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Lie Series and the Realization Problem

by

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Abstract

The realization of analytic input-output maps by linear-analytic systems is considered by identifying the Lie derivatives of the system in the output function.

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1 Introduction

The theory of realization of nonlinear systems has been an important problem for many years. In [1] it is shown that a system has an analytic realization if a certain matrix has finite rank. The condition is extremely difficult to check in practice, however. Realizations of homogeneous Volterra kernels by bilinear systems has been solved in [2] where separable kernels are shown to occur in the expansion of such systems. Many authors have used Lie series ([3],[4],[5]), although some problems with the approach in [5] have recently been noted ([6]). The realization of polynomial input-output maps is considered in detail in [7]. Here we shall be concerned with the realization of analytic input-output maps by linear-analytic systems. After defining the equivalence of realizations in section 2, we shall consider the zero input case in section 3 and use the Lie series to identify an analytic dynamical system with the given output map. In section 4 we shall show that by considering constant inputs, one can extend the results of section 3 to obtain linear-analytic realizations of analytic input-output maps.

2 Equivalence of Realizations

Suppose that

$$\left.\begin{aligned} &\dot{x} = f(x,u)\ ,\ x(0) = x_0 \\ &y = h(x,u) \end{aligned}\right\} \qquad (2.1)$$

and

$$\left.\begin{aligned} &\dot{z} = \overline{f}(z,u)\ ,\ z(0) = z_0 \\ &y = \overline{h}(z,u) \end{aligned}\right\} \qquad (2.2)$$

are two realizations of the same input-output map $y = S(u)$, defined on open sets $\Omega_x\ ,\ \Omega_z \subseteq \mathbf{R}^n$, respectively. (Here, we assume that $x_0 \in \Omega_x\ ,\ z_0 \in \Omega_z$.) We shall say that (2.1) and

(2.2) are **equivalent realizations** of S if there exists a diffeomorphism $k : \Omega_x \longrightarrow \Omega_z$ such that

$$\left.\begin{array}{l} \frac{\partial k}{\partial x} f(x,u) = \overline{f}(k(x),u) \\ \overline{h}(k(x),u) = h(x,u) \end{array}\right\} \quad x \in \Omega_x \tag{2.3}$$

for each $u \in U$.

These conditions are equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} \Omega_x \times U & \xrightarrow{\tilde{k}} & \Omega_z \times U \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ T\Omega_x & \xrightarrow{Tk} & T\Omega_z \end{array}$$

and

$$\begin{array}{ccc} \Omega_x \times U & \xrightarrow{\tilde{k}} & \Omega_z \times U \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{h}} \\ Y & \xrightarrow{I} & Y \end{array}$$

where $T\Omega_x$ is the tangent bundle of Ω_x , Tk is the differential of k , $\tilde{k}(x,u) = (k(x),u)$, Y is the output space and I is the identity map.

In this paper we shall also consider unforced systems of the form

$$\left.\begin{array}{l} \dot{x} = f(x) \ , \ x(0) = x_0 \\ y = h(x) \end{array}\right\} \tag{2.4}$$

This time, equivalence is defined by the commutativity of the diagrams

$$\begin{array}{ccc} \Omega_x & \xrightarrow{k} & \Omega_z \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ T\Omega_x & \xrightarrow{Tk} & T\Omega_z \end{array}$$

and

$$\begin{array}{ccc} \Omega_x & \xrightarrow{k} & \Omega_z \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \overline{h}} \\ Y & \xrightarrow{I} & Y \end{array}$$

where $k : \Omega_x \longrightarrow \Omega_z$ and

$$\frac{\partial k}{\partial x} f(x) = \overline{f}(k(x))$$
$$\overline{h}(k(x)) = h(x).$$

Recall ([4]) that the observability matrix of the system at x_0 is

$$\mathcal{O}(x_0) = [dh(x_0) \;\; L_f dh(x_0) \;\; \cdots \;\; L_f^{n-1} dh(x_0)] \; .$$

If $\mathcal{O}(x_0)$ is invertible then, in particular, $(dh(x_0))_i \neq 0$ for $1 \leq i \leq n$ and so we can choose k on some neighbourhood of x_0 such that

$$\overline{h}(\overline{x}) = \sum_{i=1}^{n} \overline{x}_i$$

where $\overline{x} = k(x)$.

