Article:

Wagg, D.J. (2002) Partial synchronization of non-identical chaotic systems via adaptive control, with applications to modelling coupled nonlinear systems. International Journal of Bifurcation and Chaos, 12 (3). 561 - 570.

https://doi.org/10.1142/S0218127402004589

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

PARTIAL SYNCHRONIZATION OF NON-IDENTICAL CHAOTIC SYSTEMS VIA ADAPTIVE CONTROL, WITH APPLICATIONS TO MODELING COUPLED NONLINEAR SYSTEMS

David J. Wagg*

Department of Mechanical Engineering, University of Bristol, Queens Building, University Walk, Bristol BS8 1TR, U.K.

May 3, 2013

Abstract

We consider the coupling of two non-identical dynamical systems using an adaptive feedback linearization controller to achieve partial synchronization between the two systems. In addition we consider the case where an additional feedback signal exists between the two systems, which leads to bidirectional coupling. We demonstrate the stability of the adaptive controller, and use the example of coupling a Chua system with a Lorenz system, both exhibiting chaotic motion, as an example of the coupling technique. A feedback linearization controller is used to show the difference between unidirectional and bidirectional coupling. We observe that the adaptive controller converges to the feedback linearization controller in the steady state for the Chua-Lorenz example. Finally we comment on how this type of partial synchronization technique can be applied to modeling systems of coupled nonlinear subsystems. We show how such modeling can be achieved where the dynamics of one system is known only via experimental time series measurements.

Running title:Partial synchronization of non-identical systems

*Email:`david.wagg@bristol.ac.uk`

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

1 Introduction

The problem of synchronizing two identical dynamical systems has been studied by many authors, for example: Ashwin *et al.* (1994); Kozlov & Shalfeev (1996), Ashwin (1998) and Yang & Duan (1998), following the work of Pecora & Carroll (1990). When this is achieved using adaptive control type methods, the process is referred to as adaptive synchronization (John & Amritkar 1994; Boccaletti, Farini & Arecchi 1997; Fradkov & Markov 1997; Dedieu & Ogorzalek 1997). More recently the concept of partial synchronization between two or more similar chaotic systems has been studied (Hasler 1998; Yanchuk, Maistrenko & Mosekilde 2001). In this paper we consider coupling two non-identical dynamical systems via partial synchronization using an adaptive synchronization technique.

A case of particular interest is when an additional feedback signal exists between the two systems such that the coupling is bidirectional and the two systems interact dynamically, giving rise to a complex dynamical behavior. This has applications to dynamic substructuring, where systems are modeled by coupling a set of interacting substructures together (Ohayon *et al.* 1997; Wagg & Stoten 2001). We demonstrate this concept using both a feedback linearization controller and an adaptive feedback linearization controller.

In addition, we demonstrate how the adaptive controller can be designed when coupling single and multiple variables from each of the non-identical nonlinear systems. We show that this type of adaptive controller is stable for such a coupled system. This is demonstrated using the example of coupling a Lorenz system with a Chua system; for similar examples see Di Benardo (1996). In this example, we observe that the steady state adaptive controller converges to the feedback linearization controller.

Finally we discuss applications to modeling dynamical systems composed of a set of coupled nonlinear dynamical systems. We discuss how partial synchronization can be used to achieve this type of modeling. We also discuss how the concepts of synchronizing dynamical systems, (Ashwin 1998) can be used to monitor the performance of the controller producing the coupling and hence the modeling process itself, using the Lorenz Chua system as an example.

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

2 Partial synchronization for non-identical systems

We consider two non-identical dynamical systems, one with state variable $x \in \mathcal{R}^p$, and the second, with state variable $y \in \mathcal{R}^q$, with governing equations of the general form

$$\begin{aligned} \dot{x}(t) &= f_1(x, t), \\ \dot{y}(t) &= f_2(y, t). \end{aligned} \tag{1}$$

In general, we consider that the dynamics of the two systems are nonlinear and that there is no cross coupling between the two sets of state variables. We define a coordinate subset of x, $x_s \in \mathcal{R}^n$, and similarly $y_s \in \mathcal{R}^n$, which represent the coordinates which require synchronization to achieve coupling between the two systems. So, we will consider the class of systems for which equation 1 can be expressed as

$$\begin{aligned} \dot{x}_n(t) &= f_{11}(x_n, x_s, t) \\ \dot{x}_s(t) &= f_{12}(x_n, x_s, t) \\ \dot{y}_s(t) &= f_{21}(y_n, y_s, t) \\ \dot{y}_n(t) &= f_{22}(y_n, y_s, t) \end{aligned} \tag{2}$$

where $x_n = \{x_i \in x : x_i \notin x_s\}$ and x_i denotes the ith element of x, and likewise $y_n = \{y_i \in y : y_i \notin y_s\}$. Then if $x_s \to y_s$ as $t \to \infty$ we say that the systems is partially synchronized. When such partial synchronization occurs a coupled system is formed which is shown schematically in figure 1. The case where $x_s = x$ and $y_s = y$ is the standard synchronization problem (Pecora & Carroll 1990).

To achieve partial synchronization, we need to synchronize the dynamics of f_{12} and f_{21}. Thus we add a controller, to the coupled system, such that equation 2 can be written as

$$\begin{aligned} \dot{x}_n(t) &= f_{11}(x_n, x_s, t) \\ \dot{x}_s(t) &= f_{12}(x_n, x_s, t) + g(u, t) \\ \dot{y}_s(t) &= f_{21}(y_n, y_s, t) \\ \dot{y}_n(t) &= f_{22}(y_n, y_s, t) \end{aligned} \tag{3}$$

where u is the control signal, and $g(\cdot)$ represents the controller function. In this form, the dynamics of f_{21} can be thought of as the reference model (Landau 1979), which we want $f_{12} + g(u, t)$ to replicate and f_{12} represents the plant.

So, in the formulation of equation 3, a part of system 1 will be forced to behave like part of system 2. However, for bidirectional coupling, system 1 will also have an influence on the behavior of systems 2. In this case, an additional feedback signal between f_1 and f_2 can be used to represent the coupling. We represent it by adding a coupling function to f_{21}, such that

$$\begin{aligned} \dot{x}_n(t) &= f_{11}(x_n, x_s, t) \\ \dot{x}_s(t) &= f_{12}(x_n, x_s, t) + g(u, t) \\ \dot{y}_s(t) &= f_{21}(y_n, y_s, t) + c(x_n, x_s, t) \\ \dot{y}_n(t) &= f_{22}(y_n, y_s, t) \end{aligned} \tag{4}$$

In the case where f_1 is a physical system and f_2 is an analytical model the dynamics of f_{11} can be assumed to be unknown, and $c(x_n, x_s, t)$ would typically be a recorded time series from f_{11}. The functions f_{22} and f_{21} must be known explicitly, so that they can be computed numerically, and the *structure* of the f_{12} must be known. Knowledge of specific parameter values is not required, as the adaptive controller can be applied without this information. If $c = 0$ the coupling between the two systems (via partial synchronization) is effectively unidirectional, whereas if $c \neq 0$ the coupling is bidirectional; examples will be discussed in section 3.1, 4.1 and 5.1. We note also that the analysis in this section is for autonomous systems, however it is possible to apply this analysis to some non-autonomous systems (Wagg & Stoten 2001) which we briefly discuss in section 5.1.

