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Discrete Reconstruction of the Double Scroll Attractor

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Discrete Reconstruction of the Double Scroll Attractor

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Abstract

This work is concerned with the discrete reconstruction of the double scroll attractor. Two aspects of the reconstruction are investigated, i) the discretization of the original equations via numerical methods, and ii) the identification of discrete nonlinear models from time series generated by the original system. The quality of the models is assessed by estimating the largest Lyapunov exponent, the correlation dimension and by comparing the geometry of the reconstructed attractor to the original double scroll. The major aim of the paper is to investigate how parameters such as the discretization period, sampling rate and the number of terms in an estimated model affect the dynamics and also to point out if there is any relationship among such parameters. Conclusions in this direction are believed to be especially relevant in determining adequate model structures in identification applications. This is known to be critical in the identification of nonlinear systems.

I. INTRODUCTION

Although most real systems are continuous in time, it is often desirable to derive discrete models which faithfully represent the dynamics of such systems. Some of the reasons for this are i) in practice, measurements are usually carried out at specific time intervals, ii) digital processing and control is becoming increasingly common, and iii) digital simulations can be performed quickly and easily.

Moreover, it is well recognized that the choice of the discrete interval is critical in reconstructing the dynamics from a set of measurements. In particular, such an interval affects dynamic invariants such as the geometry of attractors, the largest Lyapunov exponent, λ_1, and the correlation dimension, D_c. Thus the proper choice of the discrete interval is crucial when discretizing continuous-time differential equations for numerical integration and also when sampling a time series from which a discrete model will be identified. In the former case such an interval is referred to as the discretization period, T_d, and in the latter as the sampling period, T_s.

A related issue is the choice of the variable from which the dynamics will be reconstructed. Sometimes only one variable of a system is measured and recorded. It is not evident that it would be possible to reconstruct the dynamics of an nth-order system from measurements of r $(r<n)$ variables. Fortunately, Takens has shown that, provided that the

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reconstructed (embedded) space has a sufficiently large dimension, in many cases only one measured variable will suffice in reconstructing the original dynamics [1].

Nonlinear discrete models are identified from data lying on the double scroll attractor. The comparison of such models with discretized counterparts illustrate a number of issues concerning model structure selection. An interesting result is that the piecewise linear function of the Chua system is completely incorporated in the identified models at the expense of an increase in the number or terms. It has also been verified that identification is possible from measurements of a single variable.

It is believed that the identified models cited in this paper are the only discrete and nonlinear (note, not piecewise linear) models available for the double scroll attractor.

II. Preliminaries

Chua's circuit is certainly one of the most well studied nonlinear circuits and a great number of papers ensure that the dynamics of this circuit are also well documented [2]-[4] The normalized equations of Chua's circuit are

$$\begin{cases} \dot{x} = \alpha(y - h(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases} \tag{1}$$

where

$$h(x) = \begin{cases} m_1 x + (m_0 - m_1) & x \geq 1 \\ m_0 x & \mid x \mid \leq 1 \\ m_1 x - (m_0 - m_1) & x \leq -1 \end{cases} \tag{2}$$

In what follows $m_0 = -1/7$ and $m_1 = 2/7$. Varying the parameters α and β the circuit displays several regular and chaotic regimes. The famous double scroll attractor, for instance, is obtained for $\alpha = 9$ and $\beta = 100/7$ and has the largest Lyapunov exponent and the Lyapunov dimension equal to $\lambda_1 = 0.23$ and $D_L = 2.13$, respectively [4]. Such values of α and β will be used henceforth. In the examples the correlation dimension, D_c, will be used instead of D_L. It is known that in general $D_c < D_L$ [5]. In fact, for the original attractor the following value was estimated, $D_c = 1.99 \pm 0.023$.

Equations (1)–(2) were simulated using a fourth-order Runge-Kutta algorithm with integration interval equal to 10^{-3}. The resulting data were sampled at the rate defined by T_s. The attractor can then be reconstructed using just one of the variables as shown in Figs. 1a–c.

The way models are compared and validated is crucial. Because it is desired to investigate how T_s influences the dynamics and also to verify the feasibility of identifying discrete nonlinear maps from data on the double scroll attractor, it seems necessary to use invariants which characterize the dynamics quantitative and qualitatively. This is especially true for chaotic systems of which Chua's circuit is a well known example.