3 The Lie Series and Dynamical System Realization

In this section we shall first consider an analytic differential equation

$$\dot{x} = f(x) \; , \quad x(0) = x_0 \tag{3.1}$$

together with a measurement equation

$$y = h(x) \; . \tag{3.2}$$

We can solve for y by using the Lie series (see [3]) in the following way. Define the new variables ϕ_i , $i \geq 1$ by

$$\begin{aligned} \phi_1 &= y = h(x) \\ \phi_2 &= \frac{\partial \phi_1}{\partial x} f = L_f \phi_1 \end{aligned}$$

$$\phi_3 = \frac{\partial \phi_2}{\partial x} f = (L_f)^2 \phi_1$$

$$\vdots$$

$$\phi_i = (L_f)^{i-1} \phi_1$$

Then,

$$\dot{\phi}_i = \frac{\partial \phi_i}{\partial x} f = L_f \phi_i = \phi_{i+1}$$

and so

$$\dot{\Phi} = A\Phi \quad , \quad \Phi(0) = \Phi_0$$

where

$$\Phi = (\phi_1, \phi_2, \cdots)^T$$

$$A = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

$$\Phi_0 = (h(x_0), (L_f h)(x_0), ((L_f)^2 h)(x_0), \cdots)^T.$$

Hence,

$$\Phi = e^{At}\Phi_0$$

and

$$y = \sum_{i=0}^{\infty} \frac{t^i}{i!} \left[(L_f)^i h\right](x_0). \tag{3.3}$$

To carry out the converse process of realization, first note that the Lie derivatives of h at x_0 can be found from

$$\left[(L_f)^i h\right](x_0) = \frac{d^i y}{dt^i}(0) \quad , \quad i \geq 0. \tag{3.4}$$

Now consider the determination of $f(x)$ from (3.4). For $i = 0$ we have

$$h(x_0) = y(0). \tag{3.5}$$

From section 2, if (f, h) is observable at x_0 we can assume that $h(x_0) = x_{10} + x_{20} + \cdots + x_{n0}$. Thus (3.5) determines x_0 to within $n-1$ real parameters. We can choose any x_0 which satisfies (3.5). For $i = 1$ we have

$$(L_f h)(x_0) = \frac{dy}{dt}(0)$$

i.e.

$$f(x_0)\frac{\partial h}{\partial x}(x_0) = f_1(x_0) + f_2(x_0) + \cdots + f_n(x_0) = \frac{dy}{dt}(0),$$

since $h(x) = x_1 + x_2 + \cdots + x_n$. Hence, again, $f(x_0)$ is determined up to $n-1$ real parameters. In fact, $f(x_0)$ lies on the hyperplane

$$H_1 : \xi_1 + \cdots + \xi_n = \frac{dy}{dt}(0)$$

in $\mathbf{R}^n$. Next, if $i = 2$ we have

$$\begin{aligned}\frac{d^2y}{dt^2}(0) = ((L_f)^2 h)(x_0) &= \left(f\frac{\partial}{\partial x}\left(f\frac{\partial}{\partial x}\right)h\right)(x_0) \\ &= f(x_0)\frac{\partial f}{\partial x}(x_0)\frac{\partial h}{\partial x}(x_0) + f^2(x_0)\frac{\partial^2 h}{\partial x^2}(x_0)\end{aligned}$$

with an obvious interpretation for $f^2(x_0)$). Since $h = x_1 + \cdots + x_n$ it follows that $\frac{\partial f}{\partial x}(x_0)$ lies on the hyperplane

$$H_2 : f_1(x_0)(\xi_{11} + \cdots + \xi_{1n}) + f_2(x_0)(\xi_{21} + \cdots + \xi_{2n}) + \cdots + f_n(x_0)(\xi_{n1} + \cdots + \xi_{nn}) = \frac{d^2y}{dt^2}(0)$$

