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A Topological Interpretation of Stability in Linear Optimal Control Problems

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Abstract— A simple linear optimal control problem is interpreted as that of finding Jacobi fields along a geodesic on a Riemannian manifold. In this context, stability can be interpreted as a Morse index condition on the initial manifold for the geodesic problem. It is indicated how this stability condition can be generalised to non-linear problems via the Maslov index.

Keywords— Stability, Optimal Control, Riccati Equation, Jacobi Equation, Morse Index

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1 Introduction

Consider the following simple linear optimal control problem: minimise

$$S = \frac{1}{2}\int_{t_0}^{t_f} (\mathbf{x}^T Q(t)\mathbf{x} + \mathbf{u}^T R(t)\mathbf{u})dt + \frac{1}{2}\mathbf{x}(t_f)^T F\mathbf{x}(t_f)$$

subject to

$$\dot{\mathbf{x}} = B(t)\mathbf{u}$$

where $Q(t)\epsilon\mathbf{R}^{n\times n}$, $R(t)\epsilon\mathbf{R}^{m\times m}$, $B(t)\epsilon\mathbf{R}^{n\times m}$, $F\epsilon\mathbf{R}^{n\times n}$, $\mathbf{x}\epsilon\mathbf{R}^n$ and $\mathbf{u}\epsilon\mathbf{R}^m$. We shall assume throughout that Q, R and F are symmetric and that B and R have full rank.

Following the method of Lagrange multipliers we introduce a vector $\mathbf{y}\epsilon\mathbf{R}^n$ and a Lagrangian

$$\overline{L} = \mathbf{y}^T(\dot{\mathbf{x}} - B\mathbf{u}) + \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \frac{1}{2}\mathbf{u}^T R\mathbf{u}.$$

This gives the associated Hamiltonian

$$H = \mathbf{y}^T \dot{\mathbf{x}} - \overline{L} = \mathbf{y}^T B\mathbf{u} - \frac{1}{2}\mathbf{x}^T Q\mathbf{x} - \frac{1}{2}\mathbf{u}^T R\mathbf{u}.$$

Pontryagin's maximum principle then says that the optimal control is that which maximises H. Since H has a unique critical point with respect to $\mathbf{u}$, the optimal control is given by

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \Leftrightarrow \mathbf{u} = R^{-1}B^T\mathbf{y}.$$

We are therefore led to maximising the following Hamiltonian

$$H = \frac{1}{2}\mathbf{y}^T B(t)R^{-1}(t)B^T(t)\mathbf{y} - \frac{1}{2}\mathbf{x}^T Q(t)\mathbf{x}. \tag{1}$$

The standard approach is to look for a solution of this Hamilton-Jacobi (or Bellman's) equation as a symmtric quadratic form in $\mathbf{x}$

$$S(\mathbf{x}, t) = \frac{1}{2}\mathbf{x}^T P(t)\mathbf{x}$$

where $\mathbf{y} = \partial S/\partial \mathbf{x} = P(t)\mathbf{x}$ and $H = -\partial S/\partial t = -\frac{1}{2}\mathbf{x}^T(\partial P/\partial t)\mathbf{x}$. Substituting into equation (1) and equating coefficients in $\mathbf{x}$ then gives the Riccati equation for the matrix P

$$\begin{aligned} -\frac{\partial P}{\partial t} &= PB(t)R^{-1}(t)B^T(t)P - Q(t) \\ P(t_f) &= F. \end{aligned}$$

Essentially one is solving for the n-plane $P(t)$ in phase space which meets F at time t_f. Given a state $\mathbf{x}(t)$, the plane $\mathbf{y} = P(t)\mathbf{x}$ determines which phase trajectory above $\mathbf{x}(t)$ meets F at time t_f. The subsequent evolution of $(\mathbf{x}(t), \mathbf{y}(t))$ coincides with that of $P(t)$.

Generally in a control problem one takes R to be positive definite and Q and F to be positive semi-definite. The larger $||F||$, the better the resulting stabilization in time t_f.

For stabilization over an infinite time period subject to time invariant dynamics $\dot{\mathbf{x}} = B\mathbf{u}$, one takes

$$S = \frac{1}{2}\int_0^\infty (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})dt.$$

If Q and R are positive definite, then an asymptotically stabilizing controller exists such that S converges. All the above can be found for instance in [2] or [5].

