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Abstract: A recursive algorithm which maps a general class of nonlinear rational model, defined as the ratio of two polynomial functions, into the frequency domain is derived using the harmonic expansion method. The new algorithm provides, for the first time, a direct analytic map from the time domain rational model parameters to the higher order frequency response functions. Complex nonlinear time domain behaviours can be analysed and interpreted in the frequency domain and simulated examples are included to illustrate the concepts involved.

1. Introduction

Ever since Norbert Wiener applied the Volterra functional series to nonlinear system analysis (Wiener, 1942; 1958), the Volterra/Wiener theory of nonlinear systems has been extensively studied and has become well known to both scientists and engineers. The theory characterises nonlinear systems using either the Volterra kernels in the time-domain or equivalently by a transformation of the Volterra kernels into the frequency-domain. The latter are commonly called generalised frequency response functions (GFRF) and represent obvious extensions to the well known linear frequency response functions. The GFRF of a nonlinear system provides an intuitive representation of the frequency domain properties of the system and many nonlinear phenomena can be studied using this approach. The initial development of these concepts was conducted in the late 1960s and early 1970s (Brillinger and Rosenblatt, 1967; Bedrosian and Rice, 1971; Bussgang, Ehrman and Graham, 1974) but progress was been hindered by the difficulties of obtaining the GFRF for practical systems. The classical method of estimation utilises multi-dimensional correlation or FFT techniques and has often been limited by the complexity of multidimensional windowing and smoothing, the requirements for special inputs and very long record lengths(Schetzen, 1980; Vinh et al, 1988; Kim and Powers, 1988).

An alternative indirect approach is to estimate a time-domain model from the sampled input-output data and then to use this model to derive the GFRF(Billings, Tsang and Tomlinson, 1988). The model used for the identification is usually a polynomial NARMAX or Nonlinear Auto-Regressive Moving Average with eXogeneous inputs model (Leontaritis and Billings 1985). The main advantage of using a NARMAX model instead of direct estimation based on the Volterra series is a large reduction in the number of parameters and the length of data set required for identification. The nonlinear GFRF can then be computed by extending the ideas of the probing method to derive the map from the NARMAX model to the GFRF's. The probing method has been used by several authors for simple examples (Bedrosian and Rice, 1971; Bussgang, Ehrman and Graham, 1974; Chua and Ng, 1979a, 1979b) but the analysis becomes awkward as the order of nonlinearity increases. Peyton-Jones and Billings(1989) extended these concepts and developed a recursive algorithm for computing the GFRF for the identified nonlinear polynomial models.

Two classes of NARMAX models have been widely studied, the polynomial NARMAX model and the rational NARMAX model. The polynomial NARMAX model is well known and can be estimated with relative ease. The rational NARMAX model, which is expressed as a ratio of two polynomials, is much more different to estimate but has better extrapolation properties and can approximate a much wider class of severely nonlinear systems with only a small number of model parameters(Sontag, 1979; Billings and Chen, 1989; Billings and Zhu, 1991). Although the excellent approximation properties of the rational model have been well known and exploited in static function approximation for many years these results could not be extended to the dynamic system case because of the inherent problems of noise and bias. These problems arise because of the denominator terms in the rational model and are not present in numerator only expansions such as polynomial models. Recent work on the development of new parameter estimation routines for the rational model have led to the introduction of a new class of algorithms which can accommodate these effects(Billings and Zhu, 1991; 1993; Zhu and Billings 1991; 1993). The restrictions which currently limit frequency domain analysis to mildly nonlinear and polynomial systems can therefore be lifted by studying the frequency domain properties of the rational model.

In the present paper a recursive relationship is derived which provides a direct map from the parameters of a time domain nonlinear rational model to the generalised frequency response functions. This provides, for the first time, the opportunity to study how complex nonlinear time domain effects are characterised in the frequency domain. Generalised frequency response functions up to arbitrary order can be readily computed for severely nonlinear systems. Because the resulting algorithm is essentially just an algebraic relationship the structural form of the map is exposed and all the disadvantages of the classical FFT type algorithms such as multi-dimensional windowing and smoothing, and excessive data lengths are avoided. The paper begins in Section 2 with definitions of the Volterra series and the associated GFRF. The general form of the rational model is presented in Section 3. The harmonic expansion method for computing the GFRF's is formalized and investigated in Section 4 as the foundation for the derivation of the recursive algorithm for computing the nth order GFRF in Section 5. Simulated examples are given in Section 6.

2. Nonlinear System Representations in the Time and Frequency Domain

The classical input/output representation for nonlinear systems is based on the Volterra functional series expansion(Volterra 1930)

$$y(t) = \mathbf{V}[u(t)] = \sum_{n=1}^{\infty} y_n(t) \tag{1}$$

which relates the system output $y(t)$ to the input $u(t)$ by the nonlinear causal operator $\mathbf{V}$. The n-th order output of the system $y_n(t)$ is defined by

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \qquad n > 0 \tag{2}$$

and $h_n(\cdot)$ is known as the nth order Volterra kernel. The kernel $h_n(\cdot)$ can be recognised as a generalised impulse response function of order n with the linear case given when n=1. The Volterra series has been extensively studied by numerous researchers over many decades(Billings, 1980), but the most recent study has focused on the frequency domain description which is defined in terms of the multiple Fourier transform of $h_n(\cdot)$

$$H_n(j\omega_1, \cdots j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n)\, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\tau_1 \cdots d\tau_n \tag{3}$$

This alternative description is commonly called the nth order generalised frequency response function(GFRF) or simply nth order transfer function(Zhang and Billings, 1992). Notice that (3) reduces to the standard linear transfer function $H_1(j\omega_1)$ for the case n=1. The importance of the higher order GFRF's has been realised since the early 1960s (Brillinger, 1965; Bedrosian and Rice, 1971) because of the role this plays in the frequency domain analysis of nonlinear systems. Both $h_n(\cdot)$ and $H_n(\cdot)$ provide invariant descriptions and are independent of the excitation. Indeed, since the n-th order impulse response $h_n(\cdot)$ and n-th order transfer function $H_n(\cdot)$ are Fourier transform pairs eqn.(2) may also be written as

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots j\omega_n) \prod_{i=1}^{n} U(j\omega_i)\, e^{j(\omega_1 + \cdots + \omega_n)t}\, d\omega_i \tag{4}$$

where $U(j\omega_i)$ represents the input spectrum.

Observe that in eqn.(2) the nth order kernel $h_n(\cdot)$ and hence associated transform $H_n(\cdot)$ are not unique in the sense that changing the order of the arguments may give different kernels but will still yield the same output $y_n(t)$. For convenience of analysis it is common practice to define a symmetrised function by summing the asymmetric function over all possible permutations of the arguments and dividing by the number to give

$$H_n^{sym}(j\omega_1, \cdots j\omega_n) = \frac{1}{n!} \sum_{\substack{all\ permutations \\ of\ \omega_1 \cdots \omega_n}} H_n(j\omega_1, \cdots j\omega_n) \tag{5}$$

This symmetric GFRF is then unique and independent of the order of the arguments.

3. The Nonlinear Rational Model and System Identification

In this paper we will consider a class of nonlinear discrete-time models which are called nonlinear rational models. The whole class of nonlinear rational models can be expressed in a general form as

$$y(t) = \frac{\mathbf{Y}_a(t;\theta_a,\mathbf{y},\mathbf{u})}{\mathbf{Y}_b(t;\theta_b,\mathbf{y},\mathbf{u})} \tag{6}$$

where $\mathbf{Y}_a(t;\theta_a,\mathbf{y},\mathbf{u})$ and $\mathbf{Y}_b(t;\theta_b,\mathbf{y},\mathbf{u})$ are used to denote polynomials in the numerator and denominator, respectively. These are defined as

$$\mathbf{Y}_a(t;\boldsymbol{\theta}_a,\mathbf{y},\mathbf{u}) = \sum_{m=1}^{M_a}\left[\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K_a}\alpha_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i)\right] \tag{7}$$

and

$$\mathbf{Y}_b(t;\boldsymbol{\theta}_b,\mathbf{y},\mathbf{u}) = \sum_{m=0}^{M_b}\left[\sum_{p=0}^{m}\sum_{k_1,k_{p+q}=1}^{K_b}\beta_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i)\right] \tag{8}$$

where M_a and M_b are the maximum degrees of nonlinearities, K_a and K_b are the maximum lags in the input and output, $\alpha(\cdot)$ and $\beta(\cdot)$ are the parameters associated with the various terms in the two polynomials (corresponding to the parameter sets $\boldsymbol{\theta}_a$ and $\boldsymbol{\theta}_b$ respectively), $p+q = m$; and

$$\sum_{k_1,k_n=1}^{K} \equiv \sum_{k_1=1}^{K}\cdots\sum_{k_n=1}^{K}$$

Notice that the lower limit on the first summation of $\mathbf{Y}_b$ is zero, which implies a nonzero constant term $\beta_{0,0}$ may be included in the denominator. In the present analysis a d.c. component or offset in the model expression is excluded from the the numerator polynomial $\mathbf{Y}_a$ on the assumption that the d.c component can be removed from the data.

Consider a specific nonlinear rational model to illustrate the notation

$$y(t) = \frac{1.94y(t-1) - 0.93y(t-2) + 0.01u^2(t-2) + 0.3u(t-3)}{1 + 0.42y(t-1)u(t-2) + 0.02y^2(t-1)}$$

This may be obtained from the general form (6) by setting the coefficients as

$\alpha_{0,1}(3) = 0.3$; $\alpha_{1,0}(1) = 1.94$; $\alpha_{1,0}(2) = -0.93$; $\alpha_{0,2}(1,1) = 0.01$;

$\beta_{0,0} = 1$; $\beta_{1,1}(1,2) = 0.42$; $\beta_{2,0}(1,1) = 0.02$; *else* $\alpha_{p,q}(\cdot)$, $\beta_{p,q}(\cdot) = 0$;

with $K_a= 3$, $K_b= 2$, $M_a= M_b= 2$.

