Monograph:
Banks, S.P., Salamci, M.U. and McCaffrey, D. (1996) Global Stabilization of Nonlinear Systems via Switching Manifolds. Research Report. ACSE Research Report 647 . Department of Automatic Control and Systems Engineering

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Global Stabilization of Nonlinear Systems via Switching Manifolds

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Research Report No 647

Abstract: The global stabilization of nonlinear systems is considered by reducing the problem to a lower dimensional switching manifold which is made globally attracting. The method generalizes the standard Lyapunov approach.

Keywords: Stabilization, Nonlinear Systems, Switching Manifolds.

Acknowledgement: This work was partially supported by the EPSRC under grant numbers GR/H 73066 and GR/J 75241.

I. INTRODUCTION

The stabilization of systems using switching surfaces has been investigated thoroughly for linear systems using variable structure control [1], [2]. The idea of the present paper comes from attempts to stabilize dissipative systems in Hilbert space [3], [4]. Here we shall use a generalized Lyapunov-like theory to develop switching surfaces directly, which are globally attracting by construction. If these surfaces can be designed around the stable manifold of the unforced system, global stabilization is guaranteed. If the unforced system has no stable manifold, then part of the control may be used to create one and then the remaining part of the control can be used to drive the system to this manifold.

In the next section we shall study local systems defined on $\mathbf{R}^n$ by a linear analytic structure, i.e.

$$\dot{x} = f(x) + ug(x) \tag{1}$$

The basic idea of the control is simple. We choose a function $\sigma(x)$ such that $\{x: \sigma(x) = 0\}$ is a smooth manifold through $x = 0$ and for which

$$\langle g, grad\sigma \rangle \neq 0$$

Then we choose the control u so that $\sigma \to 0$ as $t \to \infty$. Thus σ is a generalized Lyapunov function, although we shall now require

$$\dot{\sigma}(x) < 0 \text{ if } \sigma(x) > 0$$
$$\dot{\sigma}(x) > 0 \text{ if } \sigma(x) < 0$$

so that $\sigma \to 0$.

In section 3, some methods are given for smoothing the control chattering. The theory is extended to MIMO systems in section 4.

The global theory for systems on analytic manifolds will be given in section 5. Thus, systems are defined by vector fields V and W on a manifold X which are locally of the form *f(x)* and *g(x)*. The theory of manifolds we require can be found in [5] and Morse theory in [6].

Finally in section 6 we shall discuss the solutions of the partial differential equation

$$\langle grad\sigma, g \rangle = constant$$

which must be solved in general to find a suitable function σ.

II. LOCAL SYSTEMS

In this section we shall consider a local system of the form

$$\dot{x} = f(x) + ug(x) \tag{2}$$

We shall assume first that $g(x) \neq 0$ for all x (this condition will be relaxed later). Let $\sigma(x)$ be a smooth function such that all the level surfaces $\sigma(x) = const$ are *(n-1)*-dimensional smooth manifolds in $\mathbf{R}^n$, and consider the control

$$u = \frac{-\langle f, grad\sigma \rangle + c}{\langle g, grad\sigma \rangle} = \frac{-L_f\sigma + c}{L_g\sigma} \tag{3}$$

where L_f is the Lie derivative with respect to f and $c < 0$ if $\sigma > 0$ and $c > 0$ if $\sigma < 0$. Then we have our first result:

2.1 Theorem *Suppose that we can find a function σ such that*

$$L_g\sigma(x) \neq 0$$

for all x. Then the level surfaces $\sigma(x)$ is globally attracting with the control (3)

Proof We have

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$$\dot{\sigma} = \frac{\partial \sigma}{\partial x}\dot{x} = \langle f, grad\sigma\rangle + u\langle g, grad\sigma\rangle = c$$

if u is given by (3). Hence, if $\sigma(x_0) > 0$ then $c < 0$ and $\sigma \to 0$ as $t \to -\sigma(x_0)/c$. Similarly, if $\sigma(x_0) < 0$ then $c > 0$ and $\sigma \to 0$ again as $t \to -\sigma(x_0)/c$. □

The most obvious function σ to choose is

$$\sigma(x) = \|x\|^2 - r \tag{4}$$

for some $r > 0$. Then we have

2.2 Corollary *Suppose that the system (2) is locally controllable in the open set U near 0 and*

$$B_r = \{x : \|x\| \le r\} \subseteq U.$$

If

$$\langle g(x), x\rangle \ne 0 \quad \textit{for all } x \in \overline{\mathbf{R}^n \setminus B_r},$$

then the system (2) is globally stabilizable.

Proof Define σ as in (3); then $grad\ \sigma = 2x$ and

$$u = \frac{-\langle f, grad\sigma\rangle + c}{\langle g, grad\sigma\rangle} = \frac{-2\langle f, x\rangle + c}{2\langle g, x\rangle}$$

and since $\langle g, x\rangle \ne 0$ the control is well-defined. This control will drive any point $x \in \overline{\mathbf{R}^n \setminus B_r}$ to B_r and then we can use the local controllability. □