in $\mathbf{R}^{n^2}$. (H_2 depends, of course, on the choice of $f(x_0)$ on H_1.) For $i = 3$,

$$\frac{d^3y}{dt^3}(0) = ((L_f)^3 h)(x_0) = f^2\frac{\partial^2 f}{\partial x^2}\frac{\partial h}{\partial x} + f\left(\frac{\partial f}{\partial x}\right)^2\frac{\partial h}{\partial x}$$

where

$$f^2 \frac{\partial^2 f}{\partial x^2} \frac{\partial h}{\partial x} = \sum_i \sum_j \sum_k f_i f_j \frac{\partial^2 f_k}{\partial x_i \partial x_j} \frac{\partial h}{\partial x_k} \quad ,$$

and a similar interpretation for the second term. Thus, $\frac{\partial^2 f}{\partial x^2}$ lies on the hyperplane

$$H_3 : \sum_i \sum_j \sum_k f_i(x_0) f_j(x_0) \xi_{ijk} = \frac{d^3 y}{dt^3}(0) - \left(f \left(\frac{\partial f}{\partial x} \right)^2 \frac{\partial h}{\partial x} \right)(x_0)$$

in $\mathbf{R}^{n^3}$. In general, it is clear that $\frac{\partial^\ell f}{\partial x^\ell}$ lies on the hyperplane

$$H_{\ell+1} : \sum_{i_1} \cdots \sum_{i_{\ell+1}} f_{i_1}(x_0) f_{i_2}(x_0) \cdots f_{i_\ell}(x_0) \xi_{i_1 i_2 \cdots i_{\ell+1}} = \frac{d^{\ell+1} y}{dt^{\ell+1}}(0) - \Delta_\ell \tag{3.6}$$

where Δ_ℓ depends on the choices for $f(x_0), \frac{\partial f}{\partial x}(x_0), \cdots, \frac{\partial^{\ell-1} f}{\partial x^{\ell-1}}(x_0)$ on the hyperplanes $H_1, \cdots, H_\ell$. Choosing a point $\xi_{i_1, \cdots, i_\ell}(x_0)$ on each hyperplane H_k, $1 \leq k < \infty$, we can define formally the series

$$\overline{f}_k(x) = \sum_{\ell=0}^{\infty} \sum_{i_1=1}^{n} \cdots \sum_{i_\ell=1}^{n} \xi_{i_1 \cdots i_\ell k}(x_{i_1} - x_{i_1}(0))^{i_1}(x_{i_2} - x_{i_2}(0))^{i_2} \cdots (x_{i_\ell} - x_{i_\ell}(0))^{i_\ell} \tag{3.7}$$

for any choice of $x(0) = x_0$, which satisfies

$$h(x_0) = x_{10} + \cdots + x_{n0} = y(0).$$

We have therefore proved:

Theorem 3.1 The analytic function $y(t)$ has a realization, of dimension n, in the form of the dynamical system (3.1,3.2) if and only if the series $\overline{f}_k(x)$, $1 \leq k \leq n$, defined by (3.7) converges in some neighbourhood of x_0, for some choice of $\xi_{i_1 \cdots i_\ell}(x_0)$ in H_ℓ, $1 \leq \ell < \infty$. □

Note that, for any n, if we truncate (3.6) to a finite number of terms, then

$$\overline{f}^p_k(x) = \sum_{\ell=0}^{p} \sum_{i_1=1}^{n} \cdots \sum_{i_\ell=1}^{n} \xi_{i_1 \cdots i_\ell k}(x_{i_1} - x_{i_1}(0))^{i_1}(x_{i_2} - x_{i_2}(0))^{i_2} \cdots (x_{i_\ell} - x_{i_\ell}(0))^{i_\ell} \tag{3.8}$$

is a well-defined function and the system

$$\begin{aligned} \dot{x} &= \overline{f}^p(x) \quad , \quad x(0) = x_0 \\ y &= x_1 + \cdots + x_n \end{aligned}$$

has the same output function $y(t)$ as (3.1,3.2) up to order p in t. In particular, we can find a system of dimension 1 which has the same output function up to any desired finite order.