Nonlinear rational models provide compact representations of complex nonlinear effects and are vastly superior to standard polynomial expansions. The simple rational model

$$y(t) = \frac{1}{(1 + u(t-1))}$$

for example provides an efficient representation of the complex polynomial representation

$$y(t) = 1 - u(t-1) + u(t-1)^2 - \cdots + u(t-1)^{2n} - \cdots$$

which includes an infinite number of terms.

While the properties of the rational model have been recognised and exploited in static function approximation dynamic rational model have been totally neglected until very recently. The dynamic rational model can be identified by using either a prediction error algorithm(Billings and Chen, 1989) or an extended least squares routine(Billings and Zhu, 1991).

4. Computing the Frequency Response Using the Harmonic Expansion Method

Consider a time-domain model $\mathbf{M}(\cdot)$ which is expressed as

$$\mathbf{M}(t;\, \boldsymbol{\theta}\,,\, \mathbf{y}\,,\, \mathbf{u}) = 0 \tag{9}$$

where $\mathbf{M}(\cdot)$ is a functional of the input $\mathbf{u}$, output $\mathbf{y}$ and $\boldsymbol{\theta}$ is the set of model parameters. In the discrete-time case, $\mathbf{u}$ and $\mathbf{y}$ contain both the current and previous sampled values so that

$$\mathbf{u} \equiv \{u(t), u(t-1), \cdots\}$$

$$\mathbf{y} \equiv \{y(t), y(t-1), \cdots\}$$

It is important to note here that the main assumption of previous analysis by Peyton-Jones and Billings(1989) that there is an explicit $y(t)$ term in the model expression (9) is no longer necessary in the present study. In the model expression $\mathbf{M}(\cdot)$ all the outputs have been eliminated by substituting the Volterra functional representation (4) into the expression to give

$$\mathbf{M}(t;\boldsymbol{\theta},\, H,\, \mathbf{u}) = 0 \tag{10}$$

So that y is now replaced by the GFRF's $H \equiv \{H_1, H_2, \cdots\}$ in the equation. For some special cases the $H_i(\cdot)$ may be obtained directly by manipulating equation (10) but for most nonlinear systems this is not a realistic approach since equation (10) will be a very complicated integral equation. However, for a wide class of nonlinear systems, the problem can be simplified by expanding the equation for some specialised input. In the case of the harmonic expansion method the input is a sum of R complex exponentials defined as

$$u(t) = \sum_{r=1}^{R} e^{j\omega_r t} \tag{11}$$

The spectrum for this input is

$$U(j\omega) = \sum_{r=1}^{R} 2\pi\delta(j\omega - j\omega_r) \tag{12}$$

Applying the above input to the Volterra series representation (4) and performing the integration, the output $y(t)$ becomes

$$y(t) = \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{R} H_n(j\omega_{r_1}, \cdots j\omega_{r_n})\, e^{j(\omega_{r_1} + \cdots + \omega_{r_n})t} \tag{13}$$

$$= \sum_{n=1}^{N} \sum_{\substack{\text{all combinations}\\ \text{of } R \text{ frequencies}\\ \text{taken } n \text{ at a time}}} \sum_{\substack{\text{all permutations}\\ \text{of } \omega_{r_1} \cdots \omega_{r_n}}} H_n(j\omega_{r_1}, \cdots j\omega_{r_n})\, e^{j(\omega_{r_1} + \cdots + \omega_{r_n})t}$$

In order to find the n'th order GFRF $H_n(\cdot)$ it is convenient to consider the special case $R=n$ so that there is only one non-repetitive combination of frequencies $\{\omega_1, \ldots, \omega_n\}$ among all the possibilities. In other words, the right hand side of the above equation can be divided into two parts: a part containing distinct frequency combinations and a part containing all the repetitive combinations. Clearly the $H_n(\cdot)$ with non-repetitive but different permutations of arguments will only appear in the first part. So that eqn. (13) can be expressed as

$$y(t) = \sum_{n=1}^{N} \left[\sum_{\substack{\text{all permutations}\\ \text{of } \omega_1 \cdots \omega_n}} H_n^{asym}(j\omega_1, \cdots j\omega_n)\, e^{j(\omega_1 + \cdots + \omega_n)t} + \sum_{\substack{\text{all repetitious}\\ \text{combinations}}} H_n^{sym}(j\omega_{r_1}, \cdots j\omega_{r_n})\, e^{j(\omega_{r_1} + \cdots + \omega_{r_n})t} \right]_{n\text{th order}} \tag{14}$$

If the symmetric GFRF is used all the $H_n(\cdot)$ with different permutations of argument will be the same and $y(t)$ then becomes

$$y(t) = \sum_{n=1}^{N} \left[\begin{array}{l} n!\, H_n^{sym}(j\omega_1, \cdots j\omega_n)\; e^{j(\omega_1 + \cdots + \omega_n)t} + \\ \displaystyle\sum_{\substack{all\ repetitious \\ combinations}} H_n^{sym}(j\omega_{r_1}, \cdots j\omega_{r_n})\; e^{j(\omega_{r_1} + \cdots + \omega_{r_n})t} \end{array} \right]_{nth\ order} \tag{15}$$

Substituting eqn.(11) and (13) into (10) yields the following harmonic expansion equation

$$\mathbf{M}(t;\, \theta,\, H, \omega_r) = 0 \tag{16}$$

where $\boldsymbol{\omega}_r$ implies $\{\omega_1, \cdots, \omega_R\}$ ($R=n$ for computing H_n). $\mathbf{M}(\cdot)$ will contain many exponential terms but we are only interested in the term with non-repetitive frequencies $e^{j(\omega_1 + \cdots + \omega_n)t}$. Because eqn.(16) holds for any t and the arbitrarily chosen variables $\{\omega_1, \ldots, \omega_n\}$, the coefficient of $e^{j(\omega_1 + \cdots + \omega_n)t}$ should be zero. That is

$$\mathcal{E}_n\Big[\, \mathbf{M}(t;\theta,\, H, \boldsymbol{\omega}_r)\, \Big] = 0 \tag{17}$$

where $\mathcal{E}_n[\cdot]$ is used to denote the operation of extracting the coefficient of $e^{j(\omega_1 + \cdots + \omega_n)t}$. For a given expression the operator $\mathcal{E}_n$ actually implies a two-step operation:

i). Substitute the harmonic input (11) and the corresponding Volterra expansion (13) of the output $y(t)$ into the given expression;

ii) Extract the coefficient of $e^{j(\omega_1 + \cdots + \omega_n)t}$ from the resulting expression.

For example, $\mathcal{E}_n[y(t)\,]$ can readily be written as

$$\sum_{\substack{all\ permutations \\ of\ \omega_1 \cdots \omega_n}} H_n^{asym}(j\omega_1, \cdots j\omega_n) \qquad or \qquad n!\, H_n^{sym}(j\omega_1, \cdots j\omega_n)$$

according to eqn.(15). All the GFRF's, $H_i(\cdot)$ $i=1,2,\ldots$, appearing in equation (17) have non-repetitive arguments and they can be found by solving this equation without listing all the terms.

In order to illustrate the use of the harmonic expansion method for obtaining the GFRF's from rational models, consider a simple example

$$y(t) = \frac{a_1 u(t-2) + a_2 y(t-1)}{b_1 u(t-1) + b_2 y(t-1)} \tag{18}$$

The above model can be re-written in the form $\mathbf{M}(\cdot) = 0$ as

$$a_1u(t-2) + a_2y(t-1) - b_1y(t)u(t-1) - b_2y(t)y(t-1) = 0 \tag{19}$$

In order to find the GFRF up to 2nd order from the above time-domain model a two-exponential input is used

$$u(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \tag{20}$$

where ω_1 and ω_2 are arbitrary frequencies. The system output in terms of the Volterra model, up to 2nd order, can then be given from eqn (13) by

$$\begin{aligned} y(t;\, H,\omega_r) = \;& 2!\; H_2^{sym}(j\omega_1,j\omega_2)\; e^{j(\omega_1+\omega_2)t} \;+\; H_2^{sym}(j\omega_1,j\omega_1)\; e^{2j\omega_1 t} + H_2^{sym}(j\omega_2,j\omega_2)\; e^{2j\omega_2 t} \\ & +\; H_1(j\omega_1)\; e^{j\omega_1 t} + H_1(j\omega_2)\; e^{j\omega_2 t} \end{aligned} \tag{21}$$

where all terms of order higher than two are ignored since they are not relevant when just computing H_1 and H_2.