2.3 Remark The size of the control in corollary 2.2 can be bounded in the following way. If $\|x_0\| = \mu$, then let

$$A_{\mu,r} = \{x : r \le \|x\| \le \mu\},$$

and put

$$\gamma^+ = \inf\{\langle g(x), x\rangle : \langle g(x), x\rangle > 0,\ x \in A_{\mu,r}\}$$
$$\gamma^- = \inf\{|\langle g(x), x\rangle| : \langle g(x), x\rangle < 0,\ x \in A_{\mu,r}\}$$

and

$$\gamma = \min(\gamma^+, \gamma^-).$$

Since $A_{\mu,r}$ is compact, $\gamma > 0$. Then

$$|u| \le \frac{2\max_{A_{\mu r}} \|f\| . \mu + c}{\gamma} \quad (5)$$

$\square$

2.4 Example Consider the system

$$\dot{x} = f(x) + u \begin{pmatrix} x_1 \\ x_2^3 \end{pmatrix}$$

Then the control

$$u = \frac{-2\langle f, x\rangle - 1}{2(x_1^2 + x_2^4)}$$

will drive the system to the ball B_r for any $r > 0$ and any function *f*. The size of the control is bounded by

$$|u| \le \frac{2\max_{A_{\mu r}} \|f\| . \mu + 1}{2\inf_{B_r}(x_1^2 + x_2^4)}$$

where $\mu = \|x_0\|$.

However, we are unlikely to have a control function g in the form of that in example 2.4 and so we must choose a different control near the set where $\langle g, grad\sigma\rangle = 0$. Let

$$\Omega = \{x : \langle g, grad\sigma\rangle = 0\}$$

and let

$$\Omega_\varepsilon = \{x : dist(x, \Omega) \le \varepsilon\}$$

be an 'ε-neighbourhood' of Ω.

2.5 Theorem *Suppose there exists a function σ such that the set Ω is an m-dimensional manifold for some m<n and $\partial\Omega_\varepsilon$ is an (n-1)- dimensional manifold for each ε>0. Moreover, suppose that, for some $\bar{\varepsilon} > 0$, the set $\Omega_{\bar{\varepsilon}}$ is invariant for some feedback control u=u(x), with ω-limit set {0} (i.e. the system is stabilizable in $\Omega_{\bar{\varepsilon}}$.) Then the system (2) is globally stabilizable.*

Proof Parameterize σ so that $\Omega_{\bar{\varepsilon}}$ is the set where $\sigma = 0$ and that $\sigma > 0$ in $\mathbf{R}^n \setminus \Omega_\varepsilon$. Then the control (3) will drive the system to $\partial\Omega_{\bar{\varepsilon}}$. We can then choose a stabilizing control in Ω_ε to drive the system to 0. $\square$

2.6 Example Consider the system

$$\dot{x}_1 = -x_1 + x_2^3$$
$$\dot{x}_2 = x_2 + x_1 x_2 + u$$

For any ε>0, the strip

$$\Omega_\varepsilon = \{x : |x_2| < \varepsilon\}$$

is clearly control invariant for the system and the control

$$u = \frac{-2\langle f, x\rangle - 1}{2x_2} = \frac{-2(-x_1^2 + x_2^2 + x_1 x_2^3 + x_1 x_2^2) - 1}{2x_2}$$

drives the system to Ω_ε. Here, we have taken

$$\sigma = x_2^2 - \varepsilon^2$$

but we could have taken $\sigma = x_1^2 + x_2^2 - \varepsilon^2$, for example. □

Rather than choose the surface $\sigma = 0$ arbitrarily, we may choose it to have some relation to the dynamics of the system with no control. Thus, suppose that the system

$$\dot{x} = f(x)\ ,\ x \in \mathbf{R}^n$$

has a stable manifold $\mathcal{M} \subseteq \mathbf{R}^n$ of dimension $m<n$, and assume that g is transversal to $\mathcal{M}$ (apart, possibly, at the origin). If f (and g) are analytic then there exists a neighbourhood U of $\mathcal{M}$ in $\mathbf{R}^n$ and a function $\sigma(x)$ such that

$$\mathcal{M} = \{x : \sigma(x) = 0\}$$

and g is transversal to the level curves

$$\mathcal{M}_\varepsilon = \{x : \sigma(x) = \varepsilon\} \cap U\ ,\ \varepsilon > 0$$

The function σ is a Morse function [6] and its existence can be proved by elementary Morse theory (simply follow the dynamics determined by g). Let V denote the maximal neighbourhood of $\mathcal{M}$ on which σ can be chosen so that g is transversal to the level curves $\mathcal{M}_\varepsilon$. Then we have

2.7 Theorem *Under the above conditions, we have that the system (2) is globally stable on V.*

Proof As before define the feedback control u by

$$\frac{-\langle f, grad\sigma\rangle + c}{\langle g, grad\sigma\rangle},$$

(with $c<0$ if $\sigma>0$ and $c>0$ if $\sigma<0$). Since g is transversal to $\mathcal{M}_\varepsilon$ we have $\langle g, grad\sigma\rangle \neq 0$ on V and so the control drives the system to $\mathcal{M}$. Now, since $\mathcal{M}$ is stable manifold of

$$\dot{x} = f(x) \tag{6}$$

we can turn off the control when we reach $\mathcal{M}$ and follow the unforced system (6).

2.8 Example Consider the system

$$\left.\begin{aligned} \dot{x}_1 &= -x_2 - x_1^3 \\ \dot{x}_2 &= -x_1 + x_2^3 + u \end{aligned}\right\} = F(x,u) \tag{7}$$

where $F(x,u) = f(x) + (0,1)^T u$. The unforced system

$$\dot{x} = f(x)$$

has a unique equilibrium at (0,0) and a stable manifold of the form shown in figure 1.

