Monograph:

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Stability of Finite and Infinite Dimensional Nonlinear Delay Systems, Independent of the Delay

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Abstract:The stability of nonlinear delay systems is considered. General conditions on pseudo-linear finite- and infinite-dimensional delay systems are given for stability independent of the delay.

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1. Introduction

For finite-dimensional linear delay systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t-h)$$

it is well-known (see [[1],[2]]) that this system is stable, independently of the delay, iff

(i) A is asymptotically stable

(ii) $\rho[(j\omega I - A)^{-1}B] < 1, \forall\, \omega > 0$

and

(iii) $\rho[A^{-1}B] < 1$ or $\rho[A^{-1}B] = 1$ and $\det(A+B) \neq 0$.

This has been proved using frequency domain ideas which do not generalize easily. Here we shall give a 'state-space' proof which will generalize to nonlinear distributed systems. In particular, we shall consider systems of the form

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))x(t-h)$$

in both the finite- and infinite-dimensional cases.

The notation will generally be standard- in particular, $B_\varepsilon(\delta)$ denotes the open ball in $\mathbb{C}$ with radius ε and centre δ, and x_θ denotes the function

$$x_\theta(t) = x(t-\theta)\ ,\ 0 \leq \theta \leq h.$$

$\sigma(A), \rho(A)$ denote the spectrum and spectral radius of A, respectively.

2. Finite-Dimensional Systems

We shall first give an elementary proof of the above result , which will generalize in a number of ways. Thus, consider the linear delay system

$$\dot{x}(t) = Ax(t) + Bx(t-h)\ ,\ x(t) = \phi(t), -h \le t \le 0 \tag{2.1}$$

Then,

$$x(\tau + nh) = e^{A\tau}x(nh) + \int_{nh}^{\tau+nh} e^{A(\tau+nh-s)}Bx(s-h)ds$$

for $\tau \ge 0$. Define the functions

$$x[n](\tau) = \begin{cases} x(\tau+nh)\ ,\ n \ge 0, 0 \le \tau \le h \\ \phi(\tau)\ ,\ n = -1, -h \le \tau \le 0 \end{cases} . \tag{2.2}$$

Then we have

$$\begin{aligned} x[n](\tau) &= e^{A\tau}x[n-1](h) + \int_{nh}^{\tau+nh} e^{A(\tau+nh-s)}Bx(s-h)ds \\ &= e^{A\tau}x[n-1](h) + \int_0^{\tau} e^{A(\tau-s)}Bx[n-1](s)ds\ , \end{aligned}$$

whence

$$x[n] = Kx[n-1]$$

where $K : C^1[0, h; \mathbb{R}^n] \longrightarrow C^1[0, h; \mathbb{R}^n]$ is the operator defined by

$$(Kp)(\tau) = e^{A\tau}p(h) + \int_0^{\tau} e^{A(\tau-s)}Bp(s)ds\ . \tag{2.3}$$

It is necessary and sufficient for stability of (2.1) (independently of the delay) that

$$\sigma(K) \subseteq B_1(0),$$

for any $h > 0$. The spectrum of K is given by those λ's for which there exists a $p \neq 0$ such that

$$e^{A\tau}p(h) + \int_0^\tau e^{A(\tau-s)}Bp(s)ds = \lambda p(\tau). \tag{2.4}$$

Differentiating gives

$$\dot{p} = Ap + \frac{1}{\lambda}Bp \ , \ p(0) = \frac{1}{\lambda}p(h) \ .$$

i.e.

$$\left\{\exp\left(A + \frac{1}{\lambda}B\right)h\right\}p(0) = \lambda p(0) = p(h) \tag{2.5}$$

The result now follows easily from the above considerations. For example, the necessary condition (ii) follows from (2.4) by letting $h \to \infty$ and taking the Fourier transform. From (2.5) we get the more general result:

Theorem 2.1 *The equation (2.1) is asymptotically stable iff*

$$\lambda \notin \sigma\left(\exp\left(A + \frac{1}{\lambda}B\right)h\right)$$

for all $\lambda \in \mathbb{C}\backslash\{0\}$.$\square$

Moreover, we have

Theorem 2.2 *The equation (2.1) has a periodic orbit of period h if*

$$1 \in \sigma\left(\exp\left(A + \frac{1}{\lambda}B\right)h\right)$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$.$\square$