Therefore this paper will consider pseudo state spaces, largest Lyapunov exponents, λ_1, and correlation dimensions, D_c, of the reconstructed attractors in order to assess the quality of the models used in such reconstruction.

Plots of the pseudo state space give an idea of the attractor shape and also convey topological information. The largest Lyapunov exponent measures the local average divergence

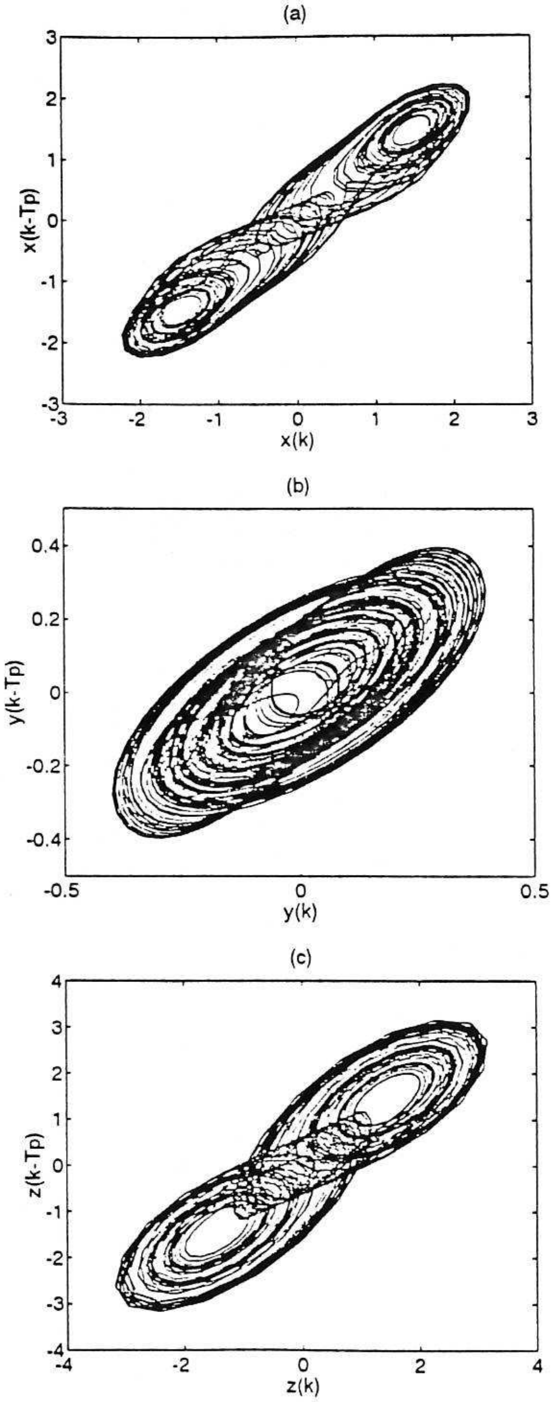

Fig. 1. Reconstruction of the double scroll attractor from the original system using (a) x component, $T_s = 0.15$ and $T_p = 2 \times T_s$, (b) y component, $T_s = 0.07$ and $T_p = 4 \times T_s$ and (c) z component, $T_s = 0.15$ and $T_p = 2 \times T_s$. The original system was integrated using a 4th-order Runge Kutta algorithm with an integration interval equal to 10^{-3}.

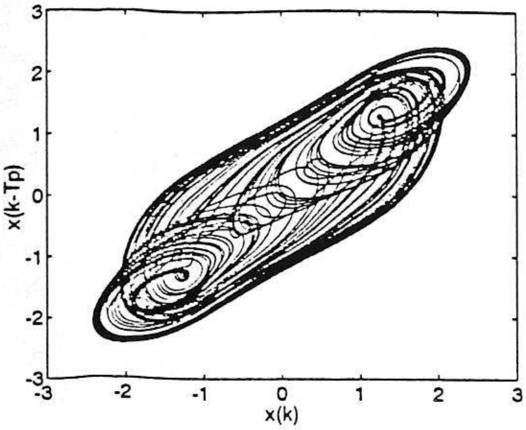

Fig. 2. Reconstructed attractor using the x component of a discretized model (forward Euler) with $T_d = 0.05$ and $T_p = 6 \times T_s$.

of nearby trajectories in state space and therefore quantifies the sensitive dependence on initial conditions. Finally, the correlation dimension quantifies the fractal structure of the attractor. These metrics have been suggested for validating chaotic models [6]-[7] and for details concerning the computation of such invariants see [8].