Substituting for the input $u(t)$ and output $y(t)$ in the model expression (19) using (20) and (21) yields

$$\begin{aligned} & a_1\left[e^{j\omega_1(t-2)} + e^{j\omega_2(t-2)}\right] + a_2\Big[2!\; H_2^{sym}(j\omega_1,j\omega_2)\; e^{j(\omega_1+\omega_2)(t-1)} \;+\; H_2^{sym}(j\omega_1,j\omega_1)\; e^{2j\omega_1(t-1)} + \\ & \qquad H_2^{sym}(j\omega_2,j\omega_2)\; e^{2j\omega_2(t-1)} + H_1(j\omega_1)\; e^{j\omega_1(t-1)} + H_1(j\omega_2)\; e^{j\omega_2(t-1)}\Big] - b_1\Big[e^{j\omega_1(t-1)} + \\ & e^{j\omega_2(t-1)}\Big]\cdot\Big[2!\; H_2^{sym}(j\omega_1,j\omega_2)\; e^{j(\omega_1+\omega_2)t} \;+\; H_2^{sym}(j\omega_1,j\omega_1)\; e^{2j\omega_1 t} + H_2^{sym}(j\omega_2,j\omega_2)\; e^{2j\omega_2 t} + \\ & H_1(j\omega_1)\; e^{j\omega_1 t} + H_1(j\omega_2)\; e^{j\omega_2 t}\Big] - b_2\Big[2!\; H_2^{sym}(j\omega_1,j\omega_2)\; e^{j(\omega_1+\omega_2)t} \;+\; H_2^{sym}(j\omega_1,j\omega_1)\; e^{2j\omega_1 t} + \\ & H_2^{sym}(j\omega_2,j\omega_2)\; e^{2j\omega_2 t} + H_1(j\omega_1)\; e^{j\omega_1 t} + H_1(j\omega_2)\; e^{j\omega_2 t}\Big]\cdot\Big[2!\; H_2^{sym}(j\omega_1,j\omega_2)\; e^{j(\omega_1+\omega_2)(t-1)} \;+ \\ & H_2^{sym}(j\omega_1,j\omega_1)\; e^{2j\omega_1(t-1)} + H_2^{sym}(j\omega_2,j\omega_2)\; e^{2j\omega_2(t-1)} + H_1(j\omega_1)\; e^{j\omega_1(t-1)} + H_1(j\omega_2)\; e^{j\omega_2(t-1)}\Big] = 0 \end{aligned}$$

Now both $H_1(\cdot)$ and $H_2(\cdot)$ explicitly appear in the above harmonic expansion equation. The first order frequency response function $H_1(\cdot)$ can therefore be obtained by equating the coefficients of either $e^{j\omega_1(t)}$ or $e^{j\omega_2(t)}$ to zero. The coefficient of $e^{j\omega_1(t)}$, for instance, is

$$a_1e^{-2j\omega_1} + a_2H_1(j\omega_1)e^{-j\omega_1} \;=\; 0$$

so that

$$H_1(j\omega_1) = -\frac{a_\Gamma}{a_2}e^{-j\omega_1} \tag{22}$$

The second order GFRF $H_2(\cdot)$ can, in a similar way, be obtained by extracting the coefficients of $e^{j(\omega_1+\omega_2)t}$ and equating these to zero to yield

$$a_2 2\, H_2^{sym}(j\omega_1,j\omega_2)\, e^{-j(\omega_1+\omega_2)} - b_1\left[e^{-j\omega_1}H_1(j\omega_2) + e^{-j\omega_2}H_1(j\omega_1)\right]$$
$$- b_2\left[H_1(j\omega_1)e^{-j\omega_2}H_1(j\omega_2) + H_1(j\omega_2)H_1(j\omega 1)e^{-j\omega_1}\right] = 0$$

So that

$$H_2^{sym}(j\omega_1,j\omega_2) = \frac{1}{2a_2e^{-j(\omega_1+\omega_2)}}\left[b_1H_1(j\omega_1)e^{-j\omega_2}+b_1H_1(j\omega_2)e^{-j\omega_1}\right.$$
$$\left. + b_2H_1(j\omega_1)H_1(j\omega_2)e^{-j\omega_2} + b_2H_1(j\omega_1)H_1(j\omega_2)e^{-j\omega_1}\right] \tag{23}$$
$$= \frac{1}{2a_2}\left[b_1H_1(j\omega_1)e^{j\omega_1} + b_1H_1(j\omega_2)e^{j\omega_2} + b_2H_1(j\omega_1)H_1(j\omega_2)e^{j\omega_1} + b_2H_1(j\omega_1)H_1(j\omega_2)e^{j\omega_2}\right]$$

In order to evaluate the higher order GFRF's $H_n(\cdot)$ the above procedure can be continued by applying more exponentials as the input and considering higher order terms in the output. However, the work rapidly becomes unwieldy as the order increases, even for this particularly simple example. Hence an easier and more efficient algorithm needs to be developed.

The model class under consideration is the nonlinear rational model given by eqn (6). Rewrite eqn.(6) as

$$\mathbf{M}(\cdot) = \mathbf{Y}_a(t;\theta_a,y,u) - y(t)\mathbf{Y}_b(t;\theta_b,y,u) = 0 \tag{24}$$

Now the model expression $\mathbf{M}(\cdot)$ becomes a polynomial in $u(t)$, $y(t)$ and the lagged values $u(t-k)$, $y(t-k)$, k=0,1,2,.... The desired frequency response functions $H_n(\cdot)$ for the rational model can then be found by applying the operator $\mathcal{E}_n$ to the model expression $\mathbf{M}$ according to eqn.(17). In order to solve equation (17) it is important to investigate some properties of the operator $\mathcal{E}_n$ for this class of models.

Remark 1: $\mathbf{M}(\cdot)$ may consist of many parts. All these parts may generate an

$e^{j(\omega_1+\cdots+\omega_n)t}$ term. The coefficient of the final $e^{j(\omega_1+\cdots+\omega_n)t}$ is the sum of the contributions from all these parts. This suggests that, for the rational model class at least, $\mathcal{E}_n[\cdot]$ is a linear operator, that is

$$\mathcal{E}_n\left[c_1\mathbf{M}_1(\cdot)+c_2\mathbf{M}_2(\cdot)\right]=c_1\mathcal{E}_n\left[\mathbf{M}_1(\cdot)\right]+c_2\mathcal{E}_n\left[\mathbf{M}_2(\cdot)\right] \tag{25}$$

This is significant because the effect of each term in the time-domain model can therefore be considered as a seperate contribution to the term $e^{j(\omega_1+\cdots+\omega_n)t}$.

Remark 2: It is known that $\mathbf{M}[\cdot]$ consists of various polynomial terms. All these polynomial terms can be divided into three types: pure inputs, pure outputs and input/output cross product terms which are generally expressed as $\prod_{i=1}^{K} u(t-k_i)$, $\prod_{i=1}^{K} y(t-k_i)$ and $\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)$, respectively. The effect for each of these after applying the operator $\mathcal{E}_n$ is given by

$$\mathcal{E}_n\left[\prod_{i=1}^{K} u(t-k_i)\right] = \begin{cases} \displaystyle\sum_{\substack{\text{all permutations}\\ \text{of } \omega_1\cdots\omega_n}} e^{-j(\omega_1 k_1+\cdots+\omega_n k_n)} & K=n \\ 0 & K\neq n \end{cases} \tag{26}$$

$$\mathcal{E}_n\left[\prod_{i=1}^{K} y(t-k_i)\right] = \begin{cases} \displaystyle\sum_{\substack{\text{all permutations}\\ \text{of } \omega_1\cdots\omega_n}} H_{n,p}(j\omega_1,\cdots j\omega_n) & K\leq n \\ 0 & K> n \end{cases} \tag{27}$$

and

$$\mathcal{E}_n\left[\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)\right] = \begin{cases} \displaystyle\sum_{\substack{\text{all permutations}\\ \text{of } \omega_1\cdots\omega_n}} \mathcal{E}_{n-q}\left[\prod_{i=1}^{p} y(t-k_i)\right]\cdot\mathcal{E}_q\left[\prod_{i=p+1}^{p+q} u(t-k_i)\right] & p+q\leq n \\ 0 & p+q> n \end{cases} \tag{28}$$

where $H_{n,p}(\cdot)$ denotes the contribution to the nth order $\mathcal{E}_n$ by a p degree nonlinear term in y. This is a recursive formula

$$H_{n,p}(\cdot) = \sum_{i=1}^{n-p+1} H_i^{asym}(j\omega_1,\cdots j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_i)k_p} \tag{29}$$

The recursion finishes with p=1 and $H_{n,1}(j\omega_1, \cdots j\omega_n)$ has the property

$$H_{n,1}(j\omega_1, \cdots j\omega_n) = H_n(j\omega_1, \cdots j\omega_n)\, e^{-j(\omega_1+ \cdots +\omega_n)k_1} \tag{30}$$

A linear output term, for example, would produce a contribution

$$\begin{aligned}\mathcal{E}_n\Big[y(t-k_1)\Big] &= \sum_{\substack{all\ permutations\\ of\ \omega_1 \cdots \omega_n}} e^{-j(\omega_1+ \cdots +\omega_n)k_1} H_n^{asym}(j\omega_1, \ldots j\omega_n)\\ &= n!\, e^{-j(\omega_1+ \cdots +\omega_n)k_1} H_n^{asym}(j\omega_1, \ldots j\omega_n)\end{aligned}$$

Eqn's (26), (27) and (28) can be obtained by replacing $u(\cdot)$ and $y(\cdot)$ with (11) and (13), respectively, and then extracting the coefficient of $e^{j(\omega_1+ \cdots +\omega_n)t}$ contained in the expression. A more detailed analysis may be found in Peyton-Jones and Billings(1989). All the above formula will be directly used to derive the GFRF for the nonlinear rational models in Sec.5.

The following important conclusions follow from the expressions in Remark 2.

Remark 3: The terms with m'th degree of nonlinearity can not produce an $e^{j(\omega_1+ \cdots +\omega_n)t}$ term with less than m $(m>n)$ non-repetitive frequencies in the harmonic expansion. In other words, the m'th nonlinearity in $y(t)$ and $u(t)$ can not contribute to $\mathcal{E}_n$, $n < m$, although this will make a contribution (recursively) to all the higher $\mathcal{E}_n$ where $n > m$.