To define σ we can simply choose it so that the level curves $\sigma = const.$ are parallel to the stable manifold σ=*0*. Thus, if the equation of the stable manifold is given by

$$x_2 = s(x_1)$$

then we take

$$\sigma = x_2 - s(x_1).$$

Hence the system (7) is globally stabilizable with control

$$u = \begin{cases} -(x_2 + x_1^3)ds/dx_1 + (x_1 - x_2^3) - 1 \\ -(x_2 + x_1^3)ds/dx_1 + (x_1 - x_2^3) + 1 \\ 0 \end{cases}$$

in the regions $x_2 >$ $(<,=)$ $s(x_1)$ respectively. In general, consider again the nonlinear system (2) which we assume can be written in the form

$$\dot{x} = f(x) + u\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{pmatrix} \tag{8}$$

where $\gamma(x) \to 1$ as $x \to 0$. Suppose that the linearized system at the origin

$$\dot{x} = \frac{\partial f}{\partial x}(0)x + u\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1(x) \end{pmatrix} \tag{9}$$

is controllable. Then we define $u = u_1 + u_2$ and choose $u_1 = u_1(x)$ to stabilize (9). Then the system (8) becomes

$$\dot{x} = f(x) + u_1(x)\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{pmatrix} + u_2\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma(x) \end{pmatrix}$$

where u_1 is a known function of x. Let $\mathcal{M}$ denote the subset of the stable manifold of the vector field $f(x) + u_1(0 \cdots 0\ \gamma(x))^T$ which is tangent to the plane $\{x\colon x_n = 0\}$ at $x = 0$. Then we have

2.9 Theorem *Let $\mathcal{M}' \subseteq \mathcal{M}$ denote the largest simply connected complete subset of $\mathcal{M}$ on which g is transversal to $\mathcal{M}$. (By* **complete** *we mean that all trajectories starting in this set remain in it) Let σ be the unique solution of the partial differential equation*

$$grad\sigma = \frac{\partial \sigma}{\partial x_1} + \cdots + \frac{\partial \sigma}{\partial x_n} = (0 \cdots 0\ \gamma(x))^T$$

with characteristic surface $\mathcal{M}'$, where g = 0. Then the system (6) is asymptotically stable in the region

$$\overline{\mathcal{M}'} = \cup_{x_0 \in \mathcal{M}'} \{x(t;x_0) : \dot{x}(t) = g(x(t)) \neq 0\}$$

i.e. $\mathcal{M}'$ is the open set consisting of all trajectories of the vector field g starting on $\mathcal{M}'$ which are continued until g = 0.

2.10 Example Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2^2 + u(1 + x_1^2) \end{aligned}$$

The linearized system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is controllable and if we take $u_1 = 2x_1$, then the origin has a stable manifold. Defining $u = u_1 + u_2$ in (8) we have

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + x_2^2 + 2x_1^3 + u_2(1 + x_1^2)\end{aligned}$$

A stable submanifold of the unforced system can be found from

$$\frac{dx_2}{dx_1} = \frac{x_1 + 2x_1^3 + x_2^2}{x_2}$$

whose solution is

$$x_2 = \pm(2e^{2x_1} - 2x_1^3 - 3x_1^2 - 4x_1 - 2)^{1/2}$$

Hence, the stable submanifold is

$$\sigma = \begin{cases} x_2 - \sqrt{2e^{2x_1} - 2x_1^3 - 3x_1^2 - 4x_1 - 2}\ , & \text{if } x_1 \geq 0 \\ x_2 + \sqrt{2e^{2x_1} - 2x_1^3 - 3x_1^2 - 4x_1 - 2}\ , & \text{if } x_1 < 0 \end{cases}$$

The control input u_2 is derived from (3) and can be expressed as

$$u_2 = \frac{1}{(1 + x_1^2)}\left\{c - (x_1 + x_2^2 + 2x_1^3) \mp \frac{(2e^{2x_1} - 3x_1^2 - 3x_1 - 2)}{\sqrt{2e^{2x_1} - 2x_1^3 - 3x_1^2 - 4x_1 - 2}} x_2\right\}$$

Figures 2, 3 and 4 give the simulation results. For this simulation $|c|$ is chosen to be 0.5, and the initial conditions are $\mathbf{x} = [0.4 \quad 0.7]^T$.

III. CHATTERING ALLEVIATION

In order to keep the system on the manifold, the control input u in (3) changes sign which brings high frequency control chattering. This can be alleviated by considering instead of (3), the control

$$u_{fn} = \frac{-c\left(\frac{\sigma}{|\sigma| + \delta}\right) - L_f\sigma}{L_g\sigma}$$

where *c>0* and δ is a positive small number.

In this case, we have, for example, if $\sigma > 0$

$$\dot{\sigma} = -c\left(\frac{\sigma}{|\sigma| + \delta}\right)$$

and so when σ is small, this is approximated

$$\dot{\sigma} = -c\frac{\sigma}{\delta}$$

i.e., $\sigma(t) = \sigma(x_0)e^{-ct/\delta}$ so that switching surface is never reached. This control is simply keeping the system close to the switching surface, which is chosen to be stable. The simulation results are given in figures 5, 6, and 7 for δ=0.02 and c=0.5.

