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Determination of a time-dependent thermal diffusivity and free boundary in heat conduction

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Abstract
In this paper, we consider the inverse problem of simultaneous determination of time-dependent leading coefficient (thermal diffusivity) and free boundary in the one-dimensional time-dependent heat equation. The resulting inverse problem is recast as a nonlinear regularized least-squares problem. Stable and accurate numerical results are presented and discussed.

Keywords: Thermal diffusivity; Free boundary; Inverse problem; Heat equation.

1 Introduction

Many heat transfer applications can be modeled by the heat equation with a fixed boundary. However, there are numerous other problems for which the domain or the boundary varies with time and such problems are known as free boundary or Stefan problems [1]. For instance, when a conductor melts and the liquid is drained away as it appears, the heat conduction problem within the remaining solid involves the heat equation in a domain that is physically changing with time. In particular, the one-phase Stefan problem can be regarded as an inverse problem.

In [2], the author investigated the heat equation with an unknown heat source in a domain with a known moving boundary. In [3, 4], the authors investigated the numerical solution of inverse Stefan problems using the method of fundamental solutions. In [5], an inverse moving boundary problem is solved using the least-squares method. In our work we consider the time-dependent nonlinear inverse one-dimensional and one-phase Stefan problem which consists in the simultaneous determination of the time-dependent thermal diffusivity and free boundary.

This paper is organized as follows: In the next section, we give the formulation of the inverse problem under investigation. The numerical methods for solving the direct and inverse problems are described in Sections 3 and 4, respectively. Furthermore, the numerical results and discussion are given in Section 5 and finally, conclusions are presented in Section 6.

2 Mathematical formulation

Consider the one-dimensional time-dependent heat equation

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \qquad (x,t) \in \Omega \qquad (1)$$

in the domain $\Omega = \{(x,t) : 0 < x < h(t), 0 < t < T < \infty\}$ with unknown free smooth boundary $x = h(t) > 0$ and time-dependent thermal diffusivity $a(t) > 0$. The initial condition is

$$u(x,0) = \phi(x), \qquad 0 \leq x \leq h(0) =: h_0, \qquad (2)$$

where $h_0 > 0$ is given, and the boundary and over-determination conditions are

$$u(0,t) = \mu_1(t), \qquad u(h(t),t) = \mu_2(t), \qquad 0 \leq t \leq T, \qquad (3)$$

$$-a(t)u_x(0,t) = \mu_3(t), \qquad \int_0^{h(t)} u(x,t)dx = \mu_4(t), \qquad 0 \leq t \leq T. \qquad (4)$$

Note that μ_1 and μ_3 represent Cauchy data at the boundary end $x = 0$, whilst μ_4 represent the specification of the energy of the heat conducting system, [6].

First we perform the change of variable $y = x/h(t)$ to reduce the problem (1)–(4) to the following inverse problem for the unknowns $a(t)$, $h(t)$ and $v(y,t) := u(yh(t), t)$:

$$\frac{\partial v}{\partial t}(y,t) = \frac{a(t)}{h^2(t)}\frac{\partial^2 v}{\partial y^2}(y,t) + \frac{yh'(t)}{h(t)}\frac{\partial v}{\partial y}(y,t) + f(yh(t),t), \qquad (y,t) \in Q \tag{5}$$

in the fixed domain $Q = \{(y,t) : 0 < y < 1, 0 < t < T\}$ with unknown time-dependent coefficients $a(t)$ and $h(t)$. The initial condition is

$$v(y,0) = \phi(h_0 y), \qquad 0 \le y \le 1, \tag{6}$$

and the boundary and over-determination conditions are

$$v(0,t) = \mu_1(t), \qquad v(1,t) = \mu_2(t), \qquad 0 \le t \le T, \tag{7}$$

$$-a(t)v_y(0,t) = \mu_3(t)h(t), \qquad h(t)\int_0^1 v(y,t)dy = \mu_4(t), \qquad 0 \le t \le T. \tag{8}$$

This model has been considered in [7]. The triplet $(h(t), a(t), v(y,t))$ is called a solution to the inverse problem (5)–(8) if it belongs to the class $C^1[0,T] \times C[0,T] \times C^{2,1}(\overline{Q})$, $h(t) > 0$, $a(t) > 0$, $t \in [0,T]$, and satisfies the equations (5)–(8). For the input data we make the following regularity and compatibility assumptions:

(A) $\mu_i(t) \in C^1[0,T]$, $\mu_i(t) > 0$ for $t \in [0,T]$, $i = 1,2,4$, $\mu_3(t) \in C^1[0,T]$, $\mu_3(t) < 0$ for $t \in [0,T]$, $\phi(x) \in C^2[0,h_0]$, $\phi(x) > 0$, $\phi'(x) > 0$ for $x \in [0,h_0]$, and $f(x,t) \in C^{1,0}([0,H_1] \times [0,T])$, $f(x,t) \ge 0$ for $(x,t) \in [0,H_1] \times [0,T]$, where

$$H_1 = \max_{[0,T]} \mu_4(t) \left(\min\left\{ \min_{[0,h_0]} \phi(x), \min_{[0,T]} \mu_1(t), \min_{[0,T]} \mu_2(t) \right\} \right)^{-1};$$

(B) $\phi(0) = \mu_1(0)$, $\phi(h_0) = \mu_2(0)$, and $\int_0^{h_0} \phi(x)dx = \mu_4(0)$.

The following existence and uniqueness of solution theorems are proved in [7].

Theorem 1. (Local existence)
If the conditions (A) and (B) are satisfied, then there exists $t_0 \in [0,T]$, (defined by the input data) such that a solution of problem (5)–(8) exists locally for $(y,t) \in [0,1] \times [0,t_0]$.

Theorem 2. (Uniqueness)
Suppose that the following conditions are satisfied:

(i) $0 \le f(x,t) \in C^{1,0}([0,H_1] \times [0,T])$;

(ii) $\phi(x) > 0$ *for* $x \in [0,h_0]$, $\mu_1(t) > 0$, $\mu_2(t) > 0$, $\mu_3(t) < 0$, *and* $\mu_4(t) > 0$ *for* $t \in [0,T]$.

Then a solution to problem (5)–(8) is unique.

3 Solution of Direct Problem

In this section, we consider the direct initial boundary value problem (5)–(7), where $a(t)$, $h(t)$, $f(x,t)$, $\phi(x)$, and $\mu_i(t)$, $i=1,2$, are known and the solution $u(x,t)$ is to be determined additionally with $\mu_i(t)$, $i=3,4$. To achieve this, we use the Crank-Nicolson finite-difference scheme [8], which is unconditionally stable and second-order accurate in space and time.