III. Discretization

Two *explicit* discretization schemes were considered, namely the *Euler* approximation and a 4th-order Runge-Kutta integration algorithm. The former, also known as the *forward Euler* algorithm, is a simple discretization method which does not need any recursive computation but, as a consequence, requires very small integration intervals in order to yield accurate results. Similar properties are shared by 2nd-order Runge-Kutta algorithms [9]. On the other hand, the fourth-order Runge-Kutta method uses the average of the function estimated at intermediate time points. Therefore accuracy is greatly improved at the expense of a far more time consuming numerical algorithm [8].

The objective of this section is to verify how the integration interval affects the geometry of the attractor, λ_1 and D_c. This will be important to compare the quality of the results obtained from identified models which are also discrete. Some results are listed in Table 1 and in Figs. 2 and 3.

Table 1. Discretized models

Algorithm	T_d	Fig.	λ_1 [a]	D_c
Euler	0.05	2	0.31	1.92 $\pm$0.028
RK	0.15	3	0.25	1.95 $\pm$0.013

[a] Calculated using $\log_e$

Table 1 shows that the *forward Euler* algorithm with $T_d = 0.05$ yields a rather inaccurate model. Figure 2 reveals that for this value of T_d the attractor is clearly distorted. For $T_d > 0.06$ the resulting models become unstable.

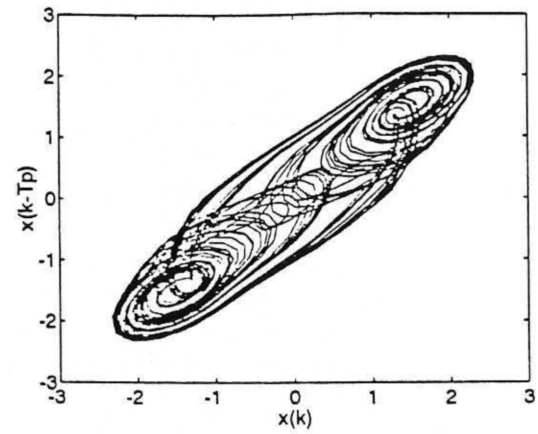

Fig. 3. Reconstructed attractor using the x component of a discretized model (4 th-order Runge-Kutta) with $T_d = 0.15$ and $T_p = 2 \times T_s$.

On the other hand, the discretization achieved using the Runge-Kutta algorithm is much better and the invariants shown in Table 1 are closer to the correct values. However, Fig. 3 illustrates that the attractor in this case already presents signs of distortion although not as severe as for the attractor shown in Fig. 2.

In comparing the plots, it should be borne in mind that what is of concern here is the overall shape of the attractors. The density of trajectories at different points on the attractor depends on the initial conditions and on the simulation time, consequently it is not a dynamical invariant. It is noted that strange attractors are space/volume-filling, hence if the simulation time is increased the trajectories in the plots become increasingly dense everywhere on the attractor.

As expected, if T_d is considerably decreased in either case, the resulting attractors become accurate.

It is worth pointing out that using the *forward Euler* relation

$$\dot{w}(k) \approx \frac{w(k+1) - w(k)}{T_d} \tag{3}$$

to approximate the derivatives in equation (1), the resulting discretized model has, in addition to the piecewise linear function, eight terms. This is believed to be a lower bound for the number of process terms in identified models if these are to reproduce the dynamics of the double scroll attractor. This will be considered in some detail in sections V and VI.

IV. Identification

Consider the nonlinear autoregressive moving average model (NARMA) [10]

$$y(t) = F^{\ell}[\, y(t-1), \ldots, y(t-n_y), e(t), \ldots\ldots, e(t-n_e)\,] \tag{4}$$

where $y(t)$ is a time series and $e(t)$ accounts for uncertainties, possible noise, unmodeled dynamics, etc. and $F^{\ell}[\cdot]$ is some nonlinear function of $y(t)$ and $e(t)$ with degree of nonlinearity $\ell \in \mathbb{Z}^+$.

