Reconstruction of MIMO Nonlinear Differential Equation Models From the Generalised Frequency Response Function Matrix

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Research Report No: 649
December, 1996

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Abstract

A new algorithm is introduced to identify differential equation models for linear and nonlinear MIMO systems from frequency response data using a weighted complex orthogonal estimator. The estimation procedure is progressive beginning with the estimation of the linear terms and then sequentially adding higher order nonlinear terms to build up the model. Simulated examples are included to demonstrate the performance of the new algorithm.

1 Introduction

Modelling of dynamic systems has always been an important issue in both science and engineering. The problem of system modelling has been studied in many different frameworks and investigated from different view points using various optimization techniques. An important class of systems that has been intensively studied by researchers in the past are linear lumped parameter systems which are described by a set of ordinary differential equations. Although most physical systems are continuous in time most of the available techniques of system modelling approximate the continuous time system by a difference equation instead of a differential equation.

Parameter estimation methods for linear difference equation (discrete) models based on sampled input-output data have been thoroughly studied (Ljung,1987; Soderstrom and Stoica,1989). The reasons for this may partly be attributed to the rapid development and wide use of digital computers. Although the approximation ability of discrete models is quite satisfactory, the parameters of nonlinear discrete models can not easily be related to the physical behaviour of the system. Hence identification of continuous time models is important for the purpose of system design and analysis.

Methods for continuous time parameter estimation using digital computers have received increasing attention in recent years. A comprehensive review of the continuous time modelling can be found in Young (1981), Unbehauen and Rao(1987,1990). A major difficulty of identification of continuous time models is associated with the numerical errors which can be induced when derivatives of the input and output signals are computed. Several techniques have been proposed by researchers in the past to minimise and possibly avoid these errors during estimation. But most of these techniques have been developed for linear systems and the extension of these techniques to nonlinear systems is much more involved.

One method of avoiding noise accentuating derivative operations on the noisy input-output signal is to fit a nonlinear discrete time model to the sampled data records and to generate the first and higher order frequency response functions and then to curve fit to these to obtain a continuous time model (Tsang and Billings,1992b; Swain and Billings,1995). Based on this approach a new estimator called the weighted complex orthogonal estimator was proposed (Swain and Billings,1995) to estimate the parameters of linear and nonlinear single input single output models. The objective of the present study is to extend this technique to fit continuous time differential equation models for multi input multi output (MIMO) systems.

The organisation of the paper proceeds as follows. Section-2 briefly describes the weighted complex orthogonal least squares estimator (Swain and Billings,1995). The concept of Volterra modelling and the generalised frequency response function matrix for MIMO systems are introduced in section-3. The estimation procedure for fitting parametric nonlinear differential equation models to frequency response data are described in section-4. In section-5, simulated examples are included to demonstrate the effectiveness of the new MIMO approach.

2 Weighted Complex Orthogonal Least Squares Estimator

Consider a system which can be modeled as

$$z(j\omega) = \sum_{i=1}^{M} \theta_i p_i(j\omega) + \xi(j\omega) \tag{1}$$

where θ_i,$i = 1, ...M$ are the real unknown deterministic parameters of the system associated with the complex regressors $p_i(j\omega)$,$i = 1, ...M$. $z(j\omega)$ is a complex dependent variable or the term to regress upon and $\xi(j\omega)$ represents the modeling error. Before any attempt is made to estimate the parameters 'θ', the complex variables involved in eqn(1) should be partitioned in to real and imaginary parts; otherwise θ could be complex. If 'N' measurements of $z(j\omega)$ and $p_i(j\omega)$ are available at $\omega_i, i = 1, ...N$ the complex system of eqn(1) can be represented after partitioning in matrix form as

$$Z = P\theta + \Xi \tag{2}$$

The weighted complex orthogonal estimator (Swain and Billings,1995) transforms eqn(2) in to an auxiliary equation

$$Z = Wg + \Xi \tag{3}$$

The properties of the matrix W are such that W^TQW is orthogonal; where 'Q' is a positive definite weighting matrix. Further let

$$V = W^TQ \tag{4}$$

The regressors of the auxiliary model of eqn.(3) can be obtained recursively from

$$\begin{aligned} w_1(\omega) &= p_1(\omega) \\ w_i(\omega) &= p_i(\omega) - \sum_{k=1}^{i-1} \alpha_{ki} w_k(\omega) \quad \text{for } k < i \end{aligned} \tag{5}$$

where

$$\alpha_{ki} = \frac{\sum_{j=1}^{N} v_k(\omega_j) p_i(\omega_j)}{\sum_{j=1}^{N} v_k(\omega_j) w_k(\omega_j)} \quad \text{for } k = 1.....i-1 \tag{6}$$

The estimates of the i-th element of the auxiliary parameter vector 'g' is given by

$$\hat{g}_i = \frac{\sum_{j=1}^{N} Z(\omega_j) v_i(\omega_j)}{\sum_{j=1}^{N} v_i(\omega_j) w_i(\omega_j)} \quad \text{for } i = 1, ...M \tag{7}$$

Once the parameters $g_i, i = 1, ...M$ are estimated,the original system parameters $\theta_i, i = 1, ...M$ can easily be recovered according to the formula

$$\hat{\theta} = \hat{g} - (T - I)\hat{\theta} \tag{8}$$

that is

$$\begin{aligned} \hat{\theta}_M &= \hat{g}_M \\ \hat{\theta}_k &= \hat{g}_k - \sum_{i=k+1}^{M} \alpha_{ki}\hat{\theta}_i, \quad \text{for } k = 1, ...M - 1 \end{aligned} \tag{9}$$

Therefore by using the above equations,the unknown parameters $\theta_i, i = 1, ...M$ can be estimated step by step. The structure of the system or which term to include in the model can be determined by using the error reduction ratio test

$$ERR_i = \frac{g_i^2 \sum_{j=1}^{N} v_i(\omega_j) w_i(\omega_j)}{\sum_{j=1}^{N} Z^T(\omega_j) Q_w Z(\omega_j)} \quad i = 1, ...M \tag{10}$$

which gives the percentage contribution that each term makes to the output variance (energy). The value of ERR indicates the significance of a candidate term. Normally at the beginning all available candidate terms are examined and the term which contributes the maximum ERR is included in the model. This is repeated until all candidate terms have been exhausted or the sum of the ERR approaches 100%. In order to avoid possible numerical ill-conditioning the normalised version of the algorithm (Swain and Billings,1995) can be used.

Before formulating the problem for estimating the parameters of continuous time nonlinear differential equation models, the concept of Volterra modelling, the generalised kernel transform and $[\omega, \beta]$ permutation needs to be introduced. These are discussed in detail in Swain and Billings(1996) but will be briefly reviewed below for completeness.

3 Volterra Modelling of MIMO Systems

The output of the 'j_1-th ' subsystem of an r-input m-output system possessing nonlinearity up to degree N_l may be expressed as

$$y_{j_1}(t) = \sum_{n=1}^{N_l} y_{j_1}^{(n)}(t) \tag{11}$$

where $y_{j_1}^{(n)}(t)$ is the n-th order component of the output $y_{j_1}(t)$. When the r-inputs are denoted as $u_{\beta_1}(t), \ldots u_{\beta_r}(t)$ eqn(11) can be expressed as

$$y_{j_1}(t) = \sum_{n=1}^{N_l} \sum_{\beta_1=1}^{r} \sum_{\beta_2=\beta_1}^{r} \cdots \sum_{\beta_n=\beta_{n-1}}^{r} \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, ..\tau_n) u_{\beta_1}(t-\tau_1)..u_{\beta_n}(t-\tau_n) d\tau_1, ..d\tau_n \tag{12}$$

where $h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, \ldots \tau_n)$ is the n-th order Volterra kernel of the j_1th subsystem. The superscripts $\beta_1, \ldots\beta_n$ in the kernel correspond to the inputs $u_{\beta_1}(t), \ldots u_{\beta_n}(t)$ that take part in the n-dimensional convolution with $h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, \ldots \tau_n)$. The kernels are called *self-kernels* when all the superscripts $\beta_1, \ldots\ldots\beta_n$ in $h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, \ldots \tau_n)$ are equal; for example $h_n^{(j_1:11\ldots1)}(\tau_1, \ldots \tau_n)$ is the n-th order *self kernel* of the system corresponding to the inputs $u_1(t)$, otherwise they are called *cross-kernels*.

The total number of kernels associated with the n-th order nonlinearity of a particular subsystem depends on the number of inputs and is denoted as $Nk_{(n)}^{(r)}$.

$$Nk_{(n)}^{(r)} = Nk_{(n-1)}^{(r)} + Nk_{(n-1)}^{(r-1)} + \ldots\ldots Nk_{(n-1)}^{(1)} \tag{13}$$

where

$Nk_{(1)}^{(r)}$ = total number of first order kernels for an r-input system = r.