The discrete form of our problem is as follows. We divide the domain $Q=(0,1)\times(0,T)$ into M and N subintervals of equal step length Δy and Δt, where $\Delta y = 1/M$ and $\Delta t = T/N$, respectively. So, the solution at the node (i,j) is $v_{i,j} := v(y_i,t_j)$, where $y_i = i\Delta y$, $t_j = j\Delta t$, and $a(t_j)=a_j$, $h(t_j)=h_j$ and $f(y_i,t_j)=f_{i,j}$ for $i=\overline{0,M}$, $j=\overline{0,N}$. Based on the Crank-Nicolson method, equation (5) can be approximated as:

$$\begin{aligned}-&A_{i,j+1}v_{i+1,j+1}+(1+B_{j+1})v_{i,j+1}-C_{i,j+1}v_{i-1,j+1}\\&=A_{i,j}v_{i+1,j}+(1-B_j)v_{i,j}+C_{i,j}v_{i-1,j}+\frac{\Delta t}{2}\left(f_{i,j}+f_{i,j+1}\right)\end{aligned} \tag{9}$$

for $i=\overline{1,(M-1)}$, $j=\overline{0,N}$, where

$$A_{i,j}=\frac{(\Delta t)\alpha_j}{2(\Delta y)^2}-\frac{(\Delta t)\gamma_j y_i}{4\Delta y},\quad B_j=\frac{(\Delta t)\alpha_j}{(\Delta y)^2},\quad C_j=\frac{(\Delta t)\alpha_j}{2(\Delta y)^2}+\frac{(\Delta t)\gamma_j y_i}{4\Delta y},$$
$$\alpha_j=\frac{a_j}{h_j^2},\quad \gamma_j=\frac{h'(t_j)}{h_j}.$$

The initial and boundary conditions (6) and (7) can also be collocated as:

$$v_{i,0}=\phi(h_0 y_i),\quad i=\overline{0,M}, \tag{10}$$
$$v_{0,j}=\mu_1(t_j),\quad v_{M,j}=\mu_2(t_j),\quad j=\overline{0,N}. \tag{11}$$

At each time step t_j, for $j=\overline{0,(N-1)}$, using the Dirichlet boundary conditions (11), the above difference equation (9) can be reformulated as a $(M-1)\times(M-1)$ system of linear equations of the form,

$$L\mathbf{u}=\mathbf{b}, \tag{12}$$

where

$$\mathbf{u}=(v_{1,j+1},v_{2,j+1},...,v_{M-1,j+1})^{tr},\quad \mathbf{b}=(b_1,b_2,...,b_{M-1})^{tr}$$

and

$$L=\begin{pmatrix}1+B_{j+1} & -C_{1,j+1} & 0 & \cdots & 0 & 0 & 0\\ -A_{2,j+1} & 1+B_{j+1} & -C_{2,j+1} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -A_{M-2,j+1} & 1+B_{j+1} & -C_{M-2,j+1}\\ 0 & 0 & 0 & \cdots & 0 & -A_{M-1,j+1} & 1+B_{j+1}\end{pmatrix},$$

$$\begin{aligned}b_1&=A_{1,j}v_{0,j}+(1-B_j)v_{1,j}+C_{1,j}v_{2,j}+A_{1,j+1}v_{0,j+1}+\frac{\Delta t}{2}(f_{1,j+1}+f_{1,j}),\\ b_i&=A_{i,j}v_{i-1,j}+(1-B_j)v_{i,j}+C_{i,j}v_{i+1,j}+\frac{\Delta t}{2}(f_{i,j+1}+f_{i,j}),\qquad i=\overline{2,(M-2)},\\ b_{M-1}&=A_{M-1,j}v_{M-2,j}+(1-B_j)v_{M-1,j}+C_{M-1,j}v_{M,j}+C_{M-1,j+1}v_{M,j+1}\\&\quad+\frac{\Delta t}{2}(f_{M-1,j+1}+f_{M-1,j}).\end{aligned}$$

As an example, consider the problem (5)–(7) with $T = \ell = 1$ and

$$a(t) = 1 + t, \quad h(t) = 1 + 2t, \quad h_0 = h(0) = 1, \quad \phi(h_0 y) = (1 + y)^2, \quad \mu_1(t) = 1 + 8t,$$
$$\mu_2(t) = (2 + 2t)^2 + 8t, \quad f(h(t)y, t) = 6 - 2t.$$

The exact solution of the direct problem (5)–(7) is given by $v(y, t) = (1+y+2yt)^2+8t$, and the desired outputs are $\mu_3(t) = -2(1+t)$ and $\mu_4(t) = \frac{(2+2t)^3-1}{3}+8t(1+2t)$. The numerical and exact solutions for $v(y, t)$ are shown in Figure 1 and very good agreement is obtained. Tables 1 and 2 give the numerical heat flux at $y = 0$ and the numerical integral in comparison with the exact values, i.e. μ_3 and μ_4. These have been calculated using the following $O(h^2)$ finite-difference approximations for derivative and trapezoidal rule for integration:

$$v_y(0, t_j) = \frac{4v_{1,j} - v_{2,j} - 3v_{0,j}}{2\Delta y}, \qquad j = \overline{1, N}, \tag{13}$$

$$\int_0^1 v(y, t_j)dy = \frac{\Delta y}{2} \left(v(0, t_j) + v(1, t_j) + 2 \sum_{i=1}^{M-1} v(y_i, t_j) \right), \qquad j = \overline{0, N}. \tag{14}$$

From these tables it can be seen that the numerical results are in very good agreement with the exact ones and that a rapid monotonic decreasing convergence is achieved.

Table 1: The exact and the numerical heat flux $-a(t)v_y(0, t)/h(t)$ for $M = N \in \{10, 20\}$, for the direct problem.

t	0.1	0.2	...	0.8	0.9	1
$M = N = 10$	-2.2000	-2.4000	...	-3.6000	-3.8000	-4.0000
$M = N = 20$	-2.2000	-2.4000	...	-3.6000	-3.8000	-4.0000
exact	-2.2000	-2.4000	...	-3.6000	-3.8000	-4.0000

Table 2: The exact and the numerical integral $h(t) \int_0^1 v(y, t)dy$ for $M = N \in \{10, 20, 40, 100\}$, for the direct problem.

t	0.1	0.2	...	0.8	0.9	1
$M = N = 10$	4.1789	6.5192	...	31.8880	38.1539	45.0450
$M = N = 20$	4.1767	6.5158	...	31.8660	38.1265	45.0113
$M = N = 40$	4.1762	6.5150	...	31.8605	38.1196	45.0028
$M = N = 100$	4.1760	6.5147	...	31.8590	38.1177	45.0005
exact	4.1760	6.5147	...	31.8587	38.1173	45.0000

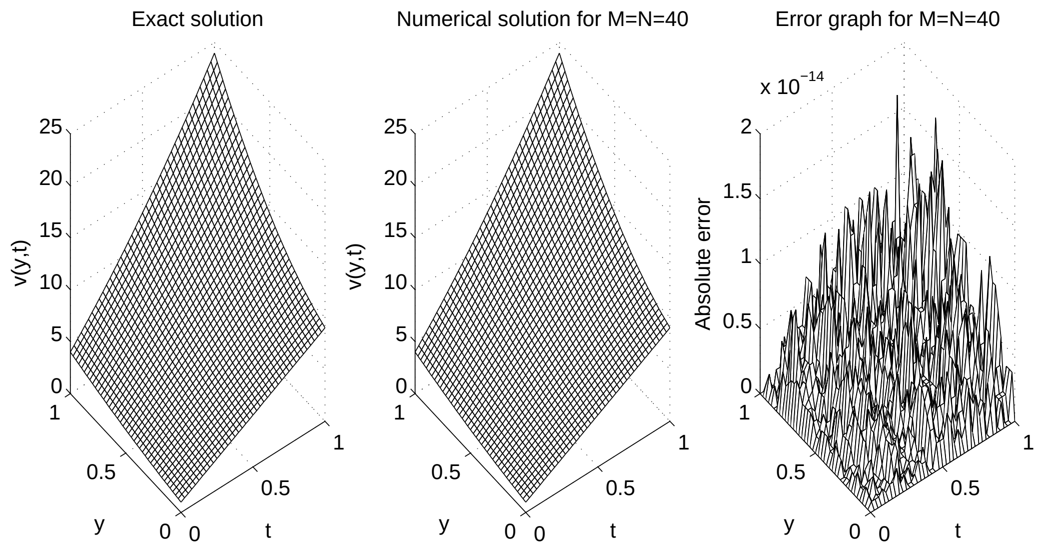


Figure 1: Exact and numerical solutions for $v(y,t)$ and the absolute error for the direct problem obtained with $M = N = 40$.

4 Numerical Approach for the Inverse Problem

In the inverse problem, we assume that the thermal diffusivity $a(t)$ and free boundary $h(t)$ are unknown. Usually, the nonlinear inverse problem (5)–(8) can be formulated as a nonlinear least-squares minimization. The regularized objective function which is minimized is given by

$$F(a,h) = \left\| -\frac{a(t)}{h(t)} v_y(0,t) - \mu_3(t) \right\|^2 + \left\| h(t) \int_0^1 v(y,t)dy - \mu_4(t) \right\|^2 + \beta \left(\|a(t)\|^2 + \|h(t)\|^2 \right), \tag{15}$$

where $\beta \geq 0$ is a regularization parameter and the norm is usually the $L^2[0,T]$-norm. The discretization of (15) is

$$F(\underline{a}, \underline{h}) = \sum_{j=0}^{N} \Big[-\frac{a_j}{h_j} v_y(0,t_j) - \mu_3(t_j) \Big]^2 + \sum_{j=0}^{N} \Big[h_j \int_0^1 v(y,t_j)dy - \mu_4(t_j) \Big]^2 + \beta \left(\sum_{j=0}^{N} a_j^2 + \sum_{j=1}^{N} h_j^2 \right). \tag{16}$$

The unregularized case $\beta = 0$ yields the ordinary nonlinear least-squares method which is usually unstable. The minimization of F subject to the physical constraints $\underline{a} > \underline{0}$ and $\underline{h} > \underline{0}$ is accomplished using the MATLAB toolbox routine *lsqnonlin*, which does not require supplying (by the user) the gradient of the objective function, [9]. The routine *lsqnonlin* attempts to find a minimum of a scalar function of several variables, starting from an initial guess, subject to constraints and this generally is referred to as a constrained nonlinear optimization.

We take bounds for the positive quantities $a(t)$ and $h(t)$ say, we seek them in the interval (10^{-10},10^3). We also take the parameters of the routine as follows:

- Number of variables $M = N = 40$.

- Maximum number of iterations = $10^2 \times$ (number of variables).
- Maximum number of objective function evaluations = $10^3 \times$ (number of variables).
- x Tolerance (xTol) = 10^{-10}.
- Function Tolerance (FunTol) = 10^{-10}.
- Nonlinear constraint tolerance = 10^{-6}.

We take the initial guess as $\underline{a}^{(0)} = \underline{h}^{(0)} = \underline{1}$. It is worth mentioning that at the first time step, i.e. $j = 0$, the derivative $v_y(0,0)$ is obtained from (10) and (13), as

$$v_y(0,0) = \frac{4\phi_1 - \phi_2 - 3\phi_0}{2\Delta y}, \tag{17}$$

where $\phi_i = \phi(h_0 y_i)$ for $i = \overline{0, M}$. In addition, when we solve the inverse problem we approximate

$$h'(t_j) = \frac{h(t_j) - h(t_{j-1})}{\Delta t} = \frac{h_j - h_{j-1}}{\Delta t}, \quad j = \overline{1, N}. \tag{18}$$

We also express $h'(0)$ as

$$h'(0) = \frac{\mu_2'(0) - a(0)\phi''(h_0) - f(h_0, 0)}{\phi'(h_0)}, \tag{19}$$

which can easily be derived from equation (3) using the chain rule technique. In (19), $a(0)$ is unknown.

If there is noise in the measured data (8), we replace $\mu_3(t_j)$ and $\mu_4(t_j)$ in (16) by $\mu_3^{\epsilon 1}(t_j)$ and $\mu_4^{\epsilon 2}(t_j)$, namely,

$$\mu_3^{\epsilon 1}(t_j) = \mu_3(t_j) + \epsilon 1_j, \quad \mu_4^{\epsilon 2}(t_j) = \mu_4(t_j) + \epsilon 2_j, \quad j = \overline{0, N}, \tag{20}$$

where $\epsilon 1_j$ and $\epsilon 2_j$ are random variables generated from a Gaussian normal distribution with mean zero and standard deviations σ_1 and σ_2, respectively, given by

$$\sigma_1 = p \times \max_{t \in [0,T]} |\mu_3(t)|, \quad \sigma_2 = p \times \max_{t \in [0,T]} |\mu_4(t)|, \tag{21}$$

where p represents the percentage of noise. We use the MATLAB function *normrnd* to generate the random variables $\underline{\epsilon 1}$ and $\underline{\epsilon 2}$ as follows:

$$\underline{\epsilon 1} = normrnd(0, \sigma_1, N+1), \quad \underline{\epsilon 2} = normrnd(0, \sigma_2, N+1). \tag{22}$$

5 Numerical Results and Discussion

The numerical results are illustrated for two different examples according to the linear or nonlinear variation of estimated coefficients. In addition, we add noise, as in (20), to the measured input data (8). To compute the thermal diffusivity $a(t)$ and the free boundary $h(t)$ we use the *lsqnonlin* routine from MATLAB optimization toolbox with the Trust-Region-Reflective algorithm, [9], to find the minimizer of the nonlinear Tikhonov regularization functional (16). We have also calculated the root mean square error (*rmse*) to analyse the error between the exact and estimated solution, defined as,

$$rmse(a(t)) = \sqrt{\frac{1}{N+1} \sum_{j=0}^{N} \left(a_{numerical}(t_j) - a_{exact}(t_j)\right)^2}, \tag{23}$$

$$rmse(h(t)) = \sqrt{\frac{1}{N} \sum_{j=1}^{N} \left(h_{numerical}(t_j) - h_{exact}(t_j)\right)^2}. \tag{24}$$

For simplicity, we take $T = 1$.

Example 1

Consider the problem (1)–(4) with unknown coefficients $a(t)$ and $h(t)$, and solve this inverse problem with the following input data:

$$\mu_1(t) = 1 + 8t, \quad \mu_2(t) = (2+2t)^2 + 8t, \quad \mu_3(t) = -2(1+t),$$
$$\mu_4(t) = \frac{(2+2t)^3 - 1}{3} + 8t(1+2t), \quad h_0 = 1, \quad \phi(x) = (1+x)^2, \quad f(x,t) = 6 - 2t.$$

One can remark that the conditions of Theorem 2 are satisfied hence, the uniqueness of solution holds. With this data the analytical solution is given by

$$a(t) = 1 + t, \quad h(t) = 1 + 2t, \quad u(x,t) = (1+x)^2 + 8t. \tag{25}$$

Then

$$a(t) = 1 + t, \quad h(t) = 1 + 2t, \quad v(y,t) = u(yh(t),t) = (1 + y(1+2t))^2 + 8t, \tag{26}$$

is the analytical solution of the problem (5)–(8).

Consider first the case where there is no noise in the input data (8). The objective function (16), as a function of the number of iterations, is represented in Figure 2. From this figure it can be seen that the convergence is rapidly achieved in a few iterations. The objective function (16) decreases rapidly and takes a stationary value of $O(10^{-8})$ in about 7 iterations. The numerical results for the corresponding unknowns $a(t)$ and $h(t)$ are presented in Figure 3. From this figure it can be seen that the retrieved thermal diffusivity $a(t)$ and free surface $h(t)$ are in very good agreement with the exact values from (26).

Next, we add $p = 2\%$ noise to the measured data μ_3 and μ_4, as in equation (20). The regularized objective function (16) is plotted, as a function of the number of iterations, in Figure 4 and convergence is again rapidly achieved. Figure 5 presents the graphs of the recovered functions, whilst the *rmse* values are given in Table 3. From this figure and table it can be seen that there is not much difference between the numerical solution obtained with $\beta = 0$ or $\beta = 10^{-3}$, but there is some slight improvement in accuracy obtained for $\beta = 10^{-1}$.

The recovered temperatures for $\beta \in \{0, 10^{-3}, 10^{-1}\}$ are shown in Figure 6. From this figure it can be seen that the temperature component of the solution is stable and is not significantly affected by the inclusion of noise in the input data.

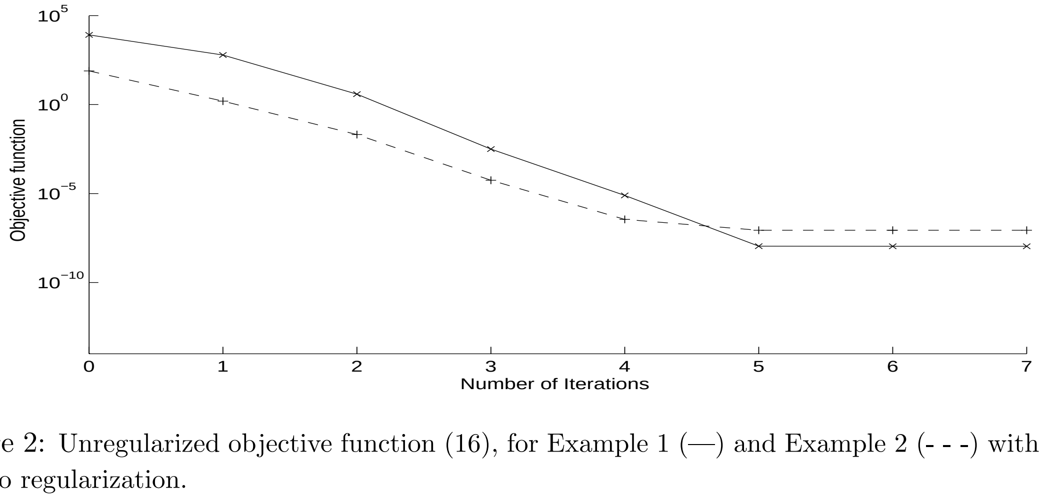
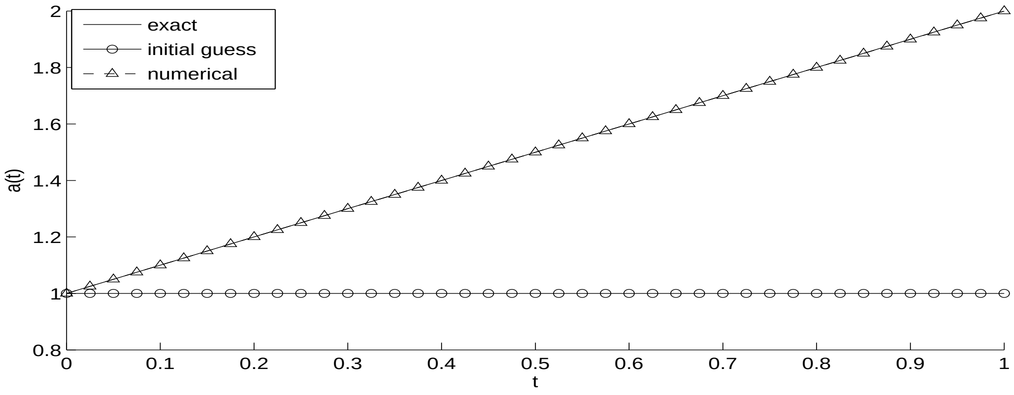
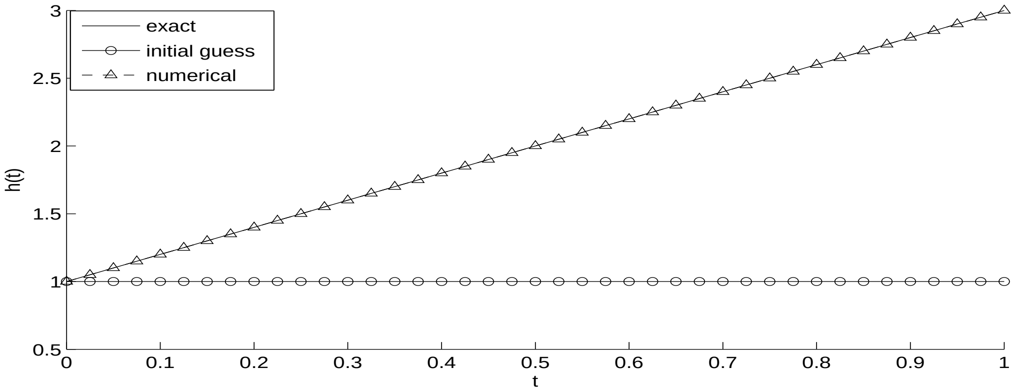


Figure 2: Unregularized objective function (16), for Example 1 (—) and Example 2 (- - -) with no noise and no regularization.



(a)



(b)

Figure 3: (a) Thermal diffusivity $a(t)$, and (b) Free surface $h(t)$, for Example 1 with no noise and no regularization.

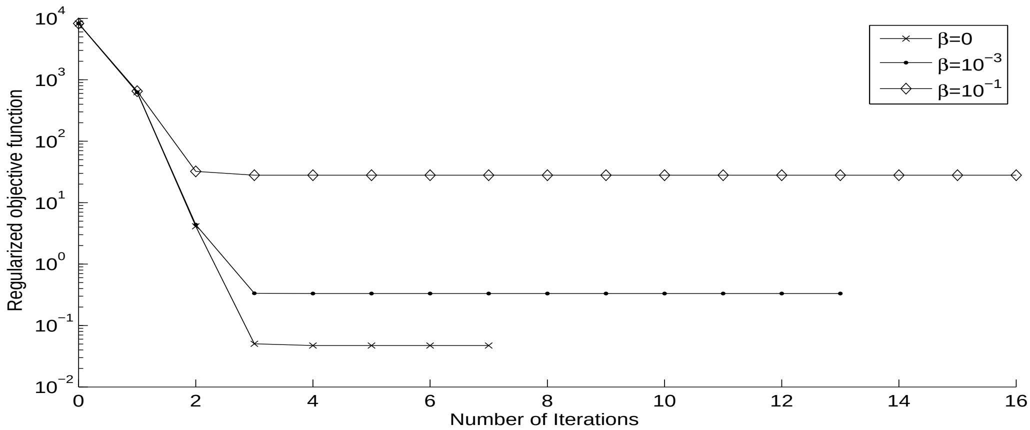
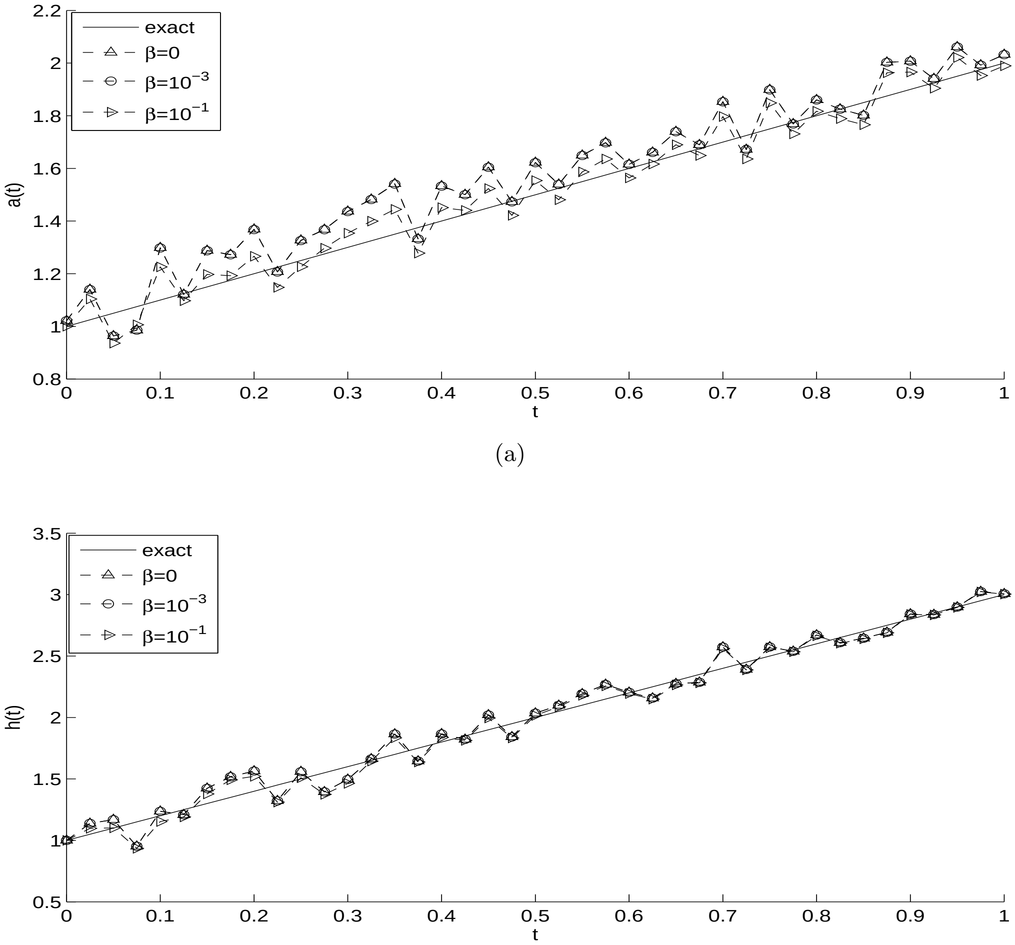


Figure 4: Regularized objective function (16), for Example 1 with $p = 2\%$ noise.



(a)

(b)

Figure 5: (a) Thermal diffusivity $a(t)$, and (b) Free surface $h(t)$, for Example 1 with $p = 2\%$ noise and regularization.

Example 2

In this example we consider the inverse problem (5)–(8) with the following input data:

$$\mu_1(t) = 1 + 8t, \quad \mu_2(t) = (1 + \sqrt{2-t})^2 + 8t, \quad \mu_3(t) = -2\sqrt{1+t},$$
$$\mu_4(t) = \frac{(1 + \sqrt{2-t})^3 - 1}{3} + 8t\sqrt{2-t}, \quad h_0 = \sqrt{2}, \quad \phi(x) = (1 + \sqrt{2}x)^2,$$
$$f(x,t) = 8 - 2\sqrt{1+t}.$$

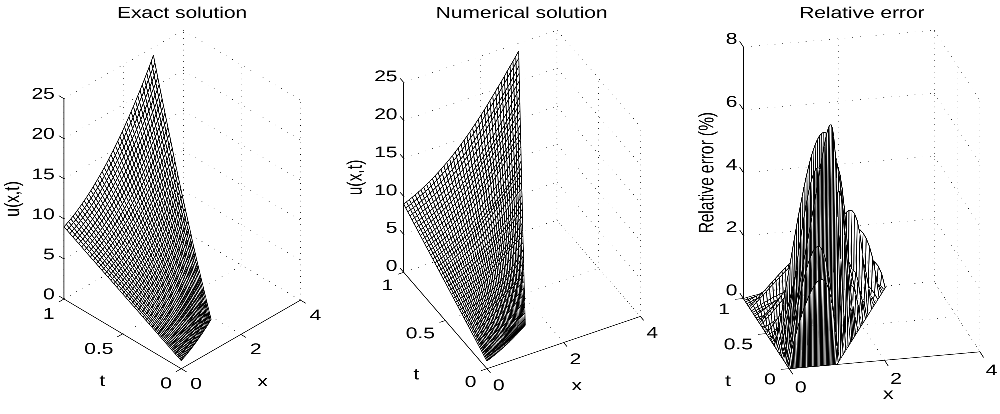
One can remark that the conditions of Theorem 2 are satisfied hence, the uniqueness of solution holds. The solution to this inverse problem is given by

$$a(t) = \sqrt{1+t}, \quad h(t) = \sqrt{2-t}, \quad u(x,t) = (1+x)^2 + 8t. \tag{27}$$

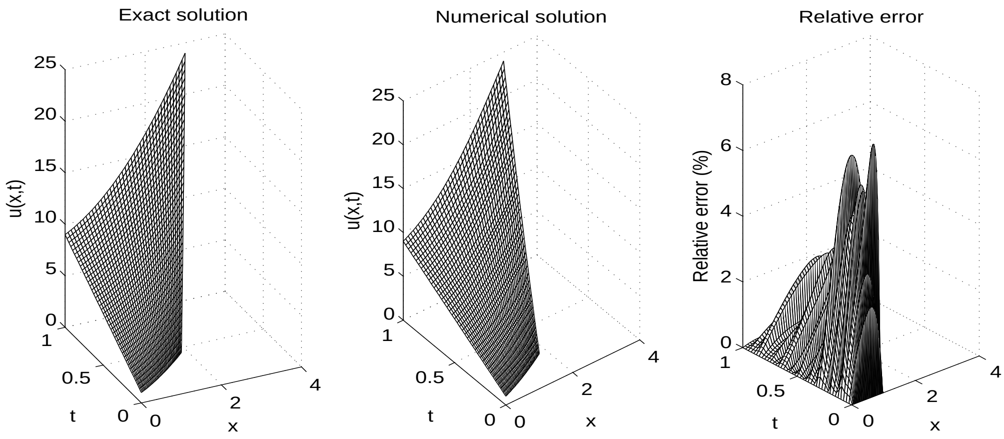
Then

$$a(t) = \sqrt{1+t}, \quad h(t) = \sqrt{2-t}, \quad v(y,t) = u(yh(t),t) = (1 + y\sqrt{2-t})^2 + 8t, \tag{28}$$

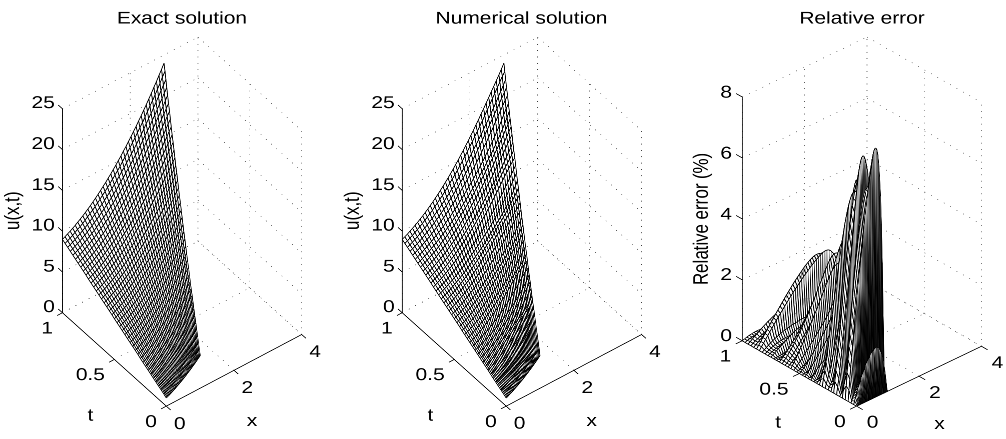
is the analytical solution of the problem (5)–(8). In this example the moving boundary is given by a nonlinear function.



(a)



(b)



(c)

Figure 6: (a) Temperature for $\beta = 0$, (b) $\beta = 10^{-3}$, and (c) $\beta = 10^{-1}$, for Example 1 with $p = 2\%$ noise.

Initially, we consider the case of noise free in the input data (8). The objective function (16), as a function of the number of iterations, is presented in Figure 2. From this figure it can be seen that the convergence is rapidly achieved in a few iterations. The objective function (16) decreases dramatically and takes a stationary value of $O(10^{-7})$ in about 7 iterations, the same as in Example 1. The numerical results for the corresponding coefficients $a(t)$ and $h(t)$ are presented in Figure 7. From this figure it can be seen that the identified coefficients are in very good agreement with the exact values from (28).

Next, we add $p = 2\%$ noise to the measured data μ_3 and μ_4, as in equation (20). The regularized objective function (16) is plotted, as a function of the number of iterations, in Figure 8 and convergence is again rapidly achieved. Figures 9 and 10 show the numerical solution $(a(t), h(t), u(x,t))$ and the $rmse$ values are given in Table 3. As in Example 1, one can observe that the inverse problem is rather stable with respect to noise included in the input data.

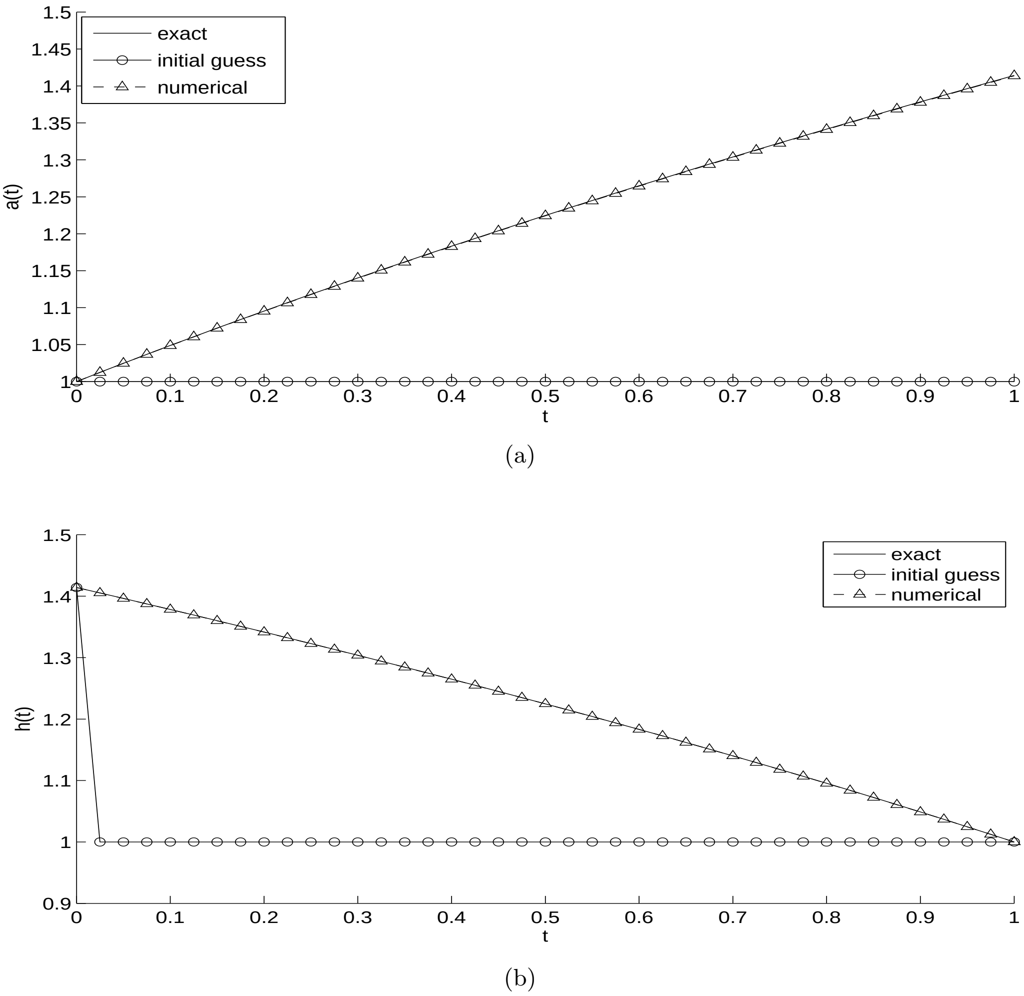


Figure 7: (a) Thermal diffusivity $a(t)$, and (b) Free surface $h(t)$, for Example 2 with no noise and no regularization.

Table 3: The $rmse$ values for Examples 1 and 2 with $p = 2\%$ noise.

	$\beta = 0$	$\beta = 10^{-3}$	$\beta = 10^{-1}$
Example 1	$rmse(a) = 0.1010$	0.1004	0.0628
	$rmse(h) = 0.0932$	0.0922	0.0872
Example 2	$rrmse(a) = 0.0336$	0.0336	0.0368
	$rrmse(h) = 0.0253$	0.0253	0.0248

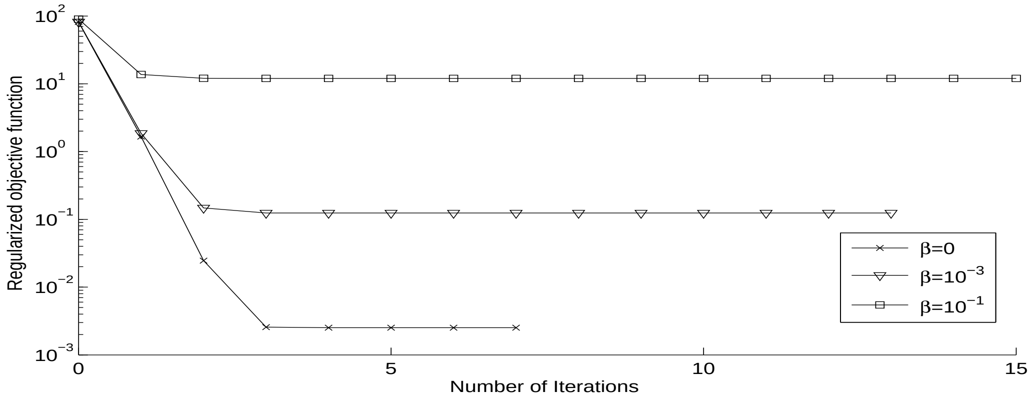
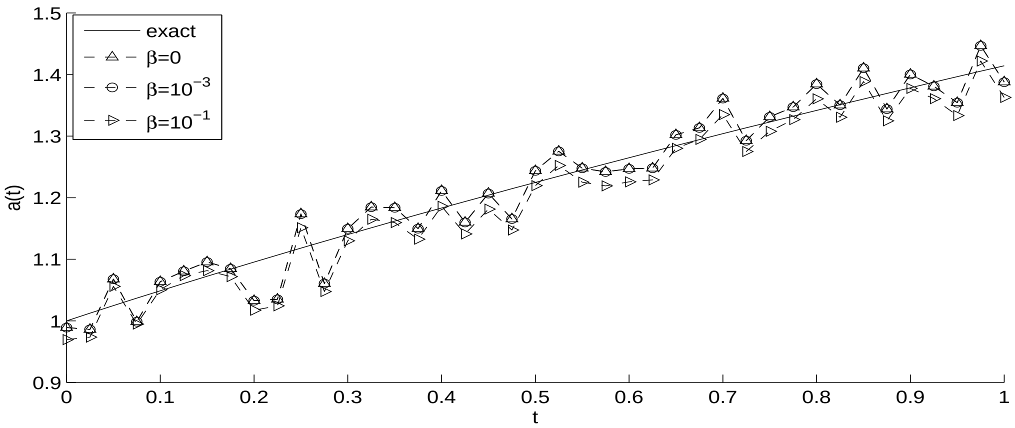
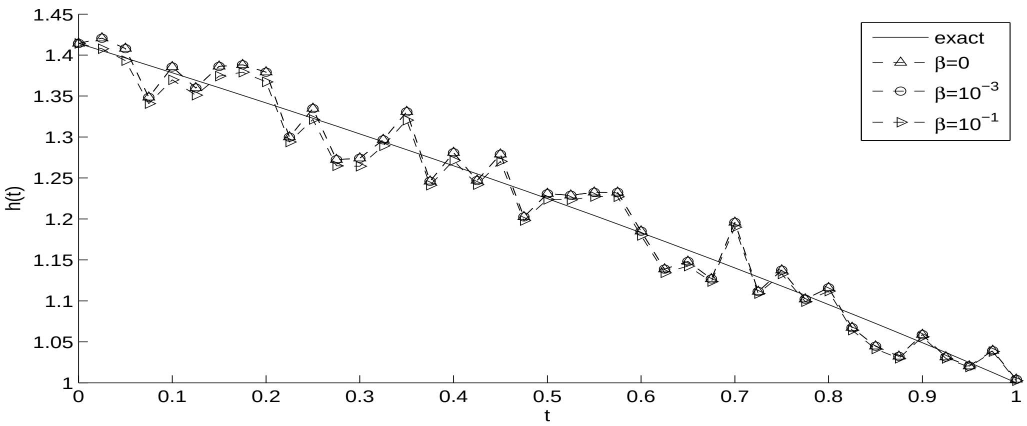


Figure 8: Regularized objective function (16), for Example 2 with $p = 2\%$ noise.



(a)



(b)

Figure 9: (a) Thermal diffusivity $a(t)$, and (b) Free surface $h(t)$, for Example 2 with $p = 2\%$ noise and regularization.

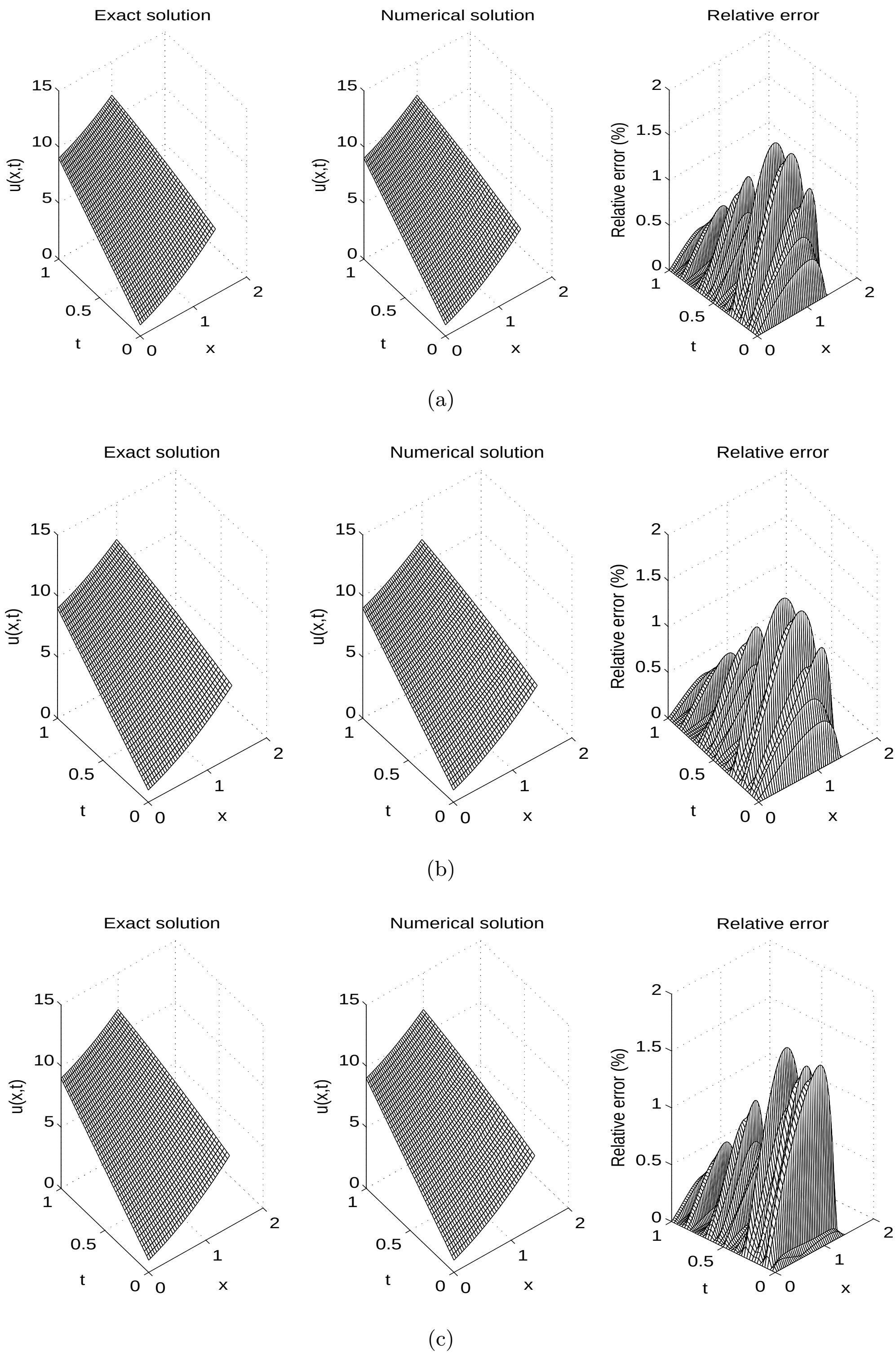


Figure 10: (a) Temperature for $\beta = 0$, (b) $\beta = 10^{-3}$, and (c) $\beta = 10^{-1}$, for Example 2 with $p = 2\%$ noise.

6 Conclusion

The inverse nonlinear problem which requires simultaneously determining the time-dependent thermal diffusivity and free boundary in the parabolic heat equation has been investigated.

The resulting inverse problem has been reformulated as a nonlinear least-squares optimization problem which produced stable and reasonably accurate numerical results. Extension of the present work to include the determination of unknown convection $b(t)u_x$ and reaction $c(t)u$ coefficients in the heat equation (1), in addition to the unknowns $a(t)$ and $h(t)$, [10], will be the subject of future work.

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References

[1] Cannon, J.R. (1984) *The One-dimensional Heat Equation*, Addison-Wesley, Menlo Park, California.

[2] Malyshev, I.G. (1975) Inverse problems for the heat-conduction equation in a domain with a moving boundary, *Ukrainian Mathematical Journal*, **27**, 568–572.

[3] Hon, Y.C. and Li, M. (2008) A computational method for inverse free boundary determination problem, *International Journal for Numerical Methods in Engineering*, **73**, 1291–1309.

[4] Johansson, B.T., Lesnic, D. and Reeve, T. (2011) A method of fundamental solutions for the one-dimensional inverse Stefan problem, *Applied Mathematical Modelling*, **35**, 4367-4378.

[5] Shidfar, A. and Karamali, G.R. (2005) Numerical solution of inverse heat conduction problem with nonstationary measurements, *Applied Mathematics and Computation*, **168**, 540–548.

[6] Lesnic, D., Elliott, L. and Ingham, D.B. (1998) The solution of an inverse heat conduction problem subject to the specification of energies, *International Journal of Heat and Mass Transfer*, **74**, 25–32.

[7] Ivanchov, M.I. (2003) Inverse problem with free boundary for heat equation, *Ukrainian Mathematical Journal*, **55**, 1086–1098.

[8] Smith, G.D. (1985) *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, Oxford Applied Mathematics and Computing Science Series, Third edition.

[9] Mathwoks R2012 Documentation Optimization Toolbox-Least Squares (Model Fitting) Algorithms, available from www.mathworks.com/help/toolbox/optim/ug /brnoybu.html.

[10] Snitko, H.A. (2012) Inverse problem for determination of time-dependent coefficients of a parabolic equation in a free-boundary domain, *Journal of Mathematical Science*, **181**, 350–365.