2.1 Controller design

To design a controller for the system we first reduce equation 4 to the form

$$\begin{aligned} \dot{x}_s(t) &= f_{12}(d_1(t), x_s, t) + g(u, t) \\ \dot{y}_s(t) &= f_{21}(d_2(t), y_s, t) + c(t) \end{aligned} \tag{5}$$

where the dynamics of x_n and y_n are now represented by the functions d_1 and d_2 respectively, which we assume act as disturbances. Then we can formulate the error dynamics for the system such that

$$\dot{e}(t) = f_{21}(d_2(t), y_s, t) - f_{12}(d_1(t), x_s, t) + c(t) - g(u, t)\ . \tag{6}$$

where the error, $e = y_s - x_s$. This can then be expressed as

$$\dot{e}(t) = \Delta f(t) + c(t) - g(u, t)\ . \tag{7}$$

where $\Delta f(t) = f_{21} - f_{12}$. For effective performance of the controller, we require that the equilibrium, $e = 0$ is stable. From equation 7 we see that the controller has to compensate for the difference between f_{12} and f_{21}, $\Delta f(t)$ and the addition feedback signal $c(t)$.

In this formulation there are two additional disturbances, d_1, d_2. These functions are not external disturbances in the ordinary sense, but signals from some other part of the coupled system. As a result, the controller must compensate for the influence of these additional signals.

3 Single variable coupling

Let us first consider the case where only a single coordinate of f_1 and f_2 is to be synchronized, and therefore e is scalar in this case. Then we can write the error dynamics as

$$\dot{e}(t) = -\lambda e + L - g(u,t), \tag{8}$$

where $L = \Delta f + \lambda e + c$, and $\lambda > 0$. This type of formulation is possible with a wide variety of both linear and nonlinear systems (Di Benardo 1996), and this requirement is therefore not overly restrictive. It is clear from equation 8 that $(L - g_1(u,t)) \to 0$, and $\lambda > 0$ will stabilize the required equilibrium, $e = 0$. Therefore L is the feedback linearization controller for the system (Di Benardo 1996).

For the class of systems considered in this work, L can be expressed as $L = k^*\xi$, where k^* represents a set of (constant) parameters, and ξ the vector of coupling variables. For such systems we use an adaptive controller which has essentially the same form as L, $g(u,t) = u = k(t)\xi$, where $k(t)$ is the adaptive gain. Using these definitions enables us to express equation 8 as

$$\dot{e}(t) = -\lambda e + \phi(t)\xi(t), \tag{9}$$

where $\phi(t) = k^* - k(t)$ is the parameter error. We then need to find an expression for $k(t)$ which stabilizes the system such that $\phi(t) \to 0$ as $t \to \infty$. This we can achieve by choosing a Lyapunov function of the form

$$V(t) = \frac{e^2}{2} + \frac{\phi\phi^T}{2\gamma}, \tag{10}$$

where γ is the controller gain. Then the derivative of V with respect to time is

$$\dot{V}(t) = e(-\lambda e + \phi(t)\xi(t)) + \frac{1}{\gamma}\phi\dot{\phi}^T, \tag{11}$$

such that choosing $\dot{\phi}^T = -\gamma e\xi$, results in $\dot{V} = -\lambda e^2$ which implies that the controller is globally asymptotically stable for $\lambda > 0$. As k^* is constant, $\dot{\phi}^T = -\dot{k}^T = -\gamma e\xi$, such that the adaptive gain becomes

$$k^T = \gamma \int_{t=t_0}^{t} e\xi \mathrm{d}t \tag{12}$$

(Sastry & Bodson 1989). Thus $k(t) \to k^*$ as $\phi \to 0$ and $e \to 0$. Note: providing $\phi \to 0$, the final adaptive gain values correspond to the unknown set of system parameters k^*. In general $k(t) \to k^*$ providing the adaptive controller has a persistently exciting signal (see for example Sastry (1999). From qualitative examination of our numerical simulations in this paper this is nearly always the case.

Finally, there is an extra effect on the stability of the partially synchronized systems due to the signals d_1, d_2 and c. For global asymptotic stability that these signals must remain bounded. As they are dependent on state variables, they can only become unbounded if the system becomes unstable. Therefore providing the system reaches a stable state with d_1, d_2 and c bounded the system will remain stable.

3.1 Example coupling the Chua and Lorenz systems

We now consider an example of coupling a Chua system with a Lorenz system. In this example (for an example of adaptive control using similar systems see Stoten & Di Bernardo (1996)), we use a Chua system defined as

$$\begin{aligned} \dot{x}_1 &= \alpha_1(x_2 - x_1) + \alpha_2 x_1 - \alpha_3(|x_1 + 1| - |x_1 - 1|) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\delta x_2 \end{aligned} \tag{13}$$

and a Lorenz system

$$\begin{aligned} \dot{y}_1 &= -\sigma(y_1 - y_2) \\ \dot{y}_2 &= r y_1 - y_2 - y_1 y_3 \\ \dot{y}_3 &= y_1 y_2 - b y_3 \end{aligned} \tag{14}$$

To ensure that both systems are chaotic, we select the parameter values: $\alpha_1 = 10$, $\alpha_2 = 0.68$, $\alpha_3 = 0.59$, $\delta = -14.87$, $\sigma = 10$, $r = 28$ and $b = 8/3$. Initial conditions for the system were selected as $x_1(0) = 1.1$, $x_2(0) = 1.0$, $x_3(0) = 7.0$, $y_1(0) = -1.1$, $y_2(0) = -1.0$ and $y_1(0) = -5.0$. This choice of parameters and initial conditions is arbitrary: control can be applied for any parameter values.

Now let us consider the case when we wish to couple (i.e. synchronize) x_3 and y_1. Thus, we define $x_n = [x_1, x_2]^T$, $x_s = x_3$, $y_s = y_1$ and $y_n = [y_2, y_3]$. Then

$$f_{11} = \begin{pmatrix} \alpha_1(x_2 - x_1) + \alpha_2 x_1 - \alpha_3(|x_1 + 1| - |x_1 - 1|) \\ x_1 - x_2 + x_3 \end{pmatrix} \tag{15}$$

and

$$f_{22} = \begin{pmatrix} ry_1 - y_2 - y_1 y_3 \\ y_1 y_2 - by_3 \end{pmatrix}. \tag{16}$$

The reference model is f_{21}, and therefore the control signal must be applied to f_{12} such that

$$f_{12} = -\delta x_2 + u. \tag{17}$$

Thus we are coupling the two systems by controlling f_{12} to follow f_{21}. So in this case we can think of the Lorenz system as the master or forcing system and the Chua as a slaved system, such that the coupling is unidirectional. We can introduce bidirectional coupling by adding a coupling function, $c(t)$, to the Lorenz system, such that the reference f_{21} can be written as

$$f_{21} = -\sigma(y_1 - y_2) + c(t) \tag{18}$$

where c is set to zero in the unidirectional case.

3.1.1 Feedback linearization controller

To demonstrate the difference between unidirectional, and bidirectional coupling, we first use a controller based on feedback linearization (see for example Di Benardo (1996)). To design such a controller we need to know Δf explicitly, which in this example is

$$\Delta f = f_{21} - f_{12} = -\sigma y_1 + \sigma y_2 + c + \delta x_2 - u. \tag{19}$$

The error variable $e = y_s - x_s = y_1 - x_3$, so that equation 19 can be expressed as

$$\Delta f = f_{21} - f_{12} = -\lambda e + \sigma(y_2 - x_3) + c + \delta x_2 - u, \tag{20}$$

where $\lambda = \sigma$ in this case. So, in this example as $\sigma > 0$, we can write

$$\Delta f = f_{21} - f_{12} = -\sigma e + L - u, \tag{21}$$

which is equivalent to the right hand side of equation 9. Thus for feedback linearization we set $u = L = \sigma(y_2 - x_3) + c + \delta x_2$.

