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Lie Algebras, Structure of Nonlinear Systems and Chaotic Motion

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ABSTRACT

The structure theory of Lie algebras is used to classify nonlinear systems according to a Levi decomposition and the solvable and semisimple parts of a certain Lie algebra associated with the system. An approximation theory is developed and a new class of chaotic systems is introduced.

1. Introduction

The use of linear eigenstructure theory for linear systems is well-known and enables one to write a linear system

$$\dot{x} = Ax$$

in the form

$$\dot{y} = \Lambda y$$

where $y = P^{-1}x$ and $\Lambda = P^{-1}AP$ is the Jordan form of A. The phase-space theory of nonlinear systems uses this structure locally, in a neighbourhood of each equilibrium point. In this paper we shall consider a large scale structure theory for systems of the form

$$\dot{x} = A(x)x \ .$$

We shall define the Lie algebra associated with this system to be the Lie subalgebra of $g\ell(n, \mathbb{C})$ generated by all the matrices $A(x)$, $x \in \mathbb{R}^n$, and we shall denote it by $\mathcal{L}_A$. Using the structure theory of Lie algebras (see [1,2]), we can write the system in the form

$$\dot{x} = S(x)x + \Sigma(x)x$$

where $S(x)$ is in a solvable subalgebra of $\mathcal{L}_A$ and $\Sigma(x)$ is in a semisimple subalgebra (this is the Levi decomposition). Using the theory of semisimple Lie algebras, we can further decompose the system in the form

$$\dot{x} = S(x)x + H(x)x + \sum_{\phi\in\Delta} e_\phi(x)E_\phi$$

where Δ is the set of nonzero roots and $H(x)$ belongs to a Cartan subalgebra, so that all the matrices $H(x)$, $x \in \mathbb{R}^n$, are simultaneously diagonalisable.

A preliminary study of these systems was made in [3] and applications to stability in [4]. Here we shall give a more detailed study of solvable and simple systems and give an approximation technique which leads to a new stability result. The ideas will be illustrated by a number of examples generated by simple systems and similar systems with a solvable perturbation. This will lead to a class of chaotic systems similar to the Lorentz attractor.

2. The Lie Algebra of a Differential Equation

Consider a semilinear equation of the form

$$\dot{x} = A(x)x \tag{2.1}$$

where $A(x)$ is analytic. We can write

$$A(x) = \sum_{|\mathbf{i}|\geq 0} A_{\mathbf{i}} x^{\mathbf{i}} \tag{2.2}$$

where $\mathbf{i} = (i_1, \cdots, i_n)$, $x^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ and $A_{\mathbf{i}}$ is a constant $n \times n$ matrix. It is possible to associate two natural Lie algebras with (1.1), namely

$$\begin{aligned} \mathcal{L}_{\{A(x)\}} &= \text{Lie subalgebra of } g\ell(n, \mathbb{C}) \text{ generated by } A(x),\ x \in \mathbb{R}^n \\ \mathcal{L}_{\{A_{\mathbf{i}}\}} &= \text{Lie subalgebra of } g\ell(n, \mathbb{C}) \text{ generated by } A_{\mathbf{i}},\ |\mathbf{i}| \geq 0 \end{aligned}$$

The first result shows that these are, in fact, identical:

Lemma (2.1) $\mathcal{L}_{\{A(x)\}} = \mathcal{L}_{\{A_{\mathbf{i}}\}}$.

Proof Let $\{E_i\}_{1\leq i\leq m}$ be a basis of $\mathcal{L}_{\{A_{\mathbf{i}}\}}$. By (2.2) we can write

$$A(x) = \sum_{i=1}^{m} p_i(x) E_i$$

where $p_i(x)$ exists for all $x \in \mathbb{R}^n$ by the analyticity of $A(x)$. Hence, $A(x) \in \mathcal{L}_{\{A_{\mathbf{i}}\}}$ and so $\mathcal{L}_{\{A(x)\}} \subseteq \mathcal{L}_{\{A_{\mathbf{i}}\}}$.

Conversely, we shall show that $A_{\mathbf{i}} \in \mathcal{L}_{\{A(x)\}}$ for each $\mathbf{i}$. Clearly $A_{\mathbf{0}} = A(0)$ so it is certainly true for $\mathbf{i} = \mathbf{0}$. We shall prove that $A_{1,0,\cdots,0} \in \mathcal{L}_{\{A(x)\}}$, the others being similar. Now,

$$A_{1,0,\cdots,0} = \frac{\partial}{\partial x_1} A(x) \mid_{x=0} = \lim_{h\to 0} \frac{A(he_i) - A(0)}{h}.$$

If $\{F_i\}_{1\leq i\leq m'}$ is a basis of $\mathcal{L}_{\{A(x)\}}$, then

$$A_{1,0,\cdots,0} = \lim_{h\to 0} \sum_{i=1}^{m'} q_i(h) F_i$$

for some functions $q_i(h)$. Since the F_i's are linearly independent each limit $\lim_{h\to 0} q_i(h)$ must exist, so $A_{1,0,\cdots,0} \in \mathcal{L}_{\{A(x)\}}$.$\square$

In view of this lemma we shall use whichever formulation of the Lie algebras is most appropriate, and without loss of generality, we may write

$$\mathcal{L}_A = \mathcal{L}_{\{A(x)\}} = \mathcal{L}_{\{A_\mathfrak{i}\}}$$

and call $\mathcal{L}_A$ *the* Lie algebra generated by the nonlinear equation (2.1).
Theorem (2.2) *Any nolinear system of the form (2.1) may be written in the form (with respect to a suitable basis)*

$$\dot{x} = S(x)x + \begin{pmatrix} \Gamma_1(x) & & & \\ & \Gamma_2(x) & & \\ & & \ddots & \\ & & & \Gamma_r(x) \end{pmatrix} x \tag{2.3}$$

where $(\tilde{S}(x), \tilde{S}(x)) = 0$ *, with* $\tilde{S}(x) = [S(x), S(x)]$*, and* Γ_i *belongs to one of the simple Lie algebras* $A_m, B_m, C_m, D_m, G_2, F_4, E_6, E_7, E_8$ *where* $(\ ,\ )$ *denotes the Killing form on* $\mathcal{L}_A$*.*
Proof Let $\mathcal{L}_A = \mathfrak{r} + \mathfrak{m}$ be a Levi decomposition of $\mathcal{L}_A$ (see [5]), where $\mathfrak{r}$ is the radical of $\mathcal{L}_A$ and $\mathfrak{m}$ is a semisimple subalgebra. (Note that this is not a direct sum so this decomposition is not unique.) Any semisimple algebra $\mathfrak{m}$ may be written as a direct sum of simple ideals:

$$\mathfrak{m} = \mathfrak{m}_1 \ominus \cdots \ominus \mathfrak{m}_r$$

for some r and since the sum is direct and $A(x) \in \mathfrak{r} + \mathfrak{m}$ we may choose coordinates so that (2.1) can be written in the form (2.3). Since $\mathfrak{r}$ is solvable the condition $(\tilde{S}(x), \tilde{S}(x)) = 0$ follows from Cartan's criterion for semi-simplicity.□
Corollary (2.3) *If* $\mathcal{L}_A$ *is solvable we may write (2.1) in the form*

$$\dot{x} = \begin{pmatrix} s_{11}(x) & \cdots & \cdots & s_{1n}(x) \\ & s_{22}(x) & \cdots & s_{2n}(x) \\ & & \cdots & \cdots \\ & & & s_{nn}(x) \end{pmatrix} x \tag{2.4}$$

Proof This follows since any solvable linear Lie algebra may be written in upper triangular form with respect to a suitable basis.□
Corollary (2.4) *If* $\mathcal{L}_A$ *is semisimple we may write (2.1) in the form*

$$\dot{x} = \begin{pmatrix} \Gamma_1(x) & & & \\ & \Gamma_2(x) & & \\ & & \ddots & \\ & & & \Gamma_r(x) \end{pmatrix} x \tag{2.5}$$

□

Next we would like a condition on the matrices $A_\mathfrak{i}$, $|\mathfrak{i}| \geq 0$, so that the Lie algebra $\mathcal{L}_A$ is semisimple. By Cartan's criterion this holds if and only if the Killing form of $\mathcal{L}_A$ is nondegenerate. Let $E_1, \cdots, E_m$ be a basis of $\mathcal{L}_A$ which can clearly be taken to consist of real matrices (since the $A_\mathfrak{i}$ are real). Hence the quadratic form (X, X) is real on this basis and so we can define the norm

$$\|E_i\| = (E_i, E_i)^{1/2}.$$

By the Gram-Schmidt orthogonalisation procedure, we define

$$\begin{aligned} F_1 &= E_1/\|E_1\| \\ \tilde{F}_{i-1} &= E_{i-1} - \sum_{j=1}^{i} F_j(E_{i-1}, F_j) \\ F_{i-1} &= \tilde{F}_{i-1}/\left\|\tilde{F}_{i-1}\right\| \end{aligned} \tag{2.6}$$

for $1 \leq i < m$. Then $(\cdot,\cdot)$ is nondegenerate if and onlt if $\left\|\tilde{F}_i\right\| \neq 0$ for each $i \in \{1, \cdots, m\}$. Hence we have

Lemma (2.5) *$\mathcal{L}_A$ is semisimple if and only if, for any (real) basis $E = \{E_1, \cdots, E_m\}$ of $\mathcal{L}_A$ the matrices $\tilde{F}_1, \cdots, \tilde{F}_m$ associated with E by (2.6) satisfy*

$$\sum_{k=1}^{m} {}^i\lambda_k^2 \neq 0$$

where ${}^i\lambda_k$, $1 \leq k \leq m$ are the eigenvalues of the operator ad $\tilde{F}_i$.

Proof This follows directly from

$$\left\|\tilde{F}_i\right\| = \mathrm{Tr}\ (\mathrm{ad}\ \tilde{F}_i)^2$$

and the invariance of Tr.□

3. The Solvable Case

We consider in this section the case where $\mathcal{L}_A$ is solvable. It follows from Lie's theorem that $\mathcal{L}_A$ is solvable if and only if $\mathfrak{D}_{\mathcal{L}_A} = [\mathcal{L}_A, \mathcal{L}_A]$ (the derived algebra of $\mathcal{L}_A$) is nilpotent. Morever, by Engel's theorem, a Lie algebra $\mathfrak{g}$ is nilpotent if and only if ad X is nilpotent for each $X \in \mathfrak{g}$. Hence $\mathcal{L}_A$ is solvable if and only if ad X is nilpotent for each $X \in \mathfrak{D}_{\mathcal{L}_A}$. Since the sum of nilpotent operators is nilpotent, we can test this condition on a basis of $\mathfrak{D}_{\mathcal{L}_A}$. Hence we have

Lemma (3.1) *$\mathcal{L}_A$ is solvable if and only if all the eigenvalues of the operators ad $[E_i, E_j]$, $1 \leq i, j \leq m$ are zero, where $\{E_1, \cdots, E_m\}$ is a maximal linearly independent set of $\mathcal{L}_A$.*□

If $\mathcal{L}_A$ is solvable, then by Lie's theorem we can choose a basis so that the equation (2.1) takes the form

$$\dot{x} = \begin{pmatrix} \lambda_1(x) & \nu_{12}(x) & \cdots & \nu_{1n}(x) \\ & \lambda_2(x) & & \\ & & \ddots & \nu_{n-1n}(x) \\ & & & \lambda_n(x) \end{pmatrix} x = \Lambda(x)x + N(x)x \tag{3.1}$$

where $\Lambda(x) = \mathrm{diag}\ (\lambda_1(x), \cdots, \lambda_n(x))$ and $N(x) = \begin{pmatrix} 0 & \nu_{12}(x) & \cdots & \nu_{1n}(x) \\ & 0 & & \\ & & \ddots & \nu_{n-1n}(x) \\ & & & 0 \end{pmatrix}$.

We shall first prove a stability result from this (see also [6]).