The multidimensional Fourier transform of the n-th order Volterra kernel results in the n-th order kernel transform and constitutes the elements of the n-th order generalised frequency response function matrix (GFRFM) denoted as $GFRFM^{(n)}$. The elements of the j_1-th row of $GFRFM^{(n)}$ are

$$H_n^{(j_1:\beta_1,\ldots,\beta_n)}(j\omega_1, \ldots\ldots j\omega_n) \quad ;\text{for} \quad \begin{aligned} \beta_1 &= 1, \ldots r \\ \beta_2 &= \beta_1, \ldots r \\ \beta_n &= \beta_{n-1}, \ldots r \end{aligned}$$

(14)

The elements of the GFRFM can be computed using the harmonic probing technique (Bedrosian and Rice,1971). The procedure of computing each column of the GFRFM differs from another column and requires the harmonic inputs to be configured differently depending on which column of the GFRFM is being analysed. Instead of deriving an expression for each column it is legitimate to define a generalised kernel transform and find the procedure for computing this. Other kernel transforms can be considered to be special cases of the generalised kernel transform.

The Generalised Kernel Transform is denoted as

$$\mathrm{GKERT} = \mathrm{H}_{\mathrm{n}}^{(\mathrm{j}_1:\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{\mathrm{n_d}} \text{times}})}(\mathrm{j}\omega_1,\ldots\ldots\mathrm{j}\omega_{\mathrm{n}})$$

where

$\mathrm{n_d}$ = the number of distinct inputs present in the kernel

γ_1 = number of times β_1 occurs in the superscript and

$\gamma_{\mathrm{n_d}}$ = number of times $\beta_{\mathrm{n_d}}$ appears in the kernel

For example the kernel transform $\mathrm{H}_5^{(\mathrm{j}_1:\beta_1,\beta_1,\beta_2,\beta_3,\beta_3)}(.)$ would be represented in the above form with $\mathrm{n_d} = 3, \gamma_1 = 2, \gamma_2 = 1$ and $\gamma_3 = 2$.

Average Generalised Kernel Transform and $[\omega, \beta]$ Permutation

The frequency domain behaviour of a single input single output (SISO) system can be described in terms of symmetric versions of the generalised frequency response functions. Analogously the frequency domain behavior of MIMO systems is described in terms of the *average generalised kernel transform.* The symmetric GFRF is computed by an averaging operation over all permutations of the frequency arguments of asymmetric GFRFs. But the average kernel transform is computed by the permutation of both the arguments and the superscripts of the GKERT by using $[\omega, \beta]$ permutation (Swain and Billings,1996). To explain this consider the configuration of an n-tone input while computing the GKERT. The n-tone input is split according to

$$
\begin{aligned}
u_{\beta_1}(t) &= e^{j\omega_1 t} + \ldots\ldots + e^{j\omega_{\gamma_1} t} \\
u_{\beta_2}(t) &= e^{j\omega_{1+\gamma_1} t} + \ldots\ldots + e^{j\omega_{\gamma_1+\gamma_2} t} \\
\vdots \quad &\quad \vdots \qquad\qquad \vdots \\
u_{\beta_{n_d}}(t) &= e^{j\omega_{1+\gamma_1+\ldots+\gamma_{n_d-1}} t} + \ldots\ldots + e^{j\omega_n t}
\end{aligned}
\tag{15}
$$

This implies that the frequencies $\{\omega_1, \ldots, \omega_{\gamma_1}\}$ belong to the input point β_1 ; $\omega_{1+\gamma_1}, \ldots, \omega_{\gamma_1+\gamma_2}$ belongs to the input point β_2 and so on. The *Average Generalised Kernel Transform* is defined as

$$
H_{n_{avg}}^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{ times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}})}(j\omega_1,\ldots\ldots j\omega_n)
$$

$$
= \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ [\omega,\beta]}} H_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{ times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}})}(j\omega_1,\ldots\ldots j\omega_n) \tag{16}
$$

where the $[\omega, \beta]$ permutation means that when any argument of a kernel transform $H_n^{(.)}(.)$, ω_i (say) which belongs to input point β_i if is permutated to ω_k (say), β_i must change to β_k such that $e^{j\omega_k t}$ belongs to the input point β_k.

As an example

$$
\begin{aligned}
3!H_{3_{avg}}^{(j_1:122)}(j\omega_1, j\omega_2, j\omega_3) &= H_3^{(j_1:122)}(j\omega_1, j\omega_2, \omega_3) + H_3^{(j_1:122)}(j\omega_1, j\omega_3, \omega_2) \\
&+ H_3^{(j_1:221)}(j\omega_2, j\omega_3, \omega_1) + H_3^{(j_1:212)}(j\omega_2, j\omega_1, \omega_3) \\
&+ H_3^{(j_1:221)}(j\omega_3, j\omega_2, \omega_1) + H_3^{(j_1:212)}(j\omega_3, j\omega_1, \omega_2) \\
&= \sum_{\text{permutation } [\omega,\beta]} H_3^{(j_1:122)}
\end{aligned}
\tag{17}
$$

With the above background the expression for the GKERT of a system will be given by mapping the differential equation models of the system into the frequency domain.

3.1 Frequency Domain Mapping of Differential Equation Models

Consider an r-input m-output nonlinear system. The dynamics of the 'j_1-th' subsystem can be represented by

$$\sum_{n=1}^{N_l}\sum_{p=0}^{n}\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{\beta_1=1}^{r}\sum_{\beta_2=\beta_1}^{r}\cdots\sum_{\beta_q=\beta_{q-1}}^{r}$$
$$\sum_{l_1,.l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1:l_1,..l_{p+q})\prod_{i=1}^{p}D^{l_i}y_{\alpha_i}(t)\prod_{i=p+1}^{p+q}D^{l_i}u_{\beta_{i-p}} = 0 \quad (18)$$

where $p+q=n$ and the operator D^{l_i} is defined as

$$D^{l_i}x(t) = \frac{d^{l_i}x(t)}{dt} \quad (19)$$

'L' is the order of the maximum derivative. The parameter $c_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1:l_1,..l_{p+q})$ is associated with the term $\prod_{i=1}^{p}D^{l_i}y_{\alpha_i}(t)\prod_{i=p+1}^{p+q}D^{l_i}u_{\beta_{i-p}}$ in the j_1-th subsystem.

For example the equation

$$1.5\dot{y}_1(t)+y_1(t)+1.67\dot{y}_2(t)+1.1y_2(t)+u_1(t)+20.0y_1^2(t)+$$
$$15.0y_1(t)y_2(t)+2.3\dot{y}_1(t)\dot{u}_2(t)+3.3y_2(t)\dot{u}_1(t) = 0 \quad (20)$$

which may describe the first subsystem of a 2-input 2-output system would be represented in the above form as $c_{10}^{1}(1:1)=1.5$ $c_{10}^{1}(1:0)=1.0$ $c_{10}^{2}(1:1)=1.67$ $c_{10}^{2}(1:0)=1.1$, $c_{01}^{1}(1:0)=1.0$ $c_{20}^{11}(1:00)=20.0$, $c_{20}^{12}(1:00)=15.0$, $c_{11}^{12}(1:11)=2.3$, $c_{11}^{21}(1:01)=3.3$.

The frequency domain equivalent of eqn(18) corresponding to the generalised kernel transform can be expressed as

$$\underbrace{-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L}c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+\ldots j\omega_n)^{l_1}\right]n!H_{n_{avg}}^{(\alpha_1:\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)}_{\text{contribution from linear output terms}}$$

$$=\underbrace{\sum_{l_1,.l_n=0}^{L}c_{0,n}^{\underbrace{\beta_1,\ldots}_{\gamma_1 \text{ times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}}}(j_1:l_1,..l_q)H_U}_{\text{contribution from pure input nonlinear terms}}$$

$$+\underbrace{\sum_{p=2}^{n}\sum_{\alpha_1=1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_p=0}^{L} c_{p,0}^{\alpha_1,..\alpha_p}(j_1:l_1,..l_p)H_Y}_{\text{contribution from pure output nonlinear terms}}$$

$$+\underbrace{\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\dots}_{n_1\text{ times}}\underbrace{\beta_2,\dots}_{n_2\text{ times}}\underbrace{\beta_{n_d},\dots}_{n_{n_d}\text{ times}}}(j_1:l_1,..l_{p+q})H_{YU}}_{\text{contribution from input-output cross-product terms}} \tag{21}$$

