Monograph:

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NOISE ANALYSIS IN TRACKING THE FEATURES OF A SPATIALLY DISTRIBUTED FIELD

by

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AC & SE Research Report No. 501

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3. Robustness of the Single Scan Algorithm

This section examines the robustness of the single scan algorithm, when the observer of the physical quantity to be tracked is imperfect. Consider a field generated by an nth order polynomial bearing coefficients $a_1, a_2, \ldots, a_n$. Then, given n samples, H_i, $i=1,2\ldots,n$, captured in a particular scan and spaced at interval Δy, the single scan algorithm determines the position $y(\overline{\mathrm{H}})$, of first sample H_1, using the relation [1],

$$\frac{\Delta^{n-1}H}{\Delta y^{n-1}} = D^{n-1}H + \frac{(n-1)}{2}D^{n}H\Delta y \tag{1}$$

Where

$$D^{n-1}H = (n-1)!\,a_{n-1} + n!\,a_n y(\overline{\mathrm{H}}) \tag{2}$$

and

$$D^{n}H = n!\,a_n, \tag{3}$$

Now using the above equations y can be written as

$$y(\overline{\mathrm{H}}) = \frac{1}{n!\;a_n}\;\frac{\Delta^{n-1}H}{\Delta y^{n-1}} - \frac{a_{n-1}}{na_n} - \frac{n-1}{2}\Delta y \tag{4}$$

Here the argument $\overline{\mathrm{H}}$ signifies that the value of y depends upon the vector of the sampled values. Apparently in the above equation, only $\Delta^{n-1}H$ relies on the sampled values and thus will be effected by the noise. It has been found that $n-1$ th order difference for H_i, $i=1,2,..,n$ follows the closed form relation.

$$\Delta^{n-1}H = \sum_{i=1}^{n}(-1)^{n+i}H_i\overset{n-1}{\underset{i-1}{C}}\ {}^{1} \tag{5}$$

Now let the observation process is corrupted by a Gaussian white noise sequence ω_i, $i=1,\ldots,n$ to give noisy observations H'_i such that

$$H'_i = H_i + \omega_i \quad \text{for} \quad i = 1,\ldots,n \tag{6}$$

Hence, $n-1$ th difference for n noisy samples of H' is written as

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[1] Here and throughout this report $\overset{p}{\underset{q}{C}} = \frac{p!}{(p-q)!\,q!}$

$$\Delta^{n-1}H' = \sum_{i=1}^{n}(-1)^{n+i}(H_i+\omega_i)\binom{n-1}{i-1} = \sum_{i=1}^{n}(-1)^{n+i}H_i\binom{n-1}{i-1} + \sum_{i=1}^{n}(-1)^{n+i}\omega_i\binom{n-1}{i-1}$$
$$= \Delta^{n-1}H + e(\Delta^{n-1}H) \qquad (7)$$

Here $\Delta^{n-1}H$ is the exact *n-1* th difference and $e(\Delta^{n-1}H)$ is the error introduced due to imperfection of the observation process. Thus

$$e(\Delta^{n-1}H) = \sum_{i=1}^{n}(-1)^{n+i}\omega_i\binom{n-1}{i-1} \qquad (8)$$

In order to analyse the distribution of this error when ω_i is normally distributed zero mean white noise sequence with variance σ^2, we use the expectation of sum rule [2, p. 283]

$$E\{\sum_{i=1}^{m}a_i x_i\} = \sum_{i=1}^{m}a_i E\{x_i\} \qquad (9)$$

to find the expectation of $e(\Delta^{n-1}H)$ as

$$E\{e(\Delta^{n-1}H)\} = E\{\sum_{i=1}^{n}(-1)^{n+i}\omega_i\binom{n-1}{i-1}\}$$
$$= \sum_{i=1}^{n}(-1)^{n+i}\binom{n-1}{i-1}E\{\omega_i\} = 0 \qquad (10)$$

Thus if the noise introduced in the observation process is zero mean, the error included in *n-1* th difference will also be zero mean. Furthermore, being a linear combination of the normally distributed white noise, $e(\Delta^{n-1}H)$ will also be normal. Assuming that the noise is uncorrelated, the variance of the error $e(\Delta^{n-1}H)$ can be calculated using *Bienaymé* Equality [2, p. 288]:

$$Var\{\sum_{i=1}^{m}a_i x_i\} = \sum_{i=1}^{m}a_i^2 Var\{x_i\} \qquad (11)$$

Therefore the variance of the error $e(\Delta^{n-1}H)$ is

$$Var\{e(\Delta^{n-1}H)\} = Var\{\sum_{i=1}^{n}(-1)^{n+i}\omega_i\binom{n-1}{i-1}\}$$
$$= \sum_{i=1}^{n}\left[(-1)^{n+i}\binom{n-1}{i-1}\right]^2 Var\{\omega_i\} = \sum_{i=1}^{n}\left[\binom{n-1}{i-1}\right]^2\sigma^2 \qquad (12)$$