Theorem (3.2) *Suppose that $\|N(x)\| < \nu$ for $\|x\| \le K$ for some $\nu > 0, K \ge 0$ and that $\lambda_i(x) \le -\mu$ for $1 \le i \le n$ and $\|x\| \le K$ with $\mu > 0$. Then the system (3.1) is asymptotically stable in the open ball $B = \{x : \|x\| < K\}$, if $\mu > \nu$.*

Proof We have

$$x(t) = e^{\Lambda(x_0)t}x_0 + \int_0^t e^{\Lambda(x_0)(t-s)}(\Lambda(x(s)) - \Lambda(x_0) + N(x(s)))x(s)ds$$

where $x(0) = x_0$. Let $x_0 \in B$ and put $\gamma = K - \|x_0\|$. Since Λ and N are continuous and B is compact, we have

$$\|N(x)\| < \nu - \varepsilon/2 \text{ , for } \|x\| \le K$$

and

$$\|\Lambda(x) - \Lambda(x_0)\| < \varepsilon/2$$

for $\|x - x_0\| < \gamma/2$, say. By continuity of solutions, we have $\|x(t) - x_0\| < \gamma/2$ when $t \in [0, \tau]$ for some $\tau > 0$. Then $x(t) \in B$ and so

$$\begin{aligned} \|x(t)\| &\le e^{-\mu t}x_0 + \int_0^t e^{-\mu(t-s)}(\|\Lambda(x(s)) - \Lambda(x_0)\| + \|N(x(s))\|)\, \|x(s)\|\, ds \\ &\le e^{-\mu t}\, \|x_0\| + \int_0^t e^{-\mu(t-s)}\nu\, \|x(s)\|\, ds \quad , \quad \text{for } t \in [0, \tau] \end{aligned}$$

By Gronwall's inequality we have

$$\|x(t)\| \le e^{(-\mu-\nu)t}\, \|x_0\|$$

and so $x(t)$ is strictly decreasing in B.□

Remark (3.3) We can clearly refine this result and require only that $\lambda_i(x) \le -\mu(x)$, where $\mu(x) > 0$ and $\|N(x)\| \le \nu(x)$ with $\mu(x) > \nu(x)$ for $x \in B$.□

The next result shows that, in certain cases, it is easy to prove complete integrability of solvable systems. Suppose that such a system is written in the form

$$\dot{x} = \begin{pmatrix} \lambda_1(x) & & & \\ & \lambda_2(x) & & \\ & & \ddots & \\ & & & \lambda_n(x) \end{pmatrix} + N(x)x \tag{3.2}$$

where $N(x)$ is nilpotent and upper triangular. Assume that $\lambda_i(x)$ is a function only of $x_{i+1}, \cdots, x_n$ (so that λ_n is constant) and $\nu_{ij}(x)$ is a function only of $x_{i-1}, \cdots, x_n$ where $N = (\nu_{ij})$. Then we have a system of the form

$$\begin{aligned} \dot{x}_n &= \lambda_n x_n \\ \dot{x}_{n-1} &= \lambda_{n-1}(x_n)x_{n-1} + \nu_{n-1,n}(x_n)x_n \\ \dot{x}_{n-2} &= \lambda_{n-2}(x_n, x_{n-1})x_{n-2} + \nu_{n-2,n}(x_n, x_{n-1})x_n + \nu_{n-2,n-1}(x_n, x_{n-1})x_{n-1} \\ & \cdots \\ \dot{x}_1 &= \lambda_1(x_2, \cdots, x_n)x_1 + \sum_{j=2}^{n} \nu_{1,j}(x_2, \cdots, x_n)x_j \end{aligned}$$

These equations may be integrated recursively:

$$\begin{aligned}
x_n(t) &= e^{\lambda_n} x_{n0} \\
x_{n-i}(t) &= e^{\int_0^t \lambda_{n-i}(x_n(\tau),\cdots,x_{n-i+1}(\tau))d\tau} x_{n-i0} + \\
& \int_0^t e^{\int_0^{t-s} \lambda_{n-i}(x_n(\tau),\cdots,x_{n-i+1}(\tau))d\tau} \sum_{j=n-1-i}^{n} \nu_{n-i,j}(x_n(s),\cdots,x_{n-i-1}(s))x_j(s)ds
\end{aligned}$$

Given a general solvable system we can check if it is completely integrable in the following way:

Lemma(3.4) *Given a system of the form* $\dot{y} = A(y)y$ *where* $\mathcal{L}_A$ *is solvable the system is completely integrable if there exists a basis* $f_1, \cdots, f_n$ *of* $\mathbb{R}^n$ *such that* $sp\{f_1, \cdots, f_n\}$ *is invariant under* A *and* $A|_{sp\{f_1,\cdots,f_n\}}$ *is independent of* $x_1 = \sum p_{1j}y_j, \cdots, x_{n-i} = \sum p_{n-i,j}y_j, 0 \le i \le n-2$, *where* $x = Py$ *and* P *is the matrix of the change of basis from the standard basis* $e_1, \cdots, e_n$ *to* $f_1, \cdots, f_n$.

Proof Let $\bar{A}^{(i)} : \mathbb{R}^n/\{f_n, \cdots, f_{n-i}\} \to \mathbb{R}^n/\{f_n, \cdots, f_{n-i}\}$ be the induced map,$0 \le i \le n-2$. Since sp$\{f_n\}$ is invariant under $A(x) = A(f(y))$ and $A|_{sp\{f_n\}}$ is independent of $x_1, \cdots, x_n$, we have $A(x)f_n = \lambda_n f_n$ for some constant λ_n.If the result is true for i, then

$$\bar{A}^{(i-1)} : \mathbb{R}^n/\{f_n, \cdots, f_{n-i-1}\} \to \mathbb{R}^n/\{f_n, \cdots, f_{n-i-1}\}$$

is independent of $x_1, \cdots, x_{n-i-1}$ and so

$$\bar{A}^{(i-1)}\bar{f}_{n-i-2} = \lambda_{n-i-2}\bar{f}_{n-i-2}$$

where λ_{n-i-2} is independent of $x_1, \cdots, x_{n-i-2}$ and $\bar{f}_j$ is the image of f_j in $\mathbb{R}^n/\{f_n, \cdots, f_{j-1}\}$ under the canonical map. The result follows now by expressing the equation in the form (3.2).□

Example (3.5) Consider the system

$$\begin{aligned}
\dot{y}_1 &= -y_1y_2 + y_2^2 \\
\dot{y}_2 &= y_1 - y_1y_2 + y_2^2 - y_2
\end{aligned} \tag{3.3}$$

i.e.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & -y_1 + y_2 \\ 1 & -y_1 + y_2 - 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

so that

$$A(y) = \begin{pmatrix} 0 & -y_1 + y_2 \\ 1 & -y_1 + y_2 - 1 \end{pmatrix}.$$

$\mathcal{L}_A$ is generated by $E_1 = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ so has basis $E_1, E_2, E_3 = [E_1, E_2] = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. It is easy to check that the eigenvalues of ad$[E_i, E_j]$ are all zero for $1 \le i, j \le 3$

and so by lemma (3.1), $\mathcal{L}_A$ is solvable. Consider the basis elements E_1, E_2, E_3. These have a common eigenvector $f_1 = (1,1)^T$. We then seek another vector f_2 such that $\bar{E}_i\ \bar{f}_2 = \lambda_2(E_i)\bar{f}_2$ for $1 \le i \le 3$, i.e. $(E_i - \lambda_2(E_i))f_2 = \alpha_i f_1$ for some $\alpha_1, \alpha_2, \alpha_3$. A solution is easily found as $f_2 = (1,-1)^T$. Consider the transformation $x = P^{-1}y$ where $P = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$. Then we obtain the system

$$\dot{x} = \begin{pmatrix} -x_2 & x_2+1 \\ 0 & -1 \end{pmatrix} x.$$

This equation has solution

$$\begin{aligned} x_2(t) &= e^{-t}x_{20} \\ x_1(t) &= e^{(1-e^{-t})}x_{10} + \int_0^t e^{(e^{-t}-e^{-s})x_{20}}(e^{-s}x_{20}+1)e^{-s}x_{20}ds \end{aligned}$$

and using $x = P^{-1}y$ we can write the integral of equation (3.3) as

$$\begin{aligned} y_1(t) &= \frac{e^{-t}}{2}(y_{10}-y_{20}) + \frac{e^{(1-e^{-t})}}{2}(y_{10}+y_{20}) + \\ & \int_0^t e^{(e^{-t}-e^{-s})(y_{10}-y_{20})/2}(e^{-s}(y_{10}-y_{20})/2+1)\frac{e^{-s}}{2}(y_{10}-y_{20})ds \\ y_2(t) &= -\frac{e^{-t}}{2}(y_{10}-y_{20}) + \frac{e^{(1-e^{-t})}}{2}(y_{10}+y_{20}) + \\ & \int_0^t e^{(e^{-t}-e^{-s})(y_{10}-y_{20})/2}(e^{-s}(y_{10}-y_{20})/2+1)\frac{e^{-s}}{2}(y_{10}-y_{20})ds \end{aligned}$$

4. The Semisimple Case

We first discuss the decomposition of a system into simple components. Let $E_1, \cdots, E_m$ be a basis of $\mathcal{L}_A$ and let $F_1, \cdots, F_m$ be the orthogonalisation of this basis with respect to the Killing form as in section 2. We have

Lemma (4.1) *If $E = \{E_1, \cdots, E_m\}$ is a basis of $\mathcal{L}_A$ and $V_1, \cdots, V_r \subseteq \mathbb{R}^n$ are minimal (nontrivial) invariant subspaces of $\mathbb{R}^n$ under E (i.e. $E_iV_j \subseteq V_j$ for all i, j) then we may write the system (2.1) in the form*

$$\dot{x} = \begin{pmatrix} \Gamma_1(x) & & 0 \\ & \ddots & \\ 0 & & \Gamma_r(x) \end{pmatrix} x$$

where $\mathcal{L}_{\Gamma_i(x)}$ is simple, $1 \le i \le r$.

Proof Since E is a basis of $\mathcal{L}_A$, each V_i is invariant under $\mathcal{L}_A$ and so we choose a basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$ such that $e_{p_i}, \cdots, e_{p_{i+1}-1}$ is a basis of V_i, $1 \le i \le r$. Now each subspace $0 \ominus \cdots \ominus \mathcal{L}_{\Gamma_i} \ominus 0 \ominus \cdots \ominus 0$ is an ideal in $\mathcal{L}_A$, although it may not be minimal, of course. However, every simple matrix Lie algebra is a subalgebra of A_k (the Lie algebra of trace zero matrices) for some k and so $\mathcal{L}_{\Gamma_i(x)}$ is certainly in a minimal simple Lie algebra and the result follows.□

Of course, the main problem here is finding the minimal invariant subspaces of a number of (noncommuting) operators. Consider the case of two-dimensional invariant subspaces of two linear operators E, F acting on $\mathbb{R}^n$; the general case of several operators and higher-dimensional invariant subspaces is similar, but much more computationally involved. If E, F have a common two-dimensional invariant subspace spanned by w_1, w_2, then

$$Ew_i = \sum_{j=1}^{2} \alpha_{ij} w_j \ , \ Fw_i = \sum_{j=1}^{2} \beta_{ij} w_j \ , \ i = 1, 2$$

for some 2×2 matrices $\alpha = (\alpha_{ij}), \beta = (\beta_{ij})$. It is easy to see that we have

$$V_E \triangleq \begin{pmatrix} (E - \alpha_{11} I) & -\alpha_{12} I \\ -\alpha_{21} I & (E - \alpha_{22} I) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0 \tag{4.1}$$

$$V_F \triangleq \begin{pmatrix} (F - \beta_{11} I) & -\beta_{12} I \\ -\beta_{21} I & (F - \beta_{22} I) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

Hence α and β satisfy

$$\begin{vmatrix} (E - \alpha_{11} I) & -\alpha_{12} I \\ -\alpha_{21} I & (E - \alpha_{22} I) \end{vmatrix} = 0 \ , \ \begin{vmatrix} (F - \beta_{11} I) & -\beta_{12} I \\ -\beta_{21} I & (F - \beta_{22} I) \end{vmatrix} = 0.$$

The simplest way to solve (4.1) is to minimise numerically the objective function

$$\|V_E\|^2 + \|V_F\|^2$$

subject to the constraint $\|w_1\|^2 + \|w_2\|^2 = 1$.

Example (4.2) Consider the differential equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 2 + 2x_1 & -3 - 3x_1 & 3 + 3x_1 \\ -2x_1 & 1 + x_1 & -x_1 \\ -1 - 2x_1 & 3 + 3x_1 & -2 - 3x_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Then $\mathcal{L}_A$ is generated by the matrices $A_{00} = \begin{pmatrix} 2 & -3 & 3 \\ 0 & 1 & 0 \\ -1 & 3 & -2 \end{pmatrix}$ and $A_{10} = \begin{pmatrix} 2 & -3 & 3 \\ -2 & 1 & -1 \\ -2 & 3 & -3 \end{pmatrix}$.

It can be seen (using MAPLE, for example) that these matrices are put into block diagonal form by $P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ so that the equation becomes

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} 1 + 2x_1 & x_1 & 0 \\ 1 & -1 - 2x_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where $y = P^{-1}x$, giving a system with $\begin{pmatrix} 1+2x_1 & x_1 \\ 1 & -1-2x_1 \end{pmatrix}$ generating a C_1-type system.