Example 3.1 Consider the simple, one-dimensional system

$$\begin{aligned} \dot{x} &= x^2 \ , \ x(0) = x_0 = 1 \\ y &= e^x \ . \end{aligned} \tag{3.9}$$

By (3.3) the Lie series is given by

$$y(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!} [(L_f)^i h](x_0)$$

where

$$(L_f)^i h(x_0) = \left(x^2 \frac{d}{dx} \right)^i e^x \bigg|_{x=1} \ .$$

The first few terms are given by

$$y(t) = e + et + \frac{3}{2} et^2 + 2et^4 + \cdots \ . \tag{3.10}$$

Now, given (3.10) we want to realize a system of the form (3.9) with (3.10) as its output function. Suppose we assume a state variable z with $h(z) = z$. Then we have

$$\begin{aligned} z_0 &= y(0) = e \\ f(z_0) &= \frac{dy}{dt}(0) \bigg/ \frac{dh}{dz}(z_0) = \frac{dh}{dt}(0) = e \\ \frac{df}{dz}(z_0) &= \frac{dy^2}{dt^2}(0) \bigg/ f(z_0) = 3 \\ \frac{df^2}{dz^2}(z_0) &= \left(\frac{d^3 y}{dt^3}(0) - f(z_0) \left(\frac{df}{dz}(z_0) \right)^2 \right) \bigg/ f^2(z_0) = 4/e. \end{aligned}$$

Continuing in this way we obtain

$$\begin{aligned} f(z) &= \sum_{i=0}^{\infty} f^{(i)}(z_0) \frac{(z - z_0)^i}{i!} \\ &= z(\log z)^2. \end{aligned}$$

Hence we realize (3.9) with the system

$$\begin{aligned} \dot{z} &= z(\log z)^2 \\ y &= z. \end{aligned} \tag{3.11}$$

The map

$$z = h(x) = e^x$$

is a local diffeomorphism near $x = 1$ and so (3.11) is equivalent to (3.9) in the sense of section 2.

Example 3.2 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2^3 \\ \dot{x}_2 &= x_2^3 \\ y &= h(x) = x_1 + x_2 \ , \quad x_{10} = 1 \ , \ x_{20} = 1. \end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned} \mathcal{O}(x_0) &= [dh(x_0) \ \ L_f dh(x_0)] \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \end{aligned}$$

so the system is observable at x_0.

The Lie series for $y(t)$ is given by

$$y(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!}[(L_f)^i h](x_0)$$

where

$$f(x) = (x_1 + x_2^2, x_2^3).$$

Computing $(L_f)^i h$ gives

$$\begin{aligned}
(L_f)^0 h(x_0) &= h(x_0) = 2 \\
(L_f)h(x_0) &= x_1(0) + x_2^2(0) + x_2^3(0) = 3 \\
(L_f)^2 h(x_0) &= x_1(0) + x_2^2(0) + 2x_2^4(0) + 3x_2^5(0) = 7 \\
(L_f)^3 h(x_0) &= x_1(0) + x_2^2(0) + 2x_2^4(0) + 8x_2^6(0) + 15x_2^7(0) = 27 \\
(L_f)^4 h(x_0) &= x_1(0) + x_2^2(0) + 2x_2^4(0) + 8x_2^6(0) + 48x_2^8(0) + 105x_2^9(0) = 165 \\
&\vdots
\end{aligned}$$

Hence

$$y(t) = 2 + 3t + \frac{7t^2}{2} + \frac{27t^3}{3!} + \frac{165t^4}{4!} + \cdots$$

Defining $\xi_{i_1 \cdots i_k} = \frac{\partial_k^{k-1}}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}}(x_0)$ as before, we can write the equations (3.6), for $n = 2$, in the form

$$\begin{aligned}
\xi_1 + \xi_2 &= 3 \quad \left(= \frac{dy}{dt}(0)\right) \\
\xi_1(\xi_{11} + \xi_{12}) + \xi_2(\xi_{21} + \xi_{22}) &= 7 \quad \left(= \frac{d^2y}{dt^2}(0)\right) \\
\xi_1^2(\xi_{111} + \xi_{112}) + 2\xi_1\xi_2(\xi_{121} + \xi_{122}) + \xi_2^2(\xi_{221} + \xi_{222}) &= 27 - \\
\xi_1(\xi_{11}^2 + \xi_{11}\xi_{12} + \xi_{12}\xi_{21} + \xi_{12}\xi_{22}) & \\
+\xi_2(\xi_{22}^2 + \xi_{22}\xi_{21} + \xi_{21}\xi_{11} + \xi_{21}\xi_{12}) & \\
\cdots &
\end{aligned}$$