In this paper we will show that the above control problem can be interpreted as a variational problem along a geodesic on a Riemannian manifold. In this context, the stability condition for the infinite time problem can be interpreted as a Morse index condition on the initial manifold of the geodesic problem. In particular the stable solution can be associated with a particular initial manifold. We will then indicate how this can be generalised to give an index condition for the stability of non-linear Hamiltonian problems; in particular for optimal control problems.

2 Manifolds, Geodesics and Jacobi's Equation

Much of the following can be found in any book on differential geometry, for example [6] or Chapter 2 of [1]. However, no one reference seems to cover all that we require, in particular the connection between distance spheres on a manifold and initial conditions for Jacobi's equation, so we give a brief description here. Details of Morse theory can be found in [4].

Let M be a Riemannian manifold with metric g and local coordinates $x^1, \ldots, x^n$ in an open set $U \subseteq M$. Then $\partial/\partial x^1, \ldots, \partial/\partial x^n$ form a basis for the tangent space $T_p(M)$ at $p \epsilon U$.

A connection on the tangent bundle $T(M)$ is a rule for differentiating vector fields. It is defined by specifying the covariant derivatives of basis vectors with respect to one another

$$\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \Gamma^k_{ij}\partial/\partial x^k$$

for some functions Γ^k_{ij}. If $X = X^i\partial/\partial x^i$ and $Y = Y^i\partial/\partial x^i$ are vector fields on M then the covariant derivative of Y with respect to X is

$$\nabla_X(Y) = (d/dt(Y^k) + \Gamma^k_{ij}X^iY^j)\partial/\partial x^k$$

where d/dt denotes differentiation along X.

A vector field is parallel along a curve if its covariant derivative is zero along the curve. The parallel translation of a vector along a curve is the parallel vector field along the curve which coincides with the given vector initially. There is a unique symmetric (i.e. $\nabla_{\partial/\partial x^i}(\partial/\partial x^j) = \nabla_{\partial/\partial x^j}(\partial/\partial x^i)$) connection on M which preserves the Riemann metric under parallel translation.

Geodesics are extremals of the energy of the Riemann metric. They satisfy $\nabla_{c'}c' = 0$ where c' is the field of tangent vectors to the curve $c(t)$ on M. The linearisation of this equation along c gives Jacobi's equation

$$\nabla_{c'}(\nabla_{c'}J) + R(J, c')c' = 0$$

where R is the curvature tensor associated with the connection . A vector field J along $c(t)$ satisfying this equation is called a Jacobi field. It describes how variations of the initial conditions for the geodesic problem evolve under the geodesic flow. It can be shown ([6], Chapter IV) that the curves $J(t)$ in $T(M)$ are themselves extremals of the integral of a function on $T(M)$ which is quadratic on each fibre.

The geodesic equation is a second order differential equation on M. Initial conditions are given by specifying an n-dimensional initial submanifold T^0 of $T(M)$. If $x^1, \ldots, x^n$ are local coordinates on M then we can take $x^1, \ldots, x^n, X^1, \ldots, X^n$ to be coordinates on $T(M)$ where a tangent vector at a point $p = (x^1, \ldots, x^n)$ has components $X^1\partial/\partial x^1 + \ldots + X^n\partial/\partial x^n$. The two most common examples of initial manifolds are then:

i) $T^0 = T_p(M)$ i.e. all geodesics emanating from p in any direction. Coordinates on T^0 are given by $X^1, \ldots, X^n$ or, alternatively, by $t, s^1, \ldots, s^{n-1}$ where t is the length of a vector in $T_p(M)$ and $s^1, \ldots, s^{n-1}$ parameterise its direction.