In this paper, the map $F^\ell[\cdot]$ is taken to be a polynomial of degree ℓ. In order to estimate the parameters of this map, equation (4) has to be expressed in prediction error form as

$$
\begin{aligned}
y(t) &= \left[\Psi_y^{\mathrm{T}}(t-1)\ \Psi_{y\xi}^{\mathrm{T}}(t-1)\ \Psi_\xi^{\mathrm{T}}(t-1)\right] \begin{pmatrix} \hat{\Theta}_y \\ \hat{\Theta}_{y\xi} \\ \hat{\Theta}_\xi \end{pmatrix} + \xi(t) \\
y(t) &= \Psi^{\mathrm{T}}(t-1)\hat{\Theta} \ + \ \xi(t)
\end{aligned}
\tag{5}
$$

and the parameter vector Θ can be estimated by minimizing the following cost function [11]

$$
J_{LS}(\hat{\Theta}) \doteq \| \, y(t) - \Psi^{\mathrm{T}}(t-1)\hat{\Theta} \, \| \tag{6}
$$

where $\| \cdot \|$ is the Euclidean norm. Moreover, least squares minimization is performed using orthogonal techniques in order to effectively overcome two major difficulties in nonlinear model identification, namely i) numerical ill-conditioning and ii) structure selection.

To be able to determine the correct structure of a nonlinear model is crucial mainly because of two reasons. Firstly, the number of candidate terms in a polynomial model such as equation (4) becomes impractical even for moderate values of ℓ and n_y. Secondly, the inclusion of too many terms in a model can affect drastically the dynamical properties of such a model [12]. In this paper an algorithm based on the *error reduction ratio* (ERR) was used in order to choose the n_p terms in $\Psi_y^{\mathrm{T}}(t-1)$ [13].

Equations (1) and (2) were integrated using a 4 th-order Runge-Kutta algorithm. The resulting data (one time series for each component) were subsequently sampled with T_s. Each sampled time series contained 1900 data points and was in turn used to identify one independent model.

The choice of T_s was made based on the correlation time, τ_c, defined as the first minimum of the autocorrelation function. The following rule-of-thumb was used $\tau_c/10 < T_s < \tau_c/5$. In this case $\tau_c \approx 1.2$ for the three components and hence $T_s = 0.15$ was used.

It is interesting to note that, although $T_s = 0.15$ was a good choice for identifying models from the time series of the x and z components, no reasonable models were identified from the data of the y component sampled at this rate. After a number of simulations it became apparent that, for this component, T_s had to be decreased. Thus for such data $T_s = 0.07$ was used. However, it should be pointed out that the identification from the y component data was far more difficult and somewhat less accurate than for the other data. This is illustrated in Table 2 and in Figs. 4–6 which show that successful reconstruction is possible from any of the variables of the original system.

Table 2. Identified models

Comp.	T_d	n_y	n_p	Fig.	$\lambda_1{}^a$	D_c
x	0.15	4	17	4	0.25	2.00 ± 0.009
y	0.07	5	14	5	0.22	2.27 ± 0.052
z	0.15	5	16	6	0.23	1.99 ± 0.006

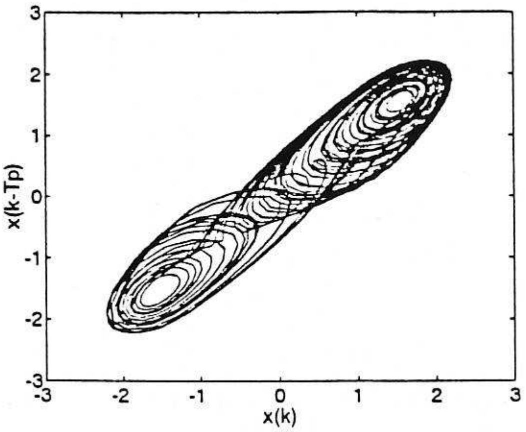

Fig. 4. Reconstructed attractor of the model identified from records of the x component sampled with $T_s = 0.15$.