where

$$\begin{aligned} H_U \;=\; & \sum_{\substack{\text{all permutations of}\\ \omega_1,\dots\omega_{\gamma_1}}} (j\omega_1)^{l_1}\dots(j\omega_{\gamma_1})^{l_{\gamma_1}} \sum_{\substack{\text{all permutations of}\\ \omega_{1+\gamma_1},\dots\omega_{\gamma_1+\gamma_2}}} (j\omega_{1+\gamma_1})^{l_{1+\gamma_1}}\dots(j\omega_{\gamma_1+\gamma_2})^{l_{\gamma_1+\gamma_2}} \\ & \dots \sum_{\substack{\text{all permutations of}\\ \omega_{1+\gamma_1+\dots\gamma_{n_d-1}},\dots\dots\omega_n}} (j\omega_{1+\gamma_1+\dots+\gamma_{n_d-1}})^{l_{1+\gamma_1+\dots+\gamma_{n_d-1}}}\dots(j\omega_n)^{l_n} \end{aligned} \tag{22}$$

$$H_Y \;=\; \sum_{\substack{\text{all permutations of}\\ [\omega,\beta]}} H_{n,p}^{\alpha_p,\dots\dots\alpha_1}(j\omega_1,\dots j\omega_n) \tag{23}$$

and $H_{n,p}^{\alpha_p,\dots\dots\alpha_1}(\,.\,)$ denotes the contribution of p-th order nonlinear output terms of the form $\prod_{i=1}^{p} D^{l_i}y_{\alpha_i}(t)$ to the n-th order nonlinearity. This is estimated recursively as

$$H_{n,p}^{\alpha_p,\alpha_{p-1},\dots\alpha_1}(.) = \sum_{i=1}^{n-p+1} H_i^{(\alpha_p:\beta_{\sigma_1},\dots\beta_{\sigma_i})}(j\omega_1,\dots j\omega_i)H_{n-i,p-1}^{\alpha_{p-1}\dots\alpha_1}(j\omega_{i+1},..j\omega_n)(j\omega_1+\dots+j\omega_i)^{l_p}. \tag{24}$$

where β_{σ_1} corresponds to the input point to which $e^{j\omega_1 t}$ belongs and β_{σ_i} corresponds to the input point to which $e^{j\omega_i t}$ belongs.

The recursion finishes with $p = 1$ and $H_{n,1}^{\alpha_1}(j\omega_1,\dots.j\omega_n)$ has the property

$$H_{n,1}^{\alpha_1}(.) = H_n^{(\alpha_1:\beta_{\sigma_1},\dots\beta_{\sigma_n})}(j\omega_1,\dots j\omega_n)(j\omega_1+\dots.+j\omega_n)^{l_1} \tag{25}$$

and

$$H_{UY} = \sum_{\substack{\text{all } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},\dots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \cdots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\dots\gamma_{n_{d-1}}},\dots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} H^{(U)}_{n,qc}(\Omega_q) \sum_{\text{perm } [\omega,\beta]} H^{\alpha_p,\dots\alpha_1}_{n-q,p}(\Omega_{n-q}) \tag{26}$$

where

$$\begin{aligned} H^{(U)}_{n,qc}(\Omega_q) \;=\; & \Big(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_1},\dots\Omega_{n_1}^{\gamma_1}}} (j\Omega_1^{\gamma_1})^{l_{p+1}} \dots (j\Omega_{n_1}^{\gamma_1})^{l_{p+n_1}} \Big) \\ & \Big(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_2},\dots\Omega_{n_2}^{\gamma_2}}} (j\Omega_1^{\gamma_2})^{l_{p+n_1+1}} \dots (j\Omega_{n_2}^{\gamma_2})^{l_{p+n_1+n_2}} \Big) \dots \\ & \dots \Big(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_{n_d}},\dots\Omega_{n_{n_d}}^{\gamma_{n_d}}}} (j\Omega_1^{\gamma_{n_d}})^{l_{p+n_1+\dots+n_{d-1}+1}} \dots (j\Omega_{n_d}^{\gamma_{n_d}})^{l_q} \Big) \end{aligned} \tag{27}$$

$\Omega_1^{\gamma_1}, \dots \Omega_{n_1}^{\gamma_1}$ represent the variable of each combination of γ_1 frequencies belonging to the input point $u_{\beta_1}(t)$ taken n_1 at a time and similarly $\Omega_1^{\gamma_i}, \dots \Omega_{n_i}^{\gamma_i}$ represent the variables of each combination of γ_i frequencies belonging to the input point $u_{\beta_i}(t)$ taken n_i at a time and

$$\begin{aligned} \Omega_q &= [\Omega_1^{\gamma_1}, \dots \Omega_{n_1}^{\gamma_1}, \dots \Omega_1^{\gamma_{n_d}}, \dots \Omega_{n_d}^{\gamma_{n_d}}] \\ \Omega_{n-q} &= [\omega_1, \omega_2, \dots \omega_n] \bigcap [\Omega_q] \end{aligned} \tag{28}$$

These equations can be used to derive regression equations for estimating the unknown parameters of the system. Before describing the general procedure of deriving regression equations associated with the MIMO nonlinear differential equation model the following example demonstrates how the information of the GFRFM of a system can be put in a least squares based framework to estimate the parameters of a MIMO nonlinear differential equation model.

Example

Consider a MISO system described by

$$\begin{aligned} a_1\ddot{y}_1(t) + a_2\ddot{y}_1(t) + y_1(t) = {} & b_0u_1(t) + b_1\dot{u}_1(t) + c_0u_2(t) + c_1\dot{u}_2(t) + d_1\dot{y}_1(t)\ddot{y}_1(t) + d_2\dot{y}_1(t)\dot{u}_1(t) \\ & + d_3y_1(t)\ddot{u}_2(t) + d_4\dot{u}_1(t)\ddot{u}_1(t) + d_5\dot{u}_1(t)\dot{u}_2(t) + d_6\dot{u}_2(t)\ddot{u}_2(t) \end{aligned} \tag{29}$$

The elements of the first and second order generalised frequency response function matrices (GFRFM) of this system are given as

$$\mathrm{GFRFM}^{(1)} = [H_1^{(1:1)}(j\omega_1), H_1^{(1:2)}(j\omega_1)]$$
$$\mathrm{GFRFM}^{(2)} = [H_2^{(1:11)}(j\omega_1, j\omega_2), H_2^{(1:12)}(j\omega_1, j\omega_2), H_2^{(1:22)}(j\omega_1, j\omega_2)] \tag{30}$$

The estimation process to determine the nonlinear differential equation model of the system utilizes the information in all the columns of the GFRFM sequentially by starting with the columns of first order GFRFM , then the columns of the second order GFRFM and so on. The procedure consists of several steps as described below.

Estimation of the Linear Terms

Step-1.1:

The frequency domain equivalent of the system corresponding to the first column of $\mathrm{GFRFM}^{(1)}$ is derived from eqn(29) with $u_1(t) = e^{j\omega_1 t}$ and $u_2(t) = 0$ by equating the coefficients of $e^{j\omega_1 t}$. This gives

$$[a_1(j\omega_1)^2 + a_2(j\omega_1) + 1]H_1^{(1:1)}(j\omega_1) = b_0 + j\omega_1 b_1 \tag{31}$$

This can be written as

$$-H_1^{(1:1)}(j\omega_1) = a_1(j\omega_1)^2 H_1^{(1:1)}(j\omega_1) + a_2(j\omega_1)H_1^{(1:1)}(j\omega_1) - b_0 - j\omega_1 b_1 \tag{32}$$

The parameters a_1, a_2, b_0 and b_1 can be solved using least squares based techniques by inserting values for the frequency response function $H_1^{(1:1)}(j\omega_1)$.

Step-1.2 :

The frequency domain equivalent of the system corresponding to the second column of the $\mathrm{GFRFM}^{(1)}$ is

$$[a_1(j\omega_1)^2 + a_2(j\omega_1) + 1]H_1^{(1:2)}(j\omega_1) = c_0 + j\omega_1 c_1 \tag{33}$$

Note that the parameters a_1, a_2 on the left hand side of eqn(33) have been estimated in step-1.1 above. Thus the parameters c_0 and c_1 can be estimated using least squares techniques once values for $H_1^{(1:2)}(j\omega_1)$ have been inserted.

Estimation of the Second Order Nonlinear Terms

Step-2.1 :

The frequency domain equivalent of the system corresponding to the first column of $GFRFM^{(2)}$ is computed by applying $u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t}$ and $u_2(t) = 0$ in equation(29) by equating the coefficients of $e^{j(\omega_1+\omega_2)t}$. This gives

$$[a_1(j\omega_1 + j\omega_2)^2 + a_2(j\omega_1 + j\omega_2) + 1]2!H_{2_{avg}}^{(1:11)}(j\omega_1, j\omega_2) = d_4 \sum_{\substack{\text{all perm.}\\ \omega_1,\omega_2}} (j\omega_1)^1(j\omega_2)^2$$
$$+d_2 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_2)^1(j\omega_1)^1 H_1^{(1:1)}(j\omega_1) + d_1 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_1)^1 H_1^{(1:1)}(j\omega_1)(j\omega_2)^2 H_1^{(1:1)}(j\omega_2) \qquad (34)$$

The parameters d_1, d_2 and d_4 can be estimated from eqn(34) once the values of the $H^{(\cdot)}(.)$ have been inserted.