That is, if the observation process is corrupted by a Gaussian white noise $N(0,\sigma)$, the error introduced in *n-1* th difference will follow a Gaussian distribution $N(0,\sigma_e)$ such that

$$\sigma_e^2 = \sum_{i=1}^{n} \left[\overset{n-1}{\underset{i-1}{C}} \right]^2 \sigma^2$$

(13)

Since the quantity $\sum_{i=1}^{n} \left[\overset{n-1}{\underset{i-1}{C}} \right]^2$, called the *error coefficient*, is greater than unity for all values of n>1, the variance of the error in the finite difference is always greater than the variance of noise introduced in the sampled values. In fact, the variance is amplified many times. Fig. 1 shows how does the error coefficient increase with *n*. It is obvious to note that lower the order of polynomial to be tracked, the smaller will be the variance of errors introduced in *n-1* th finite difference.

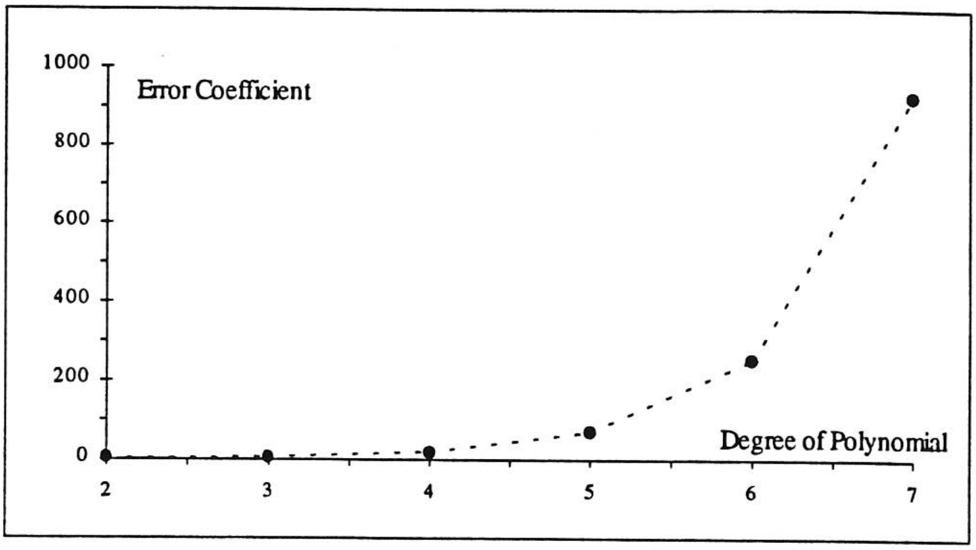

Fig. 1. Increase in Error with polynomial order

We will now see effect of this error in determination of the initial scanning position $y(\overline{H})$. Using equations (9) and (11), the mean and the variance of $y(\overline{H})$ may be written as

$$E\{y(\overline{H})\} = y(\overline{H}) \tag{14}$$

and

$$Var\{y(\overline{H})\} = \frac{\sum_{i=1}^{n} \left[\overset{n-1}{\underset{i-1}{C}} \right]^2 \sigma^2}{(\Delta y^{n-1} n! a_n)^2} \tag{15}$$

Hence the variance of $y(\overline{H})$ depends upon σ, n, Δy and a_n only (and not on $a_1, a_2, \ldots, a_{n-1}$). It is noteworthy that increasing the sampling interval will result in a smaller variance. Moreover, apparently with larger n, the variance will be smaller provided the other quantities stay constant. A useful measure of the robustness is *noise amplification factor* defined by

$$\phi = Noise\ Amplification\ Factor = \sqrt{\frac{Var\{Output\ Noise\}}{Var\{Input\ Noise\}}} \tag{16}$$

In this case ϕ can be written using equation (15) as

$$\phi = \frac{\sqrt{\sum_{i=1}^{n}\left[C_{i-1}^{n-1}\right]^2}}{\Delta y^{n-1} n! a_n} \tag{17}$$

3.1 An Example of the Single Scan Polynomial Tracking (Δy=1)

As an example of the effect of the noise on the single scan tracking consider the polynomial

$$H(y) = a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 \tag{18}$$

$$\text{where} \quad \begin{aligned} a_1 &= -2.624 \\ a_2 &= 2.8790 \\ a_3 &= -0.605 \\ a_4 &= 0.0357 \end{aligned} \tag{19}$$

A plot of this curve is shown in Fig. 2.

