mathrm{grad}V.
\end{aligned}$$

□

Corollary 3.10. *For $t > 0$, J does not vanish before the boundary of the region $V < e$ and hence there are no focal points corresponding to the subspace L_1 in this region.*

The conditions imposed on V in the above discussion are that its level surfaces be $(n-1)$-dimensional and grad$V$ be parallel to $\dot{c}$. If c starts orthogonal to a submanifold K, then a necessary condition for this is that K be tangent to a level surface of V at $c(0)$. We end this section with an example illustrating the application of Corollary 2.6 to a problem involving the Jacobi metric.

Example 3.11. Consider a cost function of the form

$$l = \frac{1}{2}u_1^2 + \frac{1}{2}\frac{u_2^2}{\sin^2(x^1)} - (1 - \cos(x^1))$$

with dynamics

$$\dot{x}^1 = u_1 \qquad \dot{x}^2 = \frac{u_2}{\sin^2(x^1)}.$$

Forming the Hamiltonian and applying the maximum principle gives

$$h = \frac{1}{2}y_1^2 + \frac{1}{2}\frac{y_2^2}{\sin^2(x^1)} + (1 - \cos(x^1))$$

which is the Hamiltonian corresponding to motion on a sphere of radius 1 in the presence of a gravitational potential acting in the positive z-direction with x^1 and x^2 interpreted as in Example 2.1. The corresponding phase trajectories are those of a non-linear spherical pendulum with lowest point at $x^1 = 0$. They satisfy

$$\dot{x}^1 = y_1 \qquad \dot{x}^2 = \frac{y_2}{\sin^2(x^1)}$$

$$\dot{y}_1 = \frac{y_2^2\cos(x^1)}{\sin^3(x^1)} - \sin(x^1) \qquad \dot{y}_2 = 0.$$

There is a family of solutions, which we denote C, corresponding to motion in a fixed vertical plane satisfying

$$x^2 = \text{const.} \qquad \ddot{x}^1 = -\sin(x^1).$$

Intuitively, the first conjugate point to $x^1 = 0$ along this family of solutions occurs at $x^1 = \pi$. We will verify this by applying Corollary 2.6 to the Jacobi metric for this problem.

The metric on the sphere corresponding to the kinetic energy is

$$g = (dx^1)^2 + \sin^2(x^1)(dx^2)^2$$

and so the arc length parameter with respect to g along C is dx^1. For a fixed value of $h = e$, the curves in C are geodesics in the Jacobi metric

$$\hat{g} = (e - 1 + \cos(x^1))((dx^1)^2 + \sin^2(x^1)(dx^2)^2)$$

after reparameterisation with energy 1. The tangent vector field to these curves under the reparameterisation is therefore

$$\hat{e}_1 = \frac{dx^1}{d\tau}\frac{\partial}{\partial x^1} = \sqrt{\frac{2}{e - 1 + \cos(x^1)}}\frac{\partial}{\partial x^1}.$$

The Christoffel symbols and curvature tensor components can be calculated by hand or in Maple. The relevant non-zero ones for this example are

$$\Gamma^1_{11} = -\frac{1}{2}\frac{\sin(x^1)}{e - 1 + \cos(x^1)}$$

$$\Gamma^2_{21} = -\frac{1}{2}\frac{(\sin^2(x^1) - 2\cos(x^1)e + 2\cos(x^1) - 2\cos^2(x^1))}{\sin(x^1)(e - 1 + \cos(x^1))}$$

$$R_{1212} = g_{22}R^2_{112} = \frac{1}{2}\frac{\sin^2(x^1)(3 - 4e + 2e^2 + 6\cos(x^1)e - 6\cos(x^1) + 3\cos^2(x^1))}{(e - 1 + \cos(x^1))}.$$

It can be checked directly that the curves in C with tangent vector $\hat{e}_1$ are geodesics in $\hat{g}$

$$\hat{\nabla}_{\left(\frac{dx^1}{d\tau}\frac{\partial}{\partial x^1}\right)}\left(\frac{dx^1}{d\tau}\frac{\partial}{\partial x^1}\right) = \left[\left(\frac{dx^1}{d\tau}\frac{\partial}{\partial x^1}\right)\left(\frac{dx^1}{d\tau}\right) + \Gamma^1_{11}\left(\frac{dx^1}{d\tau}\right)^2\right]\frac{\partial}{\partial x^1} = 0.$$