Step-2.2 :

The frequency domain equivalent of the system for column-2 of $GFRFM^{(2)}$ is

$$[a_1(j\omega_1 + j\omega_2)^2 + a_2(j\omega_1 + j\omega_2) + 1]2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_2) = d_5(j\omega_1)^1(j\omega_2)^1$$
$$+d_3 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_2)^2(j\omega_1)^1 H_1^{(1:1)}(j\omega_1) + d_1 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_1)^1 H_1^{(1:1)}(j\omega_1)(j\omega_2)^2 H_1^{(1:2)}(j\omega_2) \qquad (35)$$

Note that the parameter d_1 has been estimated before which can be brought to the left hand side to give

$$\begin{aligned}
&[a_1(j\omega_1 + j\omega_2)^2 + a_2(j\omega_1 + j\omega_2) + 1]2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_2) \\
&\quad -d_1 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_1)^1 H_1^{(1:1)}(j\omega_1)(j\omega_2)^2 H_1^{(1:2)}(j\omega_2) \\
&= d_5(j\omega_1)^1(j\omega_2)^1 + d_3 \sum_{\substack{\text{all perm.}\\ [\omega,\beta]}} (j\omega_2)^2(j\omega_1)^1 H_1^{(1:1)}(j\omega_1)
\end{aligned} \qquad (36)$$

Again this can be solved to give estimates of d_3 and d_5 once values for the $H^{(\cdot)}(.)$ have been inserted.

Step-2.3 :

The frequency domain equivalent of the system corresponding to the last column of $GFRFM^{(2)}$

is

$$[a_1(j\omega_1+j\omega_2)^2+a_2(j\omega_1+j\omega_2)+1]2!H_{2_{avg}}^{(1:22)}(j\omega_1,j\omega_2)=d_6\sum_{\substack{\text{all perm}\\ \omega_1,\omega_2}}(j\omega_1)^1(j\omega_2)^2$$
$$+d_3\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_2)^2(j\omega_1)^1H_1^{(1:2)}(j\omega_1)+d_1\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^1H_1^{(1:2)}(j\omega_1)(j\omega_2)^2H_1^{(1:2)}(j\omega_2) \tag{37}$$

Since the parameters d_3 and d_1 have been estimated before these can be brought to the left hand side and the resulting equation can be solved to yield an estimate of d_6.

4 Problem Formulation

In order to estimate the parameters of the system the information from all the columns of the GFRFM will be utilized . The parameters of a particular subsystem of an n-th order nonlinear MIMO system are estimated in n- stages. At the i-th stage the parameters associated with the i-th order nonlinear terms are estimated by utilizing the information of the i-th order generalised frequency response function matrix $GFRFM^{(i)}$. Each stage consists of a number of steps where at the k-th step of the i-th stage the frequency domain equivalent (FDE) of the system corresponding to the k-th column of the $GFRFM^{(i)}$ is put in a regression formulation to estimate the parameters of the system. The number of steps in the i-th stage is equal to the number of columns in $GFRFM^{(i)}$. This will be explained in more detail in the following section.

The last example illustrates the normal procedure of estimating the parameters of a simple MISO system. A more general procedure is described in this section.

4.1 Stage-1 : Estimation of the Parameters Associated with the Linear Terms

The total number of steps in this stage is equal to 'r' where r is the number of inputs to the system and is equal to the number of columns of the first order generalised frequency response function matrix $GFRFM^{(1)}$ denoted as $ncol_1$

Step-1.1 :

The frequency domain equivalent of eqn(18) corresponding to the first column of the $GRFRFM^{(1)}$

is computed by setting $n = 1, \gamma_1 = 1, \beta_1 = 1$ in eqn(21) which gives

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1)^{l_1}\right] H_1^{(\alpha_1:1)}(j\omega_1) = \sum_{l_1=0}^{L} c_{0,1}^{1}(j_1; l_1)(j\omega_1)^{l_1} \tag{38}$$

Note that multiplying by a constant on both sides of eqn(38) has no effect on the form of the equation except all the parameters will be changed which suggests that all the parameters can not be uniquely estimated. Hence it is necessary to fix a parameter before the estimation begins. Without loss of generality it is assumed that the parameter corresponding to the linear output term of the j_1-th subsystem $c_{1,0}(j_1; 0)$ is unity. Moving all other terms to the right hand side (RHS) gives

$$-H_1^{(j_1:1)}(j\omega_1) = \sum_{\alpha_1=1}^{m}\left[\sum_{\substack{l_1=0 \\ l_1\neq 0 \text{ for } \alpha_1=j_1}}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1)^{l_1}\right] H_1^{(\alpha_1:1)}(j\omega_1) + \sum_{l_1=0}^{L} c_{0,1}^{1}(j_1; l_1)(j\omega_1)^{l_1} \tag{39}$$

which is a linear in the parameters expression and can be solved by applying the weighted complex orthogonal estimator (Swain and Billings,1995).

Step-1.2 :

To estimate the parameters associated with the linear terms of input $[u_2]$; consider the FDE of eqn(18) corresponding to the second column of $GFRFM^{(1)}$ by setting $\beta_1 = 2, \gamma_1 = 1$ in eqn(21).

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1)^{l_1}\right] H_1^{(\alpha_1:2)}(j\omega_1) = \sum_{l_1=0}^{L} c_{0,1}^{2}(j_1; l_1)(j\omega_1)^{l_1} \tag{40}$$

Note that all the parameters associated with linear output terms $c_{1,0}^{\alpha_1}(j_1 : l_1); l_1 = 0, \ldots L$, $\alpha_1 = 1, \ldots m$ on the left hand side (LHS) have been estimated as linear terms in the first substep 1.1. Thus the parameters on the R.H.S. of eqn(40) can be estimated using the weighted complex orthogonal estimator (Swain and Billings,1995).

Similarly at **step 1.r** the parameters of the linear terms consisting of inputs $[u_r]$ can be estimated from the FDE of the system corresponding to the r-th column of $GFRFM^{(1)}$ which is given as

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1)^{l_1}\right] H_1^{(\alpha_1:r)}(j\omega_1) = \sum_{l_1=0}^{L} c_{0,1}^{r}(j_1; l_1)(j\omega_1)^{l_1} \tag{41}$$

4.2 Stage-2 : Estimation of the Parameters of the Second Order Nonlinear Terms

Analogous with the linear case, the information from all the columns of the second order GFRFM ; $GFRFM^{(2)}$ will be progressively utilized to estimate all the parameters associated with the second order nonlinear terms. The total number of steps required to search all the nonlinear second order parameters of the system equals the total number of columns of the $GFRFM^{(2)}$ denoted as $ncol_2$. For an r-input system $ncol_2 = Nk_{(2)}^{(r)} = \frac{r(r+1)}{2}$. Note that the estimation of the parameters will not be affected by the order of selection of the columns of the GFRFM. However without loss of generality it is assumed that the columns of the $GFRFM^{(2)}$ are arranged as $H_2^{(j_1:\beta_1,\beta_2)}(j\omega_1, j\omega_2)$; $\beta_1 = 1, \ldots r, \beta_2 = \beta_1, \ldots r$. For example for a system having two inputs $u_{\beta_1}(t)$, and $u_{\beta_2}(t)$, the elements of $GFRFM^{(2)}$ are given as

$$GFRFM^{(2)} = [H_2^{(j_1:\beta_1\beta_1)}, H_2^{(j_1:\beta_1\beta_2)}, H_2^{(j_1:\beta_2\beta_2)}] \tag{42}$$

Step 2.1 :

The frequency domain equivalent of eqn(18) corresponding to column-1 of the $GFRFM^{(2)}$ derived from eqn(21) by setting $\gamma_1 = 2$ is given as

$$\begin{aligned}
-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2)^{l_1}\right] 2!H_{2_{avg}}^{(\alpha_1:\beta_1,\beta_1)}(j\omega_1,j\omega_2) \\
= \sum_{l_1,.l_n=0}^{L} c_{0,2}^{\beta_1\beta_1}(j_1;l_1,l_2)[(j\omega_1)^{l_1}(j\omega_2)^{l_2}+(j\omega_2)^{l_1}(j\omega_1)^{l_2}] \\
+\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_1}(j_1:l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_2)(j\omega_1)^{l_2}] \\
+\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1\alpha_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}H_1^{(\alpha_2:\beta_1)}(j\omega_2) \\
+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_2)(j\omega_1)^{l_2}H_1^{(\alpha_2:\beta_1)}(j\omega_1)]
\end{aligned} \tag{43}$$

The parameters $c_{10}^{\alpha_1}(j_1:l_1), l_1 = 0, \ldots L,\ \alpha_1 = 1, \ldots m$ in L.H.S. of eqn(43) have been estimated as linear terms in **step 1.1** while all the parameters on the R.H.S. of the above equation can be estimated from eqn(43) by replacing the first and second order kernel transforms by their estimates and applying the new estimator.