Clearly the choices

$$x_{01} = 1 \ , \ x_{02} = 1$$

$$\xi_1 = 2 \ , \ \xi_2 = 1$$

$$\xi_{11} = 1 \ , \ \xi_{21} = 2 \ , \ \xi_{12} = 0 \ , \ \xi_{22} = 3$$

$$\xi_{111} = \xi_{112} = \xi_{121} = \xi_{122} = 0 \ , \ \xi_{221} = 2 \ , \ \xi_{222} = 6$$

will give the correct terms up to order 2, i.e.

$$\begin{aligned} \dot{x}_1 &= 2 + 2(x_2 - 1) + (x_2 - 1)^2 + (x_1 - 1) \\ \dot{x}_2 &= 1 + 3(x_2 - 1) + 3(x_2 - 1)^2 \ \{+(x_2 - 1)^3\} \end{aligned}$$

The term in braces will be given by the next equation in $\xi_{ij\ell k}$, and all other terms can be taken to be zero. Of course, we are unlikely to be able to guess $x_0, \xi_1, \xi_2, \cdots$ correctly; however, any other choice gives us a dynamical system with the same outputs. Thus, for example, choose

$$z_{01} = 2 \ , \ z_{02} = 0$$

$$\xi_1 = 1 \ , \ \xi_2 = 2$$

$$\xi_{11} = 0 \ , \ \xi_{12} = 1 \ , \ \xi_{21} = 1 \ , \ \xi_{22} = 2$$

$$\xi_{111} = 1 \ , \ \xi_{112} = 0 \ , \ \xi_{121} = \xi_{211} = -2 \ , \ \xi_{122} = \xi_{212} = -1 \ , \ \xi_{221} = 1 \ . \ \xi_{222} = 1$$

$\cdots$

We then obtain the system

$$\begin{aligned} \dot{z}_1 &= 1 + z_2 + \frac{1}{2}(z_1 - 2)^2 - 2(z_1 - 2)z_2 + \frac{1}{2}z_2^2 + \cdots \\ \dot{z}_2 &= 2 + (z_1 - 2) + 2z_2 - (z_1 - 2)z_2 + \frac{1}{2}z_2^2 + \cdots \\ y &= z_1 + z_2 \end{aligned} \tag{3.13}$$

which, as can easily be checked, has the same Lie series as (3.12). If the terms in the expansion (3.13) can be chosen so that the formal series has nonzero radius of convergence, then the system must be equivalent to the system (3.12) in the sense of section 2. by the analyticity of the systems.

Note that one can also find a one-dimensional system with the same solution as (3.12) for $x_{10} = 1$, $x_{20} = 1$; namely,

$$\begin{aligned} \dot{x} &= 3 + \frac{7}{3}(x-2) + \frac{2}{3}(x-2)^2 + \cdots \quad , \quad x_0 = 2 \\ y &= x. \end{aligned}$$

Of course, this would not be correct for other initial conditions x_0– this gives a practical test for the minimal dimension of the realization.

4 Realizations of Input-Output Maps

Now consider the realization of an input-output map $S : \mathcal{U} \longrightarrow \mathcal{Y}$ (for some input and output spaces $\mathcal{U}, \mathcal{Y}$ respectively), in terms of a linear-analytic system

$$\begin{aligned} \dot{x} &= f(x) + ug(x) \quad , \quad x(0) = x_0 \\ y &= h(x). \end{aligned} \tag{4.1}$$