Using lemma (4.1) we can classify semisimple systems in terms of the classical simple Lie algebras:

Theorem (4.3) *Every system*

$$\dot{x} = A(x)x$$

where $\mathcal{L}_A$ generates a simple Lie algebra can be written in one of the following forms:

$$\begin{aligned}
\dot{x} &= A(x)x \ , \ \mathrm{tr}(A(x)) = 0 \ \ (\text{type A}) \\
\dot{x} &= \begin{pmatrix} 0 & u(x) & v(x) \\ -v^T(x) & A_{11}(x) & A_{12}(x) \\ -u^T(x) & A_{21}(x) & -A_{11}^T(x) \end{pmatrix} x \ , \ A_{12}^T(x) = -A_{12}(x), A_{21}^T(x) = -A_{21}(x) \ \ (\text{type B}) \\
\dot{x} &= \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & -A_{11}^T(x) \end{pmatrix} x \ , \ A_{12}^T(x) = A_{12}(x), A_{21}^T(x) = A_{21}(x) \ \ (\text{type C}) \\
\dot{x} &= \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & -A_{11}^T(x) \end{pmatrix} x \ , \ A_{12}^T(x) = -A_{12}(x), A_{21}^T(x) = -A_{21}(x) \ \ (\text{type D}) \\
\dot{x} &= \begin{pmatrix} 0 & -\sqrt{2}b_1(x) & -\sqrt{2}b_2(x) & -\sqrt{2}b_3(x) & \sqrt{2}a_1(x) & \sqrt{2}a_2(x) & \sqrt{2}a_3(x) \\ -\sqrt{2}a_1(x) & \lambda_1(x) & c_1(x) & c_3(x) & 0 & b_3(x) & -b_2(x) \\ -\sqrt{2}a_2(x) & c_2(x) & \lambda_2(x) & c_5(x) & -b_3(x) & 0 & b_1(x) \\ -\sqrt{2}a_3(x) & c_4(x) & c_6(x) & -(\lambda_1(x)+\lambda_2(x)) & b_2(x) & -b_1(x) & 0 \\ \sqrt{2}b_1(x) & 0 & -a_3(x) & a_2(x) & -\lambda_1(x) & -c_2(x) & -c_4(x) \\ \sqrt{2}b_2(x) & a_3(x) & 0 & -a_1(x) & -c_1(x) & -\lambda_2(x) & -c_6(x) \\ \sqrt{2}b_3(x) & -a_2(x) & a_1(x) & 0 & -c_3(x) & -c_5(x) & \lambda_1(x)+\lambda_2(x) \end{pmatrix} \\
& \quad (\text{type } G_2) \\
\dot{x} &= \left(\sum_{i=1}^{r} a_i(x)X_i + \sum_{i=1}^{s} b_i(x)Y_i \right) x \ \ (\text{types } F_4, E_6, E_7, E_8)
\end{aligned}$$

where X_i, Y_j satisfy

$$\begin{aligned}
[X_i, Y_j] &= \sum_{p=1}^{s} x_{pj}^i Y_p \ , \ 1 \le i \le r, 1 \le j \le s \qquad (4.2) \\
[Y_\alpha, Y_\beta] &= \sum_{i=1}^{r} x_{\alpha\beta}^i X_i \ , \ 1 \le \alpha, \beta \le s
\end{aligned}$$

where $X_i = (x_{\alpha\beta}^i)$ and X_i, Y_j can be realized on a 16-dimensional space for type F_4 (with $r = 36, s = 16$) or a 2^7-dimensional space for types E_6, E_7, E_8. For E_8, $r = 120, s = 128$ and E_6, E_7 are subalgebras of dimension 78 and 133, respectively.

Proof The proof follows directly from the classification theorem of simple Lie algebras (see [7]) and the following lemma of Witt [8]. (In applying the lemma we have simply taken the identty representation.)□

Lemma (4.4) *(E. Witt) Let* $\mathfrak{g}$ *be an* r*-dimensional simple Lie algebra with basis* $\{X_1, \cdots, X_r\}$ *and* ρ *an irreducible matrix representation of* $\mathfrak{g}$ *on a vector space* V *of dimension* s *which is nontrivial and not the regular representation of* $\mathfrak{g}$. *Let* $\{Y_1, \cdots, Y_s\}$ *be a basis of* V. *If* $D_i \triangleq \rho(X_i)$ *is real and skew symmetric,* $TrD_iD_j = -s\delta_{ij}$ *and* $Tr\sum_{i,k}(D_iD_k)^2 = \frac{1}{2}rs^2$, *then* $\mathfrak{h} = \mathfrak{g} \ominus V$ *is a simple Lie algebra with the commutation relations*

$$\begin{aligned} [X_i, Y_j] &= \sum_{p=1}^{s} d_{pj}^{i} Y_p \ , \ 1 \le i \le r, 1 \le j \le s \\ [Y_\alpha, Y_\beta] &= \sum_{i=1}^{r} d_{\alpha\beta}^{i} X_i \ , \ 1 \le \alpha, \beta \le s \end{aligned}$$

where $D_i = (d_{\alpha\beta}^i)$.□

5. Inhomogeneous Equations and the Variation of Constants Formula

Before proceeding with the theory of semisimple systems, we must first obtain some general results on linear approximations of differential equations of the form

$$\dot{x}(t) = A(x(t))x(t) + f(t) \ . \ x(t_0) = x_0 \tag{5.1}$$

We shall consider the following approximating sequence of linear differential equations:

$$\begin{aligned} \dot{x}^{[1]}(t) &= A(x_0)x^{[1]}(t) + f(t) \ , \ x^{[1]}(t_0) = x_0 \\ \dot{x}^{[i]}(t) &= A(x^{[i-1]}(t))x^{[i]}(t) + f(t) \ , \ x^{[i]}(t_0) = x_0 \end{aligned} \tag{5.2}$$

Let $\Phi^{[i-1]}(t, t_0)$ denote the transition matrix generated by $A(x^{[i-1]}(t))$. It is well-known (Brauer [9]) that

$$\left\|\Phi^{[i-1]}(t, t_0)\right\| \le \exp\left[\int_{t_0}^{t} \mu(A(x^{[i-1]}(\tau))d\tau\right]$$

where $\mu(A)$ is the logarithmic norm of A. We next require an estimate for $\Phi^{[i-1]} - \Phi^{[i-2]}$.

Lemma (5.1) *Suppose that* $\mu(A(x)) \le \mu$ *for some constant* μ *and for all* x *and that*

$$\|A(x) - A(y)\| \le \alpha \|x - y\| \ , \ \forall \, x, y \in \mathbb{R}^n$$

Then

$$\left\|\Phi^{[i-1]}(t, t_0) - \Phi^{[i-2]}(t, t_0)\right\| \le \alpha e^{\mu(t-t_0)}(t - t_0) \sup_{s\in[t_0,t]} \left\|x^{[i-1]}(s) - x^{[i-2]}(s)\right\|$$

Proof $\Phi^{[i-1]}, \Phi^{[i-2]}$ are solutions of the respective equations

$$\begin{aligned} \dot{z} &= A(x^{[i-1]}(t))z \ , \ z(t_0) = I \\ \dot{w} &= A(x^{[i-2]}(t))w \ , \ w(t_0) = I \end{aligned}$$

Hence,

$$\frac{d}{dt}(z-w) = A(x^{(i-1)}(t))(z-w) + \left[A(x^{(i-1)}(t)) - A(x^{(i-2)}(t))\right] w$$

and so

$$z - w = \int_{t_0}^{t} \Phi^{(i-1)}(t,s)\left[A(x^{(i-1)}(s)) - A(x^{(i-2)}(s))\right] w(s)ds$$

i.e.

$$\begin{aligned}\|z-w\| &\le \int_{t_0}^{t} \exp\left(\int_{s}^{t} \mu(A(x^{(i-1)}(\tau)))d\tau\right) \exp\left(\int_{t_0}^{s} \mu(A(x^{(i-2)}(\tau)))d\tau\right) \\ &\quad \times\alpha \left\|x^{(i-1)}(s) - x^{(i-2)}(s)\right\| ds \\ &\le \exp(\mu(t-t_0))\alpha(t-t_0) \sup_{s\in[t_0,t]} \left\|x^{(i-1)}(s) - x^{(i-2)}(s)\right\| . \ \square\end{aligned}$$

From (5.2) we have

$$x^{(i)}(t) = \Phi^{(i-1)}(t,t_0)x_0 + \int_{t_0}^{t} \Phi^{(i-1)}(t,s)f(s)ds. \tag{5.3}$$

Let

$$\xi^{(i)}(t) = \sup_{s\in[t_0,t]} \left\|x^{(i)}(s) - x^{(i-1)}(s)\right\| .$$

Then from (5.3),

$$\begin{aligned}\xi^{(i)}(t) &\le \alpha \exp(\mu(t-t_0))(t-t_0)\xi^{(i-1)}(t)\,\|x_0\| \\ &\quad +\alpha \int_{t_0}^{t} \exp(\mu(t-s))\alpha(t-s)\xi^{(i-1)}(s)\,\|f(s)\|\, ds \\ &\le \alpha \exp(\mu(t-t_0))(t-t_0)\xi^{(i-1)}(t)\left[\|x_0\| + \int_{t_0}^{t} \exp(-\mu(s-t_0))\,\|f(s)\|\, ds\right]\end{aligned}$$

Suppose that, for some $T > t_0$ (possibly ∞), $\exp(-\mu(t-t_0))\,\|f(t)\| \in L^1[t_0,T]$ and define

$$K = \int_{t_0}^{T} \exp(-\mu(s-t_0))\,\|f(s)\|\, ds.$$

Then,

$$\xi^{(i)}(t) \le \alpha\left(\|x_0\| + K\right)(T-t_0)\exp(\mu(t-t_0))\xi^{(i-1)}(t)$$

for $t \in [t_0,T]$. Suppose that

$$\nu \triangleq \alpha\left(\|x_0\| + K\right)(T-t_0)\exp(\mu(t-t_0)) < 1.$$

Then

$$\xi^{i}(t) \leq \nu \xi^{i-1}(t)\ ,\ t \in [t_0, T]$$

and so

$$\xi^{i}(t) \leq \nu^{i-2} \xi^{2}(t). \tag{5.4}$$

Hence we have

Theorem (5.2) *Let $A(x)$ satisfy*

$$\begin{array}{ll} \text{(A1)} & \mu(A(x)) \leq \mu \\ \text{(A2)} & \|A(x) - A(y)\| \leq \|x - y\| \quad \forall\, x, y \in \mathbb{R}^n \end{array}$$

and suppose that

$$\nu = \alpha \left(\|x_0\| + \int_{t_0}^{T} \exp(-\mu(s - t_0)) \|f(s)\| \, ds \right) (T - t_0) \exp(\mu(T - t_0)) < 1.$$

Then the equation (4.1) has a unique solution on $[t_0, T]$ which is given by the limit of the solutions of the approximating equations (4.2) on $C([t_0, T], \mathbb{R}^n)$.