Note that after the execution of step 2.1 the terms that would be estimated are

- all the second order nonlinear output terms of the j_1-th subsystem
- second order nonlinear input terms of the form $D^{l_1}u_{\beta_1}(t)D^{l_2}u_{\beta_1}(t)$, $l_1, l_2 = 0, \ldots L$. Other second order nonlinear input terms do not contribute to the computation of $H_2^{(\alpha_1:\beta_1,\beta_1)}(j\omega_1, j\omega_2)$ and therefore can not be estimated using eqn(43).
- The subset of second order input-output cross product terms of the form $D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_1}(t)$, $\alpha_1 = 1, \ldots m$, $l_1, l_2 = 0, \ldots L$.

In order to estimate other second order nonlinear terms, the FDE of the system corresponding to other columns must be utilized. In order to demonstrate the procedure consider a two input system having inputs $u_{\beta_1}(t)$ and $u_{\beta_2}(t)$.

Step-2.2 : The FDE of eqn(18) corresponding to the second column of GFRFM$^{(2)}$ is

$$
\begin{aligned}
-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2)^{l_1}\right] 2!H_{2_{avg}}^{(\alpha_1:\beta_1,\beta_2)}(j\omega_1,j\omega_2) \\
= \sum_{l_1,.l_n=0}^{L} c_{0,2}^{\beta_1\beta_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}(j\omega_2)^{l_2}] \\
+\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_1}(j_1:l_1,l_2)[(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}] \\
\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_2}(j_1:l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}] \\
+\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1\alpha_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}H_1^{(\alpha_2:\beta_2)}(j\omega_2) \\
+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}H_1^{(\alpha_2:\beta_1)}(j\omega_1)]
\end{aligned}
\tag{44}
$$

Note that while executing this step the second order nonlinear output terms and the input-output cross product terms which contain a factor $D^{l_2}u_{\beta_1}(t)$ have been estimated in step-2.1 and can be brought to the L.H.S. This gives the equation

$$
-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2)^{l_1}\right] 2!H_{2_{avg}}^{(\alpha_1:\beta_1,\beta_2)}(j\omega_1,j\omega_2)
$$

$$-\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_1}(j_1:l_1,l_2)[(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}]$$

$$-\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1\alpha_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}H_1^{(\alpha_2:\beta_2)}(j\omega_2)$$

$$+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}H_1^{(\alpha_2:\beta_1)}(j\omega_1)]$$

$$=\sum_{l_1,.l_n=0}^{L} c_{0,2}^{\beta_1\beta_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}(j\omega_2)^{l_2}]$$

$$+\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_2}(j_1:l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)(j\omega_2)^{l_2}] \quad (45)$$

This equation may be solved to give the estimates of the parameters $c_{0,2}^{\beta_1\beta_2}(j_1;l_1,l_2)$, $c_{11}^{\alpha_1\beta_2}(j_1:l_1,l_2)$, $l_1,l_2=0,\ldots L$.

Step-2.3 : The FDE of eqn(18) corresponding to column-3 of GFRFM is

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2)^{l_1}\right]2!H_{2_{avg}}^{(\alpha_1:\beta_2,\beta_2}(j\omega_1,j\omega_2)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{l_1=0}^{L} c_{11}^{\alpha_1\beta_2}(j_1:l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_1)(j\omega_2)^{l_2}+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}]$$

$$-\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1\alpha_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_1)(j\omega_2)^{l_2}H_1^{(\alpha_2:\beta_2)}(j\omega_2)$$

$$+(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_2)(j\omega_1)^{l_2}H_1^{(\alpha_2:\beta_2)}(j\omega_1)]$$

$$=\sum_{l_1,.l_n=0}^{L} c_{0,2}^{\beta_2\beta_2}(j_1;l_1,l_2)[(j\omega_1)^{l_1}(j\omega_2)^{l_2}+(j\omega_2)^{l_1}(j\omega_1)^{l_2}]$$

(46)

4.3 Stage-n : Estimation of the Parameters of the n-th Order Nonlinear Terms

The total number of steps required to estimate all the n-th order parameters of j_1-th subsystem of an r-input system is equal to $Nk_{(n)}^{(r)}$ where $Nk_{(n)}^{(r)}$ is the number of n-th order kernels of the j_1-th subsystem. This is also equal to the total number of columns of $GFRFM^{(n)}$, $Nk_{(n)}^{(r)} = ncol_n$.

Step n.1 :

Find the frequency domain equivalent of eqn(18) corresponding to the first column of $GFRFM^{(n)}$. This is obtained from eqn(21) by putting $\gamma_1 = n$ and $\beta_1 = 1$. Note that while estimating using the n-th order $GFRFM^{(n)}$ some parameters which have been estimated in previous stages appear on the right hand side (R.H.S.). These are

- lower order pure output terms of order p ;$p < n$
- input-output cross product terms consisting of p-th order factors of the output and q-th order factors of the input $[u_{\beta_1}(t)]$ such that $p + q < n$

Bringing the contribution of these terms to the L.H.S gives

$$
\begin{aligned}
-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+\ldots+j\omega_n)^{l_1}\right] n! H_{n_{avg}}^{\overbrace{(\alpha_1:\beta_1,\ldots\beta_1)}^{n-times}}(j\omega_1,\ldots j\omega_n) \\
-\sum_{p=2}^{n-1}\sum_{\alpha_1=1}^{m}\ldots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_p=0}^{L} c_{p,0}^{\alpha_1,..\alpha_p}(j_1:l_1,..l_p)H_Y \\
-\sum_{q=1}^{n-1}\sum_{p=1}^{n-q-1}\sum_{\alpha_1=1}^{m}\ldots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\overbrace{\beta_1,..\beta_1}^{q\ times}}(j_1:l_1,..l_{p+q})H_{UY} \\
= \sum_{l_1,.l_n=0}^{L} c_{0,n}^{\overbrace{\beta_1,\ldots\beta_1}^{n-\ times}}(j_1:l_1,..l_n)H_U \\
+\sum_{\substack{q=1\\ ;p=n-q}}^{n-1}\sum_{\alpha_1=1}^{m}\ldots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_p=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\overbrace{\beta_1,..\beta_1}^{q\ times}}(j_1:l_1,..l_{p+q})H_{UY} \\
+\sum_{\alpha_1=1}^{m}\ldots\sum_{\alpha_n=\alpha_{n-1}}^{m}\sum_{l_1,l_n=0}^{L} c_{n,0}^{\alpha_1,..\alpha_n}(j_1:l_1,..l_n)H_Y \qquad (47)
\end{aligned}
$$

The parameters on the R.H.S. of the above equation can now be estimated by applying the weighted complex orthogonal estimator.

Step n.i; $i = 2,\ldots\ldots Nk_{(n)}^{(r)}$

Let the columns corresponding to the i-th column of $GFRFM^{(n)}$ be represented as $H_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 times},\underbrace{\beta_2,\ldots}_{\gamma_2 times}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} times})}(j\omega_1,\ldots\ldots j\omega_n)$.

Find the FDE of the system corresponding to the i-th column of the n-th order GFRFM and bring the contribution of the terms which have been estimated before to the L.H.S. These terms are

- all output terms having nonlinearity up to degree n .
- a subset of input-output cross product terms. In order to find the cross product terms that have been estimated prior to the **n.i-th step**, the following notation is introduced.

Let $\mathbf{TUY}_{\mathbf{cont}}$ represents the set of all input-output cross product terms that contribute to the kernel transform corresponding to the i-th column of the n-th order GFRFM. This consists of a p-th order factor of the output denoted as Y^p and a q-th order factor of the input denoted by U^q such that $p + q \leq n$. U^q consists of

$$\begin{gathered}
n_1\text{-th order factor of } u_{\beta_1}, \\
n_2\text{-th order factor of } u_{\beta_2}, \\
\ldots\ldots \;\ldots\ldots \;\ldots\ldots \;\ldots\ldots \\
n_{n_d}\text{-th order factor of } u_{\beta_{n_d}}, \text{ subject to the following constraints} \\
n_1 + n_2 + \ldots\ldots + n_{n_d} = q \text{ and} \\
\max(n_1) \leq \gamma_1 \\
\max(n_2) \leq \gamma_2 \\
\max(n_{n_d}) \leq \gamma_{n_d}
\end{gathered}$$

Let $\mathbf{TUY}_{\mathbf{est}}$ represent the subset of $\mathbf{TUY}_{\mathbf{cont}}$ which has been estimated from **Step 1.1 to Step n.i-1**. That is

$$\mathrm{TUY_{est}} = \mathrm{TUY_{cont}}|_{p+q<n} \tag{48}$$

Hence the parameters of the input-output cross product terms that need to be estimated is given as $\mathbf{TUY} = \mathbf{TUY}_{\mathbf{cont}} \cap \mathbf{TUY}_{\mathbf{est}}$. That is

$$\mathrm{TUY} = \mathrm{TUY_{cont}}|_{p+q=n} \tag{49}$$

Bringing the contribution of the estimated terms to the L.H.S gives the regression equation