Remark 4: It follows from Remark 2 that among all the polynomial terms only the linear output terms, that is $y(t-k_1)$ k_1=0,1,2,..N, produce a term $e^{j(\omega_1+ \cdots +\omega_n)t}$ with $H_n(\cdot)$ appearing as a coefficient. All the other terms can only produce terms with lower order $H_i(\cdot)$, $i<n$, as the coefficients. Extracting all such terms out of $\mathbf{M}(\cdot)$ yields

$$\mathcal{E}_n\left[\sum_{k_1=0}^{K} c_{1,0}(k_1)\, y(t-k_1)\right] = -\mathcal{E}_n\Big[\mathbf{M}^{other}(\cdot)\Big] \tag{31}$$

where $c_{1,0}(k_1)$ denotes the parameters associated with the linear output terms in the model. The left hand side takes the form

$$n!\sum_{k_1=1}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1+\cdots+\omega_n)k_1}\, H_n^{sym}(j\omega_1,\cdots j\omega_n)$$

or

$$\sum_{\substack{all\ permutations\\ of\ \omega_1\cdots\omega_n}} \sum_{k_1=1}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1+\cdots+\omega_n)k_1}\, H_n^{asym}(j\omega_1,\cdots j\omega_n)$$

Now in the case of $R=n$ $H_n(\cdot)$ will only appear in the left hand side of the equation. All the $H_i(\cdot)$ on the right hand side will be of lower order, i.e. $i\leq n$. Hence for the rational model class the equation can be solved recursively starting with $H_1(\cdot)$ and incrementing to higher orders. This will be confirmed during the derivation of the next section. This conclusion is also important for defining the conditions of existance of the GFRF for nonlinear systems.

Remark 5: It is observed from Remark 2 that the contributions from all three types of terms are in the form of an all-permutation-sum of the form $\sum_{\substack{all\ permutations\\ of\ \omega_1\cdots\omega_n}}$. It will therefore be convenient to define the symmetry of the operator $\mathcal{E}_n$ as

$$\mathcal{E}_n[\cdot] = \sum_{\substack{all\ permutations\\ of\ \omega_1\cdots\omega_n}} \mathcal{E}_n^{asym}[\cdot] \tag{32}$$

where $\mathcal{E}_n^{asym}$ is the coefficient of any single $e^{j(\omega_1+\cdots+\omega_n)t}$ term. Notice that there is also a all-permutation-sum on the left hand side of eqn.(31) which contains $H_n(\cdot)$ if the asymmetric GFRF is used. Since $H_n^{sym}(\cdot)$ is given by taking the average of any asymmetric function over all possible permutations of its arguments(see (5)), an asymmetric GFRF $H_n^{asym}(\cdot)$ can be obtained by considering only the asymmetric coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$ or

$$\sum_{k_1=1}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1+\cdots+\omega_n)k_1}\, H_n^{asym}(j\omega_1,\cdots j\omega_n) = -\,\mathcal{E}_n^{asym}\Big[M^{other}(\cdot)\Big]$$

In this case the three types of polynomial terms yield

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{K} u(t-k_i)\right] = \begin{cases} e^{-j(\omega_1k_1+\cdots+\omega_nk_n)} & K=n \\ 0 & K\neq n \end{cases} \tag{33}$$

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{K} y(t-k_i)\right] = \begin{cases} H_{n,p}(j\omega_1, \cdots j\omega_n) & K \le n \\ 0 & K > n \end{cases} \tag{34}$$

and

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)\right] = \begin{cases} \mathcal{E}_{n-q}^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\right] \cdot \mathcal{E}_q^{asym}\left[\prod_{i=p+1}^{p+q} u(t-k_i)\right] & p+q \le n \\ 0 & p+q > n \end{cases} \tag{35}$$

5. Algorithm Derivation for the General Form of Rational Model

It should be possible to derive the frequency response functions $H_n(\cdot)$ for the rational model eqn (24) by applying the operator $\mathcal{E}_n[\cdot]$ to the model expression. This problem can be simplified by dividing all the monomials of the two polynomials of the rational model into three parts, a part consisting of pure input nonlinearities, a part consisting of pure output nonlinearities and a part generated from input/output cross-product-type nonlinearities, such that

$$\mathbf{Y}_a(t;\, \theta,\, \mathbf{y},\mathbf{u}) = \mathbf{Y}_a(t;\, \theta_u,\, \mathbf{u}) + \mathbf{Y}_a(t;\, \theta_y,\, \mathbf{y}) + \mathbf{Y}_a(t;\, \theta_{uy},\, \mathbf{y},\mathbf{u})$$

$$\mathbf{Y}_b(t;\, \theta,\, \mathbf{y},\mathbf{u}) = \mathbf{Y}_b(t;\, \theta_u,\, \mathbf{u}) + \mathbf{Y}_b(t;\, \theta_y,\, \mathbf{y}) + \mathbf{Y}_b(t;\, \theta_{uy},\, \mathbf{y},\mathbf{u})$$

The model (24) is then split out into many parts. Because the $\mathcal{E}_n$ is a linear operator each part can be considered seperately and then the results can be combined together as

$$\mathcal{E}_n^{asym}\left[\beta_{0,0}\, y(t) - \mathbf{Y}_a^{linear}(\cdot)\right] = \mathcal{E}_n^{asym}\left[\mathbf{Y}_a^{other}(\cdot)\right] - \mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\theta_y,\, \mathbf{u})\right] - \mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\theta_u,\, \mathbf{y})\right] - \mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\theta_{uy},\, \mathbf{y},\mathbf{u})\right] \tag{36}$$

The evaluation of each of the individual terms in eqn. (36) is considered as follows.

5.1. Linear output terms $\left[\beta_{0,0}y(t) - \mathbf{Y}_a^{linear}(\cdot)\right]$

This sub-class of terms, which contains only the linear output terms corresponds to p=1, q=0 in $\mathbf{Y}_a$ and the constant term in $\mathbf{Y}_b$, is the simpliest part in the model expression. These terms are of the form $y(t-k_1)$ with k_1=0,1,...,K_a. When the Volterra expansion (13), or more intuitively (15), is substituted for $y(t)$ the discrete time lags k_1 do not alter the term $e^{j(\omega_1+\cdots+\omega_n)t}$ but only the coefficients of exponential term. Hence the contribution from the linear output terms can readily be written as

$$\mathcal{E}_n^{asym}\left[\beta_{0,0}y(t) - \mathbf{Y}_a^{linear}(\cdot)\right] = \left[\beta_{0,0} - \sum_{k_1=1}^{K_a} \alpha_{1,0}(k_1)\, e^{-j(\omega_1+\ldots+\omega_n)k_1}\right] H_n^{asym}(j\omega_1,\ldots,j\omega_n) \tag{37}$$

5.2. Other terms in $\mathbf{Y}_a^{other}(\cdot)$

Using the results of Remark 2 the contributions from all the other terms in the numerator polynomial of eqn.(6) can easily be derived according to the type of the terms. The only thing to mension is that, for each term type (pure input, pure output and cross-product) there may exist a number of terms of the same type, although it is a simple matter to add them together, again, because of the linearity of $\mathcal{E}_n$.

The contribution of pure input numerator nonlinear terms

$$\mathbf{Y}_a(t;\theta_u,\mathbf{u}) = \sum_{m=q=1}^{M_a} \sum_{k_1,k_q=1}^{K_a} \alpha_{0,q}(k_1,\cdots,k_q) \prod_{i=1}^{q} u(t-k_i) \tag{38}$$

to the harmonic expansion equation is simply

$$\mathcal{E}_n^{asym}\left[\mathbf{Y}_a(t;\theta_u,\mathbf{u})\right] = \sum_{k_1,k_n=1}^{K_a} \alpha_{0,n}(k_1,\cdots,k_n)\, e^{-j(\omega_1k_1+\ldots+\omega_nk_n)} \tag{39}$$

Thus a pure n-th order term in $u(t)$ contributes *only* to the n-th order operator $\mathcal{E}_n$.

The pure output numerator nonlinear terms are given by

$$\mathbf{Y}_a(t;\theta_y,\mathbf{y}) = \sum_{m=p=1}^{M_a} \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1,\cdots,k_p) \prod_{i=1}^{p} y(t-k_i) \tag{40}$$

and the combined contribution from all such terms is

$$\mathcal{E}_n^{asym}\left[\mathbf{Y}_a(t;\theta_y,\,\mathbf{y})\right] = \sum_{p=1}^{n} \sum_{k_1,k_p=1}^{K} \alpha_{p,0}(k_1,\cdots,k_p)\;H_{n,p}(j\omega_1,\cdots j\omega_n) \tag{41}$$

where the recursive formula $H_{n,p}(\cdot)$ is given by (29). Also notice that the uppermost limit on the first summation has been reduced from M to n since $H_{n,p}(\cdot) = 0$ for $p > n$.

The largest sub-class of polynomial terms which contains only pure cross product terms corresponds to $p\neq0,q\neq0$, in equation (7), giving

$$\mathbf{Y}_a(t;\theta_{uy},\,\mathbf{y},\mathbf{u}) = \sum_{m=1}^{M_a} \sum_{p=1}^{m-1} \sum_{k_1,k_{p+q}=1}^{K_a} \alpha_{p,q}(k_1,\cdots,k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \tag{42}$$

Eqn.(28) in Remark 2 suggests that the contribution from the cross-product terms can be obtained by multiplying the q'th order contribution from the pure $u(t)$ component with the (recursive) $(n-q)$'th order contribution from the pure $y(t)$ component within the major summator. That is

$$\mathcal{E}_n^{asym}\left[\mathbf{Y}_a(t;\theta_{uy},\,\mathbf{y},\mathbf{u})\right] = \tag{43}$$

$$\sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1,k_n=1}^{K_a} \alpha_{p,q}(k_1,\cdots,k_{p+q})\; e^{-j(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q})}\; H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})$$

where the exponential factor relates to the input part of the nonlinearity and the recursive factor $H_{n-q,p}(\cdot)$ to the output part. Notice that the upper limits on the summations have been lowered as before because $\mathcal{E}_n[\cdot] = 0$ for $m = p+q > n$, and $H_{n-q,p}(\cdot)$ is generated using the recursive relation (29).