Another possibility of smoothing the control signal is to use saturation function, i.e.,

$$u_{sat} = \frac{-csat(\sigma) - L_f\sigma}{L_g\sigma}$$

where $c>0$, δ is a positive small number and

$$sat(\sigma) = \begin{cases} -1 & \text{if} \quad \sigma \geq \delta \\ \sigma & \text{if} \quad -\delta < \sigma < \delta. \\ 1 & \text{if} \quad \sigma \leq -\delta \end{cases}$$

By using this control, we have

$$\dot{\sigma} = -csat(\sigma) = \begin{cases} c & \text{if} \quad \sigma \geq \delta \\ \sigma & \text{if} \quad -\delta < \sigma < \delta \\ -c & \text{if} \quad \sigma \leq -\delta \end{cases}$$

which means that the control steers the system into a strip and inside the strip, the system tends to reach the stable manifold. The simulation results of this control are given in figures 8, 9, and 10 for *δ=0.02* and *c=0.5*.

IV MIMO SYSTEMS

In this section, the methodology is extended to multi-input systems. Let the nonlinear system be defined by

$$\dot{x} = f(x) + G(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i \tag{4.1}$$

where $u \in \mathbf{R}^m$. Let $\sigma(x)$ be a smooth function such that all the level curves $\sigma(x) = const$ are $(n-1)$-dimensional smooth manifolds in $\mathbf{R}^n$, and consider the control vector whose i^{-th} element defined by

$$u_i = \frac{\langle g_i, grad\sigma\rangle(-\langle f, grad\sigma\rangle + c)}{\sum_{k=1}^{m}\langle g_k, grad\sigma\rangle^2} = \frac{L_{g_i}\sigma(-L_f\sigma + c)}{\sum_{k=1}^{m}(L_{g_k}\sigma)^2} \qquad \text{for } i = 1,2,\cdots,m \tag{4.2}$$

where L_{g_i} is the Lie derivative with respect to $g_i(x)$ and $c<0$ if $\sigma>0$ and $c>0$ if $\sigma<0$.

4.1 Theorem *Assume that we can find a stable manifold σ such that*

$$\sum_{k=1}^{m}(L_{g_k}\sigma)^2 \neq 0 \qquad \forall x$$

Then the level surface $\sigma(x)=0$ is globally attracting with the control (4.2)

Proof Simply take the derivation of σ,

$$\dot{\sigma} = \frac{\partial\sigma}{\partial x}\dot{x} = \langle grad\sigma, f(x)\rangle + \sum_{i=1}^{m} u_i \langle grad\sigma, g_i(x)\rangle$$

and substitute u_i into the equation. Then,

$$\dot{\sigma} = c.$$

Hence, if $\sigma(x_0) > 0$ then $c<0$ and $\sigma \to 0$ as $t \to -\sigma(x_0)/c$. Similarly, if $\sigma(x_0)<0$ then $c>0$ and $\sigma \to 0$ again as $t \to -\sigma(x_0)/c$. □

4.2 Example Consider the system

$$\dot{x}_1 = x_2 + x_2^3 + u_1 + (1+x_1^2)u_2$$
$$\dot{x}_2 = x_1 + x_1^3 + (1-x_2^2)u_1$$

The linearized system is controllable and has a stable submanifold. The stable submanifold of the unforced nonlinear system can be generated by solving the following first order differential equation

$$\frac{dx_2}{dx_1} = \frac{x_1 + x_1^3}{x_2 + x_2^3}$$

which is

$$x_2^2 = x_1^2$$

Then the stable manifold can be easily written as

$$\sigma = x_2 + x_1 = 0$$

The control vector is derived from (4.2) and the elements are

$$u_1 = \frac{(2 - x_2^2)\left[c - (x_2 + x_2^3 + x_1 + x_1^3)\right]}{(2 - x_2^2)^2 + (1 + x_1^2)^2}$$

$$u_2 = \frac{(1 + x_1^2)\left[c - (x_2 + x_2^3 + x_1 + x_1^3)\right]}{(2 - x_2^2)^2 + (1 + x_1^2)^2}$$

Figure 11 to 14 give the simulation results for $|c|$=0.5, and $\mathbf{x}(0) = [\text{-}0.3 \quad \text{-}0.75]^T$.

To alleviate the chattering on the control signal, the same smoothing processes can be applied to the multi-input case. For instance,

$$u_{fn_i} = \frac{L_{g_i}\sigma\left(c\dfrac{\sigma}{|\sigma| + \delta} - L_f\sigma\right)}{\sum_{k=1}^{m}(L_{g_k}\sigma)^2} \quad \text{for } i = 1, \cdots, m$$

where $c>0$ and δ is a positive small number. The simulation results are given in figures 15,16, 17, and 18 for these control inputs. (δ=0.02 and c=0.5.)

The control vector can be rewritten using saturation function as follows;

$$u_{sat\,i} = \frac{L_{g_i}\sigma\left(csat(\sigma) - L_f\sigma\right)}{\sum_{k=1}^{m}(L_{g_k}\sigma)^2} \quad \text{for } i = 1, \cdots, m$$

where $c>0$, $\delta$is a positive small number and

$$sat(\sigma) = \begin{cases} -1 & \text{if} \quad \sigma \geq \delta \\ \sigma & \text{if} \quad -\delta < \sigma < \delta. \\ 1 & \text{if} \quad \sigma \leq -\delta \end{cases}$$

The simulation results are given in figures 19, 20, 21, and 22. (For this simulation, we take $\delta = 0.02$ and $c = 0.5$)