ii) $T^0 = T^{\perp}(K) = \{\mathbf{v}\epsilon T_p(M) : p\epsilon K \text{ and } g(\mathbf{v},\mathbf{w}) = 0 \;\forall\; \mathbf{w}\epsilon T_p(K)\}$ where K is a submanifold of M. This corresponds to geodesics starting on K in directions perpendicular to K. If local coordinates are chosen so that K is given by $x^{k+1} = \ldots = x^n = 0$ and the vectors $\partial/\partial x^{k+1}, \ldots, \partial/\partial x^n$ are orthogonal to K and to one another at points of K, then T^0 is given by $x^{k+1} = \ldots = x^n = X^1 = \ldots = X^k = 0$. Coordinates on T^0 are given by $x^1, \ldots, x^k, X^{k+1}, \ldots, X^n$ or, alternatively, by $x^1, \ldots, x^k, t, s^1, \ldots, s^{n-k-1}$ where t is the length of a vector at a point on K and $s^1, \ldots, s^{n-k-1}$ parameterise its direction perpendicular to K.

An infinitesimal variation of the above initial conditions will form the initial conditions for Jacobi's equation on $T(T(M))$. For example, taking a variation of the initial conditions to be a curve $x^1(u), \ldots, x^k(u), X^{k+1}(u), \ldots, X^n(u)$ in $T^0 = T^{\perp}(K)$ and taking $\dot{x}^1, \ldots, \dot{x}^n, \dot{X}^1, \ldots, \dot{X}^n$ as coordinates on $T(T(M))$, an infinitesimal variation is given by

$$\begin{array}{lllllllllll} \dot{x}^1 = \frac{\partial x^1}{\partial u} & \ldots\ldots & \dot{x}^k = \frac{\partial x^k}{\partial u} & , & \dot{x}^{k+1} = 0 & \ldots\ldots & \dot{x}^n = 0, \\ \dot{X}^1 = 0 & \ldots\ldots & \dot{X}^k = 0 & , & \dot{X}^{k+1} = \frac{\partial X^{k+1}}{\partial u} & \ldots\ldots & \dot{X}^n = \frac{\partial X^n}{\partial u}. \end{array}$$

Strictly speaking, a Jacobi field and its covariant derivatives (and hence the initial conditions above) belong to $T(M)$. The connection defines a decomposition $T(T(M)) \simeq T(M) \dotplus T(M)$ and so the covariant derivative can be projected onto $T(M)$.

Choosing geodesic or normal coordinates simplifies many of the above calculations. Define the exponential map

$$\exp : T^0 \longrightarrow M$$

by

$$\exp(p, \mathbf{v}) = c(1)$$

where c is the geodesic with $c(0) = p$ and $c'(0) = \mathbf{v}$. In a neighbourhood of the origin in $T^0 = T_p(M)$ or of K in $T^0 = T^{\perp}(K)$, exp is a diffeomorphism. So choosing coordinates $t, s^1, \ldots, s^{n-1}$ on $T^0 = T_p(M)$ or $x^1, \ldots, x^k, t, s^1, \ldots, s^{n-k-1}$ on $T^0 = T^{\perp}(K)$, $\exp^{-1}$ gives a coordinate map on a neighbourhood of p or K in M. In these coordinates, t gives the arc-length along any geodesic starting on T^0, while the remaining coordinates determine which geodesic we are on.

A solution $J(t)$ to Jacobi's equation along a geodesic $c(t)$ is expressed in terms of a frame for $T(M)$ along the geodesic. If t is the arc-length parameter described above then $c'(t) = \partial/\partial t$. Suppose we take one of the frame vectors to be $\partial/\partial t$ and examine the component of $J(t) = f(t)\partial/\partial t$ in this direction. Since $t = 0$ determines the initial point p or the initial manifold K, a variation of the initial conditions in this direction can only consist of a variation of the magnitude of c'. So the initial conditions for f are $f(0) = 0$ and $f'(0) = a$ for some constant a. Now the curvature tensor is skew-symmetric, so $R(J, c')c' = fR(c', c')c' = 0$. Also, since c is a geodesic, $\nabla_{c'}c' = 0$ and so

$$\nabla_{c'}J = \frac{df}{dt}c' + f\nabla_{c'}c' = \frac{df}{dt}c'.$$

Thus Jacobi's equation reduces to $d^2f/dt^2 = 0$ with solution $J(t) = at\partial/\partial t$.