Proof The proof follows directly from (4.4) since this implies that $x^{i}(t)$ is a Cauchy sequence in the Banach space $C([t_0, T], \mathbb{R}^n)$.□

Corollary(5.3) *If $\mu < 0$ and*

$$\alpha \left(\|x_0\| + \int_{t_0}^{T} \exp(-\mu(s - t_0)) \|f(s)\| \, ds \right) \left(-\frac{1}{\mu} \right) e^{-1} < 1$$

then the solution exists for all $T > 0$.□

We can find a bound on $\sup_{t \in t_0, T} \|x(t)\|$ where $x(\cdot) = \lim_{i \to \infty} x^{i}(\cdot)$, and the limit is taken in $C([t_0, T], \mathbb{R}^n)$ as follows:

$$\begin{aligned} \sup_{t \in t_0, T} \left\| x^{i}(t) \right\| &= \sup_{t \in t_0, T} \left\| x^{i}(t) - x^{i-1}(t) + x^{i-1}(t) - \cdots - x^{1}(t) + x^{1}(t) \right\| \\ &\leq \sum_{j=2}^{i} \xi^{j}(t) + \sup_{t \in t_0, T} \left\| x^{1}(t) \right\| \\ &\leq \sum_{j=2}^{i} \nu^{j-2} \xi^{2}(t) + \sup_{t \in t_0, T} \left\| x^{1}(t) \right\| \\ &= \frac{1 - \nu^{i-1}}{1 - \nu} \xi^{2}(t) + \sup_{t \in t_0, T} \left\| x^{1}(t) \right\|. \end{aligned}$$

Hence, as $i \to \infty$,

$$\sup_{t \in t_0, T} \|x(t)\| \leq \frac{1}{1 - \nu} \xi^{2}(t) + \sup_{t \in t_0, T} \left\| x^{1}(t) \right\|. \tag{5.5}$$

We can bound x^{1} and ξ^{2} in a similar way to that used in lemma 5.1; we obtain

Lemma (5.4) *Bounds on x^{1} and ξ^{2} are given by*

$$\sup_{t\in[t_0,T]} \left\|x^{1}(t)\right\| \leq \sup_{t\in[t_0,T]} e^{(t-t_0)\bar{\mu}} \left(\|x_0\| + \int_{t_0}^{T} e^{-(s-t_0)\bar{\mu}} \|f(s)\| \, ds \right)$$

where $\bar{\mu} = \mu(A(x_0))$ and

$$\xi^{2}(t) \leq \alpha \sup_{t\in[t_0,T]} \left\|x^{1}(t)\right\| \left\{ \Gamma + \sup_{t\in[t_0,T]} e^{(t-t_0)\bar{\mu}} \int_{t_0}^{T} e^{-(s-t_0)\bar{\mu}} \|f(s)\| \, ds \right\}$$

where

$$\Gamma = \sup_{t\in[t_0,T]} \left| \frac{1}{\mu} \left(1 - e^{\mu(t-t_0)}\right) \right| \left\| e^{A(x_0)(t-t_0)} - I \right\| \cdot \|x_0\| \ . \square \tag{5.6}$$

From theorem (5.2) and lemma (5.4) we obtain results for the unforced system

$$\dot{x}(t) = A(x(t))x(t) \ , \ x(t_0) = x_0 \ . \tag{5.7}$$

In fact, setting $f = 0$, we obtain

Corollary (5.5) *If $A(x)$ satisfies the conditions (A1),(A2) in theorem (5.2) and x_0 is sufficiently small, so that*

$$\|x_0\| < \frac{1}{\alpha(T - t_0)} e^{-\mu(t-t_0)} \tag{5.8}$$

then the matrix function $A(x(t))$ generates a nonlinear (semi-) group $S(t, x_0)$ which exists on $[t_0, T]$ and $S(t, x_0) = \lim_{k\to\infty} x^{k}(t)$ (in $C([t_0, T], \mathbb{R}^n)$) where

$$\dot{x}^{k}(t) = A(x^{k-1}(t))x^{k}(t) \ , \ x^{k}(t_0) = x_0 \ .$$

Moreover, we have

$$\sup_{t\in[t_0,T]} \|S(t, x_0)\| \leq \left(1 + \frac{1}{1-\nu}\alpha\Gamma\right) \sup_{t\in[t_0,T]} e^{(t-t_0)\bar{\mu}} \|x_0\|$$

where $\nu = \alpha \|x_0\| (T - t_0)e^{(T-t_0)\mu}$.$\square$

The stability of the semigroup $S(t, x_0)$ can now be proved in the case where $\mu < 0$.

Theorem (5.6) *Under the conditions of corollary (5.5), if $\mu < 0$, then the semigroup $S(t, x_0)$ is stable.*

Proof We know that the sequence of functions x^{k} defined by

$$\dot{x}(t) = A(x(t))x(t) \ , \ x(t_0) = x_0 \ . \tag{5.9}$$

converges to the solution of (5.7) if x_0 satisfies (5.8). The solution of (5.9) satisfies

$$\begin{aligned} \left\|x^{k}(t)\right\| &\leq \exp\left[\int_{t_0}^{t} \mu\left(A(x^{k-1}(s))\right) ds\right] \\ &\leq e^{\mu(t-t_0)} \|x_0\| \end{aligned}$$

and the result follows.□

Next we consider the perturbed nonlinear system

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))x(t)\ ,\ x(0) = x_0$$

and the corresponding approximating sequence

$$\begin{aligned}
\dot{x}^{0}(t) &= A(x_0)x^{0}(t) + B(x_0)x^{0}(t)\ ,\ x^{0}(0) = x_0 \\
\dot{x}^{i}(t) &= A(x^{i-1}(t))x^{i}(t) + B(x^{i-1}(t))x^{i}(t)\ ,\ x^{i}(0) = x_0\ ,\ i \geq 1
\end{aligned} \tag{5.10}$$

Then we have

$$\begin{aligned}
x^{i}(t) - x^{i-1}(t) &= \int_0^t \Phi^{i-1}(t,s)B(x^{i-1}(s))(x^{i}(s) - x^{i-1}(s))ds \\
&+ \int_0^t \Phi^{i-1}(t,s)\left(B(x^{i-1}(s)) - B(x^{i-2}(s))\right)x^{i-1}(s)ds \\
&+ \int_0^t \left(\Phi^{i-1}(t,s) - \Phi^{i-2}(t,s)\right)B(x^{i-2}(s))x^{i-1}(s)ds
\end{aligned}$$

and so, by lemma (5.1), if

$$\begin{array}{lcc}
\overline{(A1)} & \mu(A(x)) \leq \mu & \\
\overline{(A2)} & \|A(x) - A(y)\| \leq \alpha\,\|x - y\| & \forall\ x, y \in \mathbb{R}^n \\
\overline{(A3)} & \|B(x) - B(y)\| \leq \beta\,\|x - y\| & \forall\ x, y \in \mathbb{R}^n \\
\overline{(A4)} & \|B(x)\| \leq \gamma & \forall\ x \in \mathbb{R}^n
\end{array}$$

then we have

$$\begin{aligned}
\xi^{i}(t) &\leq \int_0^t e^{\mu(t-s)}\gamma\xi^{i-1}(s)ds + \int_0^t e^{\mu(t-s)}\beta\xi^{i-1}(s)e^{(\mu-\gamma)s}ds \\
&+ \int_0^t \alpha e^{\mu(t-s)}\xi^{i-1}(t)\gamma e^{(\mu-\gamma)s}ds
\end{aligned}$$

since

$$\left\|x^{i}(t)\right\| \leq e^{(\mu-\gamma)t}\,\|x_0\|$$

from (5.10). Hence,

$$\begin{aligned}
\xi^{i}(t) &\leq \xi^{i-1}(t)\left\{\int_0^t e^{\mu(t-s)}\gamma ds + \int_0^t e^{\mu(t-s)}\beta e^{(\mu-\gamma)s}ds + \int_0^t \alpha e^{\mu(t-s)}\gamma e^{(\mu-\gamma)s}ds\right\} \\
&= \lambda(t)\xi^{i-1}(t)
\end{aligned}$$

where

$$\lambda(t) = \frac{\gamma}{\mu}(e^{\mu t} - 1) + \left(\frac{\beta}{\gamma} + \alpha\right)(e^{(\mu-\gamma)t} - e^{\mu t})$$

and so, if $|\lambda(t)| < 1$ for $t \in [0, T]$ we have

$$x^{i}(t) \to x(t) \text{ on } C([t_0, T], \mathbb{R}^n).$$

As in (5.5), we have

$$\sup_{t\in[0,T]} \|x(t)\| \le \frac{1}{1-\nu}\xi^{2}(t) + \sup_{t\in[0,T]} \left\|x^{1}(t)\right\| \tag{5.11}$$

where

$$\nu = \sup_{t\in[0,T]} |\lambda(t)|\,.$$

Lemma (5.7) *Under the assumptions* $\overline{(A1)} - \overline{(A4)}$ *we have*

$$\left\|x^{1}(t)\right\| \le e^{(\bar{\mu}-\gamma)t}\,\|x_0\|$$

and

$$\begin{aligned}\left\|x^{2}(t) - x^{1}(t)\right\| &\le \left(e^{(\mu-\gamma)t} + e^{(\bar{\mu}-\gamma)t}\right)\|x_0\| \\ &\le 2e^{(\mu-\gamma)t}\,\|x_0\|\end{aligned}$$

(since $\bar{\mu} \le \mu$).
Proof This follows from Gronwall's lemma and the inequality

$$\left\|x^{2}(t)\right\| \le e^{(\mu-\gamma)t}\,\|x_0\|\ .\square$$

Corollary (4.8) Under the assumptions $\overline{(A1)} - \overline{(A4)}$ we have

$$\sup_{t\in[0,T]} \|x(t)\| \le \left(\frac{2}{1-\nu} + 1\right) e^{(\mu-\gamma)t}\,\|x_0\|\ .$$

Proof This follows from lemma (4.7) and (4.11).$\square$
Remark (5.9) If $\mu + \gamma < 0$, then from corollary (5.8) we have

$$\sup_{t\in[0,T]} \|x(t)\| \le \left(\frac{2}{1-\nu} + 1\right)\|x_0\|\ .$$

Hence, although we have assumed conditions $\overline{(A1)} - \overline{(A4)}$ for all $x \in \mathbb{R}^n$, it is clear that we only require these conditions to hold in the ball

$$B_{x_0} = \left\{x : \|x\| \le \left(\frac{2}{1-\nu} + 1\right)\|x_0\|\right\}.$$

This means that the results apply to polynomial systems which, of course, do not satisfy $\overline{(A1)} - \overline{(A4)}$ on the whole of $\mathbb{R}^n$.□

We now return to the study of semisimple systems, so we assume that $\mathcal{L}_A$ is a semisimple Lie algebra and we can then write the system in the form

$$\dot{x} = H(x)x + \left(\sum_{\phi \in \Delta} e_\phi(x) E_\phi \right) x \tag{5.12}$$

where $H(x) \in \mathfrak{h}$ (a Cartan subalgebra of $\mathcal{L}_A$) and Δ is the set of (nonzero) roots of $\mathcal{L}_A$. Since $\mathfrak{h}$ is commutative we can always choose a basis so that $H(x)$ is diagonal. Moreover, by semisimplicity we can write $x = x^1 \ominus \cdots \ominus x^k$ where x^i belongs to a simple component of $\mathcal{L}_A$, so we consider a simple system of the form (dropping the superscript i on x^i):

$$\dot{x} = \begin{pmatrix} \lambda_1(x) & & \\ & \ddots & \\ & & \lambda_r(x) \end{pmatrix} x + \left(\sum_{\phi \in \Delta} e_\phi(x) E_\phi \right) x.$$

In order to apply the above results we assume that

(i)	$\max_i \lvert\lambda_i(x)\rvert < \mu$	for all $x \in \mathbb{R}^n$
(ii)	$\max_{1 \le j \le r} \lvert\lambda_j(x) - \lambda_j(y)\rvert \le \alpha \lVert x - y \rVert$	for all $x, y \in \mathbb{R}^n$
(iii)	$\lvert e_\phi(x) - e_\phi(y)\rvert \le \beta_\phi \lVert x - y \rVert$	for all $x, y \in \mathbb{R}^n$
(iv)	$\lvert e_\phi(x)\rvert \le \gamma_\phi$	for all $x \in \mathbb{R}^n$

Let

$$\beta = \sum_{\phi \in \Delta} \beta_\phi \lVert E_\phi \rVert \ , \ \gamma = \sum_{\phi \in \Delta} \gamma_\phi \lVert E_\phi \rVert .$$

Then by corollary (5.8) and the discussion before it we have

Theorem (5.10) The system (5.12) may be approximated by the sequence of equations

$$\dot{x}^{[i]}(t) = \begin{pmatrix} \lambda_1(x^{[i-1]}(t)) & & \\ & \ddots & \\ & & \lambda_r(x^{[i-1]}(t)) \end{pmatrix} x^{[i]}(t) + \left(\sum_{\phi \in \Delta} e_\phi(x^{[i-1]}(t)) E_\phi \right) x^{[i]}(t)$$

if the conditions $\overline{(A1)} - \overline{(A4)}$ hold for the constants μ, α, β and γ defined in (i)-(iv) above and, moreover, $x^{[i]}(t) \to x(t)$ in $C([0,T], \mathbb{R}^n)$ where T is such that

$$|\lambda(t)| = \left| \frac{\gamma}{\mu}(e^{\mu t} - 1) + \left(\frac{\beta}{\gamma} + \alpha \right) (e^{(\mu - \gamma)t} - e^{\mu t}) \right| < 1.$$

In this case we have

$$\sup_{t \in [0,T]} \lVert x(t) \rVert \le \left(\frac{2}{1 - \sup_{t \in [0,T]} |\lambda(t)|} + 1 \right) e^{(\mu - \gamma)t} \lVert x_0 \rVert \ .\square$$