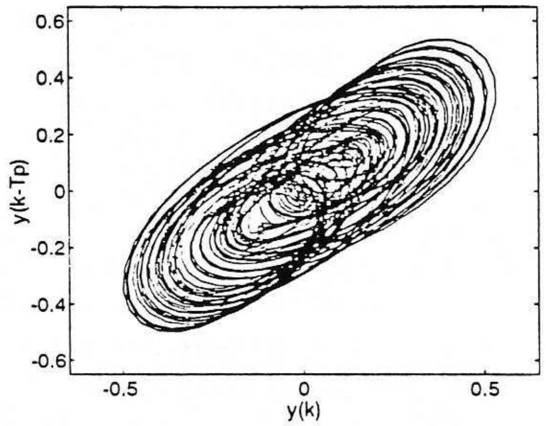

Fig. 5. Reconstructed attractor of the model identified from records of the y component sampled with $T_s = 0.07$.

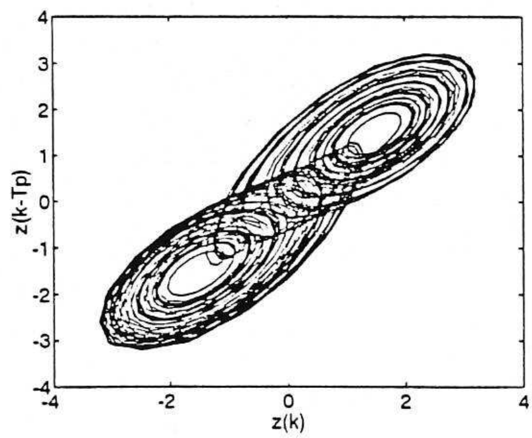

Fig. 6. Reconstructed attractor of the model identified from records of the z component sampled with $T_s = 0.15$.

In order to illustrate the kind of models identified during the course of this investigation, the model estimated from the z component is listed below

$$\begin{aligned} z(k) &= 3.2039z(k-1)-3.5558z(k-2)+1.5032z(k-3) \\ &- 0.24816\times10^{-1}z(k-4)-0.75094\times10^{-1}z(k-5) \\ &- 0.18175\times10^{-3}z(k-5)^3+0.26306\times10^{-1}z(k-1)z(k-5)^2 \\ &+ 0.24499z(k-1)^2z(k-5)-0.15721z(k-1)z(k-4)^2 \\ &- 0.79455z(k-1)^3-0.68211z(k-1)^2z(k-4) \\ &+ 1.0999z(k-1)z(k-2)z(k-4)+2.1980z(k-1)^2z(k-2) \\ &- 1.1365z(k-1)z(k-2)^2-0.29295z(k-1)z(k-2)z(k-5) \\ &- 0.52790z(k-1)^2z(k-3)+\Psi_\xi^T(t-1)\hat{\Theta}_\xi+\xi(t) \end{aligned} \tag{7}$$

where the residuals $\xi(t)$ are white and zero-mean with variance $\sigma_\xi^2=0.925\times10^{-5}$, $\Psi_\xi^T(t-1)=\xi(t-i)$ $i=1,2,\dots,20$. It is noted that the twenty terms in $\Psi_\xi^T(t-1)\hat{\Theta}_\xi$ are required in order to produce unbiased models but such terms and $\xi(t)$ are not actually used in the simulations.

V. Model Structure

In a nonlinear model such as equation (4) the number of terms increases exponentially with ℓ and n_y. For instance, the model in equation (7) has 16 process terms which were selected among 125 possible candidate terms (it is noted that in this case $\ell=3$ and $n_y=5$).

A 'brute force' approach, that is including all possible terms in the model, is not generally appropriate in nonlinear systems because, unlike the linear systems case where overparametrization usually leads to pole/zero cancellation, the overall dynamics are qualitatively and quantitatively affected by the model structure. In particular, overparametrized models are usually unstable.