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1 + \ldots j\omega_n)^{l_1}\right] n!H_{n_{avg}}^{(\alpha_1:\underbrace{\beta_1,\ldots,}_{\gamma_1\text{times}}\underbrace{\beta_2,\ldots}_{\gamma_2\text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d}\text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$$
$$-\sum_{p=2}^{n}\sum_{\alpha_1=1}^{m}\cdot\cdot\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_p=0}^{L} c_{p,0}^{\alpha_1,\ldots\alpha_p}(j_1 : l_1,\ldots l_p)H_Y$$

$$\underbrace{-\sum_{q=1}^{n-1}\sum_{p=1}^{n-q-1}\sum_{\alpha_1=1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\cdots}_{n_1\ times}\underbrace{\beta_2,\cdots}_{n_2\ times}\underbrace{\beta_{n_d},\cdots}_{n_{n_d}\ times}}(j_1:l_1,..l_{p+q})\,H_{UY}}_{\text{Terms which corresponds to } TUY_{est}}$$

$$=\sum_{l_1,.l_n=0}^{L} c_{0,q}^{\underbrace{\beta_1,\cdots}_{\gamma_1\ times}\underbrace{\beta_2,\cdots}_{\gamma_2\ times}\underbrace{\beta_{n_d},\cdots}_{\gamma_{n_d}\ times}}(j_1:l_1,..l_q)H_U$$

$$\underbrace{+\sum_{\substack{q=1\\ ;p=n-q}}^{n-1}\sum_{\alpha_1=1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,l_p=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\cdots}_{n_1\ times}\underbrace{\beta_2,\cdots}_{n_2\ times}\underbrace{\beta_{n_d},\cdots}_{n_{n_d}\ times}}(j_1:l_1,..l_{p+q})\,H_{UY}}_{\text{Terms which corresponds to } TUY} \tag{50}$$

To further clarify these expressions consider the example of a two input system. The regression equation will be formulated to estimate the parameters of the system associated with the third order nonlinear terms from $GFRFM^{(3)}$. Note that for a system with two inputs $u_{\beta_1}(t), u_{\beta_2}(t)$, the elements of the j_1-th subsystem of $GFRFM^{(3)}$ are

$$GFRFM^{(3)} = [H_3^{(j_1:\beta_1,\beta_1,\beta_1)}, H_3^{(j_1:\beta_1,\beta_1,\beta_2)}, H_3^{(j_1:\beta_1,\beta_2,\beta_2)}, H_3^{(j_1:\beta_2,\beta_2,\beta_2)}] \tag{51}$$

Stage-3 :

Step-3.1 :

The FDE of the system corresponding to the first column of $GFRFM^{(3)}$ is given as

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2+j\omega_3)^{l_1}\right]3!H_{3_{avg}}^{(\alpha_1:\beta_1,\beta_1,\beta_1}(j\omega_1,j\omega_2,j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1,\alpha_2}(j_1:l_1,l_2)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}} H_{32}^{\alpha_2,\alpha_1}(j\omega_1,..j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{l_1,l_2=0}^{L} c_{11}^{\alpha_1\beta_1}(j_1:l_1,l_2)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_2}H_{21}^{\alpha_1}(j\omega_2,j\omega_3)$$

$$=\sum_{l_1,.l_3=0}^{L} c_{0,3}^{\beta_1\beta_1,\beta_1}(j_1;l_1,l_2,l_3)[\sum_{\substack{\text{all permutations}\\ \omega_1,\omega_2,\omega_3}}(j\omega_1)^{l_1}(j\omega_2)^{l_2}(j\omega_3)^{l_3}]$$

$$+\sum_{\alpha_1=1}^{m}\sum_{l_1,.l_3=0}^{L} c_{1,2}^{\alpha_1\beta_1\beta_1}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_2}(j\omega_2)^{l_3}H_{11}^{\alpha_1}(j\omega_3)$$

$$+\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,.l_3=0}^{L} c_{2,1}^{\alpha_1\alpha_2\beta_1}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_3}H_{22}^{\alpha_2,\alpha_1}(j\omega_2,j\omega_3)$$

$$+\sum_{\alpha_1=1}^{m}\cdot\cdot\sum_{\alpha_3=\alpha_2}^{m}\sum_{l_1,.l_3=0}^{L} c_{3,0}^{\alpha_1\alpha_2\alpha_3}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}H_{33}^{\alpha_3,\alpha_2\alpha_1}(j\omega_1,..j\omega_3) \quad (52)$$

Step-3.2 :

The FDE of the system corresponding to second column of $GFRFM^{(3)}$ is given as

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1:l_1)(j\omega_1+j\omega_2+j\omega_3)^{l_1}\right]3!H_{3_{avg}}^{(\alpha_1:\beta_1,\beta_1,\beta_1}(j\omega_1,j\omega_2,j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,l_2=0}^{L} c_{2,0}^{\alpha_1,\alpha_2}(j_1:l_1,l_2)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}H_{32}^{\alpha_2,\alpha_1}(j\omega_1,..j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{l_1,l_2=0}^{L} c_{11}^{\alpha_1\beta_1}(j_1:l_1,l_2)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_2}H_{21}^{\alpha_1}(j\omega_2,j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{l_1,.l_3=0}^{L} c_{1,2}^{\alpha_1\beta_1\beta_1}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_2}(j\omega_2)^{l_3}H_{11}^{\alpha_1}(j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\sum_{l_1,.l_3=0}^{L} c_{2,1}^{\alpha_1\alpha_2\beta_1}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_3}H_{22}^{\alpha_2,\alpha_1}(j\omega_2,j\omega_3)$$

$$-\sum_{\alpha_1=1}^{m}\cdot\cdot\sum_{\alpha_3=\alpha_2}^{m}\sum_{l_1,.l_3=0}^{L} c_{3,0}^{\alpha_1\alpha_2\alpha_3}(j_1;l_1,l_2,l_3)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}H_{33}^{\alpha_3,\alpha_2\alpha_1}(j\omega_1,..j\omega_3)$$

$$=\sum_{l_1,.l_3=0}^{L} c_{0,3}^{\beta_1\beta_1,\beta_2}(j_1;l_1,l_2,l_3)[\sum_{\substack{\text{all permutations}\\ \omega_1,\omega_2}}(j\omega_1)^{l_1}(j\omega_2)^{l_2}](j\omega_3)^{l_3}$$

$$+\sum_{\alpha_1=1}^{m}\sum_{l_1,l_2=0}^{L} c_{11}^{\alpha_1\beta_2}(j_1:l_1,l_2)\sum_{\substack{\text{all perm}\\ [\omega,\beta]}}(j\omega_1)^{l_2}H_{21}^{\alpha_1}(j\omega_2,j\omega_3) \quad (53)$$

Following identical procedures regression equations for other columns of GFRFM can be formulated to identify the parameters of the system.

5 Simulated Examples

In the present section the effectiveness of the new estimator is demonstrated using three different types of systems. The first example refers to a two degree of freedom of linear system;

the second example represents a quadratically nonlinear system and the third example is a coupled Vander Pol oscillator. Note that the following guidelines have been followed in all the examples.

- No assumption has been made regarding the structure of the underlying system. An overparameterized model structure is initially specified and with no *a priori* information the algorithm is used to detect the correct structure and estimate the unknown parameters.
- The weighting matrix Q has been taken to be a diagonal matrix of the form $e^{-\omega_i^2/\lambda}$
- A normalised version of the weighted complex orthogonal estimator (Swain and Billings,1995) has been used to avoid possible numerical ill-conditioning. That is the regression equation in eqn(1) is normalised according to

$$\begin{aligned} z(j\omega) &= \textstyle\sum_{i=1}^{M} \theta_i p_i(j\omega) \\ &= \theta_1 p_1(j\omega) + \theta_2 p_2(j\omega) + \ldots. + \theta_M p_M(j\omega) \\ &= \theta_1 \|p_1(j\omega)\| \frac{p_1(j\omega)}{\|p_1(j\omega)\|} + \theta_2 \|p_2(j\omega\| \frac{p_2(j\omega)}{\|p_2(j\omega)\|} + \ldots\ldots\theta_M \|p_M(j\omega)\| \frac{p_M(j\omega)}{\|p_M(j\omega)\|} \\ &= \theta_1^{nr} p_1^{nr} + \theta_2^{nr} p_2^{nr} + \ldots. + \theta_M^{nr} p_M^{nr} \end{aligned} \quad (54)$$

where the superscript 'nr' denotes normalised term. The normalised parameters $\theta_1^{nr}, \theta_2^{nr}, \ldots..\theta_M^{nr}$ are initially estimated. The original parameters of the model are then recovered directly from the normalised parameters.