A numerical simulation of the unidirectional (master-slave) system, $c = 0$, is shown in figure 2. Here the response of x_3 from the Chua system is shown as a solid line, while the response y_1 of the Lorenz system is shown as a dashed line. The controller is initially turned off $u = 0$, and the responses of the two systems are unsynchronized. Then at $t = 12$ the feedback linearization controller is turned on. The two selected coordinates from the systems quickly synchronize, with the Chua x_3 coordinate slaved to the Lorenz y_1 coordinate.

In figure 3 we show a simulation for the bidirectional case, when $c = x_3$. Here the response of x_3 from the Chua system is shown as a solid line, while the response y_1 of the Lorenz system is shown as a chain dotted line. In addition we have plotted the output from the Lorenz system (dashed line) for the $c = 0$ case as a comparison. Again the control is turned on at $t = 12$, and the two systems quickly become synchronized. This time however the dynamics are not slaved to the Lorenz system. The bidirectional coupling produces interaction between the two systems, such that the dynamical behavior is not the same as the master slave example. This can be seen in figure 3 from the deviation of the synchronized system from the Lorenz system after $t = 12$.

3.1.2 Adaptive feedback linearization control

Now we consider the same synchronization problem using an adaptive controller. To achieve this we have to express L, the feedback linearization controller, as a product of a unknown parameter vector, k^* and a coupling variable vector, ξ. Thus

$$L = \sigma(y_2 - x_3) + \delta x_2 + c = \{\sigma, \delta, \beta\} \left\{ \begin{array}{c} y_2 - x_3 \\ x_2 \\ c \end{array} \right\} \tag{22}$$

so that the coupling variable vector is $\xi = \{(y_2 - x_3), x_2, c\}^T$ and the parameter vector is $k^* = \{\sigma, \delta, \beta = 1\}$, and β is a dummy parameter variable.

The response of the system is shown in figure 4, where again the controller was initiated at time $t = 12$. The evolution of the adaptive gains $k = \{k_1, k_2, k_3\}^T$ which were all initiated at zero, is shown in figure 5 where a controller gain of $\gamma = 100$ in equation 12 was found sufficient to achieve fast adaption. We can see from figure 5 that during the first 8 seconds of adaption gains vary significantly. However, at time $t = 1000$, the adaptive gains are approximately constant with values close to $k = \{k_1 \approx 10, k_2 \approx 15, k_3 \approx 1\}^T$. Thus we see that $k_1 \to \sigma$, $k_2 \to \delta$ and $k_3 \to \beta$. Thus via the relation $u = k(t)\xi(t)$ we see that the steady state adaptive controller is the same as the feedback linearization controller, and that in effect we have identified the system parameters.

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

4 Multi-variable coupling

For multi-variable coupling between nonlinear systems controller design is more difficult. Here, we take the approach of analyzing the stability of the system in a partially decentralized form. This means that the linear error coordinates are decoupled, but nonlinear coupling exists.

Thus for a system of N error equations, the ith equation can be written in a similar form to equation 9 as

$$\dot{e}_i(t) = -\lambda_i e_i + \phi_i(t)\xi(t), \tag{23}$$

where $\phi_i(t)$ is an $(1 \times m)$ parameter error vector, and ξ is the $(m \times 1)$ coordinate coupling vector for all N error states. We choose a Lyapunov function of the form

$$V(t) = \sum_{i=1}^{N}(\frac{e_i^2}{2} + \frac{\phi_i\phi_i^T}{2\gamma_i}), \tag{24}$$

where γ_i is the controller gain. Then the derivative of V with respect to time is

$$\dot{V}(t) = \sum_{1}^{N}(e_i(-\lambda_i e_i + \phi_i(t)\xi(t)) + \frac{1}{\gamma_i}\phi_i\dot{\phi}_i^T), \tag{25}$$

such that choosing $\dot{\phi}_i^T = -\gamma_i e_i \xi$, results in $\dot{V} = \sum_1^N -\lambda_i e_i^2$ which implies that the controller is globally asymptotically stable. Therefore $\dot{\phi}_i^T = -\dot{k}^T = -\gamma_i e_i \xi$.

4.1 Example of multi-variable coupling

We now discuss an example of coupling more than a single variable again using the Chua and Lorenz systems as an example. In this example, two variables from each system are coupled simultaneously.

To demonstrate this, we select x_2 and y_1 as the first pair of variables, and x_3 and y_2 as the second pair of variables for coupling. This means that we now have two error variables, $e_1 = y_1 - x_2$ and $e_2 = y_2 - x_3$. Unidirectional coupling only is considered, such that $c = 0$. In this case the coupling functions are

$$f_{12} = \begin{pmatrix} x_1 - x_2 + x_3 \\ -\delta x_2 \end{pmatrix} \tag{26}$$

and

$$f_{21} = \begin{pmatrix} -\sigma(y_1 - y_2) \\ r y_1 - y_2 - y_1 y_3 \end{pmatrix}. \tag{27}$$

Therefore

$$\Delta f = f_{21} - f_{12} = \begin{pmatrix} -\sigma(y_1 - y_2) - x_1 + x_2 - x_3 - u_1 \\ ry_1 - y_2 - y_1 y_3 + \delta x_2 - u_2 \end{pmatrix}, \tag{28}$$

which can be expressed as

$$\Delta f = f_{21} - f_{12} = \begin{pmatrix} -\sigma e_1 + \sigma y_2 + (1-\sigma)x_2 - x_3 - x_1 - u_1 \\ -e_2 + ry_1 + \delta x_2 - x_3 - y_1 y_3 - u_2 \end{pmatrix}, \tag{29}$$

such that we can write

$$\begin{aligned} \dot{e}_1(t) &= -\sigma e_1 + L_1 - u_1, \\ \dot{e}_2(t) &= -e_2 + L_2 - u_2 \end{aligned} \tag{30}$$

Where the feedback linearization controllers are

$$\begin{aligned} L_1 &= \sigma y_2 + (1-\sigma)x_2 - x_3 - x_1 \\ L_2 &= ry_1 + \delta x_2 - x_3 - y_1 y_3 \end{aligned} \tag{31}$$

For adaptive feedback linearization we write each $L_i = k_i^* \xi$ such that in this case

$$\begin{aligned} k_1^* &= [0, \sigma, (1-\sigma), -1, -1, 0] \\ k_2^* &= [r, 0, \delta, -1, 0, -1] \end{aligned} \tag{32}$$

and $\xi = [y_1, y_2, x_2, x_3, x_1, y_1 y_2]^T$. Then finally we can express equation 30 in the required format of equation 23 by substituting $u_i = k_i(t)\xi$, giving

$$\begin{aligned} \dot{e}_1(t) &= -\sigma e_1 + \phi_1(t)\xi(t), \\ \dot{e}_2(t) &= -e_2 + \phi_2(t)\xi(t) \end{aligned} \tag{33}$$

Then for this system stabilizing controllers can be applied using the gain vectors given by $k_i^T = \gamma_i \int_{t=t_0}^{t} e_i \xi \mathrm{d}t$. The results of simulating this example are shown in figure 6. As with the previous examples, the control is started at time $t = 12$. From figure 6 (a) we see that x_2 become synchronized with y_1 very quickly after the control starts. However, the x_3, y_2 synchronization takes significantly longer; approximately 2 seconds. We also find that after 1000 seconds $k_1 \approx k_1^*$, but that k_2 has not completely converged to k_2^*. This behavior occurs because of our choice of the λ_i values in this example; equation 30. For the x_2, y_1 synchronization $\lambda_1 = \sigma = 10$, but for the x_3, y_2 synchronization $\lambda_1 = 1$. Considering the convergence when $L_1 - u_1 = 0$ and $L_2 - u_2 = 0$, $e_1 = \exp(-10t)$ while $e_2 = \exp(-t)$. Therefore the error convergence of e_1 will be greater than that of e_2.