To illustrate this point, consider the family of models identified from a time series of the x component for which $T_s=0.25$ with the maximum lag equal to $n_y=6$. It is noted that although models estimated from data sampled faster present improved overall dynamical characteristics, the deleterious effects of overparametrization are equivalent. The model with twelve process terms, $n_p=12$, reproduces the major features of the attractor shown in Fig. 1a. As n_p is increased the geometry of the reconstructed attractors obtained from the respective models clearly deteriorates. Finally, models with $n_p\geq16$ are unstable.

Similar results have been obtained using other values of T_s to sample the time series of other components. The general pattern observed as the number or process terms is increased was

PERIODIC $\longrightarrow$ CHAOTIC $\longrightarrow$ UNSTABLE

Based on this overall pattern, it is conjectured that periodic models are obtained when n_p is relatively low and therefore the model structure is not sufficiently complex to reproduce the dynamics. Within a rather narrow range of values of n_p (typically of three to four values) the resulting model structures are able to produce chaos. It is noted that this does not imply that the reconstructed chaotic attractor is quantitatively correct. This has to be verified during model validation as suggested in section II. Finally, if n_p is further increased the resulting models become unstable as a consequence of overparametrization.

Hence it seems clear that the subregion, in the space of possible model structures, where chaotic models can be identified for this system is rather limited. To exemplify this statement it is pointed out that considering the three components and the parameter values $\ell = 3$, $0.05 \leq T_s \leq 2.5$, $3 \leq n_y \leq 6$ and $5 \leq n_p \leq 20$ more than 145 models were estimated. For all of the chaotic models the number of process terms was $11 \leq n_p \leq 17$, every model with $n_p < 11$ was periodic and the vast majority of models with more than 17 terms were unstable. It is noted that the values of n_p for which the models become chaotic or unstable depend on other parameters such as n_y and T_s. For instance, for a particular combination of such parameters chaotic models are found only in the range $13 \leq n_p \leq 15$.

It is interesting to observe that the same limitation verified for the number of process terms seems to hold for the maximum lag considered in the model, n_y. This can be verified by noting that no chaotic model was estimated with $n_y \leq 3$ and that the models with $n_y = 6$ are clearly less accurate than the models with $3 < n_y < 6$. Concerning these results two things should be noted, i) although a large number of models were estimated, the number of possible models is far greater, and ii) the effect of n_y on the quality of the reconstructed dynamics is less dramatic than that of n_p.

Finally, it is noted that similar results have been verified for another nonlinear circuit, namely the Duffing-Ueda oscillator [10,14]. A major difference between these two systems is that while Chua's circuit is a piecewise linear system, the Duffing-Ueda oscillator is a globally nonlinear system. The consequences of this on model structure are investigated in the next section.

VI. Comparison Between Discretized and Identified Models

Tables 1 and 2 illustrate the advantages of identified models over the discretized counterparts. The Euler approximation does not yield stable models for any of the discretization periods in Table 2. To simulate the original system using a 4th-order Runge-Kutta algorithm is far more time-consuming than to simulate equation (7) and, in addition, the latter is more accurate than the original system discretized by the Runge-Kutta algorithm with $T_d = 0.15$ which is the sampling period for (7).

The objective of the above comparison was *not* to advocate in favor of identified models but rather to point out that the accuracy of such models is enhanced. It will be argued that such an improvement seems to be related to a slight increase in the number of terms in the models.

Comparison of discretised and estimated models for the Duffing-Ueda oscillator revealed that as T_s was made increasingly shorter, the structure of the estimated models tended to the structure of the counterpart discretized using equation (3) with $T_d = T_s$. Thus both identified and discretized models had the same number of terms, namely four. Furthermore, for sufficiently small values of T_s the parameter estimates of the identified model also approximated the respective parameters in the discretized model. An explanation for this is that the inclusion of a few additional terms which is observed when the T_s is increased slightly has the effect of compensating for the loss of accuracy associated with a slower sampling. Thus for a given value of T_s the estimated model might have, say, nine terms and be quantitative and qualitatively equivalent to a four-term model estimated from data sampled at a faster rate.

In the case of Chua's circuit, the nonlinearity responsible for most of the dynamical features of the system is provided by the piecewise linear function $h(x)$. Consequently a model discretized using the approximation in equation (3) with T_d will be composed of a linear dynamical part which is a function of T_d and a piecewise linear static part which independent of T_d.