Note that we cannot apply the result in remark (5.9) to semisimple systems, since tr $H(x) = 0$ and so if some real parts of the eigenvales are negative, some must be positive. However, suppose the system has a Levi decomposition in the form

$$\dot{x} = S(x)x + H(x)x + \left(\sum_{\phi \in \Delta} e_\phi(x) E_\phi \right) x$$

where $S(x)$ is the solvable part of $\mathcal{L}_A$. Suppose that the nonsingular matrix P upper triangularises $S(x)$; then we have

$$\begin{aligned} \dot{y}(t) &= \begin{pmatrix} \zeta_1(P^{-1}y) & & \\ & \ddots & \\ & & \zeta_r(P^{-1}y) \end{pmatrix} y(t) + N(P^{-1}y)y(t) + P^{-1}H(P^{-1}y)Py(t) + \\ & \left(\sum_{\phi \in \Delta} e_\phi(P^{-1}y)P^{-1}E_\phi P \right) y(t) \end{aligned} \tag{5.13}$$

where $P^{-1}S(x)P = \text{diag }(\zeta_1(x), \cdots, \zeta_r(x)) + N(x)$ and $N(x)$ is nilpotent.To prove a stability theorem for this system we need the following simple result:

Lemma (5.11) *Consider the system*

$$\dot{x} = \begin{pmatrix} \lambda_1(x) & & \\ & \ddots & \\ & & \lambda_r(x) \end{pmatrix} x + H(x)x \ , \ x(0) = x_0$$

where all the matrices $H(x)$ commute and $\max_{1 \le i \le r} \lambda_i(x) < -\lambda < 0$ for some $\lambda > 0$. Then if Q diagonalises $H(x)$ (independently of x) and

$$\lambda > \|Q\| \cdot \left\|Q^{-1}\right\| \left(\sup_{x \in \Re^r} \max_{1 \le i \le r} |\mu_i(x)| \right)$$

where $\mu_1(x), \cdots, \mu_r(x)$ are the eigenvalues of $H(x)$, then the system is asymptotically stable and $\|x(t)\| \le e^{-2(\lambda-\mu)t} \|x_0\|$, where

$$\mu = \|Q\| \cdot \left\|Q^{-1}\right\| \left(\sup_{x \in \Re^r} \max_{1 \le i \le r} |\mu_i(x)| \right).$$

Proof Simply multiply the equation by x, i.e.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|x\|^2 &= x^T \dot{x} = x^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} x + x^T H(x) x \\ &\le -\lambda \|x\|^2 + \|x\|^2 \|H(x)\| . \square \end{aligned}$$

(Of course, if $Q = I$ and so $H(x)$ is already diagonal, then we simply require $\sup_{x\in\Re^r} \max_{1\le i\le r}(\lambda_i(x) + \mu_i(x)) < 0$.)

Now we return to equation (5.13). Suppose that

(i) $\max_{1\le i\le r} |\lambda_i(P^{-1}x) - \lambda_i(P^{-1}y)| \le \alpha \|P^{-1}\| \cdot \|x-y\|$

(ii) $\|N(P^{-1}x) - N(P^{-1}y)\| \le n_1 \|P^{-1}\| \cdot \|x-y\|$, $\|N(x)\| \le n_2$ for all $x, y \in \mathbb{R}^r$

(iii) $|e_\phi(P^{-1}x) - e_\phi(P^{-1}y)| \le \beta_\phi \|x-y\|$, $|e_\phi(x)| \le \gamma_\phi$ for all $x, y \in \mathbb{R}^r$

and set

$$\beta = \sum_{\phi\in\Delta} \beta_\phi \left\|P^{-1}E_\phi P\right\| + n_1 \left\|P^{-1}\right\| \ , \ \gamma = \sum_{\phi\in\Delta} \gamma_\phi \left\|P^{-1}E_\phi P\right\| + n_2.$$

Moreover, suppose that Q diagonalises $P^{-1}H(x)P^{-1}$, for all $x \in \mathbb{R}^r$ and that

(iv) $\max_{1\le i\le r} \zeta_i(x) < -\zeta < 0$ and

$$\bar{\zeta} \triangleq \zeta - \|Q\| \cdot \left\|Q^{-1}\right\| \left(\sup_{x\in\Re^r} \max_{1\le i\le r} |\mu_i(x)| \right) > 0$$

where $\mu_1(x), \cdots, \mu_r(x)$ are the eigenvalues of $H(x)$.Then we have

Theorem (5.12) *Under the conditions (i)-(iv), if $\bar{\zeta} > \gamma$ and $\frac{\gamma}{\bar{\zeta}} + \frac{\beta}{\gamma} + \alpha \|P^{-1}\| < 1$, then the equation (4.1) is asymptotically stable.*□

6. Example Systems

In this section we shall consider a number of simple examples and study their dynamical behaviour.

First we shall examine systems with Lie algebra $so(3, \mathbb{R})$.These are of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -f_1(x) & f_2(x) \\ f_1(x) & 0 & -f_3(x) \\ -f_2(x) & f_3(x) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \triangleq R(x)x \qquad (6.1)$$

and have generators

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \ M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \ M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$\begin{aligned} \frac{d}{dt}(x_1^2 + x_2^2 + x_3^2) &= 2x_1(-f_1(x)x_2 + f_2(x)x_3) + 2x_2(f_1(x)x_1 - f_3(x)x_3) + 2x_3(-f_2(x)x_1 + f_3(x)x_2) \\ &= 0 \end{aligned}$$

so these systems have invariant 'energy' $x_1^2 + x_2^2 + x_3^2$. The equilibrium points are given by the solutions of the equations

$$\begin{aligned} f_2(x)x_3 &= f_1(x)x_2 \\ f_1(x)x_1 &= f_3(x)x_3 \end{aligned}$$

(The third equation $f_3(x)x_2 = f_2(x)x_1$ is dependent on the first two.) Generically, this will be a (union) of lines although it may degenerate to a plane (or trivially a version of $\mathbb{R}^3$). Hence, generically, the intersection of the equilibrium curve with an invariant sphere will be a finite number of equilibria. For example, if

$$f_1(x) = x_1, f_2(x) = f_3(x) = 1,$$

then the equilibria are given by

$$x_1^2 = x_3 = x_1 x_2$$

i.e. by the parabola (x_1, x_1, x_1^2). This intersects each invariant sphere in two points, one of which is stable and one of which is unstable. The fact that $\|x\|^2 = x_1^2 + x_2^2 + x_3^2$ is invariant leads to the following:

Lemma (6.2) *Suppose the three-dimensional system $\dot{x} = A(x)x$ can be written in the form*

$$\dot{x} = \bar{A}(x)x + R(x)x$$

where $\bar{A}(x) + R(x)$ belongs to a Levi decomposition of $\mathcal{L}_A$ and $R(x)$ is as in equation (6.1); then the origin is asymptotically stable if $x^T \bar{A}(x)x < 0$ for all $x \neq 0$.

Proof Consider the Lyapunov function $V = \|x\|^2$. Then

$$\dot{V} = 2x^T\dot{x} = 2x^T(\bar{A}(x)x + R(x)x) < 0$$

since $x^T R(x)x = 0$.□

For example, the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\varepsilon x_1^p & -f_1(x) & f_2(x) \\ f_1(x) & -\varepsilon x_2^p & -f_3(x) \\ -f_2(x) & f_3(x) & -\varepsilon x_3^p \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is stable for any $\varepsilon > 0$, any positive integer p and any functions f_1, f_2, f_3, since in this case $x^T \bar{A}(x)x = -\varepsilon(x_1^{p-2} + x_2^{p-2} + x_3^{p-2})$. Similarly, the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} - f_1(x) & \bar{a}_{13} + f_2(x) \\ \bar{a}_{21} + f_1(x) & \bar{a}_{22} & \bar{a}_{23} - f_3(x) \\ \bar{a}_{31} - f_2(x) & \bar{a}_{32} + f_3(x) & \bar{a}_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

is stable for any negative definite matrix $\bar{A} = (\bar{a}_{ij})$.

More generally, note that if the system can be written in the form

$$\dot{x} = \begin{pmatrix} 0 & -\frac{\partial V}{\partial x_3} & \frac{\partial V}{\partial x_2} \\ \frac{\partial V}{\partial x_3} & 0 & -\frac{\partial V}{\partial x_1} \\ -\frac{\partial V}{\partial x_2} & \frac{\partial V}{\partial x_1} & 0 \end{pmatrix} x$$

then the system has invariant sets $V = const.$, since

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x_1}\left(-\frac{\partial V}{\partial x_3}x_2 + \frac{\partial V}{\partial x_2}x_3\right) + \frac{\partial V}{\partial x_2}\left(\frac{\partial V}{\partial x_3}x_1 - \frac{\partial V}{\partial x_1}x_3\right) + \\ &\quad \frac{\partial V}{\partial x_3}\left(-\frac{\partial V}{\partial x_2}x_1 + \frac{\partial V}{\partial x_1}x_2\right) \\ &= 0.\end{aligned}$$

Hence the system

$$\dot{x} = \bar{A}(x)x + \begin{pmatrix} 0 & -\frac{\partial V}{\partial x_3} & \frac{\partial V}{\partial x_2} \\ \frac{\partial V}{\partial x_3} & 0 & -\frac{\partial V}{\partial x_1} \\ -\frac{\partial V}{\partial x_2} & \frac{\partial V}{\partial x_1} & 0 \end{pmatrix} x$$

is stable if $V > 0, x \neq 0$ and $g^T(x)\bar{A}(x)x < 0$, where $g(x) = \operatorname{grad} V$, since then V is a Lyapunov function. Conversely, it is easy to see that any stable three-dimensional system may be written in this form (since, by Lyapunov's theorem, any stable system has a Lyapunov function). Hence we have proved

Lemma (6.3) *Any three-dimensional system is stable if and only if it may be written in the form*

$$\dot{x} = (\bar{A}(x) + B(x))x$$

where

$$B(x) = \begin{pmatrix} 0 & -\frac{\partial V}{\partial x_3} & \frac{\partial V}{\partial x_2} \\ \frac{\partial V}{\partial x_3} & 0 & -\frac{\partial V}{\partial x_1} \\ -\frac{\partial V}{\partial x_2} & \frac{\partial V}{\partial x_1} & 0 \end{pmatrix}$$

for some positive function V, and $\left\langle \operatorname{grad} V, \bar{A}(x)\right\rangle < 0$. The set $\{\bar{A}(x) + B(x) : x \in \mathbb{R}^3\}$ induces a Levi decomposition of $\mathcal{L}_A$ in an obvious way.□

Lemma (6.3) can be generalised to higher-dimensional systems by considering equations of the form

$$\dot{x} = \bar{A}(x)x + R(x)x$$

although we must now take $R(x)$ in the form

$$R(x) = \begin{pmatrix} 0 & \frac{\partial V}{\partial x_2}x_1 & \frac{\partial V}{\partial x_3}x_1 & \frac{\partial V}{\partial x_4}x_1 & \cdots & \frac{\partial V}{\partial x_n}x_1 \\ -\frac{\partial V}{\partial x_1}x_2 & 0 & \frac{\partial V}{\partial x_3}x_2 & \frac{\partial V}{\partial x_4}x_2 & \cdots & \frac{\partial V}{\partial x_n}x_2 \\ -\frac{\partial V}{\partial x_1}x_3 & & 0 & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \frac{\partial V}{\partial x_n}x_{n-1} \\ -\frac{\partial V}{\partial x_1}x_n & \cdots & \cdots & \cdots & -\frac{\partial V}{\partial x_{n-1}}x_n & 0 \end{pmatrix}.$$

As a further interesting example, consider the Lorentz attractor dynamics

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1+\lambda & -1 & -x_1 \\ 0 & x_1 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This has the Levi decomposition

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma+(1+\lambda) & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 & -(1+\lambda) & 0 \\ 1+\lambda & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The semisimple part

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & -(1+\lambda) & 0 \\ 1+\lambda & 0 & -x_1 \\ 0 & x_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

produces invariant dynamics on any sphere. For stability, lemma (6.1) gives the condition

$$\left(\frac{\sigma+(1+\lambda)}{2} \right)^2 < \sigma$$

which is fairly conservative. Finding V to apply lemma (6.2) is more difficult!