We can thus restrict attention to the $(n-1)$-dimensional space of variations orthogonal to $c(t)$. If we choose these remaining $(n-1)$ frame vector fields (say $P_1, \ldots, P_{n-1}$) to be orthonormal and parallel along $c(t)$ then, denoting $J(t) = \sum f^i(t)P_i(t)$,

$$\nabla_{c'}J = \sum \frac{df^i}{dt}P_i + \sum f^i\nabla_{c'}P_i = \sum \frac{df^i}{dt}P_i.$$

Jacobi's equation then becomes

$$\frac{d^2}{dt^2}J + K(t)J = 0 \tag{2}$$

where $K^i_j = g(R(P_j, c')c', P_i)$.

The final topic we wish to discuss is the Morse index. The above geodesic coordinates (and the distance minimising property of geodesics) are defined up to the first focal point (or conjugate point in the case $T^0 = T_p(M)$). A focal point is a point along a geodesic at which the differential of the exponential map becomes singular. Since the Jacobi equation describes the evolution of this differential, a focal point corresponds to a zero of a non-trivial Jacobi field. Intuitively, it is a point at which nearby geodesics starting on T^0 meet.

The Morse index of a geodesic c starting on T^0 of length t is the number of focal points, counted with multiplicity, to T^0 along the geodesic from $c(0)$ to $c(t)$. In the case of fixed initial and end points, there is an equivalent definition involving the index of inertia of the second variation of the energy of the geodesic considered as a quadratic form on the space of tangent vector fields along the geodesic which vanish at the endpoints. Up to the first conjugate point, the geodesic minimises energy and so the second variation is positive definite and has index zero. At the first conjugate point, the second variation is degenerate and has a null space of dimension equal to the multiplicity of the conjugate point. After the conjugate point, the second variation has a maximal subspace on which it is negative definite of dimension equal to the multiplicity of the conjugate point. This pattern is repeated at subsequent conjugate points with the dimension of the maximal subspace on which the second variation is negative definite equal to the sum of the multiplicities of the preceeding conjugate points.

The Morse index is a special case of a topological invariant of T^0 and M called the Maslov index. We will return to this later.

2.1 Example

Consider the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbf{R}^3$. This has Gaussian curvature 1 and coordinates u_1 measuring the angle of latitude from the North Pole and u_2 measuring the angle of longitude. The Riemann metric pulled back from $\mathbf{R}^3$ is

$$g = du_1^2 + \sin^2(u_1)du_2^2.$$

u_1 and u_2 are normal coordinates with respect to geodesics starting from the North Pole. So, parameterising the geodesics by u_1, $\partial/\partial u_1$ is tangent to the geodesic curves while $\partial/\partial u_2$ spans the infinitesimal variations of the initial conditions i.e. changes in direction. Taking an orthonormal frame $e_1 = \partial/\partial u_1$ and $e_2 = (1/\sin u_1)\partial/\partial u_2$, the variational vector field becomes $J = C\partial/\partial u_2 = C \sin u_1 e_2$ for some constant C. In this frame the covariant derivative can be shown to be

$$\begin{aligned} \nabla_{e_1} e_1 &= 0 & \nabla_{e_1} e_2 &= 0 \\ \nabla_{e_2} e_1 &= \tfrac{\cos u_1}{\sin u_1} e_2 & \nabla_{e_2} e_2 &= -\tfrac{\cos u_1}{\sin u_1} e_1. \end{aligned}$$

Hence J satisfies

$$J = 0 \quad J' = \nabla_{e_1}(C \sin u_1 e_2) = C \cos u_1 e_2 = C$$

at $u_1 = 0$ and

$$J'' = \nabla_{e_1}(C \cos u_1 e_2) = -C \sin u_1 e_2$$

i.e.

$$J'' + J = 0.$$

Note the initial conditions for the geodesic problem correspond to the axis $J = 0$ in the (J, J') phase plane. If we take a circle of equal distance $u_1 = \alpha$ from the North Pole as an initial manifold and consider normal coordinates $\overline{u}_1, \overline{u}_2$ with respect to geodesics orthogonal to $u_1 = \alpha$ then g becomes

$$g = d\overline{u}_1^2 + \sin^2(\overline{u}_1 + \alpha)d\overline{u}_2^2$$

and J becomes $J = C \sin(\overline{u}_1 + \alpha)e_2$. Now

$$J' = C \cos(\overline{u}_1 + \alpha)e_2 = \frac{\cos(\overline{u}_1 + \alpha)}{\sin(\overline{u}_1 + \alpha)} J.$$