V. GLOBAL THEORY

Let X be a compact differentiable manifold of dimension n and let V, W be vector fields on X. The controlled vector field V+uW has the local representation

$$\dot{x} = f(x) + ug(x)$$

in the coordinates $x: N \to \mathbf{R}^n$ for some open set $N \subseteq X$. If $S \subseteq X$ is a smooth submanifold of X of dimension n-1 (i.e. a hypersurface) then S and the vector W_x are **transversal** if

$$TS \oplus \mathbf{R}W_x = TX$$

Suppose that $p \in X$ is an equilibrium point of V, i.e. $V_p = 0$. It is well-known [6] that the total index of vector field V on X is given by the Euler characteristic of X, $\gamma(X)$. Then X has at least one equilibrium point if

$$\gamma(X) > 0.$$

Let

$$\dot{x} = f(x) + ug(x)$$

be a local representation for the system at p, where $x: U \to \mathbf{R}^n$ is a coordinate system in the neighbourhood U of p with $x(p) = 0$. We shall assume that *(f, g)* is linearizable and controllable at p so that we may write

$$\dot{x} = Ax + f^{(2)}(x) + u(g(0) + g^{(1)}(x)) = Ax + bu + f^{(2)}(x) + ug^{(1)}(x)$$

where

$$A = \frac{\partial f}{\partial x}(0), \ \ b = g(0), \ \ f^{(2)}(x) = f(x) - Ax, \ \ g^{(1)}(x) = g(x) - g(0)$$

We may assume that canonical coordinates have been chosen so that b has the form $(0, 0, \cdots, 0, 1)^T$. Now write $u = u_1 + u_2$ and choose $u_1 = kx$ to stabilize *(A,b)*. Then we have

$$\dot{x} = (a + bk)x + f^{(2)}(x) + (kx + u_2)g^{(1)}(x) + u_2 b$$

Now choose an *(n-1)*-dimensional stable submanifold $S \subseteq U$ of the system such that $b + g^{(1)}(x)$ is transversal to S. Then S can be defined by a function σ such that

$$S = \{x \in U : \sigma(x) = 0\}$$

If $y: U' \to \mathbf{R}^n$ is another coordinate neighbourhood such that $U \cap U' \neq \varnothing$ and $S \cap U' \neq \varnothing$ then we can extend S as follows. If

$$y = h(x)$$

then u_1 is extended to $u_1 = kh^{-1}(y)$ in U' and S is extended into U' as the union of all trajectories of the system

$$\dot{y} = (A+bk)h^{-1}(y) + f^{(2)}(h^{-1}(y)) + (kh^{-1}(y))g^{(1)}(h^{-1}(y)) + u_2(b + g^{(1)}(h^{-1}(y)))$$

in U' passing through S in $U \cap U'$. In this way we obtain the maximal extension of S to X on which $b + g^{(1)}(h^{-1}(y))$ is transversal to the submanifold. Let S_m denote this maximal $(n-1)$- dimensional stable submanifold of the system *(V, W)*. It is defined by a set of equations

$$S_m = \{x \in X : \sigma_i(x) = 0, i \in U_i\}$$

where $\{U_i\}_{1 \le i \le L}$ is a set of coordinate neighbourhoods. These functions $\sigma_i(x)$ piece together to form a section of the real line bundle over *X*. Finally we integrate the partial differential equation

$$grad\sigma = b + g^{(1)}(x)$$

from S_m and define the region $\overline{\mathcal{M}'}$ just as in the local case. Then the local control

$$u_2 = \frac{-\langle f, grad\sigma \rangle + c}{\langle g, grad\sigma \rangle}$$

will drive all the points in $\overline{\mathcal{M}'}$ to S_m, which is then a stable manifold using $u_2 = 0$.

5.1 Example Consider the simple dynamical system on the sphere in Figure 23. The *'f'* dynamics define a saddle at the north (and south) pole and the g dynamics has orbits along the latitudes. Using a neighbourhood ϕ of the north pole as shown we can define σ in a quadrant by removing a small disk at the pole. (This is a simple example just to illustrate the ideas.)

5.2 Example As a slightly less trivial example, consider Euler's equation for the control of a rigid body:

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 + u_1$$
$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 + u_2$$
$$I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2 + u_3$$

which lives on so(3), with the corresponding orthogonal matrix O satisfying $\Omega = O^T\dot{O}$ being defined on SO(3) where it is a geodesic under the obvious metric if $u_1 = u_2 = u_3 = 0$. The equilibrium point is $\Omega_1 = \Omega_2 = \Omega_3 = 0$ and there is no stable manifold, the unforced motion $(u_1 = u_2 = u_3 = 0)$ remains on the invariant ellipsoid

$$\sigma(\Omega_1, \Omega_2, \Omega_3) = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

since

$$\dot{\sigma} = \sum_{i=1}^{3} I_i \Omega_i \dot{\Omega}_i = (I_2 - I_3 + I_3 - I_1 + I_1 - I_2)\Omega_1\Omega_2\Omega_3 = 0$$