We can also obtain an explicit formula for the solution of the equation (2.1). For,

$$x[n](\tau) \quad = \quad e^{A\tau}\left[x[n-1](h) + \int_0^\tau e^{-As}Bx[n-1](s)ds\right]$$

$$= \quad e^{A\tau} \int_0^h k(s)x[n-1](s)ds$$

where $k(s)$ is the distribution

$$k(s) = \delta(s-h)I + \chi_{[0,\tau]}(s)e^{-As}B, \tag{2.6}$$

and $\chi_{[0,\tau]}(.)$ is the characteristic function of the interval $[0,T]$. Hence

$$\begin{aligned} x(\tau+nh) &= x[n](\tau) && (2.7) \\ &= e^{A\tau} \underbrace{\int_0^h \cdots \int_0^h}_{n+1} k(s_n)e^{As_n}k(s_{n-1})e^{As_{n-1}} \cdots \\ &\quad k(s_2)e^{As_2}k(s_1)e^{As_1}k(s_0)\phi(s_0)ds_0 \cdots ds_n \end{aligned}$$

for $0 \le \tau \le h$.

Next Consider the non-autonomous system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h), x(t) = \phi(t), -h \le t \le 0 \tag{2.8}$$

Defining $x[n]$ as in (2.2) we have

$$x[n](\tau) = \Phi(\tau+nh, nh)x[n-1](h) + \int_{nh}^{\tau+nh} \Phi(\tau+nh, s)B(s)x(s-h)ds$$

where $\Phi(t)$ is the transition matrix generated by $A(t)$. Hence,

$$x[n](\tau) = \Phi(\tau+nh, nh)x[n-1](h) + \int_0^\tau \Phi(\tau+nh, s+nh)B(s+nh)x[n-1](s)ds$$

i.e.

$$x[n] = \bar{K}x[n-1]$$

where $\bar{K} : C^1[0,h;\mathbb{R}^n] \to C^1[0,h;\mathbb{R}^n]$ is the operator defined by

$$(\bar{K}p)(\tau) = \Phi(\tau+nh,nh)p(h) + \int_0^\tau \Phi(\tau+nh,s+nh)B(s+nh)p(s)ds. \tag{2.9}$$

As before, a necessary and sufficient condition for stability is that

$$\sigma(\bar{K}) \subseteq B_1(0),$$

for any $h > 0$. The spectrum of $\bar{K}$ is given by

$$\Phi(\tau+nh,nh)p(h) + \int_0^\tau \Phi(\tau+nh,s+nh)B(s+nh)p(s)ds = \lambda p(\tau).$$

Since $\Phi(nh,\tau+nh) = \Phi^{-1}(\tau+nh,nh)$, we have, by differentiating,

$$\begin{aligned}\Phi(nh,\tau+nh)B(s+nh)p(\tau) &= -\lambda A(t)\Phi(nh,\tau+nh)p(\tau)\\ &\quad +\lambda\Phi(nh,\tau+nh)\dot{p}(\tau)\end{aligned}$$

i.e.

$$\dot{p}(\tau) = \left(A(t) + \frac{1}{\lambda}B(t)\right)p(\tau)\ ,\ p(0) = \frac{1}{\lambda}p(h),$$

as before. Hence,

$$\Phi_{(\lambda)}(h,0)p(0) = \lambda p(0) = p(h),$$

where $\Phi_{(\lambda)}(t,s)$ is the transition operator generated by $A(t)+\frac{1}{\lambda}B(t)$. We therefore have

Thereom 2.3 *The equation (2.8) is asymptotically stable (independently of the delay) iff*

$$\lambda \notin \sigma\left(\Phi_{(\lambda)}(h,0)\right)$$

for all $\lambda \in \mathbb{C}\backslash\{0\}$ *and all* $h \geq 0.\square$

Theorem 2.2 also generalizes to

Thereom 2.4 *The equation (2.8) has a periodic orbit of period h if*

$$1 \in \sigma\left(\Phi_{(\lambda)}(h,0)\right)$$

for some $\lambda \in \mathbb{C}\backslash\{0\}.\square$

Now consider the pseudo-linear system

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))x(t-h), x(t) = \phi(t), -h \leq t \leq 0 \tag{2.10}$$

Let $x^{(0)}$ be the solution of the equation

$$\dot{x}(t) = A(x_0)x(t) + B(x_0)x(t-h), x(t) = \phi(t), -h \leq t \leq 0$$

where $x_0 = \phi(h)$, and consider the sequence of equations

$$\dot{x}^{(i)}(t) = A(x^{(i-1)}(t))x^{(i)}(t) + B(x^{(i-1)}(t))x^{(i)}(t-h), x^{(i)}(t) = \phi(t), -h \leq t \leq 0 \tag{2.11}$$

Thus,

$$\dot{x}^{(i)} - \dot{x}^{(i-1)} \quad = \quad A(x^{(i-1)}(t))\left(x^{(i)}(t) - x^{(i-1)}(t)\right) \tag{2.12}$$

$$+\left(A(x^{(i-1)}(t)) - A(x^{(i-2)}(t))\right) x^{(i-1)}(t)$$
$$+B(x^{(i-1)}(t)) \left(x^{(i)}(t-h) - x^{(i-1)}(t-h)\right)$$
$$+\left(B(x^{(i-1)}(t)) - B(x^{(i-2)}(t))\right) x^{(i-1)}(t-h)$$

First we must bound $x^{(i-1)}(t)$ on $[-h, \infty)$. To do this we make the assumption that

$$(A) \begin{cases} \|A(x) - A(y)\| \leq \alpha \|x - y\| \\ \|B(x) - B(y)\| \leq \beta \|x - y\| \\ \left\|e^{A(x)t}\right\| \leq Me^{-\omega t} \\ \|B(x)\| \leq \Gamma \end{cases}$$

for any $x, y \in \mathbb{R}^n$ and some positive α and β. The following result is well-known ([3]); however, the simple proof below generalizes to the distributed case.

Lemma 2.5 *If* $\left\|e^{At}\right\| \leq Me^{-\omega t}$ $(\omega > 0)$ *and* $\frac{M\|B\|}{\omega} < 1$ *then the delay equation*

$$\dot{x}(t) = Ax(t) + Bx(t-h)$$

is stable (independently of the delay). Moreover,

$$\|x(t)\| \leq \sup_{t \in [0,h]} \|\phi(t)\|$$

Proof We have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} Bx(s-h) ds$$

so that

$$\|x(t)\| \leq Me^{-\omega t}\|x_0\| + Me^{-\omega t} \int_0^t e^{\omega s} \|B\| \ \|x(s-h)\| ds$$

and so if $\xi(t)$ satisfies

$$\dot{\xi}(t) = -\omega\xi(t) + M\|B\|\xi(t-h) \tag{2.13}$$

then $\|x(t)\| \le \xi(t)$. We prove that

$$\sup_{t\in[(n-1)h,nh]} \xi(t) \le \sup_{t\in[nh,(n+1)h]} \xi(t)$$

from which the result follows (since the solution is eventually monotonically decreasing or oscillates, and in both cases it is easy to see that it tends to zero as $t \to \infty$). In fact, this is easily seen to be true by considering $\dot{\xi}$ at nh and $(n+1)h$. For, at nh and $(n+1)h$, $\dot{\xi}$ is $> 0, < 0$ or $= 0$, and we have nine cases to consider. For example, if $\dot{\xi}(nh) > 0$ and $\dot{\xi}((n+1)h) < 0$ then ξ has a maximum in $(nh, (n+1)h)$ where $\dot{\xi}(t) = 0$. The result follows from (2.13).□