Using a similar Levi decomposition approach, it is easy to find new chaotic systems; for example, the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} -3 & 228 & -10 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & -3 & 3x_3 & 0 \\ 3 & 0 & x_2 & -x_1 \\ -3x_3 & -x_2 & 0 & x_4 \\ 0 & x_1 & -x_4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

has a chaotic attractor shown in various projections in figures 9.1-9.6. The semisimple part

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 3x_3 & 0 \\ 3 & 0 & x_2 & -x_1 \\ -3x_3 & -x_2 & 0 & x_4 \\ 0 & x_1 & -x_4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

has invariant spheres of the form $x_1^2 + x_2^2 + x_3^2 + x_4^2 = const.$

We next consider the case of C_n systems. Since C_n is equal to the space of symplectic matrices, it is natural to ask which systems of the form

$$\dot{x} = A(x)x \ , \ x \in \mathbb{R}^n \ , \ n \text{ even}$$

where $A(x) \in C_{n/2}$ are Hamiltonian systems. Of course, if $A(x) = A$ is constant then the system is trivially Hamiltonian. However, a general Hamiltonian system cannot be written in this form. For example, if $n = 2$, then we have

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned} \tag{6.2}$$

for some Hamiltonian $H(x, y)$. If H is analytic, then we can write

$$H(x,y) = \sum_{i,j=0}^{\infty} h_{ij} x^i y^j$$

and so

$$\frac{\partial H}{\partial x} = \sum_{i=1,,j=0}^{\infty} h_{ij} i x^{i-1} y^j \ , \ \frac{\partial H}{\partial y} = \sum_{i=0,,j=1}^{\infty} h_{ij} j x^i y^{j-1}.$$

Thus, equation (6.2) can be written in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & -a(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6.3}$$

if and only if the term in $\frac{\partial H}{\partial y}$ with no y term

$$\sum_{i=0}^{\infty} h_{i1} x^i$$

has an x factor, i.e. $h_{01} = 0$ and similarly, $h_{10} = 0$. Hence, it can be written in the form (6.3) if and only if H has no linear term. In general, the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} A(x,y) & B(x,y) \\ C(x,y) & -A^T(x,y) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6.4}$$

where $x, y \in \mathbb{R}^n$ and $B^T = B, C^T = C$, can be written in the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned}$$

if and only if

$$\begin{aligned} \frac{\partial}{\partial x_j} (A(x,y)x + B(x,y)y)_i &= \frac{\partial}{\partial x_i} (A(x,y)x + B(x,y)y)_j \\ \frac{\partial}{\partial y_j} (A(x,y)x + B(x,y)y)_i &= -\frac{\partial}{\partial x_i} \left(C(x,y)x - A^T(x,y)y\right)_j \\ \frac{\partial}{\partial y_j} \left(C(x,y)x - A^T(x,y)y\right)_i &= \frac{\partial}{\partial y_i} \left(C(x,y)x - A^T(x,y)y\right)_j \end{aligned}$$

for all $i, j \in \{1, \cdots, n\}$, by Poincaré's lemma. Hence, if we write

$$A(x,y) = \sum_{|\mathbf{p}|=0}^{\infty} \sum_{|\mathbf{q}|=0}^{\infty} A^{\mathbf{pq}} x^{\mathbf{p}} y^{\mathbf{q}}\,, \ B(x,y) = \sum_{|\mathbf{p}|=0}^{\infty} \sum_{|\mathbf{q}|=0}^{\infty} B^{\mathbf{pq}} x^{\mathbf{p}} y^{\mathbf{q}}\,, \ C(x,y) = \sum_{|\mathbf{p}|=0}^{\infty} \sum_{|\mathbf{q}|=0}^{\infty} C^{\mathbf{pq}} x^{\mathbf{p}} y^{\mathbf{q}}$$

then we obtain

Lemma (6.4) *The system (6.4) is Hamiltonian if and only if the soefficient matrices $A^{\mathbf{pq}}$, $B^{\mathbf{pq}}$ and $C^{\mathbf{pq}}$ satisfy the relations*

$$\begin{aligned}
\sum_k \left(A^{\mathbf{pq}}_{ik}\delta_{kj} + p_j B^{\mathbf{p}-\mathbf{1}_j,\mathbf{q}-\mathbf{1}_k}_{ik} + p_j A^{\mathbf{p}-\mathbf{1}_j-\mathbf{1}_k,\mathbf{q}}_{ik} \right) &= \sum_k \left(A^{\mathbf{pq}}_{jk}\delta_{kj} + p_i B^{\mathbf{p}+\mathbf{1}_i,\mathbf{q}-\mathbf{1}_k}_{jk} + p_i A^{\mathbf{p}+\mathbf{1}_i-\mathbf{1}_k,\mathbf{q}}_{jk} \right) \\
\sum_k \left(B^{\mathbf{pq}}_{ik}\delta_{kj} + q_j A^{\mathbf{p}-\mathbf{1}_k,\mathbf{q}-\mathbf{1}_j}_{ik} + q_j B^{\mathbf{p},\mathbf{q}-\mathbf{1}_j-\mathbf{1}_k}_{ik} \right) &= -\sum_k \left(C^{\mathbf{pq}}_{jk}\delta_{kj} - p_i A^{\mathbf{p}-\mathbf{1}_i,\mathbf{q}-\mathbf{1}_k}_{kj} + p_i C^{\mathbf{p}+\mathbf{1}_{ii}-\mathbf{1}_k,\mathbf{q}}_{jk} \right) \\
\sum_k \left(-A^{\mathbf{pq}}_{ki}\delta_{kj} + q_j C^{\mathbf{p}-\mathbf{1}_k,\mathbf{q}-\mathbf{1}_j}_{ik} - q_j A^{\mathbf{p},\mathbf{q}-\mathbf{1}_j-\mathbf{1}_k}_{ki} \right) &= \sum_k \left(-A^{\mathbf{pq}}_{kj}\delta_{ki} + q_i C^{\mathbf{p}-\mathbf{1}_k,\mathbf{q}+\mathbf{1}_i}_{jk} - q_i A^{\mathbf{p},\mathbf{q}-\mathbf{1}_i-\mathbf{1}_k}_{kj} \right)
\end{aligned}$$

where $\mathbf{1}_i = (0, 0, \cdots, 0, \underbrace{1}_{i}, 0, \cdots, 0)$. Moreover, the Hamiltonian is given by

$$H(x,y) = \oint_L (A(x,y)x + B(x,y)y)\, dy + \oint_L \left(-C(x,y)x + A^T(x,y)y \right) dx$$

where the integral is taken over any path L in (x,y)-space.□

For example, the conditions of lemma (6.4) can easily be checked (using MAPLE) to hold for the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y & x + 4y^2 \\ -2 & -y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with Hamiltonian $H(x,y) = x^2 + xy^2 + y^4$.

Finally we consider the G_2 system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \\ \dot{x}_7 \end{pmatrix} = \begin{pmatrix} -3 & 28 & -10 & & & & \\ & -1 & 6 & & & & \\ & & -3 & & & & \\ & & & -2 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} +$$

$$\begin{pmatrix} 0 & -3 & 3x_3 & 0 & -3 & 3x_3 & 0 \\ 3 & 0 & x_2 & -x_1 & 0 & 0 & \frac{3}{\sqrt{2}}x_3 \\ -3x_3 & -x_2 & 0 & x_4 & 0 & 0 & \frac{3}{\sqrt{2}} \\ 0 & x_1 & -x_4 & -2 & -\frac{3}{\sqrt{2}}x_3 & -\frac{3}{\sqrt{2}} & 0 \\ 3 & 0 & 0 & \frac{3}{\sqrt{2}}x_3 & 0 & 0 & 0 \\ -3x_3 & 0 & 0 & \frac{3}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & -\frac{3}{\sqrt{2}}x_3 & -\frac{3}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

(This is inspired by the four dimensional chaotic system above; clearly many more systems of this form are possible.) The dynamics are again chaotic with two views shown in figures 9.7 and 9.8.(A more detailed analysis and a proof of the chaotic behaviour of these systems will be given in a further paper [10].)

7. Conculsions

A new classification theory of nonlinear differential equations is presented, based on a Lie algebra associated with the system. Moreover, an approximation technique for the solution of the equations is introduced, which leads to time-varying linear approximations which are arbitrarily close to the true system. The semisimple part of the Lie algebra generated by the equations gives rise to invariant dynamics and the remaining solvable part can be used to create chaotic motion. A number of examples based on simple Lie algebras is given, showing that a more detailed analysis of F_4, E_6, E_7 and E_8 systems would be interesting. This will be done in a future paper.

8. References

[1] A.A.Sagle and R.E.Walde.(1973) *Introduction to the theory of Lie groups and Lie algebras*, New York:Academic Press.
[2] N.Jacobson. (1962) *Lie algebras*, New York:Wiley.
[3] S.P.Banks and S.K.Al-Jurani. (1994) Lie algebras and the stability of nonlinear systems. Int. J. Control, **60**,315-329.
[4] S.P.Banks and S.K.Al-Jurani. (1996) Pseudo-linear systems, Lie algebras and stability, IMA J. Math. Cont. & Inf., **13**,385-401.
[5] S.Varadarahan (1976) *Lie groups, Lie algebras and their representations*, New York:Wiley.
[6] S.P.Banks, A.Moser and D.McCaffrey (1995) Robust exponential stability of evolution equations, Arch. Cont. Sci., **4**(XL),261-279.
[7] Z-X Wan. (1975) *Lie algebras*, Oxford:Pergamon Press.
[8] E.Witt (1941) Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Abh. Math. Sem. Univ. Hamburg, **14**, 289-337.
[9] F.Brauer (1967) Perturbations of nonlinear systems of differential equations, II, J. Math. Anal. App., **17**, 418-434.
[10] S.P.Banks and D.McCaffrey. New chaotic systems based on simple Lie algebras. In preparation.

9. Figures

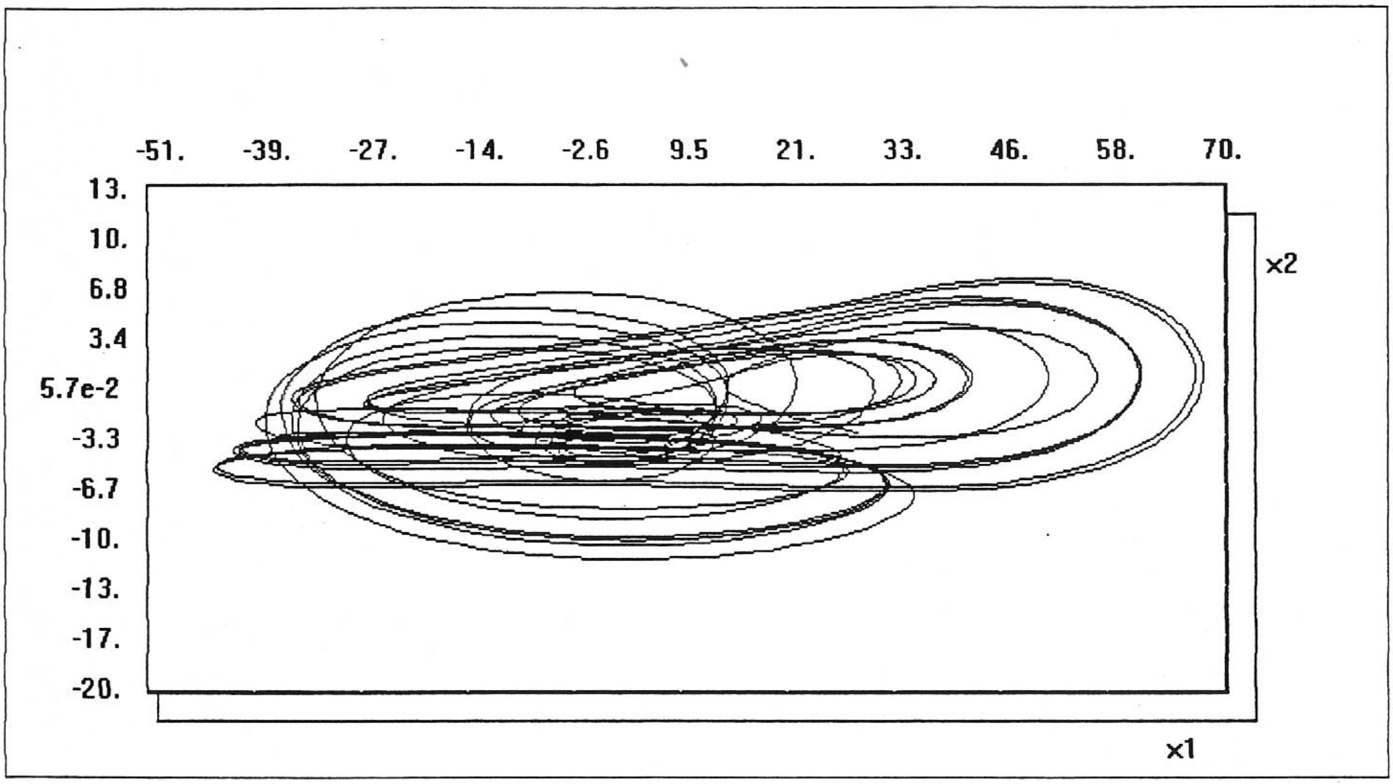

Figure 9.1: Chaotic Attractor in x1-x2

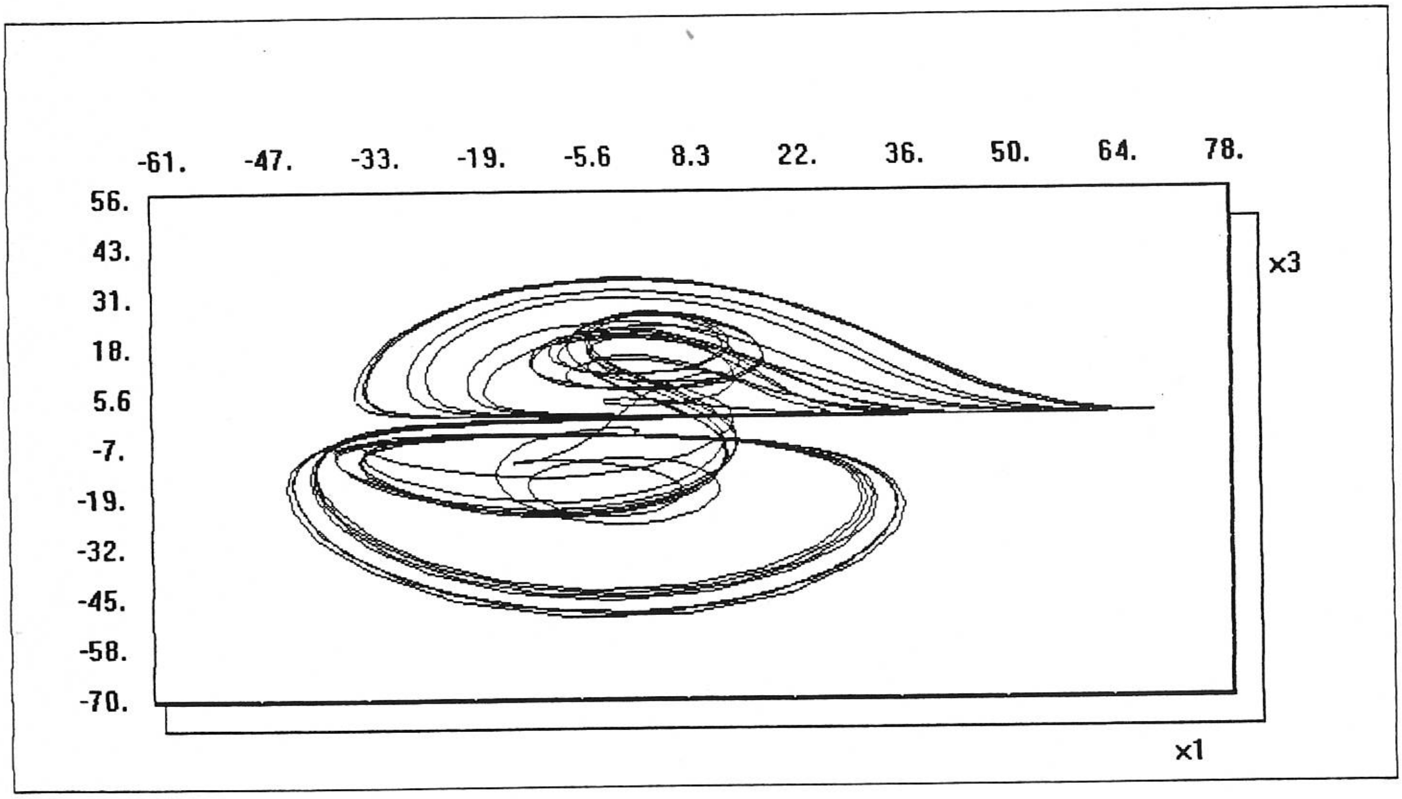

Figure 9.2: Chaotic Attractor in x1-x3

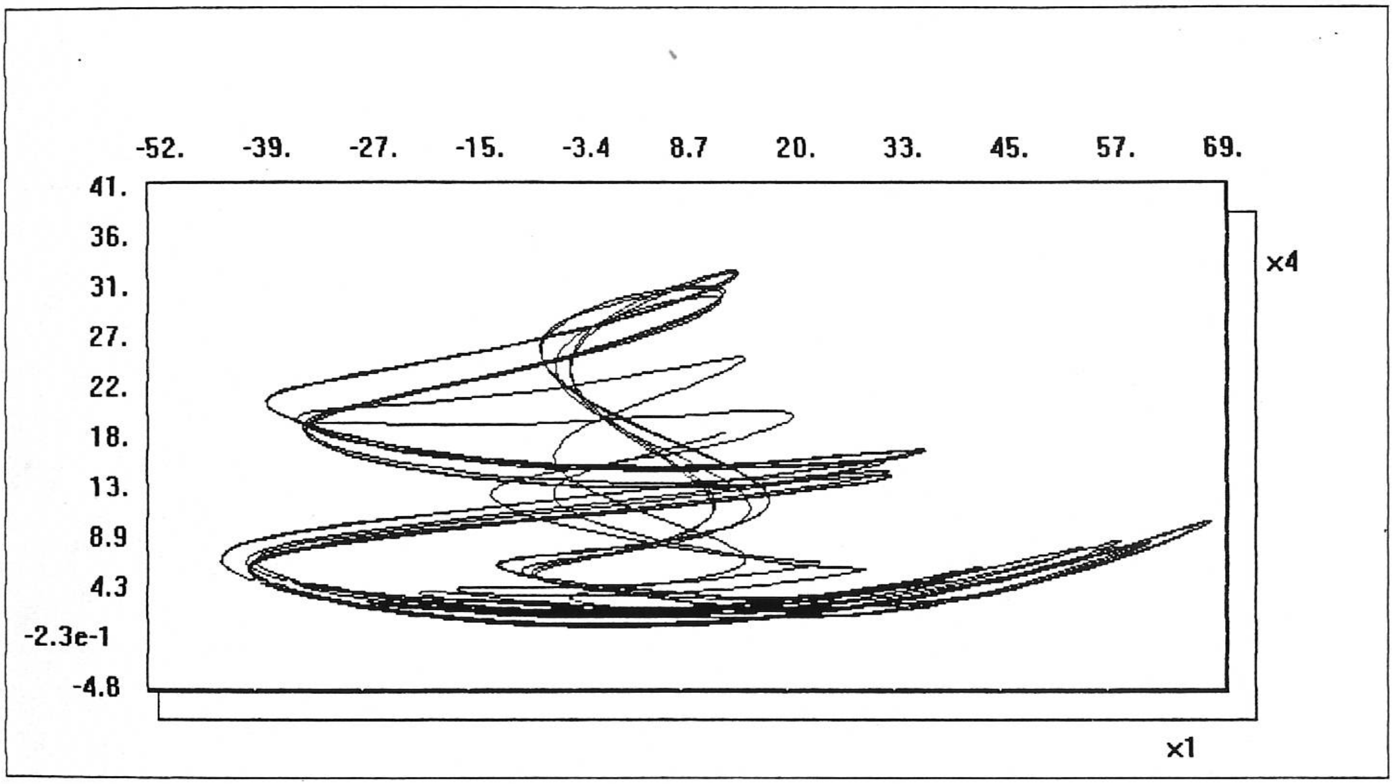

Figure 9.3: Chaotic Attractor in x1-x4

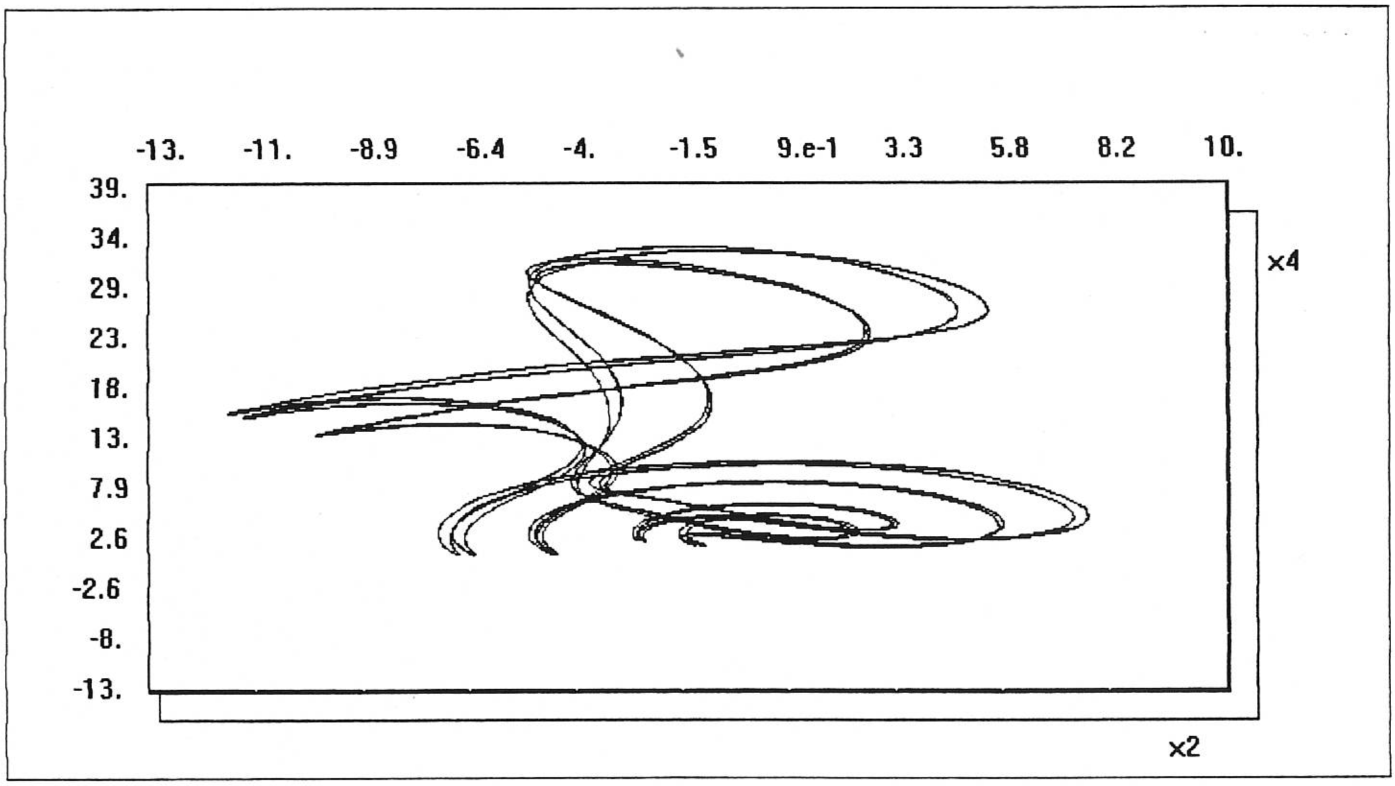

Figure 9.4: Chaotic Attractor in x2-x4