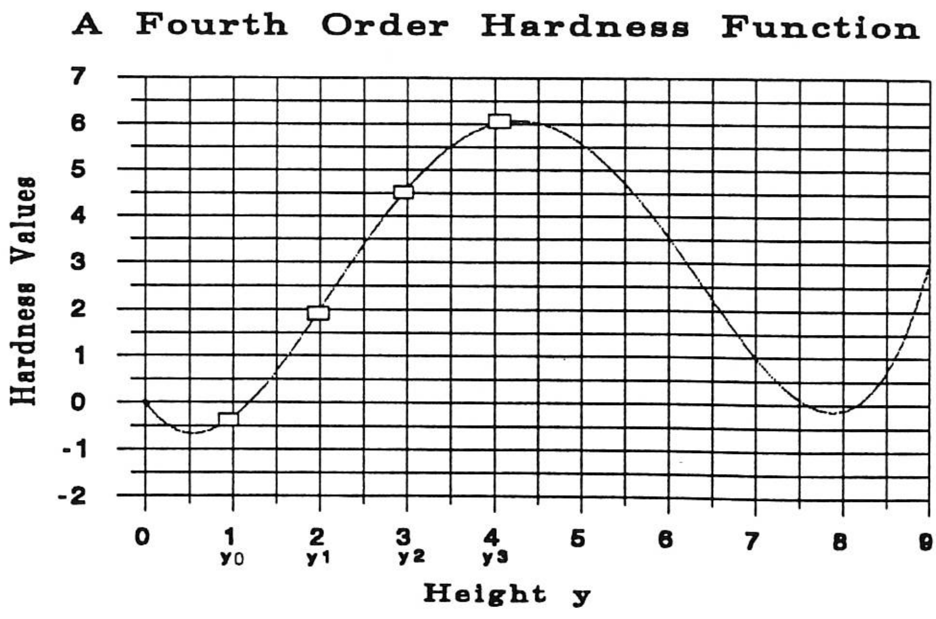

Fig. 2. Hardness Function Sampled at 4 Points

Now sampling this hardness function at points y_0, y_1, y_2 and y_3 with Δy=1 four hardness values are obtained as

$$\begin{aligned} H(y_0) &= -0.3143 \\ H(y_1) &= 2.0000 \\ H(y_2) &= 4.600 \\ H(y_3) &= 6.000 \end{aligned} \tag{20}$$

Rewriting equation (4)

$$y(\overline{H}) = \frac{1}{n!\, a_n} \frac{\Delta^{n-1} H}{\Delta y^{n-1}} - \frac{a_{n-1}}{n a_n} - \frac{n-1}{2} \Delta y \tag{21}$$

For $n=4$ and $\Delta y=1$ this becomes

$$y_0 = \frac{41}{15} + \frac{7}{6} \Delta^{n-1} H \tag{22}$$

For the deterministic case we obtain n-1 th difference using equation (5) as

$$\Delta^{n-1} H = -1.4857. \tag{23}$$

Hence

y_0=1.00 (as expected).

Now consider that the unit normal white noise $N(0,1)$ is introduced in the sampling process, then using equation (22) we can write,

$$\begin{aligned} y_0 &= \frac{41}{15} + \frac{7}{6} (\Delta^{n-1} H + e(\Delta^{n-1} H)) \\ &= 1 + \frac{7}{6} e(\Delta^{n-1} H) \end{aligned} \tag{24}$$

We are interested to determine what distribution does y_0 follow while the noise is included in the observation process of H. Calculating statistics of y_0

$$\begin{aligned} E\{y_0\} &= E\{1\} + \left(\frac{7}{6}\right) E\{e(\Delta^{n-1} H)\} \\ &= 1 \end{aligned} \tag{25}$$

and

$$Var\{y_0\} = Var\{1\} + \left(\frac{7}{6}\right)^2 Var\{e(\Delta^{n-1}H)\}$$
$$= \frac{49}{36} Var\{e(\Delta^{n-1}H)\}$$

where $Var\{e(\Delta^{n-1}H)\}$ can be calculated using equation (12). Hence

$$\begin{aligned} Var\{y_0\} &= \frac{49}{36} Var\{e(\Delta^{n-1}H)\} \\ &= \frac{49}{36}\sum_{i=1}^{4}\left[\overset{3}{\underset{i-1}{C}}\right]^2 \\ &= 27.222 \end{aligned} \tag{26}$$

Hence the standard deviation of y_0 is equal to 5.217. This has been verified experimentally by generating 3000 realisations of y_0 while sampled values of equation (20) are contaminated by zero mean unit normal white noise. Statistics of y_0 are found to be

Minimum $[y_0]$ = -16.1408
Maximum$[y_0]$ = 17.6123
Mean$\{y_0\}$ = 1.1774
Variance$\{y_0\}$ = 26.11152
Std. Dev.$\{y_0\}$= 5.11

These statistics conform to what was determined in equations (25) and (26). Here ϕ, the noise amplification factor is about 5.11. A histogram for y_0 shown in Fig. 2 illustrates the distribution of y_0 that appears normal as expected.