So J satisfies the same differential equation but the intial conditions now correspond to the line $J' = P(0)J$ in phase space where

$$P(\overline{u}_1) = \frac{\cos(\overline{u}_1 + \alpha)}{\sin(\overline{u}_1 + \alpha)}.$$

P, in fact, satisfies the Riccati equation $P' = -1 - P^2$ and is the second fundamental form of the initial manifold $u_1 = \alpha$ on the sphere. Its evolution with respect to $\overline{u}_1$ describes the evolution of the initial manifold along the geodesics. Focal points to the initial manifold occur at values of $\overline{u}_1$ where the line defined by P in phase space becomes vertical.

We can repeat the above calculation for one sheet of the two sheeted hyperboloid $x^2 + y^2 - z^2 = -1$ of Gaussian curvature -1 in $\mathbf{R}^3$ equiped with the Minkowski metric. In this case Jacobi's equation is $J'' - J = 0$ and the corresponding Riccati equation is $P' = 1 - P^2$ where

$$P(u_1) = \frac{\cosh(u_1 + \alpha)}{\sinh(u_1 + \alpha)}.$$

In the case of the sphere, the solutions to Jacobi's equation define concentric circles in phase space. The lines of initial values $P(0) = 0$ and $P'(0) = 0$ are in a degenerate position having singular projection onto one of the coordinate axes. All other lines of initial values have a focal point within distance π and another one for each increase of π in distance. The Morse index of every initial circle on the sphere therefore increases towards ∞ as the length of the geodesic tends to ∞.

In the case of the hyperboloid, the solutions define hyperbolas in phase space. Solutions starting on $P(0) < -1$ or $P(0) > 1$ correspond to closed initial manifolds with one focal point in one direction and none in the other. Solutions starting on $-1 < P(0) < 1$ correspond to open initial manifolds. These have one focal point in one direction and none in the other in the sense that there is one manifold equidistant from the initial manifold along which all the geodesics are parallel (i.e. $J' = 0$). Alternatively, write $X = J, Y = J'$ and apply the change of variables $Y = ix, X = iy$ to see that this case is the same as the first one. In both cases, the Morse index of the initial manifold (away from degenerate positions) along any geodesic is at most one.

Solutions starting on $P(0) = \pm 1$, the separatrices of the hyperbolas, correspond to the limiting case where the initial manifold is tangent to the circle at infinity. This initial manifold has no focal points and so Morse index zero with respect to geodesics of any length. In this case P is invariant and satisfies $P' = 0 = 1 - P^2$. On one separatrix the J and J' trajectories are stable, on the other unstable.

All this is summarised in Figure 1 where the hyperboloid is represented on the conformal disk.

3 Linear Quadratic Regulator and Jacobi's Equation on Manifolds

We now return to the linear quadratic regulator problem outlined in the introduction. The dynamics associated with the Hamiltonian (1) are

$$\begin{aligned} \dot{\mathbf{x}} &= B(t)R^{-1}(t)B^T(t)\mathbf{y} \\ \dot{\mathbf{y}} &= Q(t)\mathbf{x}. \end{aligned}$$

We assume that all the matrices involved are in $\mathbf{R}^{n\times n}$ and $R(t)$ is positive definite. In addition, in the time varying case, we assume that $B(t)$ is symmetric and that $B(t)$, $R^{-\frac{1}{2}}(t)$ and $\partial/\partial t(R^{\frac{1}{2}}(t)B^{-1}(t))$ all commute. Then we can change variables to

$$\begin{aligned} \mathbf{X} &= R^{\frac{1}{2}}(t)B^{-1}(t)\mathbf{x} \\ \mathbf{Y} &= R^{-\frac{1}{2}}(t)B^T(t)\mathbf{y} + \partial/\partial t(R^{\frac{1}{2}}(t)B^{-1}(t))\mathbf{x} \end{aligned}$$

to get the following system

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{Y} \\ \dot{\mathbf{Y}} &= \left[R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}} + \partial^2/\partial t^2(R^{\frac{1}{2}}B^{-1})BR^{-\frac{1}{2}}\right]\mathbf{X} \end{aligned}$$

or

$$\ddot{\mathbf{X}} - \left[R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}} + \partial^2/\partial t^2(R^{\frac{1}{2}}B^{-1})BR^{-\frac{1}{2}}\right]\mathbf{X} = 0. \tag{3}$$

Thus we can interpret our control problem as that of finding Jacobi fields along a geodesic on a Riemann manifold M where the curvature tensor evaluated on pairs of frame vectors in equation (2) is

$$K(t) = -\left[R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}} + \partial^2/\partial t^2(R^{\frac{1}{2}}B^{-1})BR^{-\frac{1}{2}}\right](t). \tag{4}$$

If we let $\overline{F} = R^{-\frac{1}{2}}B^TFBR^{-\frac{1}{2}}(t_f)$, then instead of initial conditions we have terminal conditions specified by $\frac{1}{2}\mathbf{X}(t_f)^T\overline{F}\mathbf{X}(t_f)$, i.e. $\mathbf{Y}(t_f) = \overline{F}\mathbf{X}(t_f)$. This plane corresponds to a terminal manifold for the geodesic problem on M. The solution P to the Riccati equation describes the evolution of the distance sphere on M which arrives at F at time t_f. For any state $\mathbf{X}$ at time t, the feedback required to reach F at time t_f is $\mathbf{Y} = P(t)\mathbf{X}$. This is equivalent in our interpretation to specifying any infintesimal orthogonal variation $\mathbf{X}$ of the point t on the geodesic and taking the corresponding variation of the tangent vector to be $\mathbf{Y} = P(t)\mathbf{X}$ in order to guarantee that on all the neighbouring geodesics so defined the terminal manifold is reached at distance t_f.

4 Implications for Optimal Riccati Controllability

Interpreting system (3) as a Jacobi equation with matrix of curvatures $K(t)$ defined by (4) allows one to apply Rauch's comparison theorem from differential geometry (see [3]). This is a generalisation of Sturm's result on scalar equations of type (2). We will use it in the following form.

If $\mathbf{v}$ is a unit vector orthogonal to a geodesic $c(t)$ at a point p on the geodesic then $g(R(\mathbf{v}, c')c', \mathbf{v})$ is called the sectional curvature of the plane spanned by $\mathbf{v}$ and c'. In the frame used for equation (2), $g(R(\mathbf{v}, c')c', \mathbf{v}) = \mathbf{v}^T K(t)\mathbf{v}$. Let $|.|$ denote the norm of a vector, which in this frame is just the Euclidean norm. Now suppose

$$\text{(A)} \qquad \mathbf{v}^T K(t)\mathbf{v} \geq \gamma^2 > 0$$

or

$$\text{(B)} \qquad \mathbf{v}^T K(t)\mathbf{v} \leq -\gamma^2 < 0$$

for all $t \epsilon \mathbf{R}$, for all unit vectors $\mathbf{v} \epsilon \mathbf{R}^n$ and for some constant $\gamma > 0$. Then if
i) $\mathbf{X}$ is an orthogonal Jacobi field along $c(t)$ which vanishes at $t = 0$,
ii) $\mathbf{Y}$ is an orthogonal Jacobi field along a geodesic on (A) the $(n+1)$-sphere or (B) the $(n+1)$-hyperboloid of radius γ^{-1} which vanishes at $t = 0$,
iii) $|\mathbf{X}'(0)| = |\mathbf{Y}'(0)|$,
iv) neither geodesic has a conjugate point in the interval $(0, b)$,
then

$$\text{(A)} \qquad |\mathbf{X}(b)| \leq |\mathbf{Y}(b)|$$

or

$$\text{(B)} \qquad |\mathbf{X}(b)| \geq |\mathbf{Y}(b)|.$$

Comparing with Example 2.1, we see that in case (A) $\mathbf{Y}$ has a zero every π/γ while in case (B) $\mathbf{Y}$ has no zeros other than at $t = 0$.

Now suppose we wish to control system (3) optimally to the point $\mathbf{X} = 0$ at time t_f using a Riccati derived controller $\mathbf{Y} = P\mathbf{X}$. Since in $2n$-dimensional phase space the n-plane determining the control is vertical in all n-directions at this point, the corresponding Riccati matrix $P(t_f)$ is undefined. So we take the final position $F = P(t_f)$ to be arbitrarily close to the vertical on the side of decreasing time. Running the solution to the Riccati equation backwards in time, the optimal control is defined to the first time the plane $P(t)\mathbf{X}$ meets one of the hyperplanes $X_i = 0$. At this time the plane is again vertical in the Y_i direction and $P(t)$ is undefined. This is the first conjugate point back along the geodesic and corresponds to some non-trivial Jacobi field J starting on F being zero at this point. Its position is therefore determined by Rauch's theorem. In case (A) the first zero of any J starting on F has to occur before $t = \pi/\gamma$. In case (B) J has no zeros for all non-zero t. So we have the following:

Theorem 4.1 *With the hypotheses stated at the beginning of Section 3, if $K(t)$ satisfies (A) then a solution to the Riccati equation with final condition $P(t_f) = F$ exists backwards for a time period of at most π/γ. If $K(t)$ satisfies (B) then a solution exists backwards for all time. Note that in the case of time invariant dynamics, (A) occurs when Q is negative definite and (B) when Q is positive definite.*

5 Implications for Stability

We now come to the main point of this paper which is to characterise the stability of time invariant systems of the form (3) in terms of the Morse index of the initial manifold associated with the Riccati matrix P.

In the time invariant case the linear quadratic regulator (3) becomes

$$\ddot{\mathbf{X}} - R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}}\mathbf{X} = 0. \tag{5}$$

We can assume without loss of generality that $R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}}$ has been orthogonally diagonalised. Then the system can be decomposed into n one dimensional equations which look like the harmonic or hyperbolic oscillators of Example 2.1 or $\ddot{X}_i = 0$ depending on whether the respective eigenvalue is negative, positive or zero. In the first case the phase trajectories are circles which project down to give oscillatory solutions round the origin on the x-axis. In the last case we get motion at a constant speed on the x-axis. So for asymptotic stability we need only consider the case where Q is positive definite and all the eigenvalues of $R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}}$ are positive.

In each (X_i, Y_i) subspace of $2n$-dimensional phase space we get a hyperbolic oscillator phase portrait as in Example 2.1. The n-plane $\mathbf{Y} = P\mathbf{X}$ spanned by the stable separatrix from each (X_i, Y_i) subspace is the asymptotically stabilizing feedback controller. It is one of the solutions to the Riccati equation

$$P^2 - R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}} = 0.$$

Generalising Example 2.1 we can say that the plane P corresponds to an initial manifold on the $(n+1)$-hyperboloid $X_1^2 + \ldots + X_{n+1}^2 - X_{n+2}^2 = -1$ with Morse index 0 with respect to geodesics of any length. All other n-planes in non-degenerate positions in phase space correspond to initial manifolds with Morse index one along some geodesic. In the case above where some of the eigenvalues are negative we see that all n-planes in non-degenerate position have arbitrarily large Morse index along some geodesic, while in the case where some eigenvalues are zero, all non-degenerate n-planes have Morse index one along some geodesic. Hence we have the following characterisation of stability.

Theorem 5.1 *The system (5) is asymptotically stabilizable if and only if there exists a non-degenerate n-plane P in phase space corresponding to an initial manifold on M with Morse index zero along any geodesic, where M is the Riemannian manifold with constant curvature matrix $K = -R^{-\frac{1}{2}}B^TQBR^{-\frac{1}{2}}$.*

Note, if there exists such a P then there exist 2^n such planes. One of them gives the stabilizing feedback.

6 Conclusions and Generalisations

We have shown that the stability of a simple linear system can be characterised by the vanishing of a certain topological index associated with the geodesic flow on a manifold or, equivalently, by the vanishing of an index associated with the projection of the phase flow onto the n-dimensional coordinate plane in phase space. This index, the Morse index, is a special case of one of the Maslov characteristic classes which can be defined for any Lagrangian submanifold of a symplectic manifold without reference to a phase flow. For a non-linear Hamiltonian problem, necessary conditions for stability are given by the vanishing of the Maslov classes of suitable Lagrangian manifolds passing through the origin. In particular this applies to non-linear optimal control problems. We will give the details in a forthcoming paper.