Example-1 : Two Degree Freedom Linear System

Consider a two input two output system described by the equation.

$$\frac{d^2y_1(t)}{dt^2} + 0.21\frac{dy_1(t)}{dt} + 1.2y_1(t) - 0.01\frac{dy_2(t)}{dt} - 0.2y_2(t) = u_1(t) \quad (55)$$

$$0.2\frac{d^2y_2(t)}{dt^2} + 0.01\frac{dy_2(t)}{dt} + 0.2y_2(t) - 0.01\frac{dy_1(t)}{dt} - 0.2y_1(t) = u_2(t) \quad (56)$$

where $y_1(t), y_2(t)$ represent the outputs of the systems. The magnitude plot of the first order self kernel transforms which are obtained from the frequency domain equivalent of eqn(55) and eqn(56) are shown in the Fig-1.

In order to reconstruct the system differential equation corresponding to subsystem-1 (eqn(55)), a set of 100 equally spaced frequency response data were generated from the

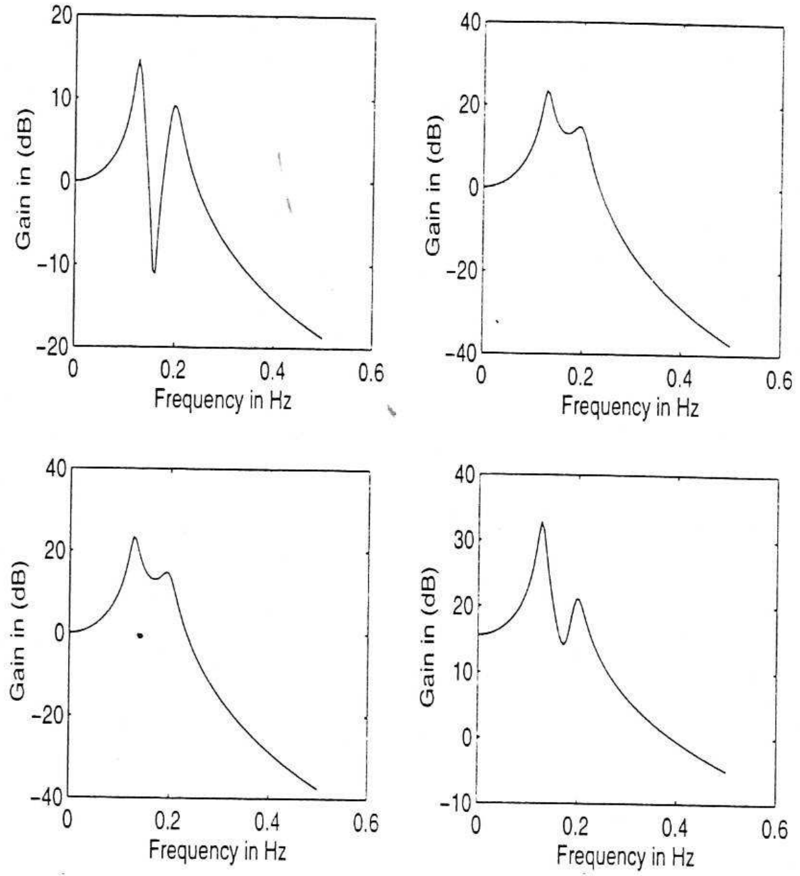

Figure 1: The Gain plot for the self kernel transforms for the two degree of freedom linear system of Example-1 (eqn(55,56)): (a) $H_1^{(1:1)}(j\omega_1)$, (b) $H_1^{(1:2)}(j\omega_1)$ (c) $H_1^{(2:1)}(j\omega_1)$ (d) $H_1^{(2:2)}(j\omega_1)$

frequency domain equivalent of eqn(55) in the frequency range 0-0.5Hz and these were used to estimate the parameters of eqn(55). An overparameterized (8th order) differential equation was initially specified for the estimation to test the capability of the proposed algorithm to detect the correct model structure. Table-1 shows the terms and the order that they were selected as the iteration proceeded together with the error reduction ratios (ERR) and the parameters of the auxiliary model. Note that the estimator initially estimates the parameters of the auxiliary model and then recovers the original system parameters from eqn(9). Recall that the ERR values correspond to the percentage contribution each term makes to the regression. The parameters of the model are shown in Table-2. To estimate the parameters of the second subsystem (eqn(56)) an identical procedure was followed except the tuning parameter was interactively chosen to be 30. The terms that were selected and the resulting parameter estimates are shown in Tables-3 and 4. The results show that the estimated parameters closely match the true parameters of the system.

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_1/dt^2	58.22	0.7829
2	u	21.71	-0.905
3	y_2	17.8	-0.2092
4	dy_1/dt	2.22	0.2077
5	dy_2/dt	0.032	-0.01

Table 1: Parameters of the Auxiliary model of Subsystem-1 for a two degree freedom linear system Example-1 (eqn(55)); $\lambda = 40.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_1/dt^2	1.00	1.00
dy_1/dt	0.21	0.21
dy_2/dt	-0.01	-0.01
y_2	-0.2	0.2
u_1	1.0	1.0

Table 2: Parameters of the Subsystem-1 for a two degree freedom linear system Example-1 (eqn(55)); $\lambda = 40.0$

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_2/dt^2	64.27	0.1824
2	u	24.84	-1.1448
3	y_1	10.78	-0.2101
4	dy_2/dt	0.0811	0.0083
5	dy_1/dt	0.016	-0.01

Table 3: Parameters of the Auxiliary model of Subsystem-2 for a two degree freedom linear system Example-1 (eqn(56)); $\lambda = 30.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_2/dt^2	0.2	0.2
dy_2/dt	0.01	0.01
dy_1/dt	-0.01	-0.01
y_1	-0.2	-0.2
u_2	1.0	1.0

Table 4: Parameters of the Subsystem-2 for a two degree freedom linear system Example-1 (eqn(56)) ; $\lambda = 30.0$

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_1/dt^2	95.05	0.9438
2	u	4.90	-0.9950
3	dy_1/dt	0.039	-0.0201
4	y_2	0.0001	0.140

Table 5: Parameters of the Auxiliary model (linear) of Subsystem-1 for the Quadratic Nonlinear System :Example-2 (eqn(57)) with $\lambda = 20.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_1/dt^2	1.00	1.00
dy_1/dt	0.02	0.02
y_2	0.14	0.14
u_1	1.0	1.0

Table 6: Parameters of Linear terms of Subsystem-1 for the Quadratic Nonlinear System :Example-2(eqn(57)) with $\lambda = 20.0$

Example-2 : A Quadratic Nonlinear System

Consider a two degree of freedom quadratically nonlinear system governed by the dynamical equation

$$\frac{d^2y_1(t)}{dt^2} + 0.020\frac{dy_1(t)}{dt} + 1.01y_1(t) + 0.14y_2(t) = u_1(t) - 0.05y_1^2(t) - 0.2y_1(t)y_2(t) - 0.15y_2^2(t) \tag{57}$$

$$\frac{d^2y_2(t)}{dt^2} + 0.02\frac{dy_2(t)}{dt} + 4.01y_2(t) + 0.072y_1(t) = u_2(t) - 0.1y_1^2(t) - 0.3y_1(t)y_2(t) - 0.2y_2^2(t) \tag{58}$$

In order to reconstruct the system differential equation for subsystem-1 (eqn(57)), initially,a set of 100 equally spaced frequency response data in the frequency range 0 -1.0Hz were generated from the systems transfer function matrix and these were used to estimate the parameters of the transfer function. An overparameterized (8th order) model was initially specified for the estimation so that the effectiveness of the proposed algorithm in detecting both the correct model structure and estimating the unknown parameters could be tested. Table-5 shows the order and the terms that were selected as the iteration proceeded together with the error reduction ratios. The parameters of the model are shown in Table-6. An identical procedure was followed to estimate the parameters of subsystem-2 (eqn(58)). Table-7 shows the terms that were selected as the iteration proceeded together with the error reduction ratios. The parameters of the model are shown in Table-8.