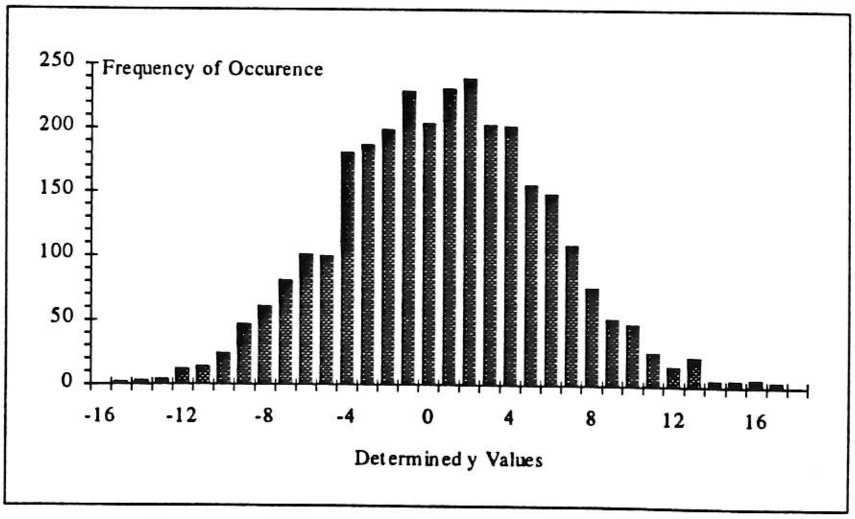

Fig 3. Histogram for positional estimate y_o (sampling interval=1)

3.2 Another Example of the Single Scan Polynomial Tracking (Δy=5)

Apparently, the disappointing results of Example 3.1 above might indicate little practical utility of the single scan algorithm. However, as we will see here, increasing the sampling interval would lead to much better consummation. While using the same polynomial field of Example 3.1 we increase sampling interval to 5. Statistics of three thousand realisations are now as below.

Minimum $[y_0]$ = 0.863
Maximum$[y_0]$ = 1.133
Mean$\{y_0\}$ = 1.002
Variance$\{y_0\}$ = 0.00167
Std. Dev.$\{y_0\}$ = 0.041

Results now are quite encouraging and ϕ, the noise amplification factor is reduced to about 0.041 compared to 5.11 with $\Delta y=1$. Histogram below shows spread of the determined position.

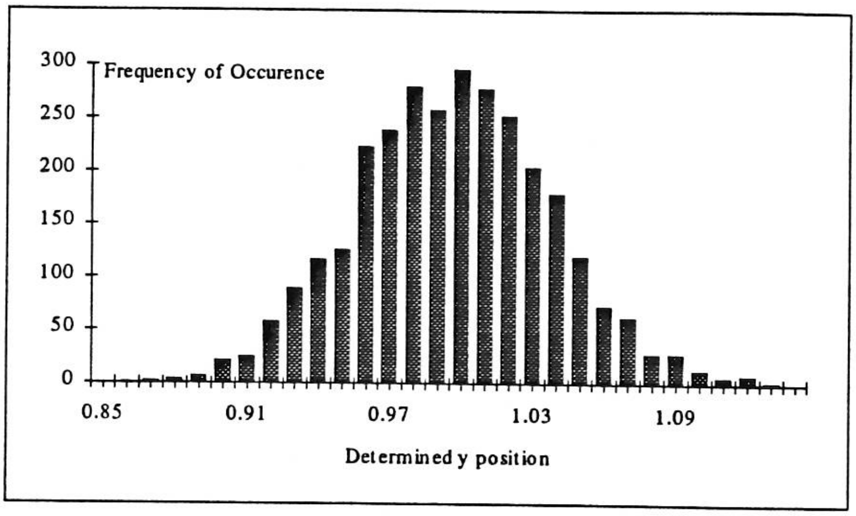

Fig 4. Histogram for positional estimate y_o (sampling interval=5)

4. Robustness of the Multiscan Algorithm

For a given polynomial field of order $n=m+r$, the multiscan algorithm uses $(r+1)(m+1)$ sampled values taken in $r+1$ scans ($m+1$ samples per scan) to determine the shift per scan normal to the field. In this case machine shift, δy, per scan is given by [1]

$$\delta y(\overline{\mathrm{H}}) = \frac{D^{n-1}H / D^{n}H + m\Delta y / 2}{\delta^{r-1}(\Delta^{m}H) / \delta^{r}(\Delta^{m}H) - (r-1)/2} \tag{27}$$

Where $\delta^r(\Delta^m H)$ and $\delta^{r-1}(\Delta^m H)$ are the double differences as outlined by equations (29). Using equations (2) and (3) again, δy can be written as

$$\delta y(\overline{\mathrm{H}}) = \frac{a_{n-1} / n\, a_n + m\Delta y / 2 + y}{\delta^{r-1}(\Delta^m H) / \delta^r(\Delta^m H) - (r-1)/2} \tag{28}$$