Using σ as the switching function, so that $\sigma = 0$ is the origin, then

$$u_1 = \frac{I_1\Omega_1\left(c - \Omega_1\Omega_2\Omega_3\left(I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)\right)\right)}{\sum_{i=1}^{3}(I_i\Omega_i)^2} = \frac{I_1\Omega_1 c}{\sum_{i=1}^{3}(I_i\Omega_i)^2}$$

$$u_2 = \frac{I_2\Omega_2 c}{\sum_{i=1}^{3}(I_i\Omega_i)^2}$$

$$u_3 = \frac{I_3\Omega_3 c}{\sum_{i=1}^{3}(I_i\Omega_i)^2}$$

(This is an example of the control in (4.2)). Note that if we choose

$$c = -\sum_{i=1}^{3}(I_i\Omega_i)^2$$

then the control is bounded and is given by

$$u_i = -I_i\Omega_i \quad , \quad 1 \le i \le 3 .$$

VI. PARTIAL DIFFERENTIAL EQUATION FOR σ

We have seen that it is necessary to solve the partial differential equation

$$grad\sigma = g \tag{6.1}$$

in order to determine a stabilizing region for a system. This equation may be written in the form

$$\frac{\partial \sigma}{\partial x_i} = g_i \,, \quad 1 \le i \le n \tag{6.2}$$

A necessary condition for the existence of a solution of this problem is

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \quad , \quad 1 \le i, j \le n \tag{6.3}$$

This condition is well-known to be necesaary and sufficient for the line integral

$$\int_0^x \nabla\sigma \cdot dx \qquad (6.4)$$

to be independent of the path from 0 to x. We have

6.1 Theorem *Let X be a compact Riemannian manifold with metric* $\gamma : TM \times TM \to \mathbf{R}$. *Then the function* σ *is given by*

$$\sigma(x) = \int_0^t \gamma\ (W(x(t)), W(x(t)))dt \qquad (6.5)$$

where $x \in X$ *and x(t) is a solution trajectory of the vector field W joining* $x(0) \in S = \{x : \sigma(x) = 0\}$ *to x.*

Proof We can extend the definition of σ systematically through a system of local neighbourhoods, so we can prove (6.5) locally. Thus, by (6.4) we define

$$\sigma(x) = \int_0^x \nabla\sigma \cdot dx = \int_0^x \gamma\,(\nabla\sigma, dx) = \int_0^t \gamma\left(\nabla\sigma, \frac{dx(t)}{dt}\right)dt$$

$$= \int_0^t \gamma\ (g(x(t)), g(x(t)))dt$$

by (VI.1). □

6.2 Example A trivial example will illustrate the method. Let X be $\mathbf{R}^n$ with the standard Euclidean metric and let $S = \{x : \sigma(x) = 0\}$ be the x_1-axis. If $g = (0,1)^T$ then $\gamma(g,g) = 1$ and so, if $x(0) = (x_1(0), 0) \in S$ and $x \in \mathbf{R}^n \setminus S$ we have

$$\sigma(x) = \int_0^{x_2} 1dt = x_2$$

6.3 Remark If g does not satisfy (6.3) then we can solve the more general equation

$$\langle grad\sigma, g\rangle = \ constant \neq 0$$

to find a suitable function σ. □

VII. CONCLUSIONS

In this paper we have shown that stabilization of many kinds of nonlinear systems can be achieved by first designing a stable manifold of codimension 1 for the system and then using a switching control to steer the system to this submanifold. The method easily extends to global systems on differentiable manifolds, giving a truly global control method for nonlinear systems.

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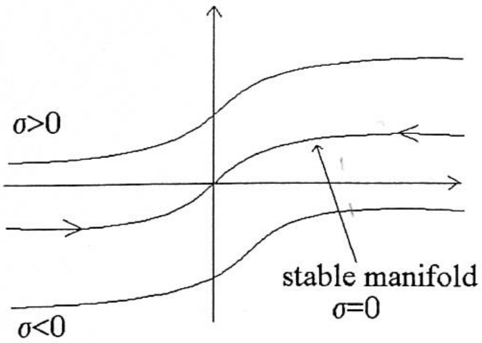

Figure 1-Definition of σ.

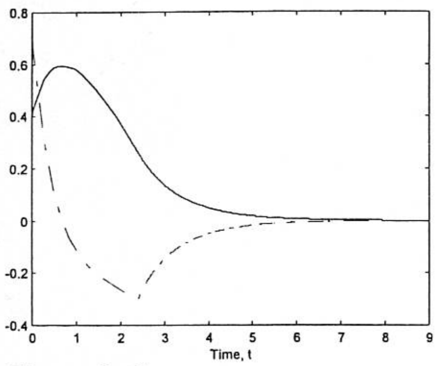

Figure 2- State responses.

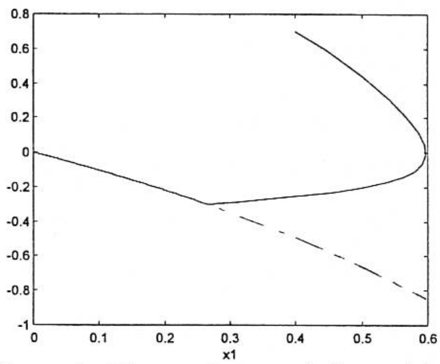

Figure 3- Phase plane and the stable manifold σ.

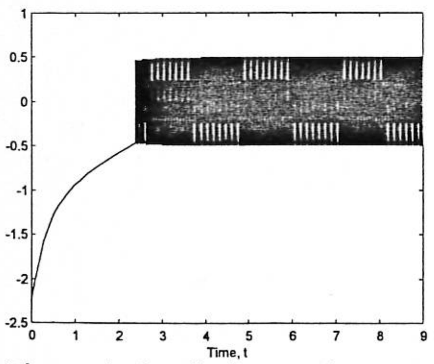

Figure 4- Applied control u.

