A LYAPANOV FUNCTION METHOD TO ESTIMATE THE STABILITY REGION OF NONLINEAR SYSTEMS USING THE LIE SERIES

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Abstract. A method to estimate the region of asymptotic stability, about an equilibrium point, of an autonomous nonlinear system is presented. The Lie series is used to approximate the solution trajectories of a general point in the stability region. The method is applied to several well-known examples in two and three dimensions.

1. Introduction. There exists no definitive method to find the region of asymptotic stability (RAS) about the equilibrium point of a general nonlinear system. Many existing methods demand considerable computational capacity or rely on the application of a complicated prescriptive procedure which is different for every system.

The approach of finding a Lyapanov function is most widely used to estimate the RAS. Typically the RAS can be estimated by the set contained within certain level hyper-surfaces of the Lyapanov function. Lyapanov methods can be divided into two main approaches: those resulting from the work of Zubov [11], and LaSalle [12].

Zubov [11] gives necessary and sufficient conditions for a certain region to be the RAS of an equilibrium point. The determination of the requisite 'optimal' Lyapanov function involves the solution of a set of nonlinear partial differential equations (pde's). Many iterative methods to solve these pde's have been proposed, for instance [3] uses the Lie series. The deficiencies of these methods are the nonuniformity of convergence of the procedures involved and the arbitrary nature of the choice of a certain function used in the method. Using an approach similar to that of Zubov, Vannelli and Vidyasagar [6] introduce the concept of a maximal Lyapanov function by considering rational function candidates. This approach, whilst requiring fewer iterations, still relies on the solution of a constrained minimization which becomes intractable in higher dimensions.

LaSalle's extension of Lyapanov theory [12] gives conditions for a set to be included in the RAS. As in this paper the typical approach falls into two parts: finding a suitable form for V (The Lyapanov function), and searching for the tangency points between V and $\dot{V}$. These methods employ a diverse range of tactics and generally lead to good results through an acceptable amount of computation. In [2] Davidson and Kurak optimize the volume enclosed by a hyper-elipse by conversion to a constrained minimisation problem. The success of this method is crucially dependent on the choice of initial conditions. Shields and Storey [1] consider three 'optimal' methods and investigate the problem of finding tangency points. In [5] Chiang and Thorp improve existing Lyapanov functions by backwards integration along trajectories to create a sequence of monotone increasing (in the sense of inclusion) sets. A similar approach is presented in [7].

[4] and [8] are non-Lyapanov methods. They both use a combination of topological

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considerations and trajectory reversing methods to give very accurate results in two dimensions. However both require the application of a complex sequence of steps which varies with each problem. For example, [8] requires the determination of the stable manifolds of an equilibrium point. Furthermore, both methods become onerous in dimensions greater than two.

2. The Lie Series. In this section we introduce the concept of the Lie series and derive bounds on its radius of convergence for polynomial systems. We use the following notation and conventions:

An n-tuple of non-negative integers, $(i_1, \ldots, i_n)$, is denoted by a boldface $\mathbf{i}$. Moreover, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} \mathbf{x}^{\mathbf{i}} &= x_1^{i_1} \cdots x_n^{i_n} \\ \sum_{\mathbf{i}=0}^{p} &= \sum_{i_1=0}^{p} \cdots \sum_{i_n=0}^{p}. \end{aligned}$$

Denote by $\mathbf{1}_k$ the n-tuple of zeros with one in the kth position:

$$\mathbf{1}_k = (0, 0, \ldots, \underbrace{1}_{k}, 0, \ldots, 0).$$

Lastly note that when dealing with a number of vector indices, say $\mathbf{i}_1, \ldots, \mathbf{i}_m$, $(\mathbf{i}_l)_j$ denotes the jth component of the vector $\mathbf{i}_l$.

Consider the nonlinear autonomous system

$$\dot{x} = f(x), \;\; x(0) = x_0 = (x_{01}, \cdots, x_{0n}). \tag{1}$$

Where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Provided it exists, the Taylor series of the solution is given by the Lie series (See [13]):

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[L_f^k x\right]_{x=x_0},$$

where

$$L_f^k = \left(f^T \frac{\partial}{\partial x}\right)^k.$$

For polynomial systems

$$f(x) = \left(\sum_{\mathbf{i}=0}^{p} a_{\mathbf{i}}^1 \mathbf{x}^{\mathbf{i}}, \cdots, \sum_{\mathbf{i}=0}^{p} a_{\mathbf{i}}^n \mathbf{x}^{\mathbf{i}}\right)^T,$$

for some $p \geq 1$. The radius of convergence, r, of the Lie series for this system is given by

$$r = 1/\lambda,$$

where

$$\lambda = \limsup_{k\to\infty} \left(\frac{1}{k!} \left| \left[L_f^k x \right]_{x=x_0} \right| \right)^{1/k}.$$

We proceed to derive a bound on λ, and thus obtain a lower bound for r. Note

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\left(\sum_{\alpha=1}^{n} \sum_{\mathbf{i}=0}^{p} a_{\mathbf{i}}^{\alpha} \mathbf{x}^{\mathbf{i}} \frac{\partial}{\partial x} \right)^k x \right]_{x=x_0} \\ &= \sum_{k=0}^{\infty} t^k \sum_{\alpha_1=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \sum_{\mathbf{i}_1=0}^{p} \cdots \sum_{\mathbf{i}_k=0}^{p} c_k a_{\mathbf{i}_k}^{\alpha_k} \cdots a_{\mathbf{i}_1}^{\alpha_1} \left[x^{\mathbf{i}_1+\cdots+\mathbf{i}_k-\mathbf{1}_{\alpha_2}-\cdots-\mathbf{1}_{\alpha_k}} \frac{\partial x}{\partial x_{\alpha_1}} \right]_{x=x_0}, \end{aligned}$$

where

$$c_k = \frac{1}{k!} \left(\mathbf{i}_1 + \cdots + \mathbf{i}_{k-1} - \mathbf{1}_{\alpha_2} - \cdots - \mathbf{1}_{\alpha_{k-1}} \right)_{\alpha_k} \cdots \left(\mathbf{i}_1 + \mathbf{i}_2 - \mathbf{1}_{\alpha_2} \right)_{\alpha_3} \left(\mathbf{i}_1 \right)_{\alpha_2};$$

and c_k and the exponent of x are taken to be zero when any of their components are negative. Using the kth term in this series, we have

$$\begin{aligned} \lambda &\le \lim_{k\to\infty} \left(\sum_{\alpha_2=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \sum_{\mathbf{i}_1=0}^{p} \cdots \sum_{\mathbf{i}_k=0}^{p} c_k a^{nk} \left| x_0^{\mathbf{i}_1+\cdots+\mathbf{i}_k-\mathbf{1}_{\alpha_2}-\cdots-\mathbf{1}_{\alpha_k}} \right| \right)^{1/k} \\ &\le \lim_{k\to\infty} \left(\sum_{\alpha_2=1}^{n} \cdots \sum_{\alpha_k=1}^{n} \sum_{\mathbf{i}_1=0}^{p} \cdots \sum_{\mathbf{i}_k=0}^{p} \frac{p^{k-1} a^{nk}}{k} \left| x_0^{\mathbf{i}_1+\cdots+\mathbf{i}_k-\mathbf{1}_{\alpha_2}-\cdots-\mathbf{1}_{\alpha_k}} \right| \right)^{1/k} \end{aligned}$$

Since

$$\begin{aligned} c_k &\le \frac{1}{k} \frac{\left[(k-1)(p,\cdots,p) - (1,\cdots,1) \right]_{\alpha_k}}{k-1} \cdots\cdots \frac{\left[2(p,\cdots,p) - (1,\cdots,1) \right]_{\alpha_3}}{2} \frac{\left[(p,\cdots,p) \right]_{\alpha_2}}{1} \\ &\le \frac{p^{k-1}}{k}, \end{aligned}$$

and letting

$$a = \max_{\mathbf{i},j} \left| a_{\mathbf{i}}^j \right|.$$

Hence r is bounded below as follows:

$$\begin{aligned} r &\ge \frac{1}{p(p+1)^n a^n n} \text{ if } |x_{0j}| < 1 \text{ for } j = 1 \cdots n, \\ r &\ge \frac{1}{p(p+1)^n a^n n \prod_j |x_{0j}|} \text{ if } |x_{0j}| \ge 1 \text{ for some } j \in 1 \cdots n, \end{aligned}$$

3. Lyapanov Functions. Consider the nonlinear autonomous system

$$\dot{x} = f(x), \; x(0) = x_0. \tag{2}$$

Where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. We assume that f is an infinitely differentiable vector field and so the sufficient condition for the existence and uniqueness of solutions to (2) is satisfied. Suppose zero is a stable equilibrium point of (2). Define the Region of Asymptotic Stability (RAS) of zero to be the set

$$R = \left\{ x_0; \lim_{t \longrightarrow \infty} x(t, x_0) = 0 \right\}. \tag{3}$$

We also assume that each solution, $x(t; x_0)$, varies continuously with x_0, ensuring that R is open and connected.

THEOREM 3.1. *[12] The equilibrium point $x = 0$ of (2) is asymptotically stable if there exists a scalar function, $V(x)$, (a Lyapanov function), with continuous first partial derivatives, and a domain, $\Omega \subseteq \mathbb{R}^n$, containing zero, such that*
1) $V(x) > 0, \forall x \in \Omega, x \neq 0$
2) $V(0) = 0$
3) $\dot{V}(x) = f^T(x) grad V(x) < 0, \forall x \in \Omega, x \neq 0$
4) $\dot{V}(0) = 0.$
Furthermore, if C is a positive constant such that the hypersurface $V(x) = C$ is contained in Ω, then the domain $V(x) < C$ is contained in R.

Once a particular V is chosen the largest value of C satisfying theorem 3.1 will yield the best approximation of R obtainable from that Lyapanov function.

Our approach is based on the theory in [9]. Suppose that the system (2) is asymptotically stable in the region R. Denote by $x(t; x_0)$ the solution at time t through $x(0) = x_0$. Then we have the following result.

LEMMA 3.2. *If the solutions of (2) are asymptotically stable in R and satisfy*

$$\|x(t; x_0)\| = O\left(\frac{1}{t^{\frac{1}{p}}}\right) \text{ for all } x \in R \text{ as } t \rightarrow \infty, \tag{4}$$

for some integer $p > 0$, then

$$V(x_0) = \int_0^\infty \|x(t; x_0)\|^{2q} \, dt, \quad x_0 \in R \tag{5}$$

is a Lyapanov function for (2) in R for any $q \geq p$.

Proof.

By condition (4) the integral certainly exists and

$$\begin{aligned} V(x_0) &> 0, \; x_0 \neq 0 \\ V(0) &= 0. \end{aligned}$$

Also, if $x_1 = x(t_1; x_0), t_1 > 0$, then, by the group property of solutions,

$$
\begin{aligned}
V(x_0) &= \int_0^\infty \|x(t;x_0)\|^{2q}\, dt \\
&= \int_0^{t_1} \|x(t;x_0)\|^{2q}\, dt + \int_{t_1}^\infty \|x(t;x_0)\|^{2q}\, dt \\
&> \int_{t_1}^\infty \|x(t;x_0)\|^{2q}\, dt \\
&= \int_0^\infty \|x(t+t_1;x_0)\|^{2q}\, dt \\
&= \int_0^\infty \|x(t;x(t_1;x_0))\|^{2q}\, dt \\
&= V(x_1).
\end{aligned}
$$

So V decreases along trajectories.□

We propose using the Lie series to estimate the solution, $x(t; x_0)$, in (5) over a finite time interval, T. Our Lyapanov function candidate is

$$
V(x_0) = \int_0^T \left\| \sum_{i=0}^m \frac{t^i}{i!} \left[L_f^i x\right]_{x_0} \right\|^{2q} dt, \tag{6}
$$

where $\|\cdot\|$ is the standard 2-norm and

$$
L_f^i = \left(f^T \frac{\partial}{\partial x} \right)^i
$$

operates on x component-wise. Suitable T, m and q must be chosen. For polynomial systems an appropriate magnitude for T may be obtained using the bound on the radius of convergence of the Lie series derived in section 2. Large m obviously yield a better approximation to the real solution. However, when increasing m one faces a conflict between attaining greater accuracy over the interval $[0, T-\delta]$, $\delta > 0$, and loosing it over $[T-\delta, T]$, due to $\sum_{i=0}^m \frac{t^i}{i!} \left[L_f^i x\right]_{x_0}$ 'blowing up' more quickly. To circumvent this problem and improve the approximation of solution trajectories we use multiple Lie series expansions about successive points on a trajectory. Put

$$
P_m(t; x_0) = \sum_{i=0}^m \frac{t^i}{i!} \left[L_f^i x\right]_{x_0},
$$

a vector of polynomials in t with coefficients depending on x_0. Put

$$
\begin{aligned}
P_m^k(t; x_0) &= \underbrace{P_m\left(t; P_m\left(t; \ldots P_m\left(t; x_0\right) \ldots\right)\right)}_{k \text{ times}}, \; k = 1, 2, \ldots \\
P_m^0(t; x_0) &= x_0,
\end{aligned}
$$

where each P_m acts componentwise. Then another candidate for V is

$$\begin{aligned} V(x) &= \int_0^T \|P_m(t;x_0)\|^{2q} dt + \int_T^{2T} \|P_m(t;P_m(T;x_0))\|^{2q} dt + \cdots \\ &= \sum_{k=0}^{N} \int_0^T \left\|P_m\left(t+kT;P_m^k(T;x_0)\right)\right\|^{2q} dt, \end{aligned} \quad (7)$$

where N is the number of additional Lie series expansions, along a trajectory, after the first.

Once a particular V has been chosen we must find the largest constant, C, such that the hyper-surface $V(x) < C$ is contained in Ω (theorem 3.1). This translates into solving the constrained optimization:

$$\min_x V(x)$$

such that

$$\begin{aligned} \dot{V}(x) &= 0 \\ V(x) &> 0. \end{aligned}$$

This gives us a set of tangency points (TPs) [1]. Choosing the TP which gives the minimum V yields C. In two and three dimensions the optimization can be carried out by sight by simply plotting $V(x) = C$ and $\dot{V}(x) = 0$.

4. Examples. *Example 1*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + x_1^3 \end{aligned}$$

This system is taken from [1]. Zero is the only stable equilibrium point. Both (-1,0) and (1,0) are unstable equilibrium points. Figure 1 shows two estimates of the RAS provided by the proposed method and a sketch of the exact RAS obtained from the phase portrait.

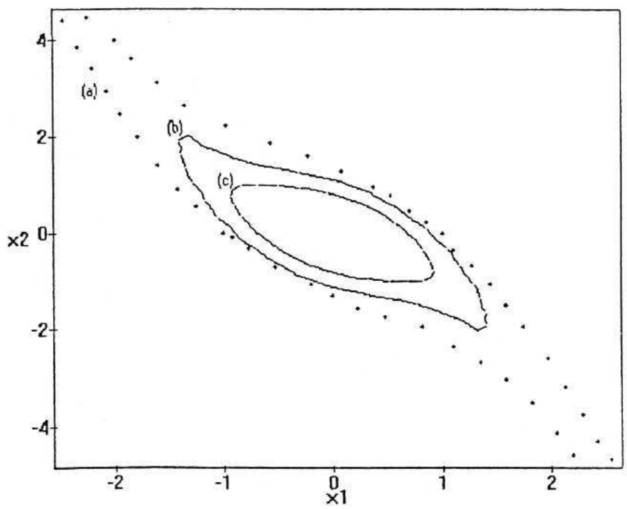

Figure 1. Estimates of RAS of example 1. (a) exact RAS, (b) estimate with m=1, N=3, T=0.5, q=1, (c) estimate with m=1, N=1, T=0.5, q=1.

Example 2

$$\dot{x}_1 = x_1^3 - x_1 - x_2$$
$$\dot{x}_2 = x_1$$

This system, the Van der Pohl Oscillator, has a single stable equilibrium point at zero. See Figure 2.

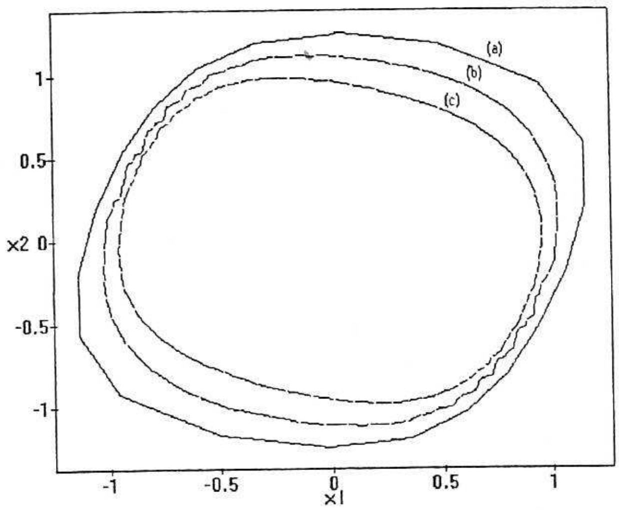

Figure 2. Estimates of RAS of example 2. (a) Exact RAS, (b) Estimate with m=2, N=3, T=0.5, q=1, (c) Estimate with m=1, N=4, T=0.5, q=1.

Example 3

$$\dot{x}_1 = x_1^3 - x_1 - x_2 - x_3$$
$$\dot{x}_2 = x_1$$
$$\dot{x}_3 = -x_3$$

This system is similar to the Van der Pohl Oscillator and has a single stable equilibrium point at zero. See Figure 3.

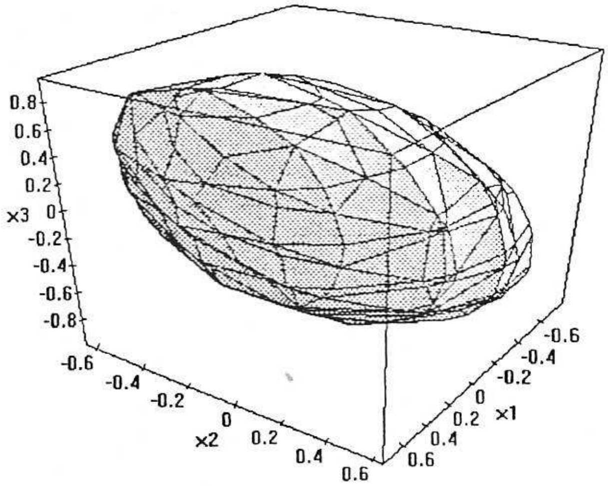

Figure 3. Estimate of RAS of Example 3 with m=1, N=1, T=0.5 and q=1.

5. Conclusions. We have presented a method to find the RAS about an equillibrium point of a nonlinear system. The merits of this method are its sound theoretical rationale, simplicity of application and reliability in producing accurate estimates of the RAS through reasonable computational effort.

REFERENCES

[1] Shields, D. N. and Storey, C., "The Behaviour of Optimal Lyapanov Functions" *Int. J. Control*, **21**, vol. 4, 561-573, 1975.

[2] Davidson, E. J. and Kurak, E. M., "A Computational Method for Determining Quadratic Lyapanov Functions for Nonlinear Systems" *Automatica*, **7**, 627-636, 1971.

[3] Kormanik, J. and Li, C. C., "Decision Surface Estimate of Nonlinear System Stability Domain by Lie Series Method" *IEEE Trans. Automat. Contr*, **AC-17**, 666-669, 1972.

[4] Genesio, R., Tartaglia, M. and Vicino, A., "On the Estimation of Asymptotic Stability Regions: State of the Art and New Proposals" *IEEE Trans. Automat. Contr.*, **AC-30**, No.8, 747-755, 1985.

[5] Chiang, H. and Thorp, J., "Stability Regions of Nonlinear Dynamical Systems: A Constructive Methodology" *Trans. Automat. Contr.*, **34**, No.12, 1229-1241, 1989.

[6] Vannelli, A. and Vidyasagar, M., "Maximal Lyapanov Functions and Domains of Attraction for Autonomous Nonlinear Systems" *Automatica*, **21**, No.1, 69-80, 1985.

[7] Kamenetskii, V. A., "Construction of Regions of Attraction by the Lyapanov Function Method" *Automation and Remote Contr.*, **55**, No. 6, 778-791, 1994.

[8] Chiang, H., Hirsch, M. W. and Wu, F. F, "Stability Regions of Nonlinear Autonomous Dynamical Systems" *IEEE Trans. Automat. Contr.*, **33**, No.1, 16-26, 1988.

[9] Banks, S. P., "Explicit Representations of Lyapanov Functions for Stable Analytic Systems" *Int. J. Systems Sci.*, **23**, No. 5, 677-682, 1992.

[10] Haddad, W. M., Kapila, V. and Chellaboina, V., "Guaranteed Domains of Attraction for Multivariable Luré Systems Via Open Lyapanov Surfaces" *IFAC 13th Triennial World Congress*, 13-18, 1996.

[11] Zubov, V. I., *Methods of A. M. Lyapanov and their Application*, The Netherlands: Noordhoff, 1964..

[12] LaSalle, J. P. and Lefschetz, S., *Stability by Lyapanov's Direct Method*, New York: Academic, 1961.
[13] Olver, P. J., *Applications of Lie Groups to Differential Equations*. New York: Springer-Verlag, 1986.