If $e_1 = (1/\sqrt{2})\hat{e}_1$ and e_2 denotes the unit vector field

$$\frac{1}{(e - 1 + \cos(x^1))^{1/2}\sin(x^1)}\frac{\partial}{\partial x^2}$$

orthogonal to C, then the sectional curvature of the tangent plane spanned by e_1 and e_2 is

$$K = \frac{1}{(e - 1 + \cos(x^1))^2\sin^2(x^1)}R_{1212}.$$

It can be checked that the sectional curvature of any other plane at the same point spanned by $\sqrt{a}e_1 + \sqrt{1-a}e_2$ and $-\sqrt{1-a}e_1 + \sqrt{a}e_2$ is also equal to K. Alternatively this follows from the observation that the sphere is symmetric under rotations and the Jacobi metric distorts the spherical metric by a factor depending only on the point in question, not on the direction.

For $e < 2$, the solutions in C have a singularity before $x^1 = \pi$ and an argument similar to the following will show there is no conjugate point before this singularity. So suppose $e = 2$ which corresponds to those trajectories which just reach the top of the swing of the pendulum $x^1 = \pi$, $\dot{x}^1 = 0$. The gradient of the potential $(1 - \cos(x^1))$ is parallel to the curves in C, so to determine the position of conjugate points to $x^1 = 0$ along these curves it is sufficient by Corollary 3.10 to apply Corollary 2.6 to $\hat{g}$. For $e = 2$

$$K = \frac{3}{2}\frac{1}{1 + \cos(x^1)} > \frac{3}{4}.$$

So any conjugate point to $x^1 = 0$ has to be within distance $2\pi/\sqrt{3}$ in $\hat{g}$. If dp is the arc length parameter with respect to $\hat{g}$, then it is related to dx^1 by

$$dp = \sqrt{1 + \cos(x^1)}dx^1 = \sqrt{2}\cos(x^1/2)dx^1.$$

So a distance $2\pi/\sqrt{3}$ in $\hat{g}$ along a curve in C starting from $x^1 = 0$ corresponds to a distance x_f in g given by

$$\frac{2\pi}{\sqrt{3}} = \int_0^{x_f} \sqrt{2}\cos\left(\frac{x^1}{2}\right) dx^1 = 2\sqrt{2}\sin\left(\frac{x_f}{2}\right).$$

This is unrealisable for x_f real, meaning that the upper bound on the distance to a conjugate point to $x^1 = 0$ is beyond the maximum distance realisable in g, namely π.

To check that the lower bound is equal to π, it is necessary to use a stepwise approach. Assume first that $x^1 \leq \pi/2$. Then $K \leq 3/2$. By Corollary 2.6 any conjugate point occurs after a distance $\sqrt{2/3}\pi$ in $\hat{g}$. This corresponds to a distance

$$x_f = 2\sin^{-1}(\pi/2\sqrt{3}) > \pi/2$$

in g. So any conjugate point occurs after $x^1 = \pi/2$.

Assuming a larger bound on x^1, say $x^1 \leq 2\pi/3$, produces a larger bound on $K \leq 3$ which pushes the lower bound on the distance to a conjugate point

$$x_f = 2\sin^{-1}(\pi/2\sqrt{6}) < 2\pi/3$$

below the original bound on x^1.

The way to proceed is to calculate a lower bound on the distance to a focal point along curves in C starting orthogonal to the submanifold $x^1 = \pi/2$. The second fundamental form of this submanifold with respect to $\hat{g}$ is

$$Pe_2 = \hat{\nabla}_{e_2} e_1 = \frac{1}{\sqrt{1+\cos(x^1)}}\Gamma^2_{21} e_2 = -\frac{1}{2}e_2$$

at $x^1 = \pi/2$. Take as the next bound $x^1 \leq 2\pi/3$ so that $K \leq 3$. Let $d = \tan^{-1}(-2\sqrt{3})$ where $\tan^{-1}$ has range $[0, \pi)$. Then by Corollary 2.6, the focal point to $x^1 = \pi/2$ occurs after a distance $(\pi - d)/\sqrt{3}$ in $\hat{g}$. This corresponds to a distance x_f in g given by

$$\frac{(\pi - d)}{\sqrt{3}} = 2\sqrt{2}\left[\sin\left(\frac{x_f}{2}\right)\right]_{\frac{\pi}{2}}^{x_f}$$