Here, quantities affected by noise in sampled values are $\delta^r(\Delta^m H)$ and $\delta^{r-1}(\Delta^m H)$ and can be written in the closed form as,

$$\begin{aligned} \delta^{r-1}(\Delta^m H) &= \sum_{j=1}^{r} \sum_{i=1}^{m+1} (-1)^{m+r+i+j+1} H_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r-1}{C}} \\ \delta^{r}(\Delta^m H) &= \sum_{j=1}^{r+1} \sum_{i=1}^{m+1} (-1)^{m+r+i+j} H_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r}{C}} \end{aligned} \tag{29}$$

Now let the observation process $H_i(j)$, $i=1,...,m+1;\ j=1,...,r+1$ is corrupted by a Gaussian white noise sequence $\omega_i(j)$, $i=1,...,m+1;\ j=1,...,r+1$ to give noisy observations H'_i such that

$$H'_i(j) = H_i(j) + \omega_i(j) \quad \text{for} \quad i = 1,...,m+1; \quad j = 1,...,r+1 \tag{30}$$

Hence, for the noisy samples

$$\begin{aligned} \delta^{r-1}(\Delta^m H') &= \sum_{j=1}^{r} \sum_{i=1}^{m+1} (-1)^{m+r+i+j+1} \left(H_i(j) + \omega_i(j)\right) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r-1}{C}} \\ &= \sum_{j=1}^{r} \sum_{i=1}^{m+1} (-1)^{m+r+i+j+1} H_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r-1}{C}} + \sum_{j=1}^{r} \sum_{i=1}^{m+1} (-1)^{m+r+i+j+1} \omega_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r-1}{C}} \\ &= \delta^{r-1}(\Delta^m H) + e(\delta^{r-1}(\Delta^m H)) \end{aligned}$$

Similarly

$$\begin{aligned} \delta^{r}(\Delta^m H') &= \sum_{j=1}^{r+1} \sum_{i=1}^{m+1} (-1)^{m+r+i+j} \left(H_i(j) + \omega_i(j)\right) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r}{C}} \\ &= \sum_{j=1}^{r+1} \sum_{i=1}^{m+1} (-1)^{m+r+i+j} H_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r}{C}} + \sum_{j=1}^{r+1} \sum_{i=1}^{m+1} (-1)^{m+r+i+j} \omega_i(j) \underset{i-1}{\overset{m}{C}} \underset{j-1}{\overset{r}{C}} \\ &= \delta^{r}(\Delta^m H) + e(\delta^{r}(\Delta^m H)) \end{aligned}$$

Where $e(\delta^r(\Delta^m H))$ and $e(\delta^{r-1}(\Delta^m H))$ are the errors introduced due to imperfection of the observation process. Now

$$e(\delta^{r-1}(\Delta^m H)) = \sum_{j=1}^{r}\sum_{i=1}^{m+1}(-1)^{m+r+i+j+1}\omega_i(j)\,\overset{m}{\underset{i-1}{C}}\,\overset{r-1}{\underset{j-1}{C}}$$
$$e(\delta^{r}(\Delta^m H)) = \sum_{j=1}^{r+1}\sum_{i=1}^{m+1}(-1)^{m+r+i+j}\omega_i(j)\,\overset{m}{\underset{i-1}{C}}\,\overset{r}{\underset{j-1}{C}} \tag{31}$$

Employing similar approach as for the single scan case while considering that $\omega_i(j)$ is uncorrelated and with Gaussian distribution $N(0,\sigma)$, the errors $e(\delta^r(\Delta^m H))$ and $e(\delta^{r-1}(\Delta^m H))$ will follow Gaussian Distribution with $N(0,\sigma_{e(\delta^r(\Delta^m H))})$ and $N(0,\sigma_{e(\delta^{r-1}(\Delta^m H))})$ respectively where

$$\sigma^2_{e(\delta^{r-1}(\Delta^m H))} = \sum_{j=1}^{r}\sum_{i=1}^{m+1}\left[\overset{m}{\underset{i-1}{C}}\,\overset{r-1}{\underset{j-1}{C}}\right]^2\sigma^2$$
$$\sigma^2_{e(\delta^{r}(\Delta^m H))} = \sum_{j=1}^{r+1}\sum_{i=1}^{m+1}\left[\overset{m}{\underset{i-1}{C}}\,\overset{r}{\underset{j-1}{C}}\right]^2\sigma^2 \tag{32}$$

The above equations show the variance of the errors introduced in $\delta^r(\Delta^m H)$ and $\delta^{r-1}(\Delta^m H)$ due to the imperfection of the measurement process. Obviously, the error variances are much amplified. For example, for tracking a polynomial of order 5 in 4 scans with 3 samples per scan, if a Gaussian noise $N(0,1)$ is present in the observation process, errors in $\delta^r(\Delta^m H)$ and $\delta^{r-1}(\Delta^m H)$ will be normal with $N(0,10.95)$ and $N(0,6)$ respectively. Now the ratio of rth and $(r-1)$th differences is

$$\zeta_\delta' = \frac{\delta^{r-1}(\Delta^m H')}{\delta^{r}(\Delta^m H')} = \frac{\delta^{r-1}(\Delta^m H) + e(\delta^{r-1}(\Delta^m H))}{\delta^{r}(\Delta^m H) + e(\delta^{r}(\Delta^m H))} \tag{33}$$

Evidently, $\delta^r(\Delta^m H)$ and $\delta^{r-1}(\Delta^m H)$ are not independent. Further, as ζ_δ' is a non-linear function of two random variables, the determination of a general expression for the variance of this quantity is not straightforward. An analytic expression for the variance of $\delta y(\overline{H})$, therefore, is not developed here. Nevertheless, experimental results reveal that the multiscan algorithm renders a non-linear relation between the variance of the observation process noise and that of the determined shift per scan . Furthermore, it has been found that distribution of $\delta y(\overline{H})$ is skewed normal (see Fig. 5) when the noise in $\overline{H}$ is standard normal. We will present an example of the multiscan polynomial tracking before further discussion about the algorithm.

4.1 An Example of Multiscan Tracking:

An example of multiscan tracking is given here. Same fourth order polynomial of the single scan example of section 3 is used here. Parameters of tracking are

Degree of polynomial	n	$= 4$
Initial Sampling Position	y	$= 1$
Sampling Interval	Δy	$= 1$

Number of Scans $r + 1$ = 3
Samples per Scam $m + 1$ = 3
Change of Height per scan δy = 1 *(To be determined)*

Result of this sampling is shown in the Table 1 below.

Sample No	Scan Number 1	Scan Number 2	Scan Number 3
1	-0.314	2.0	4.6
2	2.0	4.6	6.0
3	4.6	6.0	5.57

Table 1. Sampled Values of the Polynomial Field

With the above parameters and using the polynomial coefficients of equation (19), equation (28) implies

$$\delta y = \frac{-2.333}{\delta^{r-1}(\Delta^m H)/\delta^r(\Delta^m H) - 1} \tag{34}$$

Here two double differences can be calculated from equations (29) using sampled values in the Table 1 as

$$\delta^{r-1}(\Delta^m H) = 0.857 \tag{35}$$

$$\delta^r(\Delta^m H) = -1.486 \tag{36}$$

Now equations (34), (35) and (36) implies that

δy=1 *(As expected for noiseless sampling)*

For the purpose of the noise analysis 3000 realisations of δy are generated while sampled values of Table 1 are corrupted by a zero mean normal noise of variance=0.001. Here δy is found to observe the following statistics.

Minimum [δy] = 0.2848
Maximum[δy] = 1.339
Mean{δy} = 0.98422
Variance{δy} = 0.32032
Std. Dev{δy} = 0.13210

This results in a noise amplification factor, ϕ, equal to 4.2. Seemingly, the multiscan algorithm is more robust than the single scan version (ϕ=5.11) with similar parameters. Nevertheless, equation (17) clearly indicates that, in the case of the single scan algorithm, ϕ is a constant quantity for a given polynomial and fixed sampling interval. While in the multiscan case, as we will see shortly, ϕ is non-linear quantity that increases with increasing variance of input noise. Fig. 5 below shows histogram of δy. The distribution is negatively skewed.

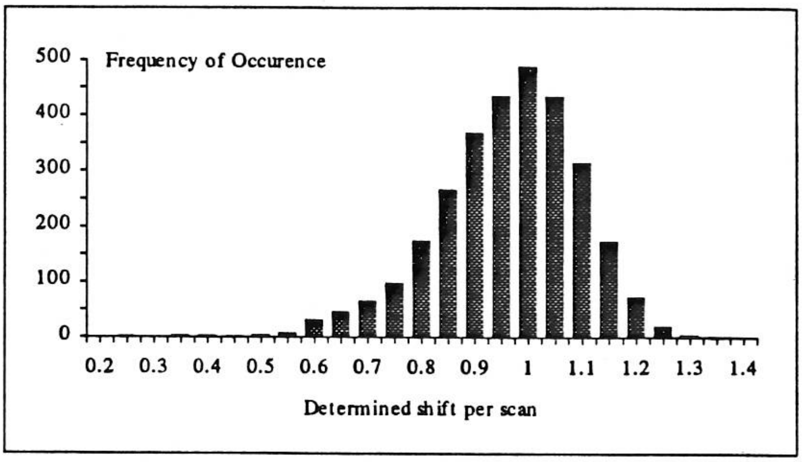

Fig. 5. Skewed distribution in the case of the multiscan tracking

Table 2 below shows effect of increasing the variance of the observation noise on the robustness of the single scan and the multiscan algorithms. Same parameters as of example 3.1 and example 4.1 are used. Each experiment delineates statistics of 3000 realisations.