In this paper, the estimated models are global as opposed to piecewise linear. Consequently it is indispensable that the nonlinear effects of the static piecewise linear function $h(x)$ be dynamically and globally represented in the structure of the final models. It seems reasonable to infer that all the nonlinear terms in an identified model for the double scroll attractor correspond to $h(x)$, otherwise a simple linear model would be adequate to model equation (1).

Therefore if the nonlinear terms are associated with $h(x)$ which in turn is independent of T_d, it is reasonable to expect that the structure of the identified models be less sensitive to T_s than in other examples such as for the Duffing-Ueda oscillator where T_d influences the entire structure of the discretized models. This, in fact, has been verified in a number of examples.

It should be noted that this also affects the total number of terms in the model, n_p. For models such as the Duffing-Ueda oscillator where the entire structure depends on T_d, the number of terms in the estimated models tend to decrease as T_s is shortened. Thus for sufficiently small values of T_s, n_p equals the number of terms in a discretized model. On the other hand, for models like the Chua circuit, the minimum number of terms in an estimated model will tend to be larger than the number of terms in the discretized counterpart because a certain number of terms have to be included to account for the piecewise linear function $h(x)$. This seems to account for the fact that no estimated model with $n_p < 11$ was chaotic. However, a slight decrease in n_p is still observed as T_s is shortened, see Table 2.

Summarizing it can be said that, within reasonable limits, adding extra terms and reestimating the parameters in a model has a compensating effect. In some examples, this effect compensates for the loss of accuracy due to, for instance, variations in the sampling period (as for the Duffing-Ueda oscillator) or to the omission of a static piecewise linearity as in the case of the double scroll attractor.

It is rather remarkable that a model such as the one in equation (7) reproduces the dynamics of the original system which is described by equations (1)–(2). The differences are evident, while the latter is continuous, piecewise linear and multivariable, the former is discrete, globally nonlinear and monovariable.

It is believed that the reconstruction is made possible by the flexibility inherent in the identification procedure. Such a flexibility is manifested in i) an increase in the number of process terms when compared to the discretized counterpart, ii) the 'additional terms' are chosen from a large number of candidate terms, and iii) parameters are reestimated according to the selected model structure.

If, on the one hand, the liberty associated with selecting the model structure enables the reconstruction of the true dynamics, on the other hand it is clear that a systematic way of choosing the correct model structure is crucial. In this respect the use of the ERR as a measure of the relevance of each candidate term has rendered satisfactory results in many cases.

Concluding, it should be stressed that although a slight increase in the number of process terms might prove beneficial, overparametrization is, for most nonlinear systems, highly detrimental to the overall dynamics. Moreover, the way the reconstructed dynamics are assessed is very important. It is noted that only a few models (of the 145) obtained in the course of the present investigation reproduced the double scroll geometry fairly well. Furthermore, some of these models failed to yield reasonable values for λ_1 and D_c and hence the number of dynamically acceptable models was only a small portion of the original set of identified models. In this respect, common procedures for model validation such as those based on model output predictions and on testing the whiteness of residuals are insufficient in revealing if the final model will reproduce dynamical invariants of the original system.

VI. Conclusions

This paper has investigated the identification of globally nonlinear models for the double scroll attractor. Algorithms based on the NARMAX model were used as a framework for obtaining the results that have been reported.

The main objectives have been to i) identify discrete models for the original system which reproduce dynamical invariants such as the geometry, the largest Lyapunov exponent and the correlation dimension of the double scroll, and not simply a model which apparently fits a particular piece of data, and ii) to investigate how the sampling period, T_s, and the model structure affect the overall dynamics of the identified models.

To gain further insight regarding the second objective above, the identified models have been compared to discretized counterparts. This comparison has suggested that, within a limited range of practical values of T_s, estimated models tend to be more accurate than the models discretized with $T_d = T_s$. Moreover, the results seem to support that such an improvement is attained thanks to a slight increase in model complexity, which in this paper has been related to the total number of process terms in the identified models.

Finally, the results reported have conferred further support to the sometimes overlooked fact that overparametrization in nonlinear models is usually highly detrimental to the dynamics. In particular, models of the double scroll attractor which include just a few more terms than the 'optimum' become unstable.

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