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

5 Applications to modeling coupled dynamical systems

Many real life dynamical systems are composed of two or more coupled systems giving rise to highly complex dynamics. Partial synchronization techniques can be applied to modeling such systems in two main ways:

1. To model systems composed of a set of coupled nonlinear subsystems, where the structure of the individual component systems is known, but the nature of the coupling is unknown.

2. To model systems composed of a set of coupled nonlinear subsystems, where information from one (or more) of the subsystems is known only in the form of a recorded time series.

The first approach can be used to synchronize two variables from the subsystems to effect coupling, without having to have explicit knowledge of the form of the coupling itself. The second method has potential uses for systems where time series data is taken from an experimental source. For example in techniques which have a numerical and experimental component to the modeling (Oomens *et al.* 1993; Donea *et al.* 1996; Wagg & Stoten 2001) These two modeling methods can be approached using the coupling techniques described in section 2.

5.1 Example: modeling a system of two coupled nonlinear dynamical systems

Consider the problem of modeling the dynamics of a complex dynamical system governed by the state equation

$$\dot{z}(t) = f(z, t), \tag{34}$$

where we have only partial knowledge of the form of $f(z, t)$. We will consider the problem where $f(z, t)$ is composed of two parts, one for which the dynamics is known explicitly, f_2, and the other, f_1 where the dynamics can be divided into; a part where the structure is known, f_{12}, and a part where the dynamics are known only via time series measurements f_{11}. This is the situation in some numerical-experimental applications, where a physical system is acted upon by some experimental apparatus and this is coupled with a numerical model (Wagg & Stoten 2001). So in this case the coordinates of the apparatus would correspond to x_s, and the dynamics of x_n would be known only implicitly from experimental measurements of the physical system.

To create a model of $f(z, t)$ the coordinates x_s and y_s need to be synchronized such that $e \to 0$. Thus if partial synchronization can be achieved then equation 34 can be written (using equation

2) as

$$\dot{z}(t) = \left\{ \begin{array}{c} \dot{x}_n(t) \\ \dot{x}_s(t) \\ \dot{y}_n(t) \end{array} \right\} = \left\{ \begin{array}{c} f_{11}(x_n, x_s, t) \\ f_{12}(x_n, x_s, t) \\ f_{22}(y_n, y_s, t) \end{array} \right\} = f(z, t). \tag{35}$$

As before, this is achieved by using an adaptive control algorithm to ensure that f_{12} tracks f_{21} i.e. $f_{12} \to f_{21}$. Thus the coupled systems form a single combined model of the overall system. In this process we will effectively reconstruct the dynamics of the experimental system (Broomhead & King 1986; Maybhate & Amritkar 1999), while simulating the dynamics known explicitly, and thus reconstruct the overall dynamics of the system $f(z, t)$.

Let us consider the case where the dynamics of x_n, represented by f_{11} are unknown in a explicit form but are known implicitly from a set of experimental measurements in the form of time series $v(t)$ and $w(t)$ such that $x_n = h_1(v(t))$ and $\dot{x}_n = h_2(w(t))$. Here $h_j(\cdot)$ are correlation functions which provide a relationship between the experimental measurements and state variables. In this case equation 3, can be written as

$$\begin{aligned} \dot{x}_n(t) &= h_2(w(t)) \\ \dot{x}_s(t) &= f_{12}(h_1(v(t)), x_s, t) + g(u, t) \\ \dot{y}_s(t) &= f_{21}(y_n, y_s, t) \\ \dot{y}_n(t) &= f_{22}(y_n, y_s, t) \end{aligned} \tag{36}$$

This set of equations can be used to form a coupled model for the overall system, by substituting $h_1 = d_1$ and $y_n = d_2$, we obtain equation 5, and for $v(t)$, $w(t)$ bounded, the stability proof follows. In addition, when f_{12} and f_{21} are synchronized, equation 36 can be reduced to the form of equation 35 to provide a combined system model.

5.1.1 *Numerical-experimental example*

Wagg & Stoten (2001) consider a numerical-experimental example where f_{11} and f_{12} are physical systems but f_{12} is (approximately) linear. A force signal, $F(t)$, between f_{11} and f_{12} is recorded experimentally which represents the coupling between the two functions. For this system equation 36 can be written in the form

$$\begin{aligned} \dot{x}_n(t) &= f_{11} \\ \dot{x}_s(t) &= Ax_s + Bu \\ \dot{y}_s(t) &= A_m y_s + B_m y_n + C_m F(t) \\ \dot{y}_n(t) &= A_z y_n + B_z y_s + C_z r(t) \end{aligned} \tag{37}$$

where $A, B, A_m, B_m, C_m, A_z, B_z$ and C_z are constant matrices and f_{11} is an unknown nonlinear function. In this example only part of the system is nonlinear, but the development of the combined model for the system uses a similar approach to that described here for nonlinear systems. We note also that this system has bidirectional coupling from the application of $F(t)$ and is non-autonomous via the forcing signal $r(t)$. Further details of this can be found in Wagg & Stoten (2001).

5.2 How effective is this modeling process?

In order to measure the degree of synchronization between the coordinates x_s and y_s, we monitor the error vector $e = y_s - x_s$. For effective modeling we require that the synchronization occurs within a certain time limitation $e \to \epsilon$ as $t \to t_s$, where ϵ is small. The effectiveness of the synchronization process can be viewed geometrically by considering the phase space for the coupled system $E = \{(x, y) \in \mathcal{R}^{p\times q}\}$. Then $\Sigma = \{(x, y) \in \mathcal{R}^{p\times q} : e = 0\}$ represents the synchronized subspace (Ashwin 1998) for the coupled system. The dynamics which are restricted to the manifold Σ correspond to that of the coupled system, equation 34. Furthermore, out of subspace dynamics correspond to failure in the synchronization (control) process. Thus we can use the synchronization subspace to monitor the performance of the controller, and hence the effectiveness or accuracy of the modeling process.

If we define the phase space of the overall system we are trying to model as $G = \{z \in \mathcal{R}^k\}$, then the combined model is a close approximation of the overall system if $\dim \Sigma \approx \dim G$. In other words the combined system, f_1 and f_2, has higher dimensional dynamics than the modeled system, $f(z, t)$, but by synchronizing the required set of coordinates we reduce the dynamics to the subspace Σ which is an approximation of the overall dynamics in G. Thus we can qualitatively identify the dynamics of the overall system by examining the dynamics in the hypersurface Σ.