(Note that we can consider the more general system

$$\dot{z} = \overline{f}(z, v) \quad ,$$

by making v into a 'state':

$$\begin{aligned} \dot{z} &= \overline{f}(z, v) \\ \dot{v} &= w \end{aligned}$$

so that we can write it in the form (4.1) with $x = (z, v)^T$, $u = w$ and

$$f(x) = (\overline{f}(z, v), 0)^T \ , \ g(x) = (0, 1)^T.)$$

To find the input-output map of (4.1) we can generalize the Lie series approach of section 3 and put

$$\begin{aligned} \phi_1 &= h(x) = y \\ \phi_i &= \begin{cases} \frac{\partial \phi_{i/2}}{\partial x} \cdot f = L_f \phi_{i/2} & i \text{ even} \\ \frac{\partial \phi_{(i-1)/2}}{\partial x} \cdot g = L_g \phi_{(i-1)/2} & i \text{ odd} \end{cases} \end{aligned} \tag{4.2}$$

Then we have

$$\begin{aligned} \dot{\phi}_i &= \frac{\partial \phi_i}{\partial x} \cdot \dot{x} \\ &= \frac{\partial \phi_i}{\partial x}(f + ug) \\ &= L_f \phi_i + u L_g \phi_i \\ &= \phi_{2i} + u \phi_{2i+1}. \end{aligned} \tag{4.3}$$

Put

$$\Phi = (\phi_1, \phi_2, \cdots)^T.$$

Then (4.3) gives

$$\dot{\Phi} = A\Phi + uB\Phi \quad , \quad \Phi(0) = \Phi_0 \tag{4.4}$$

where $A = (a_{ij})$ and $B = (b_{ij})$ are infinite matrices defined by

$$a_{ij} = \delta_{2i,j} \quad , \quad b_{ij} = \delta_{2i+1,j}.$$

The bilinear system (4.4) can be solved by Picard iteration to give the Volterra series

$$\Phi(t) = \sum_{k=0}^{\infty} \xi_k(t) \tag{4.5}$$

where

$$\xi_0(t) = e^{At}\Phi_0$$

and

$$\xi_k(t) = \int_0^t e^{A(t-s)} u(s) B \xi_{k-1}(s) ds.$$

Hence, iterating, we have

$$\begin{aligned} \xi_k(t) &= \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{k-1}} e^{A(t-\tau_1)} B e^{A(\tau_1-\tau_2)} B \cdots B e^{A(\tau_{k-1}-\tau_k)} B e^{A\tau_k} \Phi_0 \times \\ & \quad u(\tau_1 u(\tau_2) \cdots u(\tau_k) d\tau_1 \cdots d\tau_k. \end{aligned} \tag{4.6}$$

Suppose that the system maps $L^2[0,t]$ into $L^2[0,t]$ for all $t \geq 0$ and define the operator $K_k^t : L^2[0,t] \longrightarrow L^2[0,t]$ by

$$K_k^t(u) = \int_0^t \int_0^t \cdots \int_0^t p(t, \tau_1, \cdots, \tau_k) u(\tau_1) \cdots u(\tau_k) d\tau_1 \cdots d\tau_k \tag{4.7}$$

where

$$p(t, \tau_1, \cdots, \tau_k) = \begin{cases} \left(e^{A(t-\tau_1)} B e^{A(\tau_1-\tau_2)} B \cdots B e^{A(\tau_{k-1}-\tau_k)} B e^{A\tau_k} \Phi_0\right)_1 & \text{if } t \geq \tau_1 \geq \cdots \tau_k \\ 0 & \text{otherwise} \end{cases}$$

where $(L)_1$ denotes the first component of L.

Then from (4.1),(4.2),(4.5) and (4.6) we have

$$y(t) = S(u)(t) = \sum_{k=0}^{\infty} K_k^t(u)$$

where K_k^t is a k^{th} order multilinear operator.

Theorem 4.1 A necessary and sufficient condition for the input- output map S to have a linear analytic realization of the form (4.1) is that the k^{th} order Frechet derivative $\mathcal{F}_u^k(S(u))$ of S with respect to u is a k-linear operator of the form (4.7) where Φ_0 is given by (4.2) with $x = x_0$. □

However, applying theorem 4.1 will be extremely difficult in practice. A much simpler way to realize the input-output map from a linear analytic system of the form (4.1) is to assume u is constant and proceed as in section 3. Thus, if u is constant we have

$$y(t;u) = \sum_{i=0}^{\infty} \frac{t^i}{i!}[(L_{f+ug})^i h](x_0) \quad , \quad u \in \mathbf{R}.$$

In solving equations (3.13) we must be able to choose the $\xi_{i_1 \cdots i_k}$'s so that they are of the form

$$\xi_{i_1 \cdots i_k} = \eta_{i_1 \cdots i_k} + u\mu_{i_1 \cdots i_k}$$

since then the resulting power series will split into two terms with one multiplied by u.

Example 4.1 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + ux_2^2 \\ \dot{x}_2 &= x_2^3. \\ y &= = h(x) = x_1 + x_2 \quad , \quad x_{10} = 1 \ , \ x_{20} = 1. \end{aligned} \tag{4.8}$$

(This has been chosen to be similar to (3.12) for the purposes of comparison.) The output is given by

$$y(t) = \sum_{i=0}^{\infty} \frac{t^i}{i!}[(L_{f+ug})^i h](x_0)$$

where

$$f(x) = (x_1, x_2^3) \quad , \quad g(x) = (x_2^2, 0).$$

Thus,

$$\begin{aligned} (L_F)^0 h(x_0) &= h(x_0) = 2 \\ (L_F)h(x_0) &= x_1(0) + ux_2^2(0) + x_2^3(0) = 2 + u \\ (L_F)^2 h(x_0) &= x_1(0) + 3x_2^5(0) + u(x_2^2(0) + 2x_2^4(0)) = 4 + 3u \\ (L_F)^0 h(x_0) &= x_1(0) + 15x_2^7(0) + u(x_2^2(0) + 2x_2^4(0) + 8x_2^6(0)) = 16 + 11u \end{aligned}$$

$$(L_F)^0 h(x_0) \;=\; x_1(0) + 105x_2^9(0) + u(x_2^2(0) + 2x_2^4(0) + 8x_2^6(0) + 48x_2^8(0)) = 106 + 59u$$

$$\ldots \qquad \ldots$$

where $F = f + ug$. We must solve equations (3.13) where

$$\xi_{i_1 \cdots i_k} = \frac{\partial^{k-1} f_k}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}}(x_0) + u \frac{\partial^{k-1} g_k}{\partial x_{i_1} \cdots \partial x_{i_{k-1}}}(x_0).$$

Thus, we can take, for example,

$$\begin{aligned} z_{01} &= 2 \ , \ z_{02} = 0 \\ \xi_1 &= 1 + u \ , \ \xi_2 = 1 \\ \xi_{11} &= 2 \ , \ \xi_{12} = 2 \ , \ \xi_{21} = -u \ , \ \xi_{22} = 0 \\ \ldots & \quad \ldots \end{aligned}$$

which gives the system

$$\begin{aligned} \dot{z}_1 &= 1 + 2(z_1 - 2) + \cdots + u(1 - 2z_2 + \cdots) \ , \quad z_0 = (2, 0) \\ \dot{z}_2 &= 1 + 2z_2 + \cdots \\ h(z) &= z_1 + z_2 \end{aligned}$$

with the same output map as (4.8). Choosing the ξ's to make the right hand sides converge will guarantee that this equation is equivalent to equation (4.8) (in some neighbourhoods of z_0 and x_0). Note that since these are valid for any constant inputs, they must be true for any input by a simple approximation argument, which replaces such an input $u(t)$ by piecewise constant approximations.

5 Conclusions

In this paper we have described a practical method for realizing analytic input-output systems. This is achieved by choosing the system so that its associated Lie derivatives match those given by differentiating the output (with respect to t). If these Lie derivatives are chosen so that the resulting functions converge in some neighbourhood of the initial values, then the associated system is equivalent to any other with the same Lie derivatives. The resulting realizations are valid in some (perhaps very small) neighbourhood of the initial condition. In a forthcoming paper we shall demonstrate that by applying this procedure at each point on the given trajectory and then using sheaf theory one can obtain global realizations in a simple way.

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