Variance of the Observation Noise	Multiscan Case		Single Scan Case	
	Noise amplification factor	Mean	Noise amplification factor	Mean
0.001	4.17	0.9865	5.196	1.0021
0.005	5.33	0.9252	5.215	1.0046
0.010	20.88	0.790	5.187	1.0177
0.020	43.44	0.605	5.225	1.0251
0.025	104.16	0.6192	5.219	1.0284

Table 2. Sampled Values of the Polynomial Field

Clearly, for the multiscan version, the amplification factor increases explosively as the variance of the observation noise is increased. The mean also diverges with increasing noise. Consequently, above results indicate that, in general, the single scan algorithm is more noise tolerant than the multiscan algorithm as the practical range of the noise is not to be limited. However, the multiscan algorithm has a definitive edge over the single scan algorithm when number of available samples per scan are less than the degree of the polynomial field. Therefore, it is desirable to reformulate the multiscan algorithm with improved robustness.

In the multiscan algorithm, increasing the sampling interval results in a reduction of noise amplification factor. Table 3. shows results of increasing the sampling interval from 1 to 2. A zero mean normal noise with variance equal to 0.005 has been used. Other parameters are the same as example 4.1. Every experiment shows statistics of 3000 realisations.

Sampling Interval Δy	**Mean**	**Noise Amplification Factor ϕ**
1.0	0.9253	5.336
1.2	0.9667	2.951
1.4	0.9827	1.876
1.6	0.9901	1.239
1.8	0.9947	0.819
2.0	0.9975	0.539

Table 3. Reduction in Noise Amplification Factor with increasing sampling interval

Fig. 6 shows results in a graphical form. An exponential relation is apparent. The noise amplification factor rises enormously when sampling interval is further reduced below 1. For example, if sampling interval is reduced to 0.5, the noise amplification factor rises to 4540.

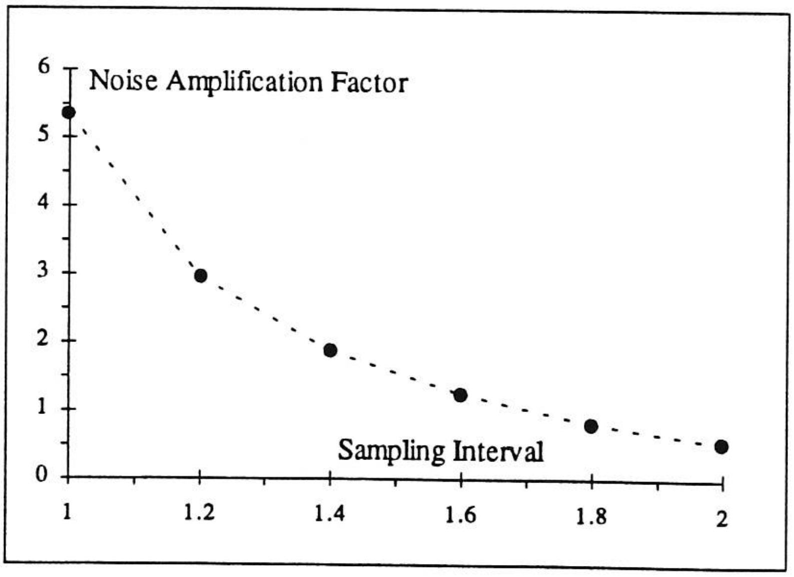

Fig. 6. Sampling Interval and Noise Amplification in Multiscan Algorithm

5. Conclusion and Discussion

An expression for the noise amplification factor for the single scan algorithm has been developed and authenticated experimentally. It has been shown that the single scan algorithm yields a normally distributed positional estimate when the observation process is contaminated with Gaussian white noise. Furthermore, as equation (15) indicates, the variance of the observation noise and that of the positional estimate are linearly related. Increasing the sampling interval results in an exponential reduction in the noise amplification factor depending on the order of field generating polynomial. It is also interesting to note that the noise amplification factor depends only on a_n, the most significant coefficient of the field generating polynomial and not on $a_1, a_2, .., a_{n-1}$.

Due to mathematical intractability, an expression for the noise amplification factor in the case of multiscan algorithm has not been developed. Nevertheless, it has been shown that the relation between the variance of the observation process noise and the variance of the estimated shift per scan is non-linear and explosive. With observation noise values as small as $N(0,0.025)$, noise amplification factor rises to 104.16 compared to 4.17 for $N(0, 0.001)$. Moreover with increasing noise, mean of the estimated shift per scan diverges rapidly as apparent from Table 2. It has been found that increase in the sampling interval reduces the noise amplification factor exponentially.

The deficient performance of the multiscan algorithm implies little utility in practical situations. However, in the circumstances where samples per scan are limited to less than the order of field generating polynomial, the multiscan version is probably the only answer. Hence it is desired to further investigate the multiscan algorithm for possible improvement in the robustness.

The foregoing analysis and discussion lead to the conclusion that for a given polynomial field the only manoeuvrable variable for improving the robustness in the single scan algorithm is sampling interval. However, maximum sampling interval is limited by the maximum available width (called *scope*) of sampling window. As might be expected, in any practical situation, available scope would be restricted. For example, in case of a coal mining shearer steered by tactile sensing and subsequent tracking of hardness profile of a coal seam, the scope is limited by the width of the cutting drum. The choice of drum width itself is restricted by the total height of the coal seam to be attacked apart from other factors. It is therefore highly desirable to improve the algorithms so as to use limited available scope while improving the robustness. This report has not considered the potential benefits of utilising numbers of samples $> n$. This is examined in the third report of this series.

6. References

[1] J. B. Edwards and R. Iqbal, "Tracking the features of a spatially distributed continuous field-The idealised 2D, deterministic case" , AC & SE research report no 492, University of Sheffield, Sheffield, UK, 1993

[2] Harold J. Larson, Bruno O. Shubert, *Probabilistic Models in Engineering Sciences, vol. 1, Random Variables and Stochastic Processes*, John Wiley and Sons, 1979.

Errata for the AC & SE Research Report No. 492:

Tracking the Features of a Spatially Distributed Continuous Field

1. Replace the Page No. 14 (6.1 Appendix A) with one attached with this note.

2. Page. No. 9, Line No. 6 (the very next line to the equation 48). Read ***sum*** instead of ***product***.

P.S. This errata is for the first report (No. 492) of this series and not for this report (No. 501)

6. Appendices

6.1 Appendix A

From Taylor's Theorem

$$\frac{\Delta H}{\Delta y} = DH + \frac{D^2 H}{2!}\Delta y + \frac{D^3 H}{3!}\Delta y^2 + \frac{D^4 H}{4!}\Delta y^3 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}\Delta y^{n-2} + \frac{D^n H}{n!}\Delta y^{n-1}$$

now

$$\begin{aligned}\Delta^2 H &= H(y+2\Delta y) - 2H(y+\Delta y) + H(y)\\ &= H + DH(2\Delta y) + \frac{D^2 H}{2!}(2\Delta y)^2 + \frac{D^3 H}{3!}(2\Delta y)^3 + \frac{D^4 H}{4!}(2\Delta y)^4 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}(2\Delta y)^{n-1} + \frac{D^n H}{n!}(2\Delta y)^n\\ &\quad -2\left(H + DH\Delta y + \frac{D^2 H}{2!}\Delta y^2 + \frac{D^3}{3!}\Delta y^3 + \frac{D^4 H}{4!}\Delta y^4 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}\Delta y^{n-1} + \frac{D^n H}{n!}\Delta y^n \right) + H\end{aligned}$$

$$\therefore \frac{\Delta^2 H}{\Delta y^2} = D^2 H + \frac{2D^3 H}{2!}\Delta y + \text{Higher order terms in } D \text{ and } \Delta y$$

$$\begin{aligned}\Delta^3 H &= H(y+\Delta 3y) - 3H(y+2\Delta y) + 3H(y+\Delta y) - H(y)\\ &= H + DH(3\Delta y) + \frac{D^2 H}{2!}(3\Delta y)^2 + \frac{D^3 H}{3!}(3\Delta y)^3 + \frac{D^4 H}{4!}(3\Delta y)^4 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}(3\Delta y)^{n-1} + \frac{D^n H}{n!}(3\Delta y)^n\\ &\quad -3\left(H + DH(2\Delta y) + \frac{D^2 H}{2!}(2\Delta y)^2 + \frac{D^3 H}{3!}(2\Delta y)^3 + \frac{D^4 H}{4!}(2\Delta y)^4 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}(2\Delta y)^{n-1} + \frac{D^n H}{n!}(2\Delta y)^n \right)\\ &\quad +3\left(H + DH\Delta y + \frac{D^2 H}{2!}\Delta y^2 + \frac{D^3 H}{3!}\Delta y^3 + \frac{D^4 H}{4!}\Delta y^4 + \ldots\ldots + \frac{D^{n-1} H}{n-1!}\Delta y^{n-1} + \frac{D^n H}{n!}\Delta y^n \right) + H\end{aligned}$$

$$\therefore \frac{\Delta^3 H}{\Delta y^3} = D^3 H + \frac{3D^3 H}{2!}\Delta y + \text{Higher order terms in } D \text{ and } \Delta y$$

Thus, finally we get

$$\frac{\Delta^{n-1} H}{\Delta y^{n-1}} = D^{n-1} H + \frac{n-1}{2!} D^n H \Delta y + (\text{no. higher order terms})$$

Therefore

$$\boxed{\frac{\Delta^{n-1} H}{\Delta y^{n-1}} = D^{n-1} H + \frac{n-1}{2!} D^n H \Delta y} \qquad (61)$$