5.2.1 Chua-Lorenz example

This can be demonstrated using the example from section 3.1. Let $E = \{(x_3, y_1, y_3) \in \mathbb{R}^3\}$ be a subset of the complete phase space, which we can use as a visualization aid. The evolution of the feedback coupled system in this space is shown in figure 7. In this example feedback linearization control was used from time $t = 0$, and the figure shows trajectories computed from $t = 40$ to $t = 50$. Lorenz-like dynamics can be observed, however these dynamics are in fact restricted to the synchronization subspace, which can be seen from viewing the x_3, y_1 plane; figure 8. Qualitatively, we observe no out of subspace dynamics, indicating a high level of accuracy in the coupling and hence modeling process, which can be expected from feedback linearization control. Similar results can be obtained using adaptive feedback linearization, however in this case some out of subspace dynamics will occur during the transient adaption phase when $t > t_s$.

6 Conclusions

We have considered how non-identical nonlinear dynamical systems can be coupled using partial synchronization with the inclusion of additional feedback coupling. For applications where two different dynamical system require coupling, a partial synchronization method can be used where one part of the system is included using only a recorded time series. We have demonstrated how both unidirectional and bidirectional coupling can be simulated in such a modeling process using a feedback linearization controller. We have also demonstrated using the example of a Chua system coupled with a Lorenz system, how an adaptive feedback linearization controller can be used to effect such coupling. The use of an adaptive controller is significant, in that it can be used to couple systems without explicit knowledge of the plant parameters, although a knowledge of the structure of the plant is required. In the steady state, we observed that the adaptive controller converged to the exact formulation of the feedback linearization controller. Finally we have discussed how the coupling techniques can be applied to modeling numerical-experimental and other coupled systems.

International Journal of Bifurcation and Chaos, Vol. 12, No. 3 (2002) 561–570

References

Ashwin, P. (1998). Non-linear dynamics, loss of synchronization and symmetry breaking. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* **212**(3), 183–187.

Ashwin, Peter, Buescu, Jorge & Stewart, Ian (1994). Bubbling of attractors and synchronisation of chaotic oscillators. *Physics Letters A* **193**, 126–139.

Boccaletti, S., Farini, A. & Arecchi, F. T. (1997). Adaptive synchronization of chaos for secure communication. *Physical Review E* **55**(5), 4979–4981.

Broomhead, D. S. & King, G. P. (1986). Extracting qualitative dynamics from experimental data. *Physica D* **20**(2-3), 217–236.

Dedieu, H. & Ogorzalek, M. J. (1997). Identifiability and identification of chaotic systems based on adaptive synchronization. *IEE Transactions on Circuits and Systems* **44**(10), 948–962.

Di Benardo, M. (1996). An adaptive approach to the control and synchronization of continuous-time chaotic systems. *International Journal of Bifurcation and Chaos* **6**(3), 557–568.

Donea, J., Magonette, P., Negro, P., Pegon, P., Pinto, A. & Verzeletti, G. (1996). Pseudodynamic capabilities of the ELSA laboratory for earthquake testing of large structures. *Earthquake Spectra* **12**(1), 163–180.

Fradkov, A. L. & Markov, A. Y. (1997). Adaptive Synchronization of chaotic systems based on speed gradient method and passification. *IEEE Transactions on Circuts and Systems* **44**(10), 905–912.

Hasler, M. (1998). Simple example of partial synchronization of chaotic systems. *Physical Review E* **58**(5), 6843–6846.

John, J. K. & Amritkar, R. E. (1994). Synchronization of unstable orbits using adaptive control. *Physical Review E* **49**(6), 4843–4848.

Kozlov, A. K. & Shalfeev, V. D. (1996). Exact synchronization of mismatched chaotic systems. *International Journal of Bifurcation and Chaos* **6**(3), 569–580.

Landau, Y. D. (1979). *Adaptive control:The model reference approach.* Marcel Dekker:New York.

Maybhate, A. & Amritkar, R. E. (1999). Use of synchronization and adaptive control in parameter estimation from a time series. *Physical Review E* **59**(1), 284–293.

Ohayon, R., Sampaio, R. & Soize, C. (1997). Dynamic substructuring of damped structures using singular value decomposition. *Transactions of the ASME* **64**, 292–298.

Oomens, C. W. J., Ratingen, M. R., Janssen, J. D., Kok, J. J. & Hendriks, M. A. N. (1993). A numerical-experimental method for a mechanical characterization of biological materials. *Journal of Biomechanics* **26**(4/5), 617–621.

Pecora, Louis M. & Carroll, Thomas L. (1990). Synchronization in chaotic systems. *Physical Review Letters* **64**(8), 821–824.

Sastry, S. (1999). *Nonlinear systems:Analysis, stability and control.* Springer-Verlag:New York.

Sastry, S. & Bodson, M. (1989). *Adaptive control:Stability, convergence and robustness.* Prentice-Hall:New Jersey.

Stoten, D. P. & Di Bernardo, M. (1996). Application of the minimal control synthesis algorithm to the control and synchronization of chaotic systems. *International Journal of Control* **65**(6), 925–938.

Wagg, D. J. & Stoten, D. P. (2001). Substructuring of dynamical systems via the adaptive minimal control synthesis algorithm. *Earthquake Engineering and Structural Dynamics* **30**, 865–877.

Yanchuk, S., Maistrenko, Y. & Mosekilde, E. (2001). Partial Synchronization and clustering in a system of diffusively coupled chaotic oscillators. *Mathematics and Computers in Simulation* **54**, 491–508.

Yang, S. S. & Duan, C. K. (1998). Generalized synchronization in chaotic systems. *Chaos, Solitions and Fractals* **9**(10), 1703–1707.

Figure Captions

- Figure 1. Schematic representation of a system formed by partial synchronization.
- Figure 2. Output from master slave system. Solid line x_3, dashed line y_1. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$. Control started at $t = 12$ seconds.
- Figure 3. Output from system with bidirectional coupling $c = x_3$. Solid line x_3, dashed line y_1, chained dotted line y_1 for the unidirectional $c = 0$ case: Note before $t = 12$ the dashed and chain dotted lines are identical. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$. Control started at $t = 12$ seconds.
- Figure 4. Output from adaptive control system. Solid line x_3, dashed line y_1. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$. Control gain $\gamma = 100$. Control started at $t = 12$ seconds.
- Figure 5. Output from adaptive control system. Solid line x_3, dashed line y_1. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$. Control gain $\gamma = 100$. Control started at $t = 12$ seconds.
- Figure 6. Multivariable coupling (synchronization) for Chua-Lorenz system using feedback linearization. (a) x_2 solid and y_1 dashed, (b) x_3 solid and y_2 dashed. Control started at $t = 12$ seconds.
- Figure 7. Output from feedback coupled system plotted in the space E. Data from $t = 40$ to $t = 50$ shown. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$
- Figure 8. Synchronization subspace from feedback coupled system $g = x_3$. Data from $t = 40$ to $t = 50$ shown. Initial conditions $x_1(0) = -1.1$, $x_2(0) = -1.0$, $x_3(0) = 1.1$, $y_1(0) = 1.1$, $y_2(0) = 1.0$ and $y_3(0) = 7.0$