After estimating the parameters associated with the linear terms, the algorithm was

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_2/dt^2	96.8	0.9770
2	u	3.09	-0.9953
3	dy_2/dt	0.009	-0.020
4	y_1	0.0007	0.072

Table 7: Parameters of the Auxiliary model (linear) of Subsystem-2 for the Quadratic Nonlinear System : Example-2 (eqn(58)) with $\lambda = 20.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_2/dt^2	1.00	1.00
dy_2/dt	0.02	0.02
y_1	0.072	0.072
u_1	1.0	1.0

Table 8: Parameters of Linear terms of the Subsystem-1 for the Quadratic Nonlinear System : Example-2 (eqn(58)) with $\lambda = 20.0$

applied to estimate the parameters of the quadratic nonlinear terms of subsystem-1. It is emphasised that the procedure of reconstruction is sequential where the parameters of the linear terms are estimated first and then the parameters associated with the second order nonlinear terms and so on. This offers significant advantages since the model can be build up term by term and thus is not limited to any degree of nonlinearity. To reconstruct the nonlinear part of the first subsystem 100 equally spaced frequency response data were generated in the frequency range of -1.0-1.0Hz. The terms that were selected from the specified model set $\{y_1^2, y_1\dot{y}_1, \dot{y}_1^2, y_2^2, y_2\dot{y}_2, y_1y_2, y_1\dot{y}_2\}$ together with the associated error reduction ratios as the iteration proceeds are shown in Tables-9-10 To reconstruct the nonlinear part of the second subsystem 100 equally spaced frequency response data were generated in the frequency range of -1.0-1.0Hz. The terms that were selected from the model set $[y_2^2, y_2\dot{y}_2, \dot{y}_2^2, y_1^2, y_1\dot{y}_1, y_1y_2, y_2\dot{y}_1]$ together with the associated error reduction ratios are shown in Tables-11- 12. The results show that the estimated parameters match with the true system parameters very well.

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	y_1^2	99.13	0.0469
2	y_1y_2	0.85	0.1953
3	y_2^2	0.001	0.15

Table 9: Parameters of the Auxiliary model (nonlinear) of Subsystem-1 for the Quadratic Nonlinear System : example-2 (eqn(57))

Candidate Terms	Estimated Parameters	True Parameters
y_1^2	0.05	0.05
y_1y_2	0.2	0.2
y_2^2	0.15	0.15

Table 10: Parameters of Nonlinear terms of the Subsystem-1 for the Quadratic Nonlinear System : Example-2 (eqn(57))

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	y_2^2	99.26	0.1814
2	y_1y_2	0.73	0.2873
3	y_1^2	0.0011	0.10

Table 11: Parameters of the Auxiliary model (nonlinear) of Subsystem-2 for the Quadratic Nonlinear System : Example-2 (eqn(58))

Example-3 : A Coupled Vander Pol Oscillator

The equations governing the dynamics of two mutually coupled Vander Pol oscillators are (Linkens,1975)

$$\begin{aligned} &\ddot{y}_1(t) + \lambda_c\ddot{y}_2(t) - c_1(1 - y_1^2(t))(\dot{y}_1(t) + \lambda_c y_2(t)) + \omega_1^2 y_1(t) = u_1(t) \\ &\ddot{y}_2(t) + \lambda_c\ddot{y}_1(t) - c_1(1 - y_2^2(t))(\dot{y}_2(t) + \lambda_c y_1(t)) + \omega_1^2 y_2(t) = u_2(t) \end{aligned} \qquad (59)$$

where λ_c is the coupling parameter. For $\lambda_c = 0.8, \omega_1 = \omega_2 = 2.0\ rad/sec, c_1 = c_2 = 0.1$, eqn(59) gives

$$\ddot{y}_1(t) - 0.1\dot{y}_1(t) + 4.0y_1(t) + 0.8\ddot{y}_2(t) - 0.08y_2(t) + 0.1y_1^2(t)\dot{y}_1(t) + 0.08y_1^2(t)y_2(t) = u_1(t) \qquad (60)$$

$$0.8\ddot{y}_1(t) - 0.08y_1(t) + \ddot{y}_2(t) - 0.1\dot{y}_2(t) + 4.0y_2(t) + 0.1y_2^2(t)\dot{y}_2(t) + 0.08y_2^2(t)y_1(t) = u_2(t) \qquad (61)$$

In order to reconstruct the linear parts of both the subsystems 100 equally spaced

Candidate Terms	Estimated Parameters	True Parameters
y_1^2	0.1	0.1
y_1y_2	0.3	0.3
y_2^2	0.2	0.2

Table 12: Parameters of Nonlinear terms of Subsystem-2 for the Quadratic Nonlinear System : Example-2 (eqn(58))

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_1/dt^2	52.04	0.8768
2	u_1	39.87	-0.7085
3	dy_2/dt	7.91	0.7783
4	dy_1/dt	0.130	-0.0956
5	y_2	0.039	-0.080

Table 13: Parameters of the Auxiliary model (linear) of Subsystem-1 for the Coupled Vander Pol Oscillator : Example-3 (eqn(60)); $\lambda = 40.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_1/dt^2	1.00	1.00
dy_1/dt	-0.1	-0.1
dy_2/dt	0.8	0.8
y_2	-0.08	-0.08
u_1	1.0	1.0

Table 14: Parameters of Linear terms of the Subsystem-1 for the Coupled Vander Pol Oscillator: Example-3 (eqn(60)) ; $\lambda = 40.0$

frequency response data were generated in the frequency range of 0-0.5Hz. To identify the parameters of subsystem-1 the tuning parameter was chosen as 40. The terms that were selected from an over parameterised model (4th order) together with the associated error reduction ratios are shown in Table-13 The linear terms of subsystem-2 were selected following an identical procedure and the results with a weighting parameter of 40 are given in the Table-15 The nonlinear part of the first subsystem was reconstructed by generating 64 equally spaced frequency response data in the frequency range -0.1 to 0.1Hz. For reconstructing the nonlinear part of the second subsystem 64 data samples were generated in the frequency range -0.08 to 0.08 Hz. The overparameterized model sets specified for subsystem-1 were [y_1^3, $y_1^2\dot{y}_1$, $y_1\dot{y}_1^2$, $\dot{y}_1^3$, $y_1^2y_2$, $y_1\dot{y}_1y_2$, $\dot{y}_1^2y_2$] and for subsystem-2 [y_2^3, $y_2^2\dot{y}_2$, $y_2\dot{y}_2^2$, $\dot{y}_2^3$, $y_2^2y_1$, $y_2\dot{y}_2y_1$, $\dot{y}_2^2y_1$]. The terms that were selected and the associated

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	d^2y_2/dt^2	52.04	0.8768
2	u_2	39.87	-0.7085
3	dy_1/dt	7.91	0.7783
4	dy_2/dt	0.130	-0.0956
5	y_1	0.039	-0.080

Table 15: Parameters of the Auxiliary model (linear) of Subsystem-2 for the Coupled Vander Pol Oscillator : Example-3 (eqn(61)); $\lambda = 40.0$

Candidate Terms	Estimated Parameters	True Parameters
d^2y_2/dt^2	1.00	1.00
dy_2/dt	-0.1	-0.1
dy_1/dt	0.8	0.8
y_1	-0.08	-0.08
u_2	1.0	1.0

Table 16: Parameters of Linear terms of the Subsystem-2 for the Coupled Vander Pol Oscillator: Example-3 (eqn(61)) ; $\lambda = 40.0$

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	$y_1^2\dot{y}_1$	99.324	0.0825
2	$y_1^2y_2$	0.675	0.0800

Table 17: Parameters of the Auxiliary model (Nonlinear) of Subsystem-1 for the Coupled Vander Pol Oscillator : Example-3 (eqn(60))

error reduction ratios are shown in Tables-17 - 20.

From the results it is obvious that the estimated parameters matched very well with the true parameters of the system.

6 Conclusions

A weighted complex orthogonal estimator has been applied to estimate continuous time nonlinear differential equation models of MIMO nonlinear systems from generalised frequency response function matrices. The estimator possesses properties and advantages of both the weighted and orthogonal least squares algorithms. The estimation procedure is progressive and involves utilizing the information in the GFRFM beginning with the first column of the first order GFRFM and continuing with the columns of the second order GFRFM and so on. There is therefore no restriction on the order and dimensions of the MIMO systems which can be investigated using this approach. The estimation procedure combines structure selection with parameter estimation and thus provides a powerful tool to identify parsimonious models for systems with unknown structure in the frequency domain. Several simulated examples

Candidate Terms	Estimated Parameters	True Parameters
$y_1^2\dot{y}_1$	0.100	0.100
$y_1^2y_2$	0.0800	0.080

Table 18: Parameters of Nonlinear terms of Subsystem-2 for the Coupled Vander Pol Oscillator : Example-3 (eqn(61))

Iteration	Selected Terms (p_i)	ERR	Par.of Aux.model
1	$y_2^2\dot{y}_2$	96.377	0.0880
2	$y_2^2y_1$	3.614	0.0802

Table 19: Parameters of the Auxiliary model (Nonlinear) of Subsystem-2 for the Coupled Vander Pol Oscillator : Example-3 (eqn(61))

Candidate Terms	Estimated Parameters	True Parameters
$y_2^2\dot{y}_2$	0.100	0.100
$y_2^2y_1$	0.0802	0.080

Table 20: Parameters of the Subsystem-2 for the Coupled Vander Pol Oscillator : Example-3 (eqn(61))

were included to demonstrate the effectiveness of the new procedure to build continuous time differential equation models from the frequency response data.

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