Next we need the logarithmic norm of a matrix A, $\mu(A)$, defined by

$$\mu(A) = \lim_{h\to 0+} \frac{\|I + hA\| - 1}{h}$$

Lemma 2.6 *Assume that*

$$\mu(A(x)) \le -\mu$$

where $\mu > 0$, *for all* $x \in \mathbb{R}^n$ *and* $\frac{\Gamma}{\mu} < 1$. *Then* $x^{(i)}(t)$ *is stable for each* $i \ge 0$ *and*

$$\left\| x^{(i)}(t) \right\| \le \sup_{t\in[-h,0]} \|\phi(t)\| .$$

Proof We have

$$x^{(i)} = \Phi^{(i-1)}(t,0)x^{(i)}(0) + \int_0^t \Phi^{(i-1)}(t,s)B(x^{(i-1)}(s))x^{(i)}(s-h)ds$$

where $\Phi^{(i-1)}(t,s)$ is the transition matrix generated by $A(x^{(i-1)}(t))$. It is well-known (see []) that

$$\left\|\Phi^{(i-1)}(t,s)\right\| \leq \exp\left[\int_s^t \mu\left(A(x^{(i-1)}(\tau))\right) d\tau\right]$$

and so

$$\left\|\Phi^{(i-1)}(t,s)\right\| \leq \exp(-\mu(t-s)).$$

Hence

$$\begin{aligned}
\left\|x^{(i)}(t)\right\| &\leq e^{-\mu t}\left\|x^{(i)}(0)\right\| + \int_0^t e^{-\mu(t-s)}\left\|B(x^{(i-1)}(s))\right\| \left\|x^{(i)}(s-h)\right\| ds \\
&\leq e^{-\mu t}\left\|x^{(i)}(0)\right\| + \int_0^t e^{-\mu(t-s)}\Gamma \left\|x^{(i)}(s-h)\right\| ds.
\end{aligned}$$

By a standard comparison argument, we see that $\left\|x^{(i)}(t)\right\| \leq \xi(t)$ where

$$\dot{\xi}(t) = -\mu\xi(t) + \Gamma\xi(t-h).$$

Hence, by lemma 2.5, we see that $\left\|x^{(i)}(t)\right\| \to 0$ as $t \to \infty$ and, moreover,

$$\left\|x^{(i)}(t)\right\| \leq \sup_{t\in[-h,0]} \|\phi(t)\|. \ \Box$$

Returning to (2.12) we have, putting $y^{(i)}(t) = x^{(i)}(t) - x^{(i-1)}(t)$,

$$\begin{aligned}
y^{(i)}(t) &= \Phi^{(i-1)}(t,0)y^{(i)}(0) + \int_0^t \Phi^{(i-1)}(t,s)[\left(A(x^{(i-1)}(s)) - A(x^{(i-2)}(s))\right) x^{(i-1)}(s) \\
&\quad + B(x^{(i-1)}(s))y^{(i)}(s-h) + \left(B(x^{(i-1)}(s)) - B(x^{(i-2)}(s))\right) x^{(i-1)}(s-h)]ds
\end{aligned}$$

and so (since $y^{(i)}(0) = 0$)

$$\begin{aligned}\left\|y^{(i)}(t)\right\| \;&\leq\; \int_0^t e^{-\mu(t-s)}\{\alpha\left\|y^{(i-1)}(s)\right\|\left\|x^{(i-1)}(s)\right\| \\ &+\Gamma\left\|y^{(i)}(s-h)\right\| + \beta\left\|y^{(i-1)}(s)\right\|\left\|x^{(i-1)}(s-h)\right\|\}ds\end{aligned}$$

under the assumptions of lemma 2.6. Suppose that

$$\sup_{t\in[-h,0]} \|\phi(t)\| = \Psi.$$

Then,

$$\begin{aligned}\left\|y^{(i)}(t)\right\| \;&\leq\; e^{-\mu t}\left\|y^{(i)}(0)\right\| + \int_0^t e^{-\mu(t-s)}\{L\left\|y^{(i-1)}(s)\right\| + \\ &\Gamma\left\|y^{(i)}(s-h)\right\|\}ds\end{aligned}$$

where $L = (\alpha+\beta)\Psi$. Again, by a comparison argument, $\left\|y^{(i)}(t)\right\| \leq \eta^{(i)}(t)$ where $\eta^{(i)}$ satisfies

$$\dot{\eta}^{(i)}(t) = -\mu\eta^{(i)}(t) + L\eta^{(i-1)}(t) + \Gamma\eta^{(i)}(t-h). \tag{2.14}$$

Now it is easy to see from lemma 2.5 that the operator P defined by

$$Pf(\theta) = \begin{cases} -\mu f(0) + \Gamma f(h), \; \theta = h \\ \frac{df}{d\theta}\;, \theta \neq h \end{cases}$$

generates a semigroup $S(t)$ such that $\int_0^\infty \|S(t)\|\,dt < \infty$. Hence, by (2.14) we have

$$\eta_\theta^{(i)}(t) \leq \int_0^t S(t-s)L\eta_\theta^{(i-1)}(s)ds$$

and so

$$\begin{aligned}\sup_{0\le s\le t}\eta_\theta^{(i)}(s) &\le L\sup_{0\le s\le t}\eta_\theta^{(i-1)}(s)\int_0^\infty \|S(t)\|\,dt\\ &= LK\sup_{0\le s\le t}\eta_\theta^{(i-1)}(s)\end{aligned}$$

where $K=\int_0^\infty \|S(t)\|\,dt$. Hence, collecting our results, we have

Theorem 2.7. *Under the assumptions of lemma 2.6, if $LK<1$, then the system (2.10) has a unique solution which is stable (independently of the delay).*$\square$

3. Multiple Delays

In this section we shall consider systems with multiple delays, which may be noncommensurate. Only an outline of the linear theory will be given since the method generalizes in a straightforward way to nonlinear systems as in the single delay case. The basic linear systems is of the form

$$\dot{x}(t)=Ax(t)+\sum_{k=1}^{m}B_kx(t-h_k)\ ,\ x(t)=\phi(t),\ -h_m\le t\le 0. \tag{3.1}$$

Here we have ordered the delays so that

$$0\le h_1\le h_2\le\cdots\le h_m.$$

Hence, as in section (2) we have

$$x(\tau+nh_1)=e^{A\tau}x(nh_1)+\int_{nh_1}^{\tau+nh_1}e^{A(\tau+nh_1-s)}\sum_{k=1}^{m}B_kx(s-h_k)ds$$

and so, if we put $\bar{s}=s-nh_1$,

$$x(\tau+nh_1)=e^{A\tau}x(nh_1)+\int_0^{\tau}e^{A(\tau-\bar{s})}\sum_{k=1}^{m}B_kx(\bar{s}+nh_1-h_k)d\bar{s}\ .$$

Now write

$$h_i = p_i h_1 + \varepsilon_i \ , \ 1 \leq i \leq m$$

where p_i is a positive integer and $\varepsilon_i < h_1$. Then we have

$$\begin{aligned} x(\tau + nh_1) &= e^{A\tau} x(nh_1) + \int_0^\tau e^{A(\tau - \bar{s})} \sum_{k=1}^m B_k x(\bar{s} + (n - p_k)h_1 - \varepsilon_k) d\bar{s} \\ &= e^{A\tau} x(nh_1) + \int_0^{\varepsilon_k} e^{A(\tau - \bar{s})} \sum_{k=1}^m B_k x(\bar{s} + (n - p_k)h_1 - \varepsilon_k) d\bar{s} \\ &\quad + \int_{\varepsilon_k}^\tau e^{A(\tau - \bar{s})} \sum_{k=1}^m B_k x(\bar{s} + (n - p_k)h_1 - \varepsilon_k) d\bar{s} \ . \end{aligned}$$

Define, as before,

$$x = [n](\tau) = \begin{cases} x(\tau + nh_1) \ , \ n \geq 0, \ 0 \leq \tau \leq h_1 \\ \phi(\tau + (n+1)h_1 \ , \ n < 0, \ 0 \leq \tau \leq h_1 \end{cases}$$

Then we have

$$x[n] = e^{A\tau} x[n-1](h) + \sum_{k=1}^m K_k x[n - p_k - 1] + \sum_{k=1}^m L_k x[n - p_k]$$

where

$$(K_k \xi)(\tau) = \int_0^{\varepsilon_k} e^{A(\tau - \bar{s})} B_k \xi(\bar{s} + h_1 - \varepsilon_k) d\bar{s}$$

and

$$(L_k \xi)(\tau) = \int_{\varepsilon_k}^\tau e^{A(\tau - \bar{s})} B_k \xi(\bar{s} - \varepsilon_k) d\bar{s} \ .$$

Put

$$\begin{aligned}
x_{p_m+1}[n] &= x[n] \\
x_{p_m}[n] &= x[n-1] \\
&\vdots \\
x_1[n] &= x[n-p_m]
\end{aligned}$$

Then,

$$\begin{aligned}
x_1[n] &= x_2[n-1] \\
x_2[n] &= x_3[n-1] \\
&\cdots \\
x_{p_m+1}[n] &= e^{A(\cdot)}x_{p_m+1}[n-1](h_1) + \sum_{k=1}^{m} K_k x_{p_m-p_k+1}[n-1] + \sum_{k=1}^{m} L_k x_{p_m-p_k+2}[n-1]
\end{aligned}$$

and so

$$y[n] = \Gamma y[n-1]$$

where

$$y[n] = (x_1[n], \cdots, x_{p_m+1}[n])^T$$

and Γ is the matrix of operators

$$\Gamma = \begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ \Xi_1 & \Xi_2 & \cdots & \cdots & \Xi_{p_m+1} + E \end{pmatrix}$$

where

$$\Xi_j = \sum_{p_m - p_k + 1 = j} K_k + \sum_{p_m - p_k + 2 = j} L_k \ ,$$

and

$$(Ev)(\tau) = e^{A\tau} v(h_1).$$

We therefore obtain, as in the single delay case,

$$y[n] = \Gamma y[n-1]$$

where $y[n] \in \oplus_{p_m+1} C^1[0, h_1; \mathbb{R}^n]$ and $\Gamma : \oplus_{p_m+1} C^1[0, h_1; \mathbb{R}^n] \to \oplus_{p_m+1} C^1[0, h_1; \mathbb{R}^n]$. As in the single delay case, therefore, a necessary and sufficient condition for stability of (3.1) (independently of the delay) is

$$\sigma(\Gamma) \subseteq B_1(0),$$

for any $h_1 > 0$.

Lemma 3.1 The spectrum of Γ is given by the numbers λ for which the equation

$$\Xi_1 v_1 + \lambda \Xi_2 v_1 + \lambda^2 \Xi_3 v_1 + \cdots + \lambda^{\ell-1}(\Xi_\ell + E)v_1 = \lambda^\ell v_1$$

has a nonzero solution v_1, where $\ell = p_m + 1$.

Proof The spectrum of Γ is given by the equation

$$\begin{pmatrix} 0 & I & & & \\ & 0 & I & & \\ & & \ddots & \ddots & \\ & & & 0 & I \\ \Xi_1 & \Xi_2 & \cdots & \cdots & \Xi_\ell + E \end{pmatrix} \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_\ell \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_\ell \end{pmatrix}.$$

The result follows since this implies

$$\begin{aligned} v_2 &= \lambda v_1 \\ v_3 &= \lambda v_2 \\ & \cdots \\ v_\ell &= \lambda v_{\ell-1} \\ \Xi_1 v_1 + \cdots + (\Xi_\ell + E)v_\ell &= \lambda v_\ell \ . \quad \square \end{aligned}$$

Example In the case of two commensurate delays we have

$$\dot{x}(t) = Ax(t) + B_1 x(t-h) + B_2 x(t-2h)\ , \ x(t) = \phi(t)\ , \ -2h \le t \le 0 \qquad (3.2)$$

Then we have the system

$$x[n] = e^{A\tau} x[n-1](h) + \int_0^\tau e^{A(\tau-s)} B_1 x[n-1](s) ds + \int_0^\tau e^{A(\tau-s)} B_2 x[n-2](s) ds.$$

Hence, by lemma 3.1, the spectrum is given by

$$e^{A\tau}p(h)+\int_0^\tau e^{A(\tau-s)}B_1p(s)ds+\lambda\int_0^\tau e^{A(\tau-s)}B_2p(s)ds=\lambda^2p(\tau)$$

and differentiating gives

$$\dot{p}=Ap+\frac{1}{\lambda^2}B_1p+\frac{1}{\lambda}B_2p\ ,\ p(0)=\frac{1}{\lambda^2}p(h).$$

Hence, theorem 2.1 generalizes to

Theorem 3.2 The equation (3.1) is asymptotically stable, independently of the delay, iff

$$\lambda\notin\sigma\left(\exp\left(A+\frac{1}{\lambda^2}B_1+\frac{1}{\lambda}B_2\right)h\right)$$

for all $h>0$ and all $\lambda\in\mathbb{C}\backslash\{0\}$.$\square$

4. Distributed Systems

In this section we shall generalize the results of section 2 to systems of the form

$$\dot{\psi}(t)=A(\psi(t))\psi(t)+B(\psi(t))\psi(t-h) \tag{4.1}$$

where $\psi(t)\in L^2(\Omega)$, $A(.)$ is an unbounded operator which satisfies:

(i) $A(\psi)$ is sectorial for each ψ and

$$\left\|(\lambda-A(\psi))^{-1}\right\|\le M/|\lambda-a|\ \text{for some}\ a$$

(ii) $A(\psi)$ generates a semigroup $T_{(\psi)}$ with $\left\|T_{(\psi)}(t)\right\|\le Ke^{-\delta t}$, $\left\|A^\alpha(\psi)T_{(\psi)}(t)\right\|\le\frac{Ke^{-\delta t}}{t^\alpha}$

and $B(.)$ is bounded. Many of the results are similar to the finite-dimensional case and so we simply state the conclusions and note the differences in the proofs.

From the conditions on $A(\psi)$ it can be proved that (see [[4]]):

Lemma 4.1 *If*

$$\left\|T_{(0)}(t-s)(A(\psi_1)-A(\psi_2))\right\| \leq \frac{Ke^{-\delta(t-s)}}{(t-s)^\alpha} L \left\|\psi_1 - \psi_2\right\|^\beta$$

for some $L, \beta > 0$, *and if*

$$\frac{\delta}{2} > \sup_{0 \leq t < 1} \ln(KE_\alpha(\lambda t))$$

where

$$\begin{aligned} \lambda &= [b\Gamma(1-\alpha)]^{1/(1-\alpha)}, b = KLM^\beta, M = \|\psi(0)\| \\ E_\alpha(x) &= \sum_{n=0}^{\infty} x^{n(1-\alpha)}/\Gamma(n(1-\alpha)+1), \end{aligned}$$

then the equation

$$\dot{\psi}(t) = A(\psi(t))\psi(t)$$

has a unique, exponentially stable solution, with

$$\|\psi(t)\| \leq Me^{-(\delta/2)t}. \square$$

Starting with the linear system

$$\dot{\psi}(t) = A\psi(t) + B\psi(t-h), \tag{4.2}$$

we have, from theorem 2.1:

Theorem 4.2 *If A satisfies the spectrum determined growth assumption, then the system (3.2) is asymptotically stable iff*

$$\lambda \notin \sigma\left(\left[T_{(A+B/\lambda)}(t)\right] h\right)$$

for all $\lambda \in \mathbb{C}\backslash\{0\}$, *where* $T_{(A+B/\lambda)}$ *is the semigroup generated by* $A + B/\lambda$.□

In the nonlinear case (equation (4.1)) we use lemma 4.1 instead of lemma 2.6, in order to bound the evolution operator generated by $A(\psi)$. The proof of theorem 2.7 then goes through virtually unchanged, but with $\delta/2$ replacing μ, etc.

Example 4.3 The system

$$\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} + (1 + 0.5\psi^2)\frac{\partial\psi}{\partial x} + \int_0^1 \psi(x, t-h)dx$$

is exponentially stable (independently of the delay), for small initial functions $\psi(0, x)$.□

5. Conclusions

In this paper we have extended some well-known results on the stability of linear delay systems, independent of the delay, to nonlinear finite- and infinite-dimensional systems. This has been achieved by writing the equations in the form of difference equations in a Hilbert space.

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