which is greater than $2\pi/3$. Since a focal point to $x^1 = \pi/2$ along curves in C is the same as a conjugate point to $x^1 = 0$ along the same curves, it follows that any conjugate point to $x^1 = 0$ occurs after $x^1 = 2\pi/3$. Proceeding in this way it can be shown that the lower bound on the conjugate point is arbitrarily close to π. In fact, $\pi/2$ is not big enough for the first step. Taking, for instance, the intermediate submanifolds to lie at $x^1 = 1.8$, 2.45, 2.85, 3.05, 3.12 and 3.13 radians shows that the conjugate point cannot occur before $x^1 = 3.14$ radians.

For the above example, grad$V$ is parallel to all extremals starting from $x^1 = 0$. For extremals starting from $x^1 = \pi/2$, $x^2 = a$ for some $a \in [0, 2\pi)$ the above argument can only be applied to the extremal $x^2 = a$ as all the other extremals starting from the given point are not parallel to grad$V$.

Note the above example can also be realised with the cost function and dynamics

$$l = \frac{1}{2}u_1^2 + \frac{1}{2}\frac{u_2^2}{\sin^2(x^1)}$$

$$\dot{x}^1 = u_1 + \sqrt{2(1+\cos(x^1))}$$

$$\dot{x}^2 = \frac{u_2}{\sin^2(x^1)}$$

although a trajectory on energy level e for the first realisation will lie on level $(e-2)$ in this realisation.

4. More General Initial Conditions

In the previous section we considered extremals starting from a point p or orthogonal to a submanifold K. More general initial conditions can be given by considering a control problem of the form: minimise

$$S = \theta(x(t_0), t_0) + \int_{t_0}^{t_f} l(x, u)dt \tag{6}$$

subject to $\dot{x} = f(x, u)$ where $x, u \in \mathbf{R}^n$. The necessary conditions for an extremal of this control problem include the condition that the adjoint vector satisfies

$$y(t_0) = \frac{\partial\theta}{\partial x}$$

giving an initial condition for all of $\mathbf{R}^n$. If the initial point lies on a k-dimensional submanifold of $\mathbf{R}^n$, $(0 \leq k < n)$, given by $M(x(t_0), t_0) = 0$ where M takes values in $\mathbf{R}^{n-k}$, then the adjoint vector satisfies

$$y(t_0) = \frac{\partial\theta}{\partial x} + \left(\frac{\partial M^T}{\partial x}\right)\xi$$

where $\xi \in \mathbf{R}^{n-k}$ is a vector of Lagrange multipliers. This initial condition allows extremals to start non-orthogonally to a submanifold.

The results of the preceding section can be applied to (6) provided there exists some $\tau_0 < t_0$ and some point p or submanifold K such that

$$\theta(x(t_0), t_0) = \int_{\tau_0}^{t_0} l(x, u)dt$$

and the extremals of (6) traced backwards start from p or orthogonal to K at τ_0. Obviously, for this to be true, θ has to satisfy $d\theta/dt = l$. A simple example will illustrate this.

Example 4.1. Suppose $l = \frac{1}{2}u_1^2 + \frac{1}{2}u_2^2$, the dynamical constraints are $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$ and $\theta = \frac{1}{2}x_1^2/t + \frac{1}{2}x_2^2/t$. This gives the equations of geodesics $\dot{y}_i = \ddot{x}_i = 0$ in the Euclidian metric on the plane, i.e. straight lines. θ is the action function $\frac{1}{2}x_1y_1 + \frac{1}{2}x_2y_2$ associated with l written in x, t variables. As functions of t, $x_i = \dot{x}_i t$ and so clearly, $d\theta/dt = l$. The initial conditions on $y_i = \dot{x}_i$ are

$$y_i(t_0) = \frac{\partial\theta}{\partial x_i} = \frac{x_i}{t_0}.$$

If we assume these apply over all $\mathbf{R}^n$ and that $t_0 > 0$, then the initial conditions simply assign to each point a tangent vector in the radial direction of the magnitude required to reach that point from the origin in time t_0.

Similar comments apply to final conditions of the form $\theta(x(t_f), t_f)$ and the comments made at the beginning of the previous section apply to the existence of extremals satisfying both initial and final conditions specified by $\theta(x(t), t)$.

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