5.3. Denominator output terms $\mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\theta_y,\,\mathbf{y})\right]$

The full class of denominator output terms can be expressed as

$$y(t)\mathbf{Y}_b(t;\theta_y,H,\omega_r) = y(t) \sum_{m=p=1}^{M_b} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\cdots,k_p) \prod_{i=1}^{p} y(t-k_i)$$

$$= \sum_{m=p=1}^{M_b} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\cdots,k_p)\; y(t)\prod_{i=1}^{p} y(t-k_i) \tag{44}$$

These can be treated just as the $(p+1)$ degree pure output terms in Remark 2. The contribution from each individual term will be

$$H^{asym}_{n,p+1}(\cdot) = \sum_{i=1}^{n-p} H^{asym}_i(j\omega_1, \cdots j\omega_i) H_{n-i,p}(j\omega_{i+1}, \cdots j\omega_n)\ e^{-j(\omega_1 + \cdots + \omega_i)k_{p+1}} \tag{45}$$

where the $(p+1)$'th lag of $y(t)$, k_{p+1}, is zero. The above formulae can also be written seperately in order to avoid notational confusion

$$H^{asym}_{n,p+1}(\cdot) = \sum_{i=1}^{n-p} H^{asym}_i(j\omega_1, \cdots j\omega_i) H_{n-i,p}(j\omega_{i+1}, \cdots j\omega_n) \tag{46}$$

where $H_{n,p}(\cdot)$ is given by eqn (29). The contribution from this whole sub-class of terms is therefore given as

$$\mathcal{E}^{asym}_n\left[y(t)\mathbf{Y}_b(t;\theta_y, \mathbf{y})\right] = \sum_{m=p=1}^{n-1} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1, \cdots ,k_p)\ H_{n,p+1}(j\omega_{i+1}, \cdots j\omega_n)$$

$$= \sum_{m=p=1}^{n-1} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1, \cdots ,k_p) \sum_{i=1}^{n-p} H^{asym}_i(j\omega_1, \cdots j\omega_i) H_{n-i,p}(j\omega_{i+1}, \cdots j\omega_n) \tag{47}$$

where the upper most limit on the first summation has been reduced from M_b to $(n-1)$ since $H_{n,p+1}(\cdot)=0$ for $p>n-1$.

5.4. Denominator input terms $\mathcal{E}^{asym}_n\left[y(t)\mathbf{Y}_b(t;\theta_u, \mathbf{u})\right]$

These type of terms can be expressed as

$$y(t)\mathbf{Y}_b(t;\theta_u,H,\omega_r) = y(t) \sum_{m=q=1}^{M_b} \sum_{k_1,k_q=1}^{K_b} \beta_{0,q}(k_1, \cdots ,k_q) \prod_{i=1}^{q} u(t-k_i)$$

$$= \sum_{m=q=1}^{M_b} \sum_{k_1,k_q=1}^{K_b} \beta_{0,q}(k_1, \cdots ,k_q)\ y(t)\prod_{i=1}^{q} u(t-k_i) \tag{48}$$

The contribution to $\mathcal{E}_n[\cdot]$ by these terms must be made jointly by $y(t)$ and the q degree nonlinear terms in $u(t)$. Thus the contribution from each individual term is given by

$$\beta_{0,q}(k_1,\ldots,k_q)e^{-j(\omega_{n-q+1}k_1 + \cdots + \omega_n k_q)} H_{n-q,1}(j\omega_1,\ldots j\omega_{n-q}) =$$

$$\beta_{0,q}(k_1,\ldots,k_q)e^{-j(\omega_{n-q+1}k_1 + \cdots + \omega_n k_q)} H_{n-q}(j\omega_1,\ldots j\omega_{n-q})$$

and the combined contribution from this sub-class of terms will be

$$\mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\boldsymbol{\theta}_u,\ \mathbf{u})\right] = \sum_{q=1}^{n-1}\ \sum_{k_1,k_p=1}^{K_b} \beta_{0,q}(k_1,...,k_q)e^{-j(\omega_{n-q+1}k_1+...+\omega_n k_q)}H_{n-q}(j\omega_1,...,j\omega_{n-q}) \tag{49}$$

5.5. Denominator cross-product terms $\mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\boldsymbol{\theta}_{uy},\ \mathbf{y},\mathbf{u})\right]$

The general form for these type of terms is

$$y(t)\mathbf{Y}_b(t;\boldsymbol{\theta}_{uy},\ H,\ \mathbf{y},\mathbf{u}) = \sum_{m=1}^{M_b}\ \sum_{p=1}^{m-1}\ \sum_{k_1,k_{p+q}=1}^{K_b} \beta_{p,q}(k_1,\ \cdots\ ,k_{p+q})\ y(t)\prod_{i=1}^{p}y(t-k_i)\prod_{i=p+1}^{p+q}u(t-k_i) \tag{50}$$

Again the qth degree pure input part can only produce q nonrepetitive frequency combinations from $\{\omega_1,\ .\ .\ .\ ,\omega_n\}$, the remaining part required to generate the term $e^{j(\omega_1+\ \cdots\ +\omega_n)t}$ is provided by the $(p+1)$ degree terms in $y(t)$. Hence

$$\mathcal{E}_n^{asym}\left[y(t)\mathbf{Y}_b(t;\boldsymbol{\theta}_{uy},\ \mathbf{y},\mathbf{u})\right] =$$

$$\sum_{q=1}^{n-1}\ \sum_{p=1}^{n-q-1}\ \sum_{k_1,k_{p+q}=1}^{K_b} \beta_{p,q}(k_1,\ \cdots\ ,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{p+1}+\ \cdots\ +\omega_n k_{p+q})}H_{n-q,p+1}(j\omega_1,...,j\omega_{n-q}) \tag{51}$$

where the lag associated with $y(t)$ should be zero. Following the analysis in section 5.3, $H_{n-q,p+1}(\cdot)$ can be written seperately as

$$H_{n-q,p+1}^{asym}(\cdot) = \sum_{i=1}^{n-q-p} H_i^{asym}(j\omega_1,\ \cdots\ j\omega_i)H_{n-q-i,p}(j\omega_{i+1},\ \cdots\ j\omega_{n-q}) \tag{52}$$

where $H_{n,p}(\cdot)$ is given by eqn.(29).

5.6. The final recursive algorithm

All the component parts of eqn.(36) have now been individually evaluated. Substituting all these, eqn's ((37), (39), (41), (43), (47), (49) and (51), into eqn (36) gives the final expression for the GFRF of the rational model eqn (6) as

$$\left[\beta_{0,0} - \sum_{k_1=1}^{K_a}\alpha_{1,0}(k_1)\ e^{-j(\omega_1+\ \cdots\ +\omega_n)k_1}\right]H_n^{asym}(j\omega_1,\ \cdots\ j\omega_n) =$$

$$+\sum_{k_1,k_n=1}^{K_a}\alpha_{0,n}(k_1,\ \cdots\ ,k_n)\ e^{-j(\omega_1 k_1\ +\ \cdots\ +\ \omega_n k_n)}$$

$$+\sum_{p=2}^{n}\ \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1,\cdots,k_p)\ H_{n,p}(j\omega_1,\cdots j\omega_n)$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\ \sum_{k_1,k_{p+q}=1}^{K_a} \alpha_{p,q}(k_1,\cdots,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{p+1}+\ldots+\ \omega_n k_{p+q})}\ H_{n-q,p}(j\omega_1,\cdots j\omega_{n-q})$$

$$-\sum_{q=1}^{n-1}\ \sum_{k_1,k_p=1}^{K_b} \beta_{0,q}(k_1,\cdots,k_q)e^{-j(\omega_{n-q+1}k_1+\ldots+\omega_n k_q)}H_{n-q}(j\omega_1,\cdots j\omega_{n-q})$$

$$-\sum_{m=p=1}^{n-1}\ \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\ldots,k_p)\left[\sum_{i=1}^{n-p} H_i^{asym}(j\omega_1,\ldots j\omega_i)H_{n-i,p}(j\omega_{i+1},\ldots j\omega_n)\right]$$

$$-\sum_{q=1}^{n-1}\ \sum_{p=1}^{n-q-1}\ \sum_{k_1,k_{p+q}=1}^{K_b} \beta_{p,q}(k_1,\cdots,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{p+1}+\ \cdots\ +\ \omega_n k_{p+q})}$$

$$\times\left[\sum_{i=1}^{n-q-p} H_i^{asym}(j\omega_1,\cdots j\omega_i)H_{n-q-i,p}(j\omega_{i+1},\cdots j\omega_{n-q})\right] \qquad (53)$$

where from eqn (29)

$$H_{n,p}(\cdot)\ =\ \sum_{i=1}^{n-p+1} H_i^{asym}(j\omega_1,\cdots j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots j\omega_n)\ e^{-j(\omega_1+\ \cdots\ +\omega_i)k_p}$$

Notice that eqn (53) yields the asymmetric GFRF. It is a simple matter to obtain a unique symmetric GFRF by applying the relation

$$H_n^{sym}(j\omega_1,\cdots j\omega_n)\ =\ \frac{1}{n!}\sum_{\substack{all\ permutations\\ of\ \omega_1\ \cdots\ \omega_n}} H_n^{asym}(j\omega_1,\cdots j\omega_n) \qquad (54)$$

as described in Section 2.