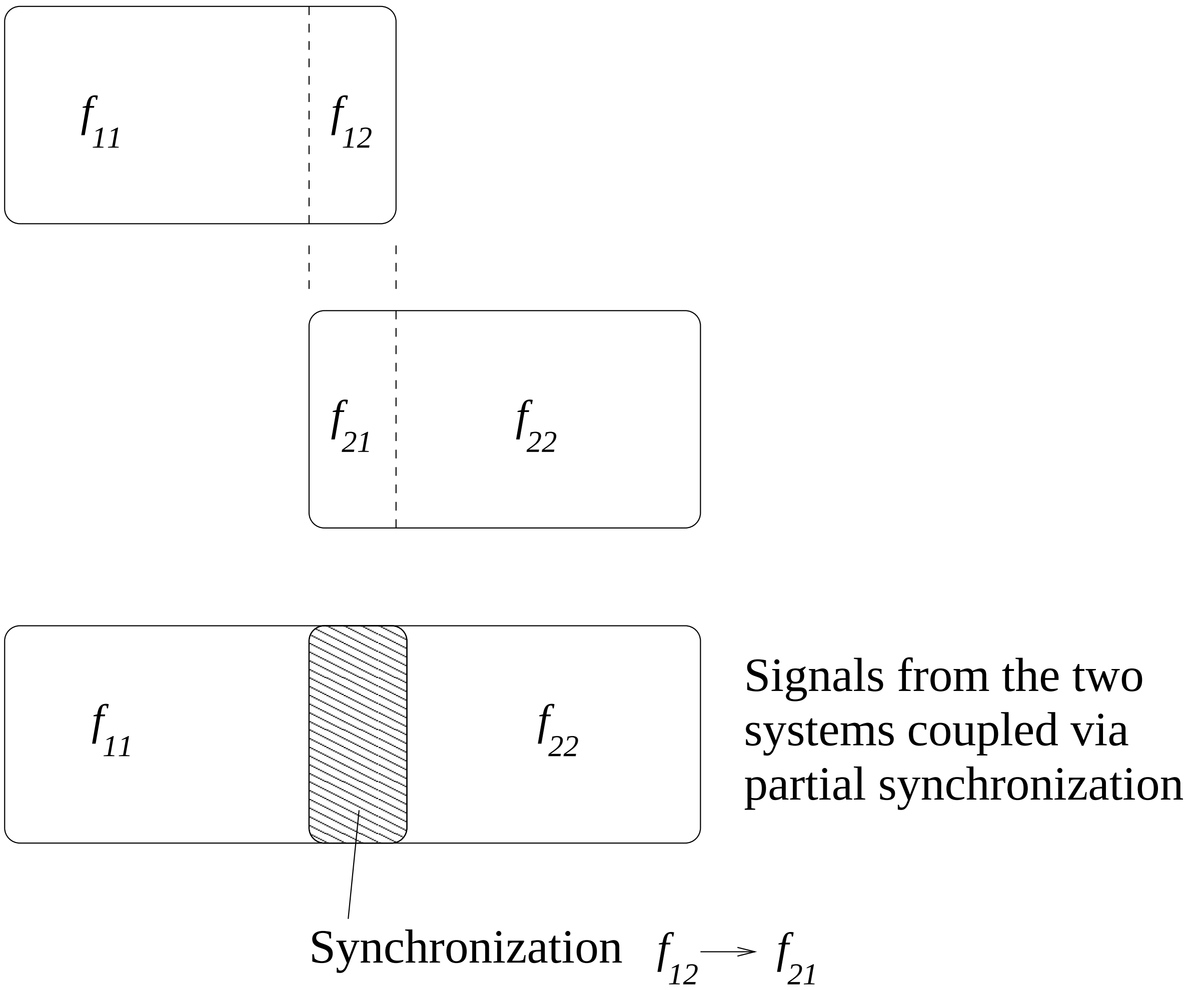

f_{11}
f_{12}
f_{21}
f_{22}
f_{11}
f_{22}
Signals from the two systems coupled via partial synchronization
Synchronization $f_{12} \longrightarrow f_{21}$

Figure 1:

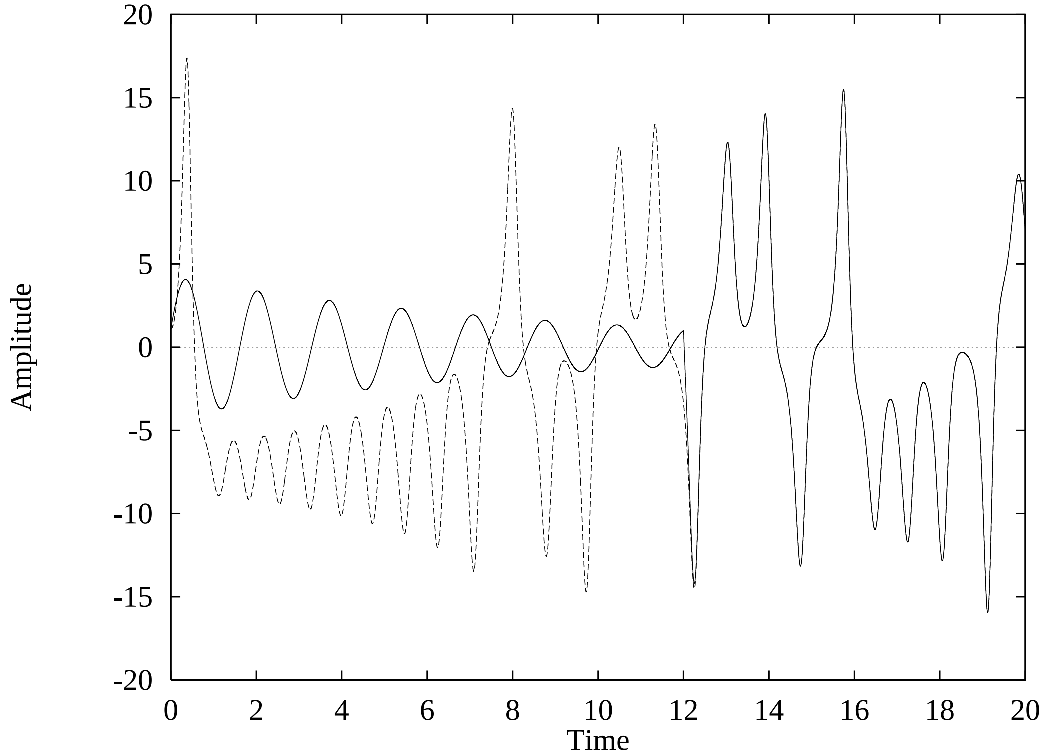


Figure 2:

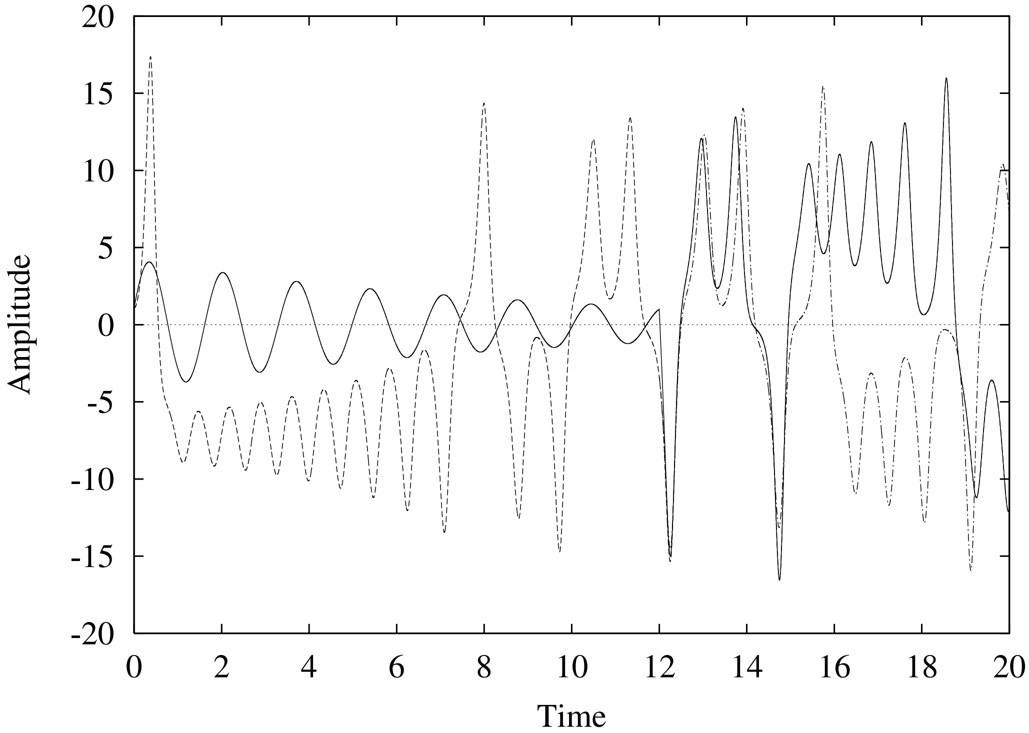


Figure 3:

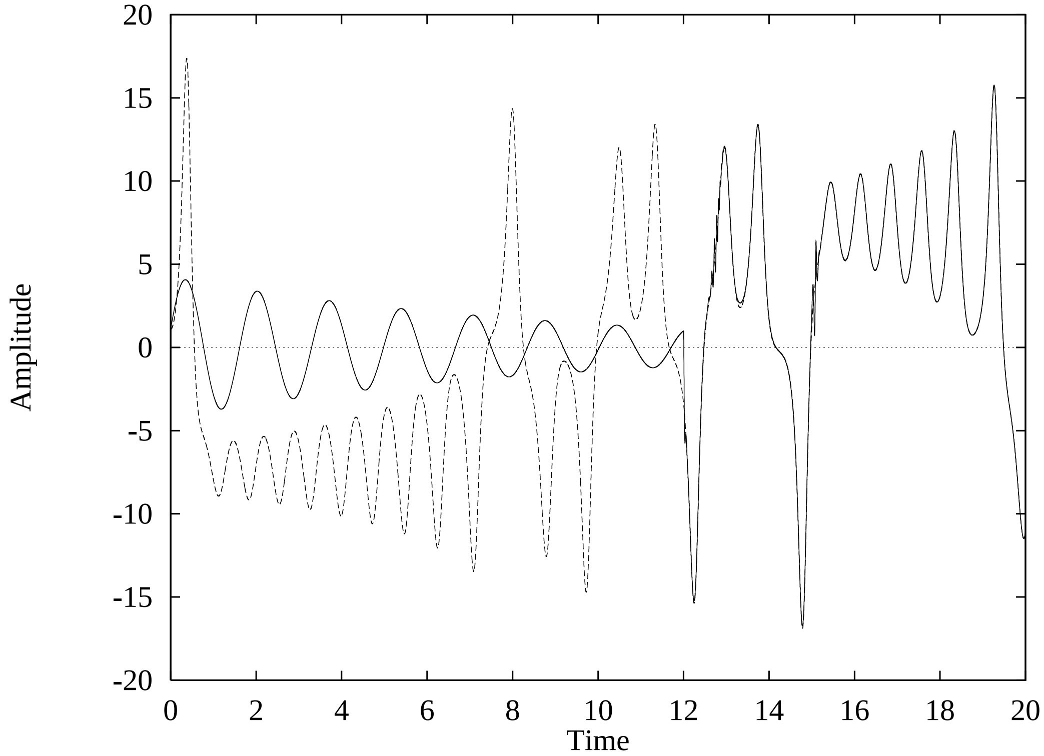


Figure 4:

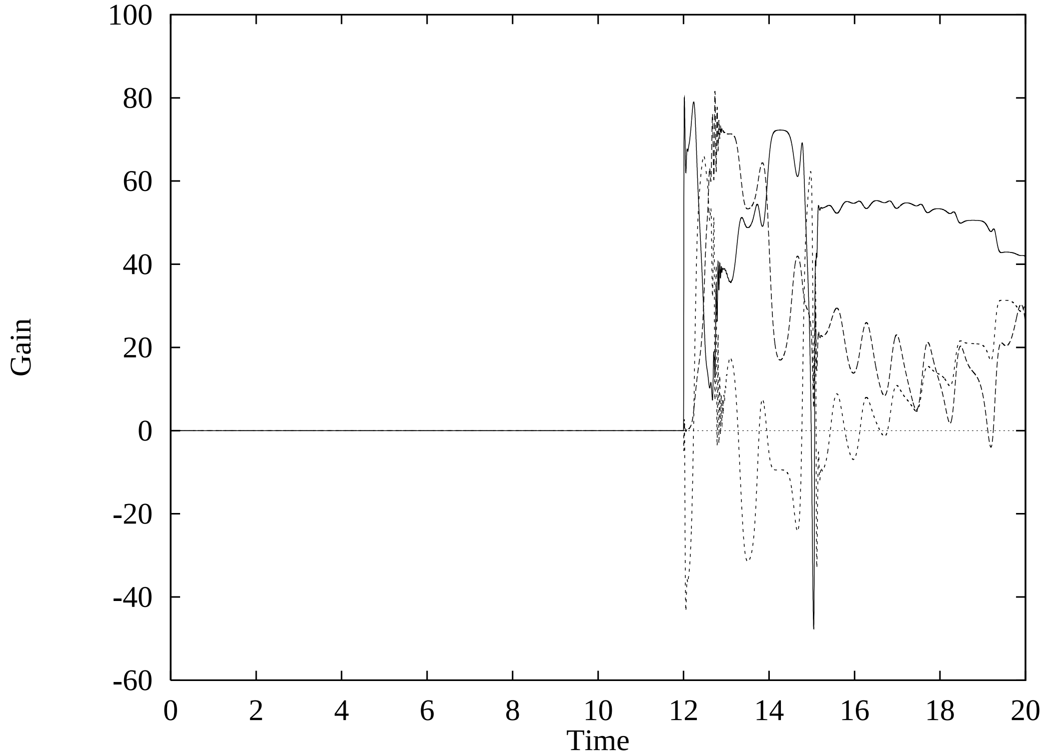
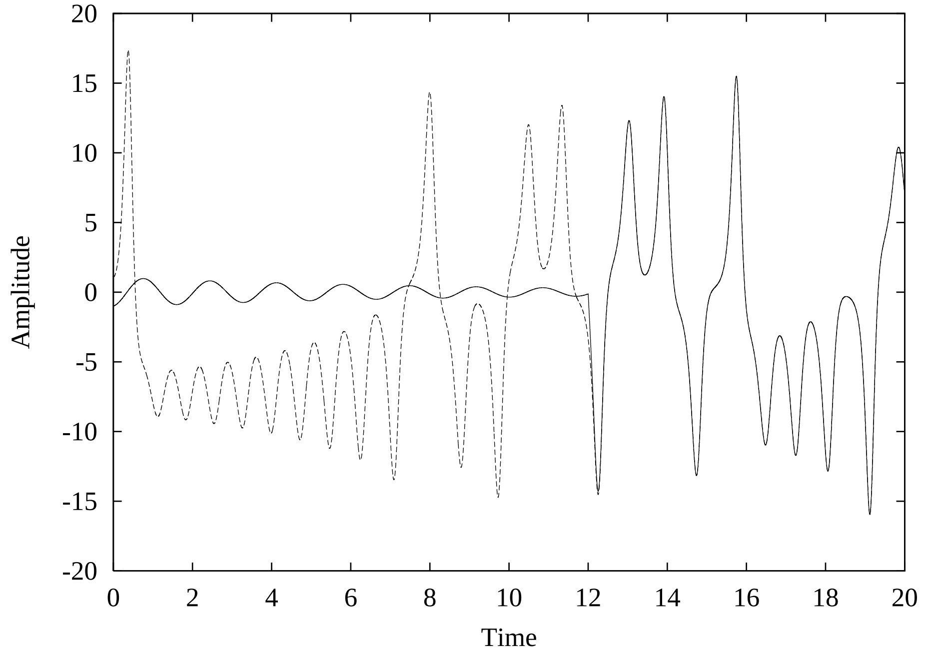
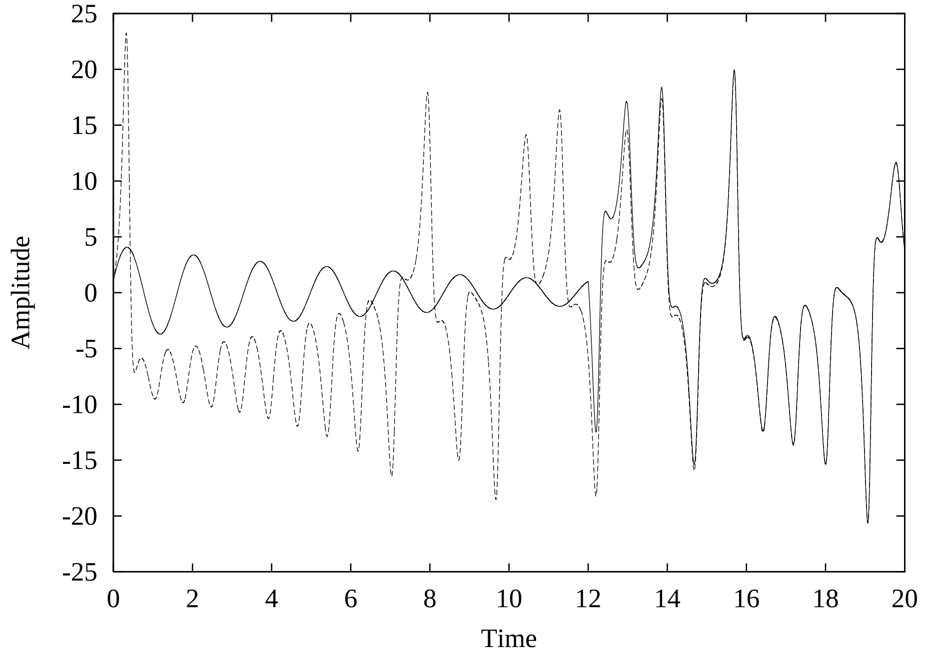