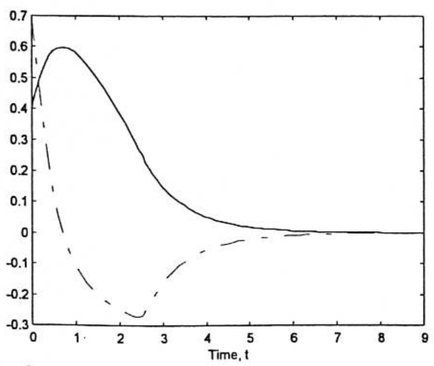

Figure 5- System response to control input u_{fn}.

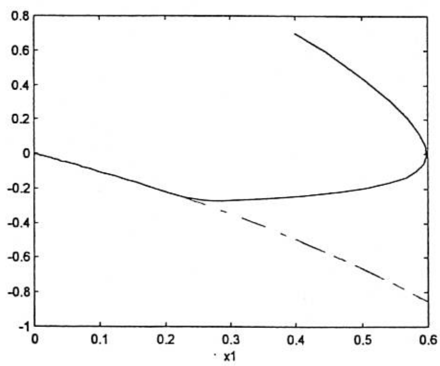

Figure 6- Phase plane and the stable manifold σ, under the control input u_{fn}.

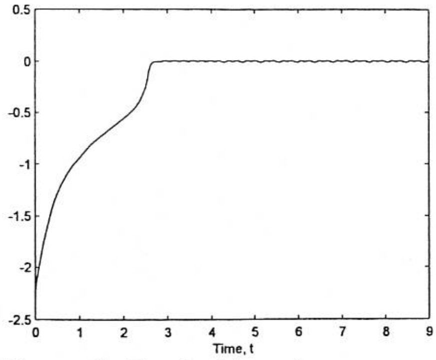

Figure 7- Control input u_{fn}.

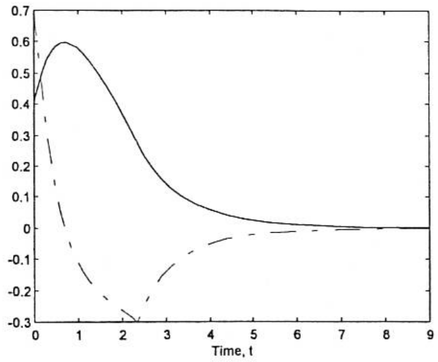

Figure 8- System response to control input u_{sat}

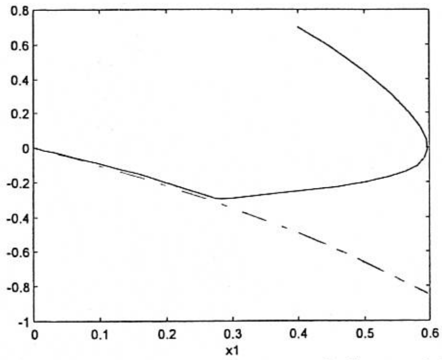

Figure 9- Phase plane and the stable manifold σ, under the control input u_{sat}.

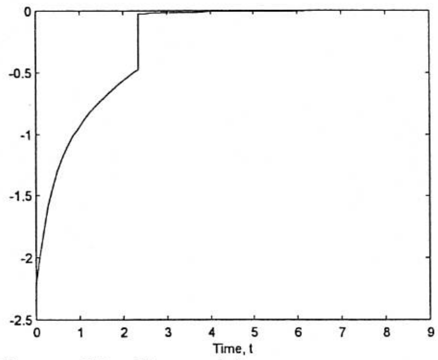

Figure 10- Control input u_{sat}.

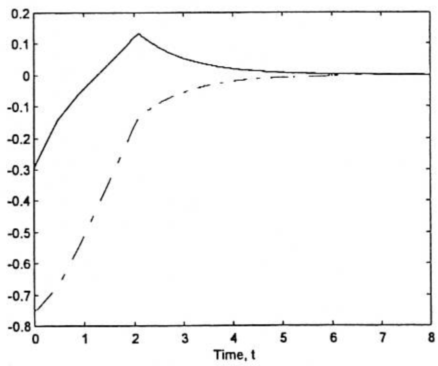

Figure 11- State responses.

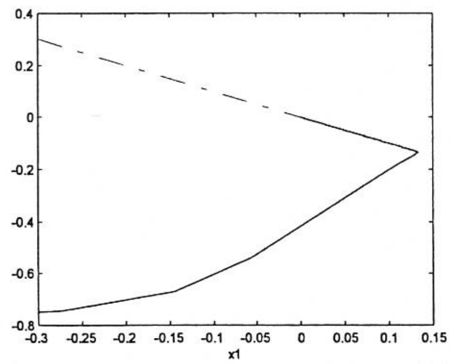

Figure 12-Phase plane and the stable manifold σ.

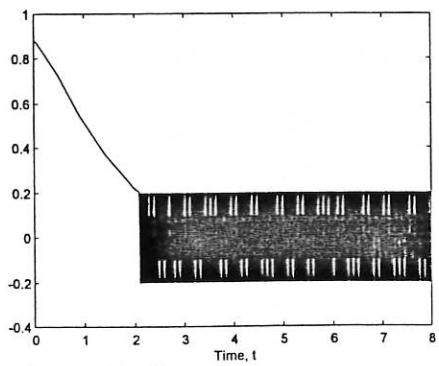

Figure 13-Control input u_1.

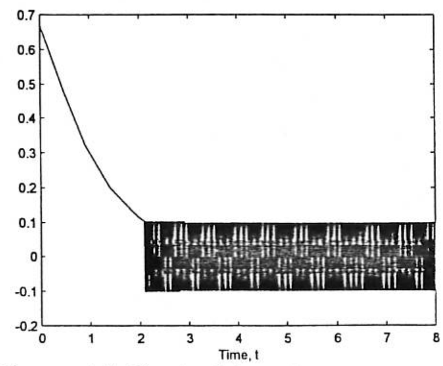

Figure 14-Control input u_2.

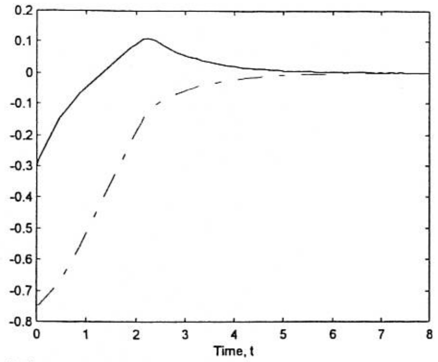

Figure 15-State responses.

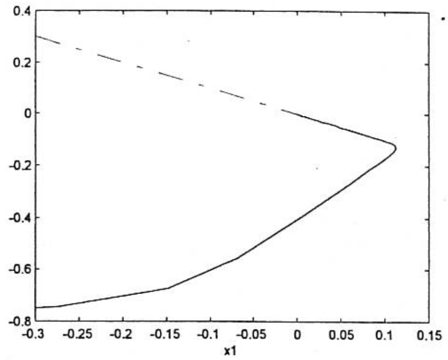

Figure 16-Phase plane and the stable manifold σ.

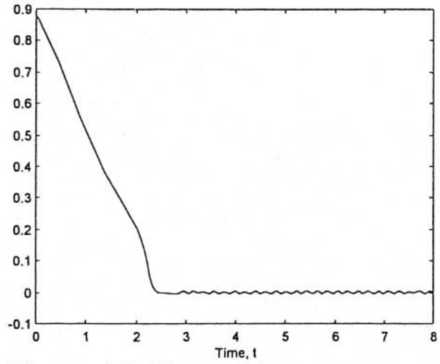

Figure 17- Control input u_{fn_1}.

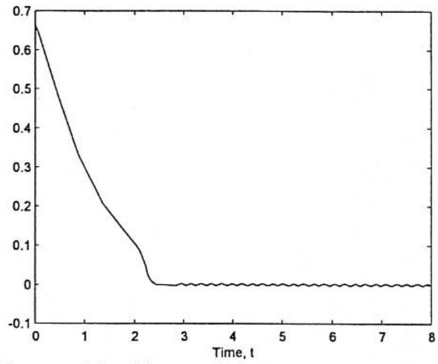

Figure 18- Control inputu_{fn_2}

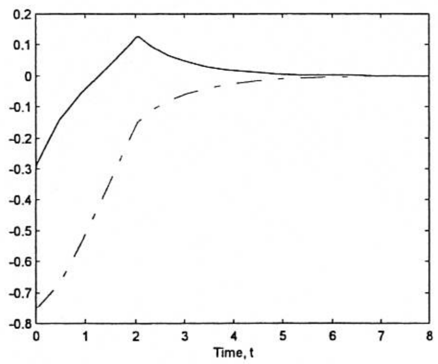

Figure 19-State responses.

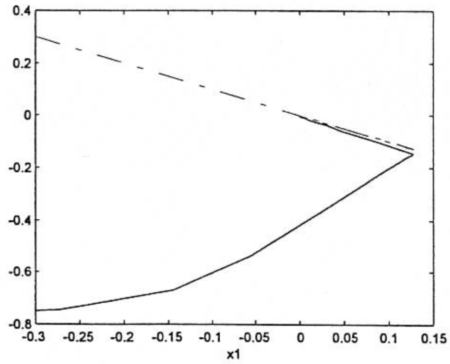

Figure 20-Phase plane and the stable manifold σ.

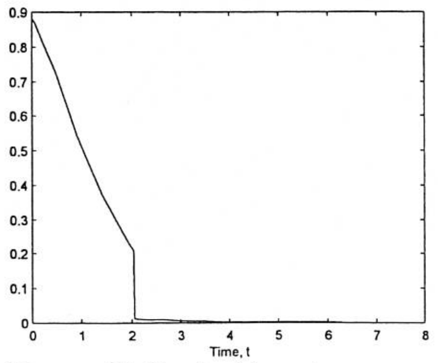

Figure 21-Control input u_{sat_1}.

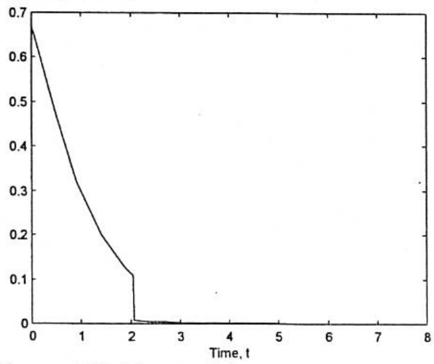

Figure 22-Control input u_{sat_2}.

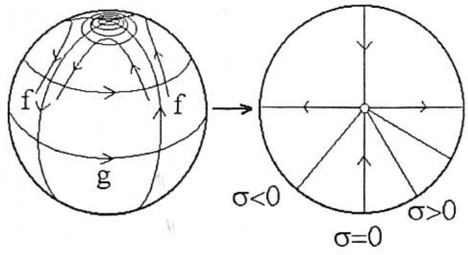

Figure 23- Simple dynamics on a sphere.