Now we can re-derive $H_1(\cdot)$ and $H_2(\cdot)$ for the simple example eqn (19) using the recursive formulae (53). For the case n=1

$$-a_2H_1(j\omega_1)e^{-j\omega_1}\ =\ a_1e^{-2j\omega_1}\quad =>\quad H_1(j\omega_1) = -\ \frac{a_1}{a_2}e^{-j\omega_1}$$

For the case n=2

$$-a_2e^{-j(\omega_1+\omega_2)}H_2^{asym}(j\omega_1,j\omega_2) = -b_2H_{2,2}(j\omega_1,j\omega_2) - b_1e^{-j\omega_2}H_{2,1}(j\omega_1)$$

$$= -b_2H_1(j\omega_1)H_1(j\omega_2) - b_1H_1(j\omega_1)e^{-j\omega_2}$$

so that

$$H_2^{asym}(j\omega_1, j\omega_2) = \frac{1}{a_2}\left[b_2 H_1(j\omega_1)H_1(j\omega_2) - b_1 H_1(j\omega_1)e^{-j\omega_2}\right]$$

After symmetrisation

$$H_2^{sym}(j\omega_1, j\omega_2) = \frac{1}{2a_2}\Big[b_1 H_1(j\omega_1)e^{j\omega_1} +$$

$$b_1 H_1(j\omega_2)e^{j\omega_2} + b_2 H_1(j\omega_1)H_1(j\omega_2)e^{j\omega_1} + b_2 H_1(j\omega_1)H_1(j\omega_2)e^{j\omega_2}\Big]$$

These are identical to the results derived by hand in Section 4 (see eqn's (22) and (23)).

Because all the terms in the numerator and denominator appear seperately in eqn (53) the evaluation of the GFRF's is relatively straightforward and not as complicated as the general expression suggests. Eqn (53) can be used both to evaluate numerical values for the GFRF's or to derive general analytical expressions for the frequency response functions for given rational model systems.

Inspection of eqn. (53) shows that the denominator, or the poles, of the frequency response are determined by the linear output terms in the numerator and the constant term in the denominator of the rational model. Therefore for the system eqn (6) to generate transfer functions there must exist at least one non-zero linear output term in the numerator, or there must be a non-zero constant term in the denominator. The zeros, the numerator of the first order (linear) frequency response function $H_1(j\omega)$ is dependent only on the pure linear input terms in the numerator polynomial $Y_a(\cdot)$ of the rational model. Hence it is quite straitghforward to write $H_1(j\omega)$ by inspection from the model as

$$H_1(j\omega) = \frac{\sum_{k_1=1}^{K_a} \alpha_{0,1}(k_1)e^{-j\omega k_1}}{\beta_{0,0} - \sum_{k_1=1}^{K_a} \alpha_{1,0}(k_1)e^{-j\omega k_1}} \tag{55}$$

Finally it is interesting to note that the GFRF for the standard polynomial model is just a special case of eqn (53) given by setting $\mathbf{Y}_b(\cdot)=1$ in eqn (6) to yield

$$\left[1 - \sum_{k_1=1}^{K_a} \alpha_{1,0}(k_1)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1}\right] H_n^{asym}(j\omega_1, \cdots j\omega_n) =$$

$$+ \sum_{k_1,k_n=1}^{K_a} \alpha_{0,n}(k_1, \cdots, k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+ \sum_{p=2}^{n} \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1, \cdots, k_p)\; H_{n,p}(j\omega_1, \cdots j\omega_n) \tag{56}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1,k_{p+q}=1}^{K_a} \alpha_{p,q}(k_1, \cdots, k_{p+q})\; e^{-j(\omega_{n-q+1}k_{p+1} + \ldots + \omega_n k_{p+q})}\, H_{n-q,p}(j\omega_1, \cdots j\omega_{n-q})$$

6. Example

As an example consider the following modified Van-der-Pol equation,

$$D^2y(t) + 2\zeta\omega_n\,(1 - y(t)^2)\,Dy(t) + \omega_n^2 y(t) - u(t) = 0 \tag{57}$$

The nonlinearity in this system is induced by the damping term so that for small displacements the damping is positive (limiting), and for large displacements the damping is negative (self-exciting). Thus the system has a stable node at the origin, with a domain of attraction which lies within the (unstable) limit cycle. Evaluating the GFRF directly from eqn (57) using the relationship for continuous-time polynomial nonlinear differential equations given by Billings and Peyton-Jones(1990), the first order frequency response which depends only on the linear terms

$$H_1(j\omega) = \frac{1}{\omega_n^2 + 2\zeta\omega_n(j\omega) + (j\omega)^2} \tag{58}$$

The gain and phase of $H_1(\cdot)$ are plotted in Figure 1(a) and (b) with the values $\zeta = 0.01$ and $\omega_n = 45\pi$. The linear frequency response exhibits a resonant peak at a frequency of 22.5Hz.

The nonlinear damping term of eqn (57) however generates frequency response functions for orders 3 and higher(note that H_2=0). In the third order case

$$H_3(j\omega_1, j\omega_2, j\omega_3) = \frac{2\zeta\omega_n}{3} \times \frac{(j\omega_1 + j\omega_2 + j\omega_3)\, H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{\omega_n^2 + 2\zeta\omega_n(j\omega_1 + j\omega_2 + j\omega_3) + (j\omega_1 + j\omega_2 + j\omega_3)^2} \tag{59}$$

which is illustrated in Figures 2(a) and (b) by fixing $\omega_3=\omega_1$.

Now consider the effects of converting eqn (57) into a discrete-time expression using the backward difference scheme

$$\frac{dy(t)}{dt} = \frac{y(t) - y(t-1)}{h} \tag{60}$$

and

$$\frac{d^2y(t)}{dt^2} = \frac{\dot{y}(t) - \dot{y}(t-1)}{h} = \frac{y(t) - 2y(t-1) + y(t-2)}{h^2} \tag{61}$$

where h is the sampling period and all the indices t on the right hand side of the eqn's (60) and (61) are discrete time intervals. Substituting eqn (60) and (61) into the continuous-time Van-der-Pol equation (57) yields

$$\frac{[y(t) - 2y(t-1) + y(t-2)]}{h^2} + 2\zeta\omega_n(y(t)^2 - 1)\frac{[y(t) - y(t-1)]}{h} + \omega_n^2 y(t) - u(t) = 0 \tag{62}$$

Rearranging gives

$$y(t) = \frac{\alpha_{1,0}(1)y(t-1) + \alpha_{1,0}(2)y(t-2) + \alpha_{0,1}(0)u(t)}{\beta_{0,0} + \beta_{2,0}(0,0)y^2(t) + \beta_{2,0}(1,0)y(t-1)y(t)} \tag{63}$$

with

$\alpha_{1,0}(1) = 2+2\zeta\omega_n h;$ $\quad\alpha_{1,0}(2) = -1;$ $\quad\alpha_{0,1}(0) = h^2;$

$\beta_{0,0} = 1+\omega_n^2 h^2-2\zeta\omega_n h;$ $\quad\beta_{2,0}(0,0) = 2\zeta\omega_n h;$ $\quad\beta_{2,0}(1,0) = -2\zeta\omega_n h;$

Eqn (63) is a typical nonlinear rational model expression which can be considered as a discrete-time approximation for the Van-der-Pol equation. Although there may be better ways of discretising the Van-der-Pol equation the approximation of Eqn (63) allows us to verify the expression for the GFRF of rational models by comparing the results obtained with the exact expression given by eqn's (60) and (61).

The first order (linear) frequency response obtained from eqn (53) is

$$H_1(j\omega) = \frac{\alpha_{0,1}(0)}{\beta_{0,0} - \alpha_{1,0}(1)e^{-j\omega} - \alpha_{1,0}(2)e^{-2j\omega}}$$

$$= \frac{h^2}{1+\omega_n^2h^2-2\zeta\omega_n h-(2+2\zeta\omega_n h)e^{-j\omega} + e^{-2j\omega}} \tag{64}$$

As expected there is no nonlinear coefficient in H_1. The second order GFRF is zero because when n=2 there are no terms which make a contribution to the right hand side of eqn (53). This is in agreement with the original system. For the third order case, eqn (53) yields

$$\left[\beta_{0,0} - \alpha_{1,0}(1)e^{-j(\omega_1+\omega_2+\omega_3)} - \alpha_{1,0}(2)e^{-2j(\omega_1+\omega_2+\omega_3)}\right]H_3(j\omega_1,j\omega_2,j\omega_3) =$$

$$- \beta_{2,0}(0,0)H_{3,3}(j\omega_1,j\omega_2,j\omega_3) - \beta_{2,0}(1,0)H_{3,3}(j\omega_1,j\omega_2,j\omega_3) \qquad (65)$$

The contributions from the two pure output terms $\beta_{2,0}(0,0)y^2(t)$ and $\beta_{2,0}(1,0)y(t-1)y(t)$ in the denominator of the model eqn (63) are given by the recursive relation (29) as

$$\beta_{2,0}(0,0)H_{3,3}(j\omega_1,j\omega_2,j\omega_3) = \beta_{2,0}(0,0)H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)$$

and

$$\begin{aligned}\beta_{2,0}(1,0)H_{3,3}(j\omega_1,j\omega_2,j\omega_3) &= \beta_{2,0}(1,0)H_1(j\omega_1)H_{2,2}(j\omega_2,j\omega_3)e^{-j\omega_1}\\ &= \beta_{2,0}(1,0)H_1(j\omega_1)H_1(j\omega_2)H_{1,1}(j\omega_3)e^{-j\omega_1}\\ &= \beta_{2,0}(1,0)H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)e^{-j\omega_1}\end{aligned}$$