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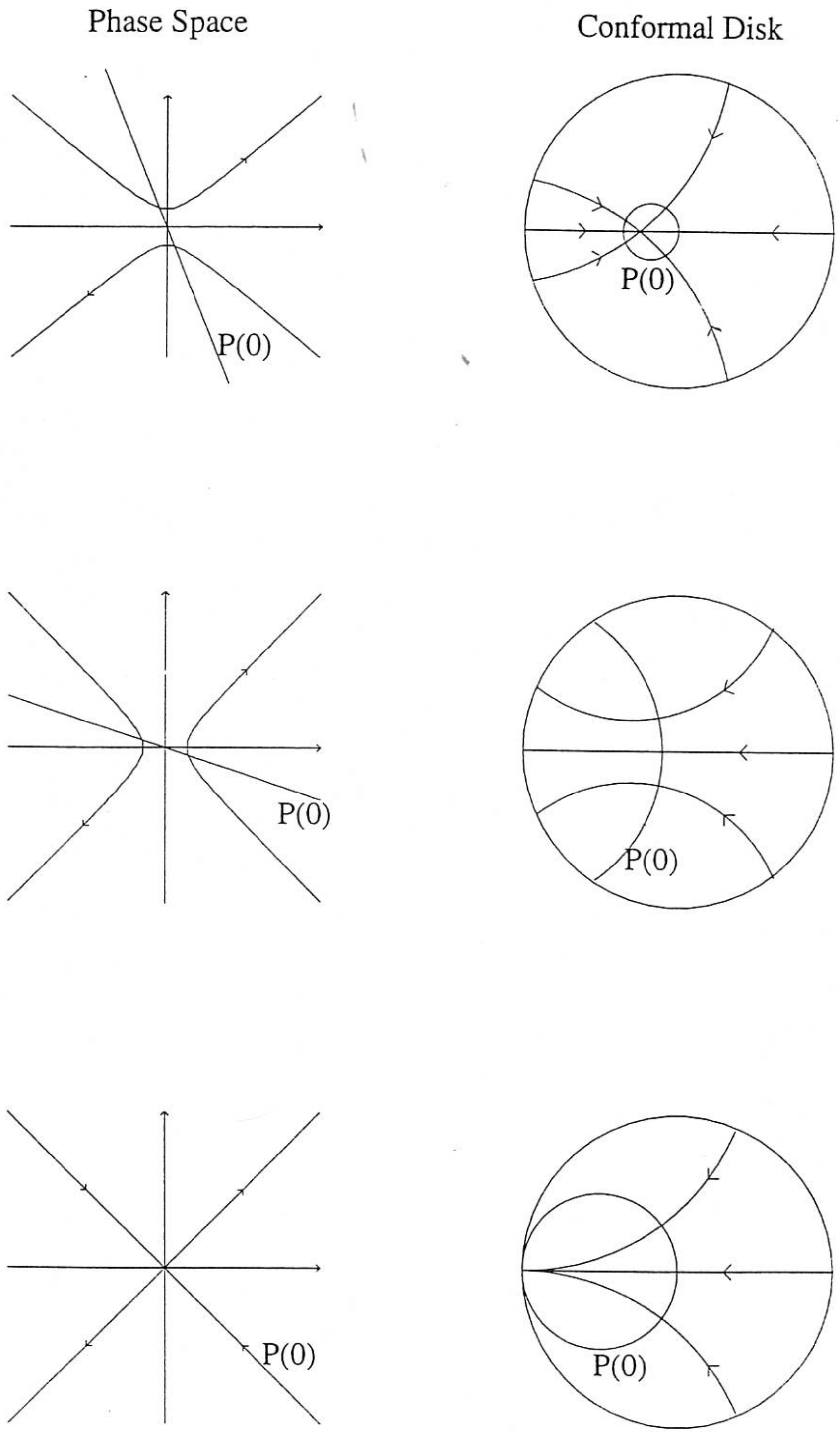

Figure 1