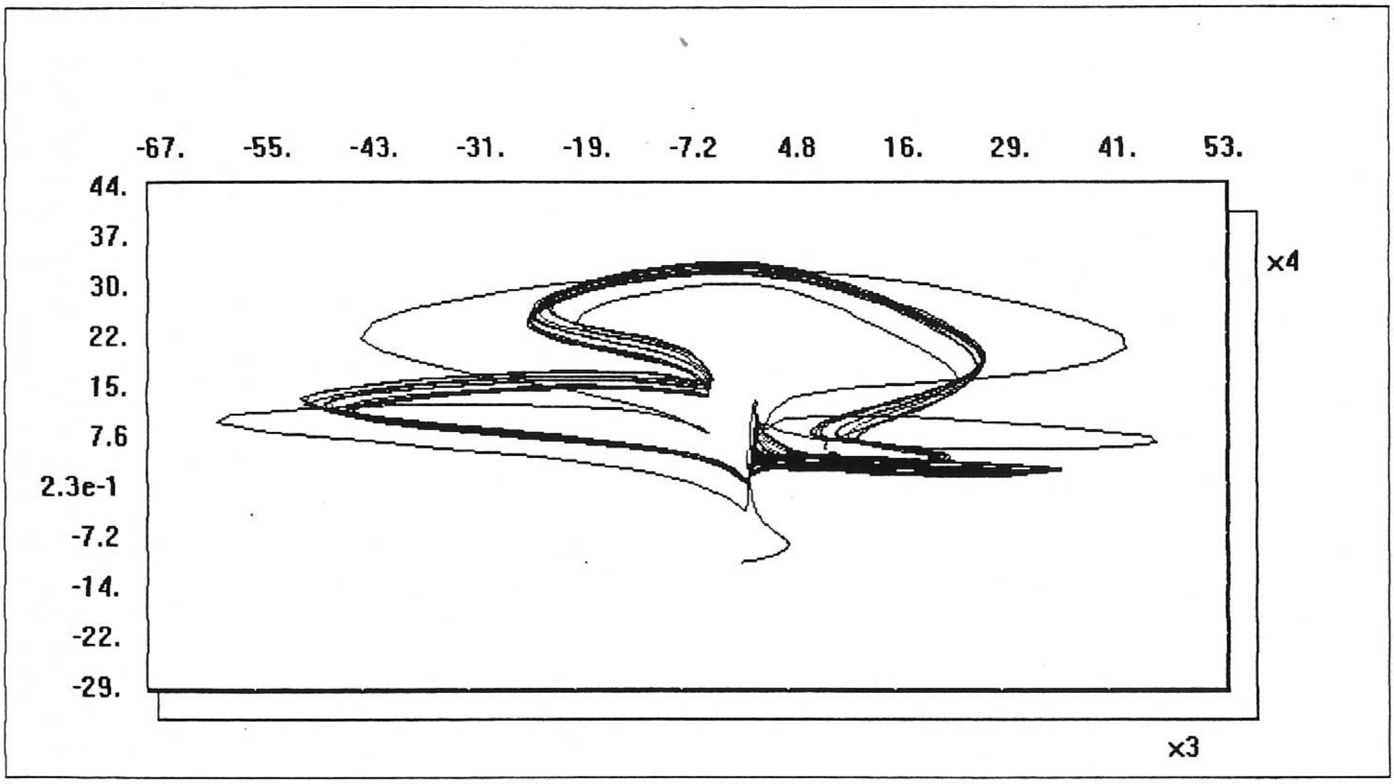

Figure 9.5: Chaotic Attractor in x3-x4

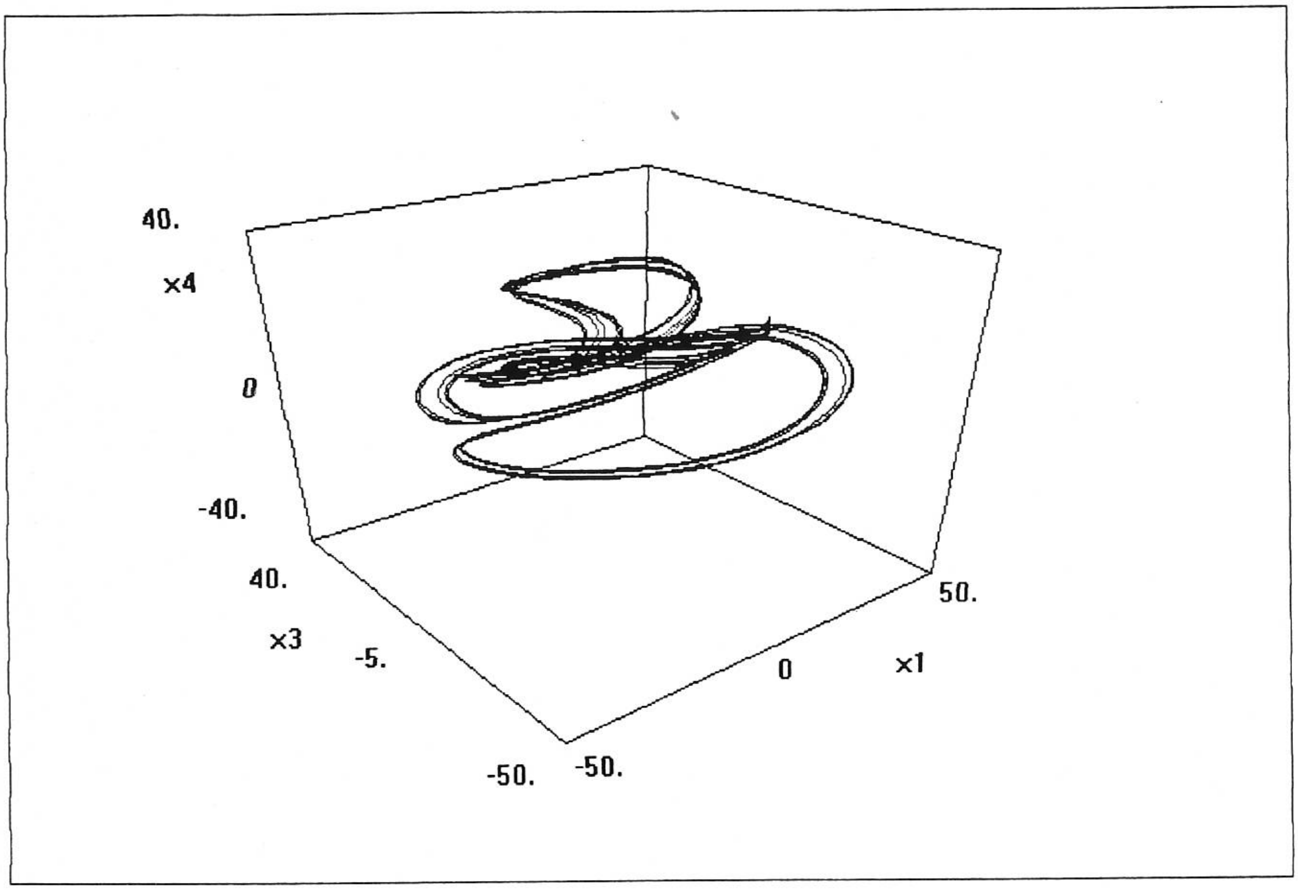

Figure 9.6: Chaotic Attractor in x1-x3-x4

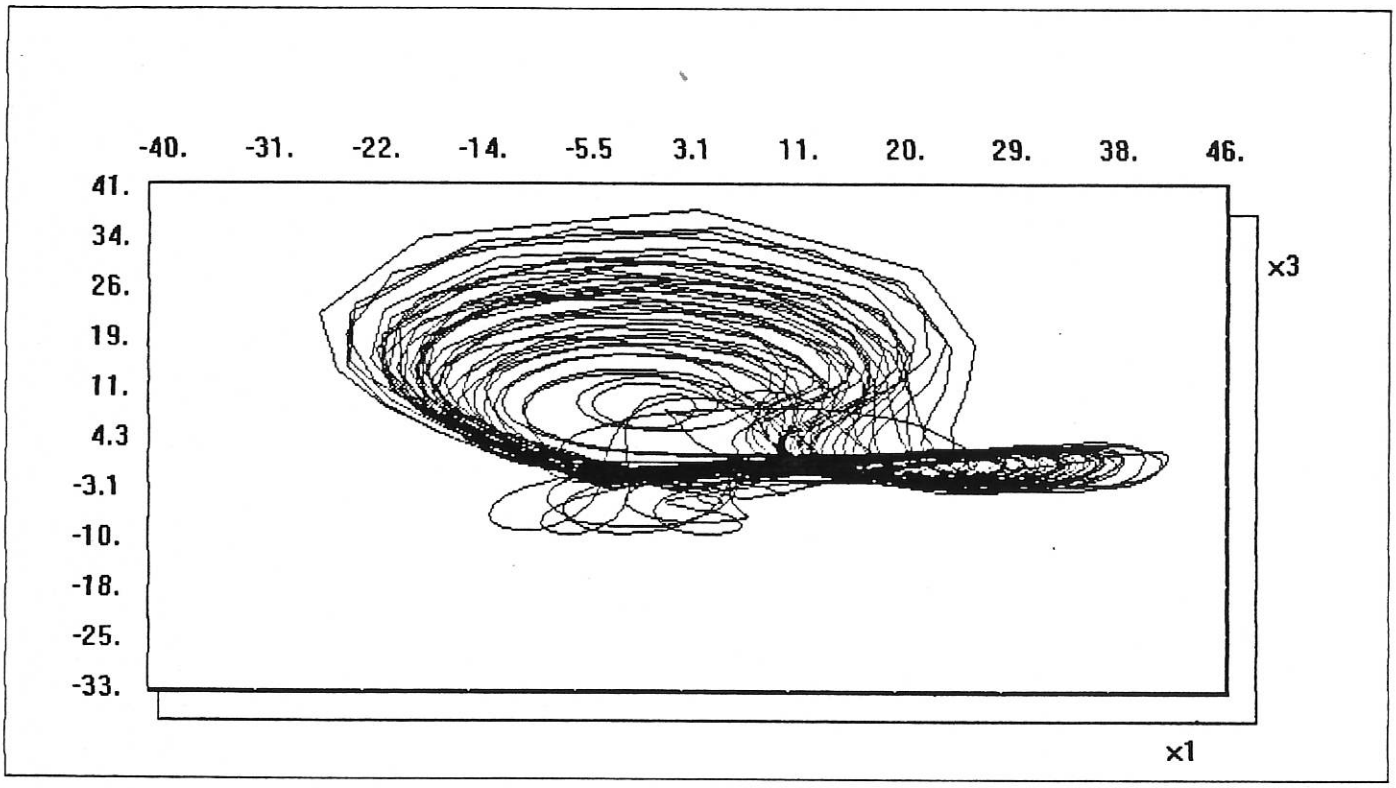

Figure 9.7: Chaotic Attractor for a G2 System

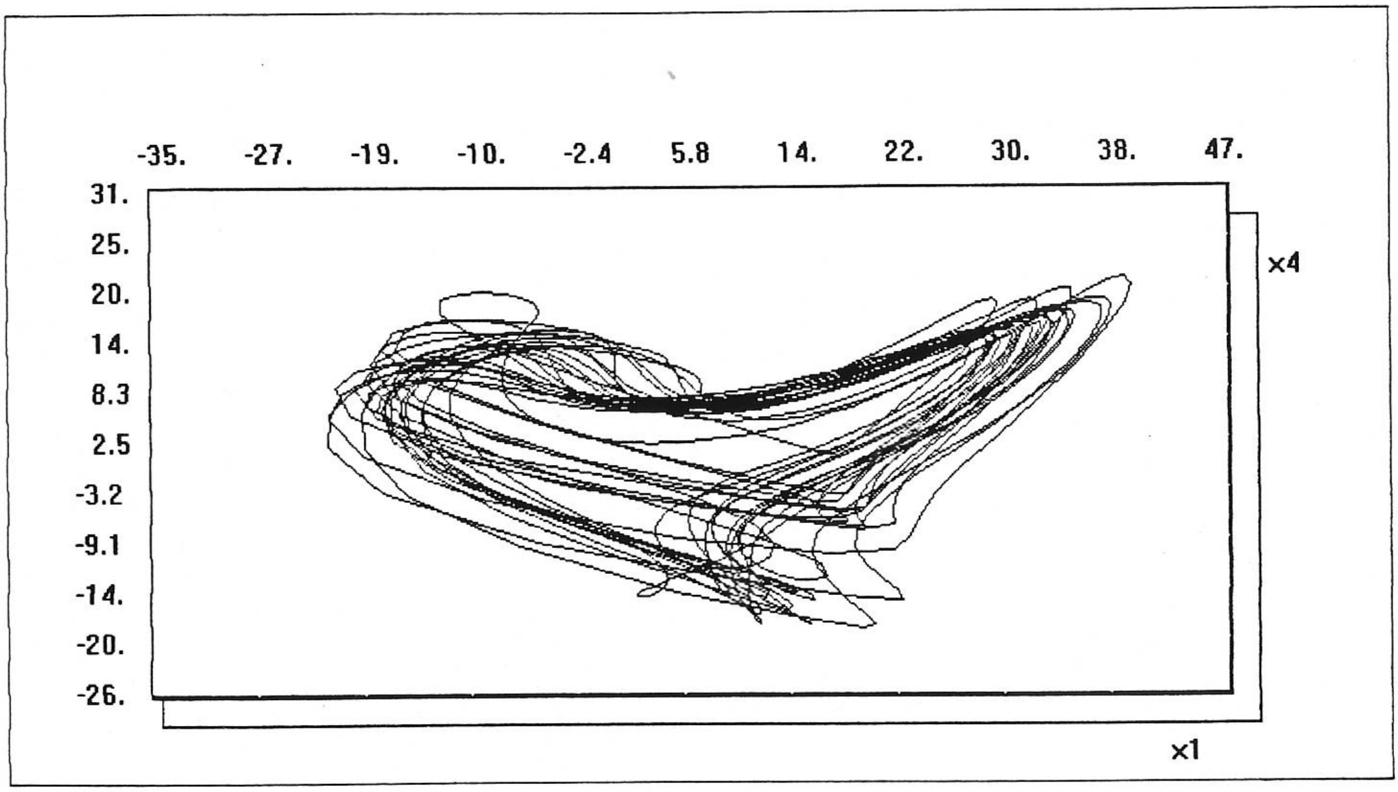

Figure 9.8: View in x1-x4