After applying the symmetrisation by (5), the symmetric GFRF is obtained as

$$\begin{aligned}H_3^{sym}(\cdot) &= -\frac{[(\beta_{2,0}(0,0) + \beta_{2,0}(1,0)(e^{-j\omega_1}+e^{-j\omega_2}+e^{-j\omega_3})/3]H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{\beta_{0,0} - \alpha_{1,0}(1)e^{-j(\omega_1+\omega_2+\omega_3)} - \alpha_{1,0}(2)e^{-2j(\omega_1+\omega_2+\omega_3)}}\\ &= -\frac{2\zeta\omega_n h[1 - (e^{-j\omega_1}+e^{-j\omega_2}+e^{-j\omega_3})/3]H_1(j\omega_1)H_1(j\omega_2)H_1(j\omega_3)}{1+\omega_n^2h^2-2\zeta\omega_n h - (2+2\zeta\omega_n h)e^{-j(\omega_1+\omega_2+\omega_3)} + e^{-2j(\omega_1+\omega_2+\omega_3)}} \qquad (66)\end{aligned}$$

Both H_1 eqn (64) and H_3 eqn (66) are plotted, in terms of gain and phase, in Fig.3, 4, 5 and 6 for the sampling frequencies $F_s = 1/h =$ 1k and 10k Hz, respectively. Comparing these plots with the true frequency responses of the original system Fig.1 and Fig 2 shows that the discrete estimates are converging to the correct GFRF as the sampling frequency increases. Ideally the frequency response of the discrete-time model, eqn (63), should be the same as that of the original system provided the discretisation is adequate and the sampling frequency is small enough.

In this example the analysis was only evaluated up to the third order but the recursive algorithm eqn (53) places no restriction on the order of the computed frequency response. A comparison with the original Van-der-Pol equation clearly demonstrates that the expression for GFRF of nonlinear rational models is correct.

7. Conclusions

An algebraic expression for the generalised frequency response functions for a large class of severely nonlinear systems has been derived from an identified time-domain rational model. This expression enables the frequency response behaviour to be related to the structure and parameters of the corresponding time-domain model. The frequency response functions exhibit invariant characteristics of the underlying system, regardless of the form of time-domain model, and it should therefore be possible to re-construct nonlinear differential equation models for practical systems from an identified discrete-time model.

Combining the parameter estimation techniques with the new frequency response function algorithm for rational models provides a powerful procedure for analysing a large class of severely nonlinear systems.

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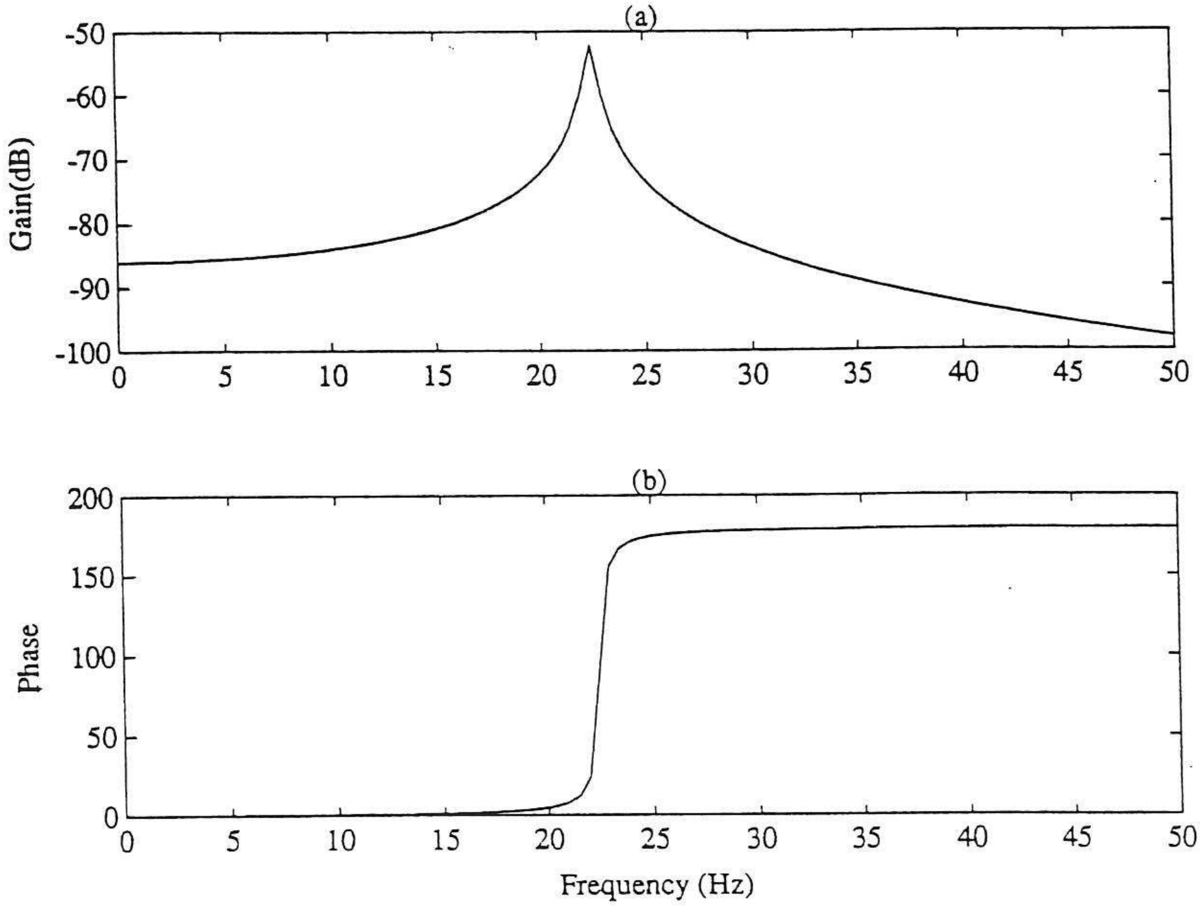

Fig.1 H_1 for the original system: (a) Gain, (b) Phase.

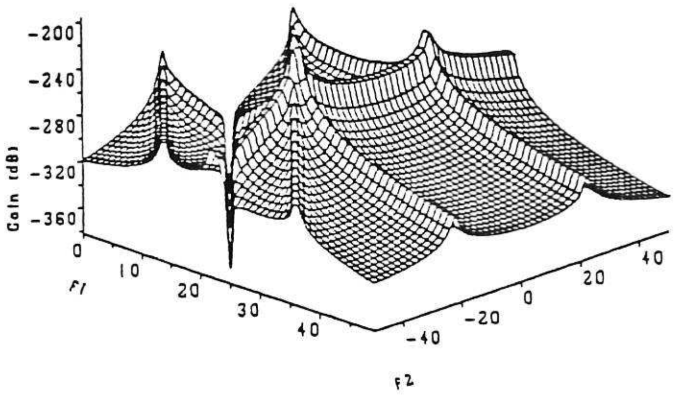

Fig.2(a)

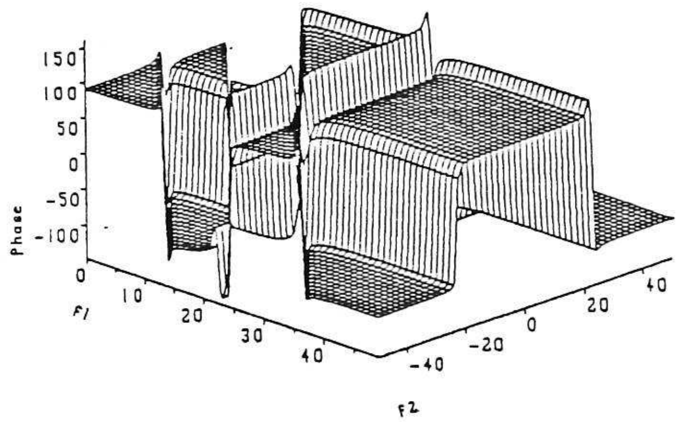

Fig.2(b)

Fig.2 H_3 for the original system: (a) Gain, (b) Phase.

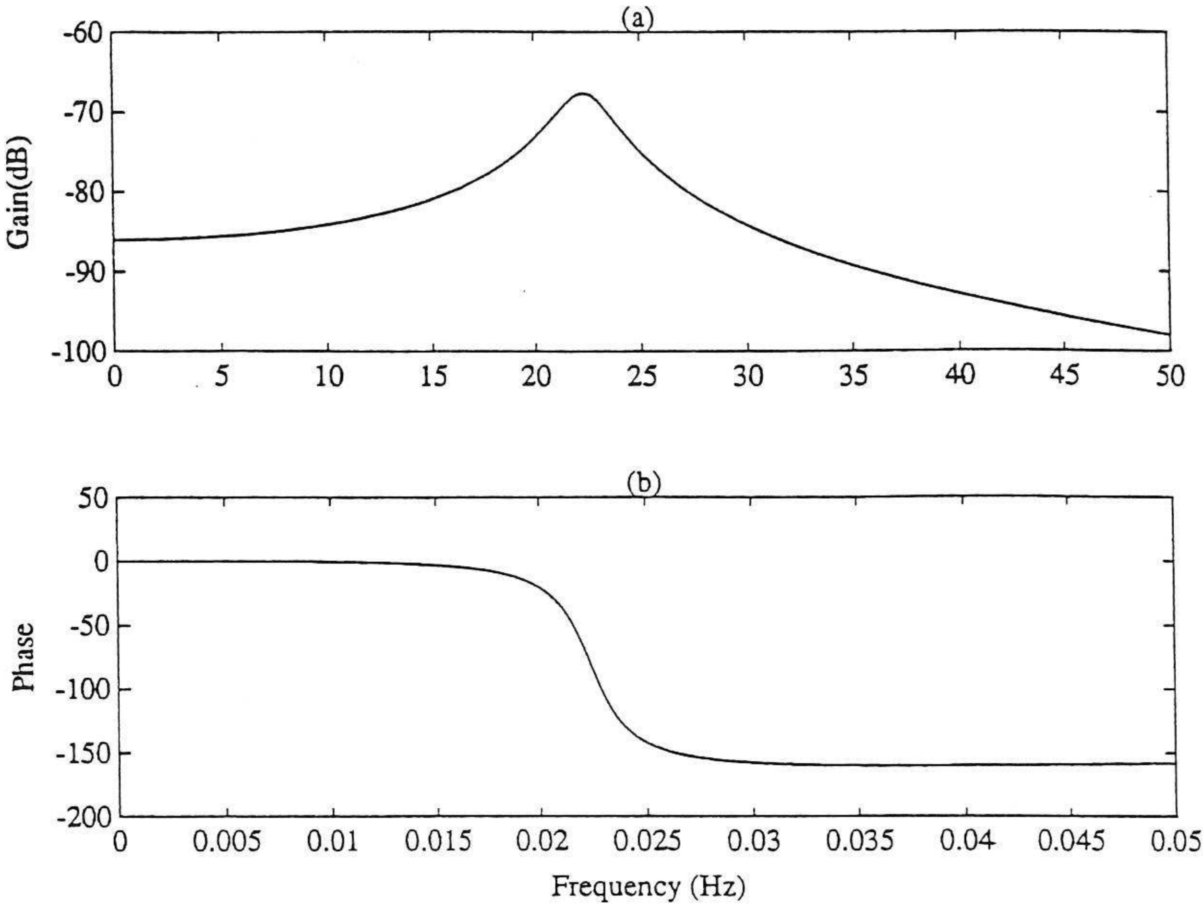

Fig.3 H_1 for the rational model with sampling frequency 1k Hz:

(a) Gain, (b) Phase.

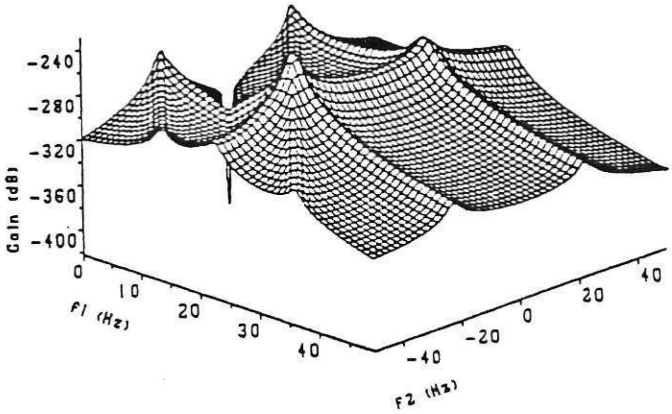

Fig.4(a)

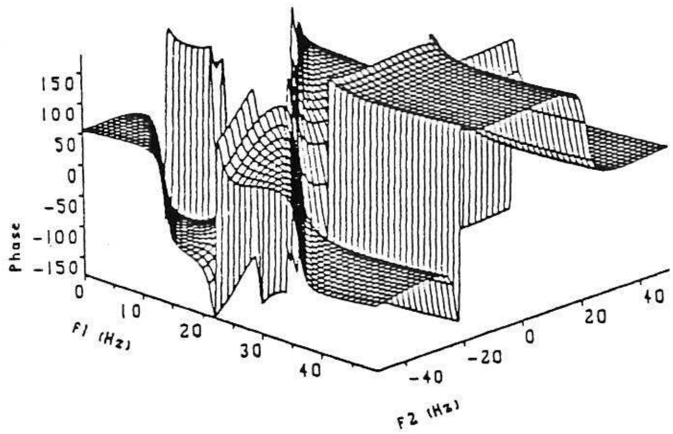

Fig.4(b)

Fig.4 H_3 for the rational model with sampling frequency 1k Hz:

(a) Gain, (b) Phase.

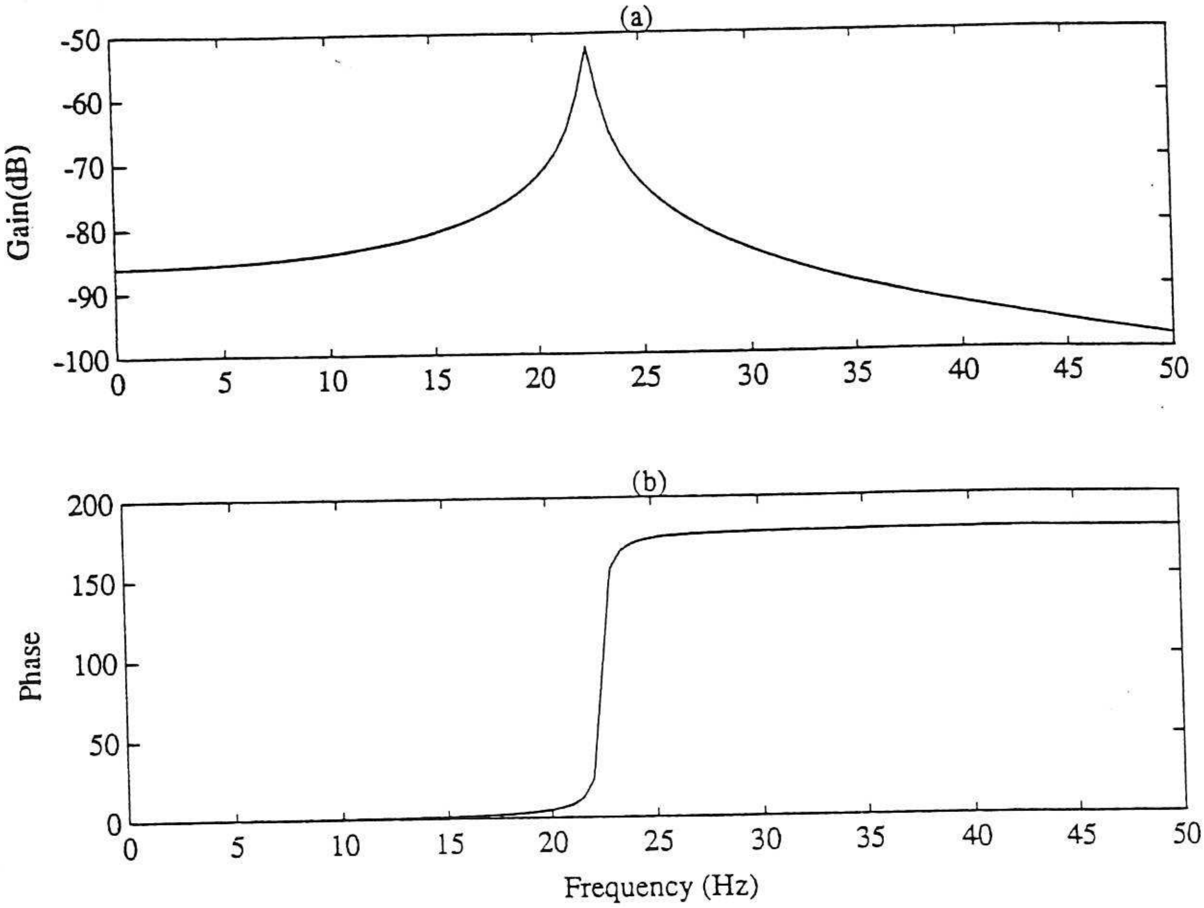

Fig.5 H_1 for the rational model with sampling frequency 10k Hz:

(a) Gain, (b) Phase.

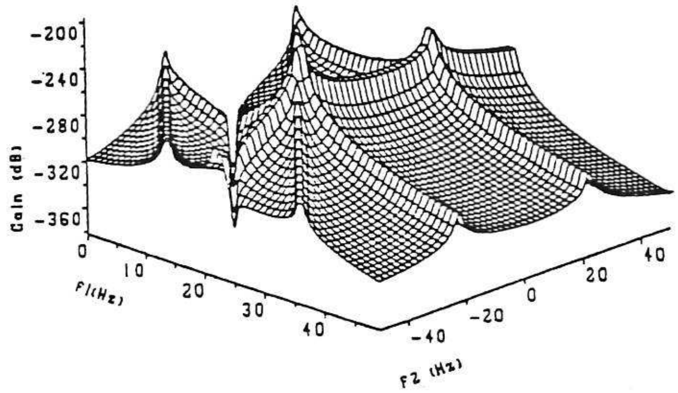

Fig6.(a)

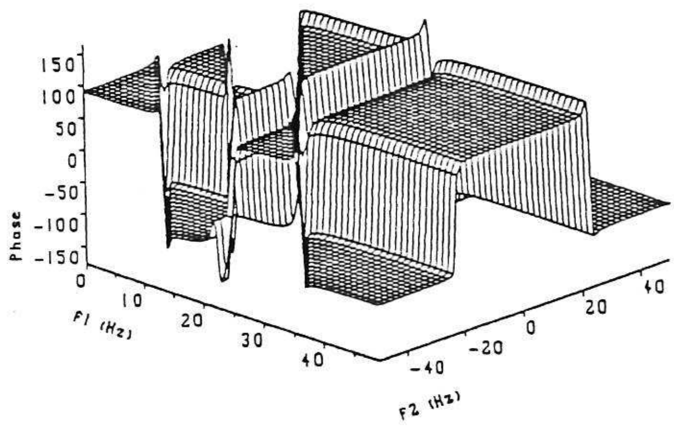

Fig.6(b)

Fig.6 H_3 for the rational model with sampling frequency 10k Hz:

(a) Gain, (b) Phase.