Figure 5:


Amplitude
Time
20
15
10
5
0
-5
-10
-15
-20
0
2
4
6
8
10
12
14
16
18
20

(a)


Amplitude
Time
25
20
15
10
5
0
-5
-10
-15
-20
-25
0
2
4
6
8
10
12
14
16
18
20

(b)

Figure 6:

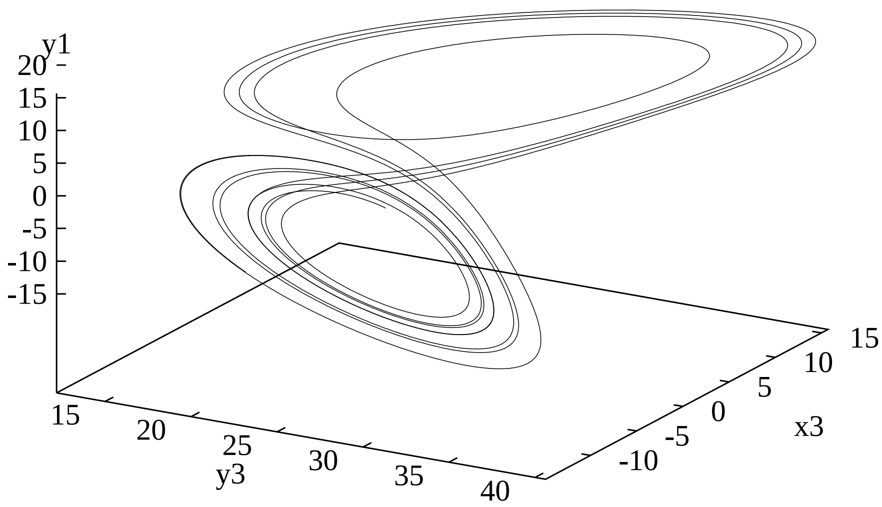


Figure 7:

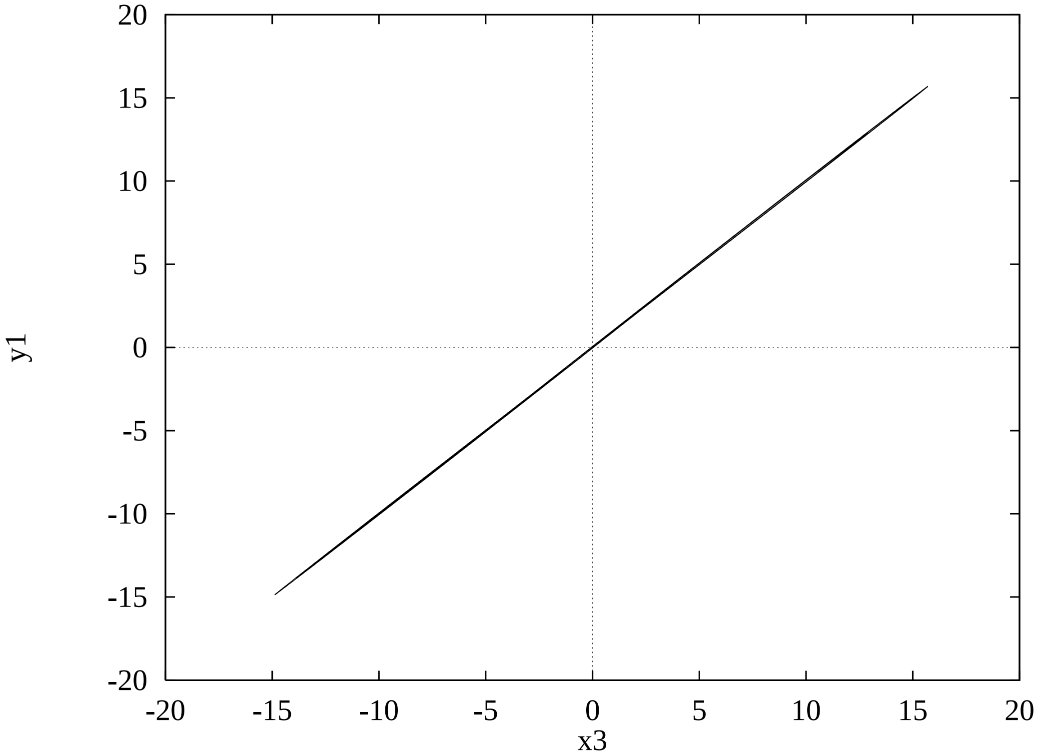


Figure 8: