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Pseudo-Linear Systems, Lie Algebras and Stability

by

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Abstract

The stability of pseudo-linear systems is considered by using a diagonal dominance approach coupled with the theory of semisimple Lie algebras and the Cartan decomposition.

Keywords : Pseudo-linear Systems, Stability, Lie Algebras .

1 Introduction

The stability of systems is, of course, fundamental to the whole of control theory and perhaps the most important result, at least for linear systems theory, is Lyapunov's basic theorem on the stability of linear equations. If we consider 'pseudo-linear' systems of the form

$$\dot{x} = A(x)x \tag{1.1}$$

then the spectrum of $A(x)$, for each x, being in the (strict) left half-plane is no longer necessary nor sufficient for stability. Control systems of a similar type have been considered in [3]. However,the global stability of the resulting feedback system is not valid, as asserted in the cited paper, without further assumptions. We therefore intend to give more precise results of the stability of pseudo-linear systems here in cases where $A(x)$ is analytic, just continuous or even discontinuous. The Lyapunov approach to such systems has already been presented ([1]) using the theory of Lie algebras. Here we shall use a diagonal dominance approach based on a comparison result in [2]. The general structure of equation (1.1) will again be reduced to a suitable form by an application of the Cartan decomposition of the Lie algebra generated by the matrices $\{A(x) : x \in \mathbf{R}^n\}$, assuming this is semisimple. (For the general theory of Lie algebras, see [4],[5].

In section 2 we show that it is natural to consider the symmetrized matrix $A + A^T$ for linear systems and in section 3 this idea is extended to pseudo-linear systems. In particular, we give conditions under which the spectrum of $A(x)$ being in the open left half-plane is sufficient for stability. It turns out, as one may expect, that the derivatives of $A(x)$, $\partial A/\partial x$ must be bounded at infinity in some way. In section4 we consider a diagonal dominance approach and show that, even in the analytic case, the eigenvalues of $A(x)$ need not be in $\mathbf{C}^-$ for asymptotic stability. The technique is extended in section 5 by the use of Lie algebras, and finally in section 6 we consider briefly the case where $A(x)$ may not even be continuous (as in variable structure systems). Here the state space is partitioned into subsets where $A(x)$ is stable or unstable. The product of the order of stability and the passage time for each region is then important.

2 Linear Systems

In this section we shall briefly review the stability of linear systems and reinterpret the Lyapunov condition. Thus, let

$$\dot{x} = Ax \tag{2.1}$$

be a linear system of equations and recall Lyapunov's theorem:

Lyapunov's Theorem The system (2.1) is stable if and only if, given any positive definite symmetric matrix Q, there exists a positive definite symmetric matrix P such that

$$PA + A^T P = -Q.$$

□

This is, of course, equivalent to the spectrum of A being in the open left half-plane. However, in the case of the nonlinear system

$$\dot{x} = A(x)x \tag{2.2}$$

it is known that the spectrum of $A(x)$ being in the open left half-plane for each x is not sufficient for stability. Hence, in order to study the stability of (2.2) we reinterpret Lyapunov's theorem in the following way. Change coordinates in (2.1) to

$$y = P^{1/2}x.$$

Then

$$\dot{y} = P^{1/2}AP^{-1/2}y$$

or

$$\dot{y} = \tilde{A}y \tag{2.3}$$

where

$$\tilde{A} = P^{1/2}AP^{-1/2}.$$

Then we have

Lemma 2.1 The system (2.3) is (asymptotically) stable if the spectrum of $\tilde{A} + \tilde{A}^T$ is in the open left half-plane.

Proof If $\sigma(\tilde{A} + \tilde{A}^T) \subseteq \mathbf{C}^-$ then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||y||^2 &= \frac{1}{2}(\dot{y}^T y + y^T \dot{y}) \\ &= \frac{1}{2}y^T(\tilde{A}^T + \tilde{A})y \\ &\leq \mu(\tilde{A}^T + \tilde{A})||y||^2 \end{aligned}$$

where

$$\mu(\tilde{A}^T + \tilde{A}) = \sup_{x \neq 0} \frac{x^T \tilde{A} x}{x^T x}$$

is the largest eigenvalue of $\frac{1}{2}(\tilde{A}^T + \tilde{A})$. □

Combining these results we have

Theorem 2.1 The system (2.3) is stable if and only if there is a change of coordinates such that the similar matrix $\tilde{A}$ in the new coordinates satisfies

$$\sigma(\tilde{A}^T + \tilde{A}) \subseteq \mathbf{C}^-. \qquad \square$$

It is clear, therefore, that it is the spectrum of $A + A^T$ for a linear system which is important, rather than that of A, and we shall see that this is also true for nonlinear systems.

3 Nonlinear Systems and the Lyapunov Equation

We shall now consider the nonlinear system

$$\dot{x} = A(x)x \quad . \tag{3.1}$$

The first result is an elementary extension of lemma (2.1).

Lemma 3.1 Suppose that $A(x)$ is continuous and let $\mu(x)$ denote the largest eigenvalue of $\frac{1}{2}(A(x) + A^T(x))$. If $\mu(x) < 0$ for all x then (3.1) is asymptotically stable.

Proof Let x_0 be arbitrary and consider the ball $B = \{x : ||x|| \leq ||x_0||\}$. Since B is compact and $\mu(x)$ is continuous, it follows that μ attains a maximum on B. Let

$$\mu_0 = \max_{x \in B} \mu(x).$$

Then $\mu_0 < 0$. Now,

$$\begin{aligned} \frac{d}{dt}||x||^2 &= \dot{x}^T x + x^T \dot{x} \\ &= x^T(A(x) + A^T(x))x \\ &\leq \mu(x)||x||^2 \end{aligned}$$

and so $||x||^2$ decreases. Hence, if $x(0) = x_0$, then

$$\frac{d}{dt}||x||^2 \leq \mu_0 ||x||^2$$

and

$$||x||^2 \leq e^{\mu_0 t}||x_0||^2$$

so that $||x|| \longrightarrow 0$, for any x_0. □

Next we consider the case where $A(x)$ has spectrum in $\mathbf{C}^-$ and see why this is not sufficient for global stability. Suppose we try the Lyapunov function $V = x^T P(x) x$ where $P(x)$ satisfies

$$P(x)A(x) + A^T(x)P(x) = -Q(x) \quad . \tag{3.2}$$

Then

$$\dot{V} = \dot{x}^T P x + x^T P \dot{x} + x^T \left(\sum_{i=1}^{n} \frac{\partial P}{\partial x_i} (A(x)x)_i \right) x$$

where $(y)_i$ denotes the i^{th} element of y. Thus,

$$\dot{V} = x^T (A^T P + PA) x + x^T \left(\frac{\partial P}{\partial x} A x \right) x$$

where

$$\frac{\partial P}{\partial x} A x = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} (A(x)x)_i \quad .$$

Hence,

$$\dot{V} = -x^T Q(x) x + x^T \left(\frac{\partial P}{\partial x} A x \right) x. \tag{3.3}$$

Now, from (3.2) we have

$$\frac{\partial P}{\partial x} A + A^T \frac{\partial P}{\partial x} + P \frac{\partial A}{\partial x} + \frac{\partial A^T}{\partial x} P = -\frac{\partial Q}{\partial x} \quad , \tag{3.4}$$

where again $\frac{\partial S(x)}{\partial x}$, for some matrix $S(x)$, denotes the vector of matrices $\left(\frac{\partial S}{\partial x_i} \right)_{1 \le i \le n}$. From (3.2) we have

$$P(x) = \int_0^\infty e^{A^T(x)t} Q(x) e^{A(x)t} dt \tag{3.5}$$

if each $A(x)$ is stable and similarly, from (3.4),

$$\frac{\partial P}{\partial x} = \int_0^\infty e^{A^T(x)t} \left(P(x) \frac{\partial A}{\partial x}(x) + \frac{\partial A^T}{\partial x}(x) P(x) + \frac{\partial Q}{\partial x} \right) e^{A(x)t} dt. \tag{3.6}$$

Let $M(x) > 0$ and $\epsilon(x) > 0$ be chosen so that

$$||e^{A(x)t}|| \leq M(x)e^{-\epsilon(x)t}.$$

Then, from (3.5),

$$\begin{aligned} ||P(x)|| &\leq \int_0^\infty ||M(x)||^2 ||Q(x)|| e^{-2\epsilon(x)t} dt \\ &= \frac{||M(x)||^2 ||Q(x)||}{2\epsilon(x)} \end{aligned}$$

and from (3.6)

$$\begin{aligned} \left\| \frac{\partial P}{\partial x_i} \right\| &\leq \frac{||M(x)||^2}{2\epsilon(x)} \left(2||P(x)|| \left\| \frac{\partial A}{\partial x_i} \right\| + \left\| \frac{\partial Q}{\partial x_i} \right\| \right) \\ &\leq \frac{||M(x)||^2}{2\epsilon(x)} \left(\frac{||M(x)||^2 ||Q(x)||}{\epsilon(x)} \left\| \frac{\partial A}{\partial x_i} \right\| + \left\| \frac{\partial Q}{\partial x_i} \right\| \right) . \end{aligned}$$

Thus,

$$\left\| \frac{\partial P}{\partial x} \right\| \leq \frac{||M(x)||^2}{2\epsilon(x)} \left(\frac{||M(x)||^2 ||Q(x)||}{\epsilon(x)} \left\| \frac{\partial A}{\partial x} \right\| + \left\| \frac{\partial Q}{\partial x} \right\| \right) .$$

where

$$\left\| \frac{\partial L}{\partial x} \right\| = \left(\sum_{i=1}^n \left\| \frac{\partial L}{\partial x_i} \right\| \right)^{1/2}$$

for a matrix function L and $||\partial L/\partial x_i||$ is the standard matrix norm. Hence, from (3.3), we have

$$\begin{aligned} \dot{V} &\leq -x^T Q(x) x + ||x||^3 \frac{||M(x)||^2}{2\epsilon(x)} \left(\frac{||M(x)||^2 ||Q(x)||}{\epsilon(x)} \left\| \frac{\partial A}{\partial x} \right\| + \left\| \frac{\partial Q}{\partial x} \right\| \right) ||A|| \\ &\leq -q(x)||x||^2 + ||x||^3 \frac{||M(x)||^2}{2\epsilon(x)} \left(\frac{||M(x)||^2 q(x)}{\epsilon(x)} \left\| \frac{\partial A}{\partial x} \right\| + \left\| \frac{\partial q}{\partial x} \right\| \right) ||A|| \end{aligned}$$

if $Q(x) = q(x)I$, where $q(x) > 0$ for all x. We can summarize the above calculation as follows:

Theorem 3.1 Suppose that the eigenvalues of $A(x)$ are in $\mathbf{C}^-$ for all x and that $Q(x) =$

$q(x)I$ can be chosen so that

$$q(x) > ||x|| \frac{||M(x)||^2}{2\epsilon(x)} \left(\frac{||M(x)||^2 q(x)}{\epsilon(x)} \left\| \frac{\partial A}{\partial x} \right\| + \left\| \frac{\partial q}{\partial x} \right\| \right) ||A||$$

then the system (3.1) is globally asymptotically stable. □

Corollary 3.1 Under the assumptions of theorem (3.1), the system (3.1) is globally asymptotically stable if

$$\left\| \frac{\partial A}{\partial x} \right\| < \frac{2\epsilon^2(x)}{||x|| ||M(x)||^4 ||A||} \quad , \quad x \neq 0. \tag{3.7}$$

Proof Take $q(x) \equiv 1$ in theorem (3.1). □

As an application of corollary (3.1) we shall consider the general equation

$$\dot{x} = f(x) \tag{3.8}$$

where f is analytic and $f(0) = 0$. Then we can write f_i in the form

$$f_i(x) = \sum_{j=1}^{n} f_{ij}(x) \cdot x_j$$

where

$$f_{ij}(x) = \sum_{\substack{|\mathbf{k}|=1 \\ k_j \geq 1}}^{\infty} \frac{f_i^{(k_1,\cdots,k_j-1,\cdots,k_n)}}{\mathbf{k}!} x_1^{k_1} \cdots x_j^{k_j-1} \cdots x_n^{k_n}.$$

(Here, $\mathbf{k} = (k_1, \cdots, k_n)$.) Thus, we can write (3.8) in the form (3.1) where

$$(A(x))_{ij} = f_{ij}(x).$$

From corollary (3.1) we have

Corollary 3.2 The system (3.8) with $f(0) = 0$ is asymptotically stable if the matrix $(f_{ij}(x))$ is stable for all x and

$$\left[\sum_i \sum_j f_{ij}^2 \right]^{1/2} \left[\sum_i \sum_j \sum_k \left(\frac{\partial f_{ij}}{\partial x_k} \right)^2 \right]^{1/2} < \frac{2\epsilon^2(x)}{||x|| \; ||M(x)||^4}$$

where

$$\left\| e^{(f_{ij}(x))t} \right\| \leq M(x) e^{-\epsilon(x)t}.$$

Proof This follows from the fact that, for any matrix M, we have

$$\|M\| \leq \left(\sum_i \sum_j m_{ij}^2 \right)^{1/2}. \tag{3.9}$$

□

If $x_i | f_j(x)$ for all i and j then we can write (3.8) in the form (3.1) where

$$A(x) = \frac{1}{n} \left(\frac{f(x)}{x} \right) \triangleq \frac{1}{n} \begin{pmatrix} \frac{f_1(x)}{x_1} & \frac{f_1(x)}{x_2} & \cdots & \frac{f_1(x)}{x_n} \\ \frac{f_2(x)}{x_1} & \frac{f_2(x)}{x_2} & \cdots & \frac{f_2(x)}{x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{f_n(x)}{x_1} & \frac{f_n(x)}{x_2} & \cdots & \frac{f_n(x)}{x_n} \end{pmatrix}.$$

We have

$$\begin{aligned} \frac{\partial A}{\partial x_k} &= \frac{1}{n} \begin{pmatrix} \frac{1}{x_1}\frac{\partial f_1}{\partial x_k} - \frac{f_1}{x_1^2}\delta_{k1} & \frac{1}{x_2}\frac{\partial f_1}{\partial x_k} - \frac{f_1}{x_2^2}\delta_{k2} & \cdots & \frac{1}{x_n}\frac{\partial f_1}{\partial x_k} - \frac{f_1}{x_n^2}\delta_{kn} \\ \frac{1}{x_1}\frac{\partial f_2}{\partial x_k} - \frac{f_2}{x_1^2}\delta_{k1} & \frac{1}{x_2}\frac{\partial f_2}{\partial x_k} - \frac{f_2}{x_2^2}\delta_{k2} & \cdots & \frac{1}{x_n}\frac{\partial f_2}{\partial x_k} - \frac{f_2}{x_n^2}\delta_{kn} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_1}\frac{\partial f_n}{\partial x_k} - \frac{f_n}{x_1^2}\delta_{k1} & \frac{1}{x_2}\frac{\partial f_n}{\partial x_k} - \frac{f_n}{x_2^2}\delta_{k2} & \cdots & \frac{1}{x_n}\frac{\partial f_n}{\partial x_k} - \frac{f_n}{x_n^2}\delta_{kn} \end{pmatrix} \\ &= \frac{1}{n} \left(\frac{1}{x_j}\frac{\partial f_i}{\partial x_k} - \frac{f_i}{x_j^2}\delta_{kj} \right)_{i,j}. \end{aligned}$$

Lemma 3.2 The following statements hold:

$$\|A(x)\| \leq \frac{1}{n} \left(\sum_{i=1}^n \sum_{j=1}^n \frac{f_i^2(x)}{x_j^2} \right)^{1/2}$$

and

$$\left\| \frac{\partial A}{\partial x} \right\| \leq \frac{1}{n} \left(\sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n \left(\frac{1}{x_j}\frac{\partial f_i}{\partial x_k} - \frac{f_i}{x_j^2}\delta_{kj} \right)^2 \right)^{1/2}.$$

Proof Use (3.9). □

Substituting these into (3.7) we get a similar result to corollary (3.2).

Example 3.1 Consider the case $n = 2$ with

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ 0 & f_{22}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{3.10}$$

Then

$$A(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ 0 & f_{22}(x) \end{pmatrix}.$$

Now, if $f_{22} \neq f_{11}$ for all x and

$$W(x) = \begin{pmatrix} 1 & f_{12}(x)/(f_{22}(x) - f_{11}(x)) \\ 0 & 1 \end{pmatrix}$$

then

$$A(x) = W(x) \begin{pmatrix} f_{11}(x) & 0 \\ 0 & f_{22}(x) \end{pmatrix} W^{-1}(x)$$

where

$$W^{-1}(x) = \begin{pmatrix} 1 & f_{12}(x)/(f_{11}(x) - f_{22}(x)) \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$e^{A(x)t} = W(x) \exp\left\{ t \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} \right\} W^{-1}(x)$$

and

$$\left\| e^{A(x)t} \right\| \leq \|W(x)\| \, \|W^{-1}(x)\| \cdot e^{-\omega(x)t}$$

assuming

$$\max\{f_{11}(x), f_{22}(x)\} \leq -\omega(x) \tag{3.11}$$

where

$$\omega(x) > 0 \text{ for all } x.$$

Hence,

$$\begin{aligned} \left\| e^{A(x)t} \right\| &\leq \left(2 + \frac{|f_{12}(x)|^2}{|f_{11}(x) - f_{22}(x)|^2}\right) e^{-\omega(x)t} \\ &= Me^{-\omega(x)t} \end{aligned}$$

where

$$M = 2 + \frac{|f_{12}(x)|^2}{|f_{11}(x) - f_{22}(x)|^2}.$$

Hence, by corollary (3.2) we see that the system (3.10) is globally asymptotically stable if (3.11) holds and

$$\left(\sum_{i,j,k=1}^{2} \left(\frac{\partial f_{ij}}{\partial x_k}\right)^2\right)^{1/2} < \frac{2\omega^2(x)}{(x_1^2 + x_2^2)^{1/2}\left(2 + \frac{|f_{12}(x)|^2}{|f_{11}(x) - f_{22}(x)|^2}\right)^4 (f_{11}^2 + f_{12}^2 + f_{22}^2)^{1/2}}$$

4 Dominance Theorems

In this section we shall consider pseudo-linear systems with a view to using some kind of 'diagonal dominance'. The first result is fundamental to this idea.

Theorem 4.1 Assume that $v_i(t)$ $(i \in I)$ are nonnegative continuous functions defined on $[t_0, \infty)$, which satisfy the inequality

$$\dot{v}_i \leq \lambda_i(t)v_i(t) + \sum_{j \in I} a_j^i(t)v_j(t) \tag{4.1}$$

where $\lambda_i(t)$ is a nonpositive continuous function, $a_j^i(t)$ are nonnegative continuous functions and I is a countable index set. If there exist $h > 0, \delta \in (0,1)$ and $d_i > 0$ such that

$$-\lambda_i(t) \geq h > 0 \ , \ -\frac{1}{d_i}\sum_{j\in I} d_j \frac{a_j^i(t)}{\lambda_i(t)} \leq \delta < 1 \ , \ \sup_{i\in I} \frac{v_i(t_0)}{d_i} < \infty \tag{4.2}$$

for all $t \geq t_0$ and $i \in I$, then there exist constants $\omega > 0$ and $m \geq 1$ such that

$$\frac{v_i(t)}{d_i} \leq m \sup_{j\in I} \frac{v_j(t_0)}{d_j} \exp(-\omega(t-t_0)) \tag{4.3}$$

for all $t \geq t_0$ and $i \in I$.

Proof See [2] □

Now consider again the system

$$\dot{x}(t) = A(x)x \quad , \quad x \in \mathbf{R}^n \tag{4.4}$$

where $A : \mathbf{R}^n \longrightarrow \mathbf{R}^{n^2}$ is a matrix-valued analytic function. In particular, we consider the case where $A(x)$ is a diagonal matrix

$$\dot{x} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} x \tag{4.5}$$

where $\lambda_i : \mathbf{R}^n \longrightarrow \mathbf{R}$, $1 \leq i \leq n$ are analytic functions. We assume that the origin is the only equilibrium point (i.e. $\lambda_i(x) \neq 0$ if $x \neq 0$). We can write the system in the form

$$\dot{x}_i = \Gamma_i(x)x_i + \sum_{j=1}^{n} a_j^i(x)x_j \tag{4.6}$$

$(1 \leq i \leq n)$ and we assume that $\Gamma_i(x) < 0$ for $x_i \neq 0$.

Theorem 4.2 The zero solution of (4.6) is globally (exponentially) stable if $\lambda_i(x) \neq 0$ if $x \neq 0$ and there exist constants $h > 0$, $\delta > 0$ and $d_i > 0$ such that

$$\Gamma_i(x) \leq -h < 0 \ , \ \frac{1}{d_i}\sum_{j=1}^{n} \frac{d_j|a_j^i(x)|}{|\Gamma_i(x)|} \leq \delta < 1 \tag{4.7}$$

for $i = 1, \cdots, n$, $x_i \neq 0$ and for all $t \geq t_0$.

Proof From (4.6) it follows that

$$x_i(t) = x_i(t_0)\exp\left(\int_{t_0}^{t}\Gamma_i(x(\tau))d\tau\right) + \sum_{j=1}^{n}\int_{t_0}^{t}a_j^i(x(s))x_j(s)\exp\left(\int_{t_0}^{s}\Gamma_i(x(\tau))d\tau\right)ds$$

and so

$$|x_i(t)| \leq |x_i(t_0)|\exp\left(\int_{t_0}^{t}\Gamma_i(x(\tau))d\tau\right) + \sum_{j=1}^{n}\int_{t_0}^{t}|a_j^i(x(s))|\;|x_j(s)|\exp\left(\int_{t_0}^{s}\Gamma_i(x(\tau))d\tau\right)ds\ .$$

If $v_i = |x_i|$ then

$$v_i(t) \leq v_i(t_0)\exp\left(\int_{t_0}^{t}\Gamma_i(x(\tau))d\tau\right) + \sum_{j=1}^{n}\int_{t_0}^{t}|a_j^i(x(s))|\;v_j(s)\exp\left(\int_{t_0}^{s}\Gamma_i(x(\tau))d\tau\right)ds\ .$$

and so

$$\dot{v}_i(t) \leq \Gamma_i(x(t))v_i(x(t)) + \sum_{j=1}^{n}|a_j^i(x(t))|\;v_j(t)$$

By theorem 4.1, there exist $\omega > 0$ and $m \geq 1$ such that

$$\frac{v_i(t)}{d_i} \leq m\sup_{j\in I}\frac{v_j(t_0)}{d_j}\exp(-\omega(t-t_0))$$

and so

$$\|v(t)\| = \|x(t)\| \leq M\exp(-\omega(t-t_0))\ ,$$

for some M. □

Example 4.1

$$\begin{aligned}\dot{x}_1(t) &= -x_1^3 + 0.5x_1^2x_2\\ \dot{x}_2(t) &= -(1+x_2^2)x_2 + x_1x_2\ .\end{aligned}$$

This system can be put in the form

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-x_1^2 + 0.5x_1x_2 & 0\\ 0 & -(1+x_2^2)+x_1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\ .$$

Note that the origin is the only equilibrium point. The equations can also be written in the form

$$
\begin{aligned}
\dot{x}_1(t) &= (-x_1^2)x_1 + 0.5(x_1^2)x_2 \\
&= \Gamma_1(x)x_1 + a_2^1(x)x_2 \\
\dot{x}_2(t) &= -(1+x_2^2)x_2 + x_1x_2 \, . \\
&= \Gamma_2(x)x_2 + a_1^2(x)x_1
\end{aligned}
$$

We have

$$
\begin{aligned}
\Gamma_1(x) &= -x_1^2 < 0 \quad , x_1 \neq 0, \\
\Gamma_2(x) &= -(1+x_2^2) < 0 \quad , x_2 \neq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{|a_2^1(x)|}{|\Gamma_1(x)|} &= \frac{|0.5x_1^2|}{|-x_1^2|} = 0.5 < 1 \\
\frac{|a_1^2(x)|}{|\Gamma_2(x)|} &= \frac{|0.5x_2|}{|-(1+x_2^2)|} = 0.5 < 1
\end{aligned}
$$

Hence the conditions of theorem are satisfied if we take $d_1 = d_2 = 1$. From this example we can see that for a nonlinear system to be asymptotically stable, it is not necessary that the eigenvalues be negative definite.

Example 4.2

$$
\begin{aligned}
\dot{x}_1(t) &= (-2x_1 + x_2)x_1^2 \\
\dot{x}_2(t) &= -(1 + x_2^2 - x_1 + x_2^2x_3^2)x_2 \\
\dot{x}_3(t) &= -(1 + x_3^2 + x_2^2 + x_1)x_3 \, .
\end{aligned}
$$

This system has the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} (-2x_1 + x_2)x_1 & 0 & 0 \\ 0 & -(1 + x_2^2 - x_1 + x_2^2 x_3^2) & 0 \\ 0 & 0 & -(1 + x_3^2 + x_2^2 + x_1) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Again, the origin is the only equilibrium point. Note that the equations have the form

$$\begin{aligned} \dot{x}_1(t) &= \Gamma_1(x)x_1 + a_2^1(x)x_2 \\ \dot{x}_2(t) &= \Gamma_2(x)x_2 + a_1^2(x)x_1 \\ \dot{x}_1(t) &= \Gamma_3(x)x_3 + a_1^3(x)x_1 \end{aligned}$$

where

$$\begin{aligned} \Gamma_1(x) &= -2x_1^2 < 0 \quad , x_1 \neq 0 \\ \Gamma_2(x) &= -(1 + x_2^2 + x_2^2 x_3^2) < 0 \quad , x_2 \neq 0 \\ \Gamma_3(x) &= -(1 + x_2^2 + x_3^2) < 0 \quad , x_3 \neq 0 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{|a_2^1(x)|}{|\Gamma_1(x)|} &= \frac{|x_1^2|}{|-2x_1^2|} < 1 \\ \frac{|a_1^2(x)|}{|\Gamma_2(x)|} &= \frac{|x_2|}{|-(1 + x_2^2 + x_2^2 x_3^2)|} < 1 \\ \frac{|a_1^3(x)|}{|\Gamma_1(x)|} &= \frac{|-x_3|}{|-(1 + x_2^2 + x_3^2|} < 1 \end{aligned}$$

and it follows that the zero solution is globally asymptotically stable.

Of course, we can also consider the general nonlinear analytic system

$$\dot{x}(t) = f(x(t))$$

where $f(0) = 0$. Then we can write the system in the form

$$\dot{x}_i(t) = a_{ii}(x)x_i + \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}(x)x_j \tag{4.8}$$

which is not necessarily diagonal. However, we still have a result similar to theorem 4.2:

Theorem 4.3 The zero solution of (4.8) is globally asymptotically stable if 0 is the unique equilibrium point of the system and there exist constants $h < 0, \delta \in (0,1)$ and $d_i > 0$ for $i = 1, \cdots, n$ such that

$$\begin{aligned} a_{ii}(x) &\leq h < 0 \ , \ x \neq 0 \\ \frac{1}{d_i}\sum_{\substack{j=1 \\ j\neq i}}^{n} d_j \frac{|a_{ij}(x)|}{|a_{ii}(x)|} &\leq \delta < 1. \end{aligned}$$

Example 4.3 Consider the system

$$\begin{aligned} \dot{x}_1(t) &= -(2+x_2^2)x_1 + x_2^2 \\ \dot{x}_2(t) &= -(1+\exp(x_1))x_2 + 0.5x_1. \end{aligned}$$

Then this system has the above form with

$$\begin{aligned} a_{11}(x) &= -(2+x_2^2) \ , \ a_{12}(x) = x_2 \\ a_{22}(x) &= -(1+\exp(x_1))x_2 \ , \ a_{21}(x) = 0.5 \end{aligned}$$

The conditions of theorem are clearly satisfied and so the system is globally asymptotically stable.

Example 4.4 Similarly, it is easy to check that the system

$$\begin{aligned} \dot{x}_1(t) &= -(1+x_2^2)x_1 + x_2^2 x_3 \\ \dot{x}_2(t) &= -(x_1^2+x_2^2+x_3^2)x_2 \\ \dot{x}_3(t) &= \frac{x_2^2 x_1}{2} + \frac{x_1^2 x_2}{2} - (2+x_1^2+x_2^2)x_3 \end{aligned}$$

is also globally asymptotically stable.

In the final part of this section, we shall consider the application of the theory of semisimple Lie algebras to the theory of stability. The use of such techniques in Lyapunov stability has already been demonstrated (see [1]). The idea here is to obtain a 'near diagonal' system given by the Cartan subalgebra of the Lie algebra generated by $\{A(x)\}$, assuming it is semisimple, and the corresponding roots.

Therefore, consider again the pseudo-linear system

$$\dot{x}(t) = A(x)x \tag{4.9}$$

where $A(x) : \mathbf{R}^n \longrightarrow \mathbf{R}^n,\ x \in \mathbf{R}^n$.

If the Lie algebra generated by $A(x)$ is semisimple, then the nonlinear system (4.9) can be written in the form

$$\dot{x}(t) = H(x)x + \sum_{\alpha\in\Delta} e'_\alpha(x)F_\alpha x$$

where $H(x)$ is the Cartan subalgebra and F_α are the roots. $H(x)$ can be diagonalized simultaneously by a linear transformation, say P:

$$\dot{y} = \Lambda(y)y + \sum_{\alpha\in\Delta} e_\alpha(y)E_\alpha y$$

where $y = P^{-1}x$, $\Lambda(y) = P^{-1}H(Py)Py$, $e_\alpha = e'_\alpha(Py)$ and $E_\alpha = P^{-1}F_\alpha P$. In general, by this transformation, the original system is transformed to a simpler one. Finally, writing the system in the form

$$\dot{y}_i = \beta_i(y)y_i + \sum_{j=1}^{n} a^i_j y_j \tag{4.10}$$

by splitting the coefficient of y_i in this equation into a negative definite part ($\beta_i(y)$) and putting the rest into the a^i_j functions we can generalize the previous results in the obvious way:

Theorem 4.4 The zero solution of (4.10) is globally asymptotically stable if $x = 0$ is the only equilibrium point and there exist constants $h > 0,\ \delta > 0$ and $d_i > 0$ such that

$$\beta_i(x) \le -h < 0 \quad , \quad \frac{1}{d_i}\sum_{j=1}^{n} d_j \frac{|a^i_j|}{|\beta_i(x)|} \le \delta < 1$$

for $j = 1, \cdots, n,\ x \neq 0$ and for all $t \geq t_0$.

Proof As in theorem 4.2. □

Example 4.5 Consider the semisimple Lie algebra A_2 (other types of semisimple Lie algebras B_n,D_n,$\cdots$ can be applied in a similar way), and consider the two-dimensional nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \begin{pmatrix} f_1(x) & f_{12}(x) \\ f_{21}(x) & -f_1(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where

$$\begin{aligned} f_1(x) &= -x_1^4 + x_2^2 \\ f_{12}(x) &= -2x_1x_2 \\ f_{21}(x) &= -2x_1^3x_2 + 0.5x_1^4 \end{aligned}$$

The functions $f_1(x), f_{12}(x)$ and $f_{21}(x)$ are functionally independent. The Lie algebra generated by $A(x)$ is semisimple of type A_2 and is spanned by the matrices

$$h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

where h_1 spans the Cartan subalgebra and E_{12}, E_{21} span the root spaces of the algebra. By splitting the coefficient of x_i in the equation for $\dot{x}_i(t)$ into a negative definite part (call it $\beta_i(x)$) and the rest into the functions a_j^i, the system can be written in the form

$$\dot{x}_1 = \beta_1(x)x_1 + a_1^1(x)x_1 + a_2^1(x)x_2,$$

where $a_j^1 = 0, j = 1, 2$ and $\beta_1(x) = -(x_1^4 + x_2^2)$, and

$$\dot{x}_2 = \beta_2(x)x_2 + a_1^2(x)x_1 + a_2^2(x)x_2,$$

where $\beta_2(x) = -(x_1^4 + x_2^2)$, $a_1^2(x) = 0.5x_1^4$ and $a_2^2 = 0$. The origin is an isolated equilibrium point and we let $d_1 = d_2 = 1$. From the above equations we have

$$\begin{aligned}
\beta_1(x) &= -(x_1^4 + x_2^2) < 0 \\
\frac{0}{|\beta_1(x)|} &< \delta < 1 \\
\beta_2(x) &= -(x_1^4 + x_2^2) < 0 \\
\frac{|a_1^2(x)|}{|\beta_1(x)|} &= \frac{0.5x_1^4}{x_1^4 + x_2^2} < \delta < 1.
\end{aligned}$$

It follows that the zero solution of the system is globally asymptotically stable.

Example 4.6 Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} \begin{pmatrix} f_{11}(x) - f_{21}(x) + f_{31}(x) & a_{12}(x) & a_{13}(x) \\ f_{21}(x) + f_{31}(x) & a_{22}(x) & a_{23}(x) \\ f_{31}(x) & f_{31}(x) + f_{32}(x) & f_{22}(x) - 3f_{31}(x) - f_{32}(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

where

$$\begin{aligned}
a_{12}(x) &= 2f_{11}(x) + f_{22}(x) + f_{12}(x) - f_{21}(x) + 2f_{31}(x) + 2f_{32}(x) \\
a_{13}(x) &= -4f_{11}(x) + f_{22}(x) - f_{12}(x) + 3f_{21}(x) + f_{13}(x) - 6f_{31}(x) - 2f_{32}(x) - f_{23}(x) \\
a_{22}(x) &= -f_{11}(x) - f_{22}(x) + f_{21}(x) + f_{31}(x) + f_{32}(x) \\
a_{23}(x) &= f_{11}(x) + 2f_{22}(x) - 3f_{21}(x) - 3f_{31}(x) - f_{32}(x) + f_{23}(x).
\end{aligned}$$

We assume that the maps $f_{ij}(x)$ are functionally independent. Then the Lie algebra generated by$A(x)$ is semisimple with the following matrices as its basis:

$$h_1 = \begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; h_2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \; F_{12} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_{21} = \begin{pmatrix} -1 & -1 & 3 \\ 1 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}, F_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F_{31} = \begin{pmatrix} 2 & 2 & -6 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{pmatrix}$$

$$F_{32} = \begin{pmatrix} 0 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, F_{23} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

where h_1 and h_2 span the Cartan subalgebra, and the matrix

$$P = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

diagonalizes this subalgebra. Let

$$\begin{aligned}
f_{11} &= -(1-(x_2-x_3)^2) \\
f_{22} &= -(1+x_3^2) \\
f_{12} &= -3(x_1+x_2-3x_3)(x_2-x_3) \\
f_{21} &= \frac{-(2+2x_3^2+(x_1+x_2-3x_3)^2)}{x_1+x_2-2x_3}(x_2-x_3) \\
f_{31} &= x_3^2 \\
f_{13} &= -0.5(1+2(x_2-x_3)^2) \\
f_{32} &= -x_3^3(x_2-x_3) \\
f_{23} &= -x_3(x_2-x_3)\ .
\end{aligned}$$

Then the above system can be put in the form:

$$\begin{aligned}
\dot{x} &= A(x)x \\
\dot{x} &= f_{11}(x)h_1 + f_{22}(x)h_2 + f_{12}(x)F_{12} + f_{13}(x)F_{13}
\end{aligned}$$

$$+f_{31}(x)F_{31} + f_{32}(x)F_{32} + f_{23}(x)F_{23}$$

$$\begin{aligned} x &= Py \\ \dot{y} &= P^{-1}\dot{x} = (P^{-1}H(x)P + \sum_\alpha f_\alpha(x)P^{-1}F_\alpha P)y \end{aligned}$$

i.e.

$$\dot{y} = (\Lambda(y) + \sum_\alpha f'_\alpha E_\alpha)y$$

or

$$\begin{aligned} \dot{y} &= f'_{11}(y)h'_{11} + f'_{22}(y)h'_{22} + f'_{12}(y)E_{12} + f'_{21}(y)E_{21} + \\ & f'_{13}(y)E_{13} + f'_{31}(y)E_{31} + f'_{32}(y)E_{32} + f'_{23}(y)E_{23} \end{aligned}$$

where

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and E_{ij} for $i \neq j$ is a matrix with 1 in the ij^{th} place and 0 elsewhere. Also, the f' are given by

$$\begin{aligned} f'_{11}(y) &= -(1 - y_2^2) \\ f'_{22}(y) &= -(1 + y_3^2) \\ f'_{12}(y) &= -3y_1 y_2 \\ f'_{21}(y) &= \frac{-(2 + 2y_3^2 + y_1^2)}{y_1} y_2 \\ f'_{31}(y) &= y_3^2 \\ f'_{13}(y) &= -0.5(1 + 2y_2^2) \\ f'_{32}(y) &= -y_3^3 y_2 \\ f'_{23}(y) &= -y_3 y_2 \ . \end{aligned}$$

Now we can test for the stability of this system using the previous theorem. First, $x = 0$ is an isolated equilibrium, and we take $d_1 = d_2 = d_3 = 1$. Then we write

$$\begin{aligned} \dot{y}_1 &= \beta_1(y)y_1 + a_1^1(y)y_1 + a_3^1(y)y_3 \\ \dot{y}_2 &= \beta_2(y)y_2 \\ \dot{y}_3 &= \beta_3(y)y_3 + a_1^3(y)y_1 \end{aligned}$$

where

$$\begin{aligned} \beta_1(y) &= -(1+3y_2^2) \ , \quad a_1^1(y) = y_2^2 \ , \quad a_3^1(y) = -0.5(1+2y_2^2) \ , \\ \beta_2(y) &= -(y_1^2 + y_2^2 + 2y_3^2) \\ \beta_3(y) &= -(1 + y_3^2 + y_3^2 y_2^2) \ , \quad a_1^3(y) = y_3^2 . \end{aligned}$$

The conditions for stability are now easy to check and so the system is globally asymptotically stable.

5 Discontinuous Systems

Not only is it not sufficient for the eigenvalues of the nonlinear system (1.1) to belong to $\mathbf{C}^-$ to be asymptotically stable, it is also not necessary, as we have seen above. In fact, if $A(x)$ is not analytic it is possible for the eigenvalues to be arbitrarily large in some regions near the origin. For example, consider the discontinuous system on $\mathbf{R}^2$:

$$\dot{x} = \begin{cases} \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x_1 \geq 0, x_2 \geq 0 \text{ or } x_1 \leq 0, x_2 \leq 0 \\ \\ \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & \text{if } x_1 \geq 0, x_2 < 0 \text{ or } x_1 \leq 0, x_2 > 0 \end{cases}$$

the order a_1 of exponential stability must be greater in magnitude than that for the unstable region. This result can be generalized:

Theorem 5.1 Consider the pseudo-linear system

$$\dot{x} = A(x)x \quad , \quad x(0) = x_0 \in \mathbf{R}^n$$

where $A : \mathbf{R}^n \longrightarrow \mathbf{R}^{n^2}$ may not be continuous, and suppose that unique (Fillipov) solutions exist for all $x_0 \in \mathbf{R}^n$. Assume that $\mathbf{R}^n$ can be partitioned into disjoint regions Ω_i, $1 \leq i \leq k$ (where k may be ∞) such that the solution $x(t)$ satisfies

$$\begin{aligned} \|x(t;x_0)\| &\leq M_i e^{-a_i t}\|x_0\| \, , a_i > 0, \text{if } \ x_0, x(t) \in \Omega_i, \ i \in S \\ \|x(t;x_0)\| &\leq M_i e^{a_i t}\|x_0\| \, , a_i > 0, \text{if } \ x_0, x(t) \in \Omega_i, \ i \in U \end{aligned}$$

where S and U form a partition of $\{1, \cdots, k\}$. If τ_i is the maximum (minimum) time spent in each region $\Omega_i, i \in U \ (i \in S)$ and

$$\tau_i < \infty \quad \text{for } i \in U$$

then the system is asymptotically stable if

$$\left\{\min_{i\in S} a_i\right\} \times \left\{\min_{i\in S} \tau_i\right\} > \left\{\max_{i\in U} a_i\right\} \times \left\{\max_{i\in U} \tau_i\right\}. \tag{5.1}$$

Proof This follows from the fact that if the solution passes through regions $i_1, i_2, \cdots, i_r$ it is given by

$$x(t) = x^{i_r}(t; x^{i_{r-1}}(\tau_{r-1}; \cdots; x^{i_2}(\tau_2, x^{i_1}(\tau_1; x_0)) \cdots))$$

where x^{i_j} is the solution of the system in Ω_{i_j} and τ_ℓ is the time spent in region i_ℓ. Thus,

$$\begin{aligned} \|x(t)\| &\leq M_{i_r} e^{\pm a_{i_r} t}\|x^{i_{r-1}}(\tau_{r-1}; \cdots; x^{i_2}(\tau_2, x^{i_1}(\tau_1; x_0)) \cdots)\| \\ &\leq M_{i_r} \cdots M_{i_1} e^{(\pm a_{i_r} t \pm a_{i_{r-1}} \tau_{r-1} \cdots \pm a_{i_1} \tau_1)}\|x_0\| \end{aligned}$$

where the $+(-)$ sign is taken for an unstable (stable) region. The result now follows from (5.1). □

Remark 5.1 The condition (5.1) is very conservative. If we know the sequence of regions through which a solution passes, then the condition can clearly be weakened.

Remark 5.2 If $A(x)$ is continuous, then there is clearly no partition of $\mathbf{R}^n$ with the required property. In this case we could assume that each unstable region Ω_i, $i \in U$ is extended to include a set K_i on which the solution is not necessarily exponentially stable, i.e. replace Ω_i, $i \in U$ by $\Omega_i \cup K_i$, $i \in U$. We then partition $\mathbf{R}^n \setminus \cup_{i \in U} (\Omega_i \cup K_i)$ into exponentially stable subsets. Then if condition (5.1) holds with each Ω_i, $i \in U$ replaced by $\Omega_i \cup K_i$ and solutions which enter Ω_i, $i \in U$ leave K_i without reentering Ω_i, then the conclusion of theorem (5.1) still holds.

6 Conclusions

The stability of pseudo-linear systems has been considered. These systems are, in fact, very general since any nonlinear system with an equilibrium point at the origin can be written in this form. After generalizing Lyapunov's theorem for linear systems we have applied a diagonal dominance approach based on a comparison result for countable systems and the theory of semisimple Lie algebras. Finally, the case of discontinuous systems has been discussed briefly. The last case should prove useful in the study of piecewise-linear systems and we shall consider this in more detail in a future paper.

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