Exponential Representation of the Solutions of Nonlinear Differential Equations

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Abstract

In this paper we shall introduce a definition of poles of a nonlinear differential equation and show that, in some cases, the solutions have an exponential representation in terms of integrals over the singular varieties defined by the poles.

Keywords:Nonlinear analytic differential equations, exponential representations.

1 Introduction

Much of the theory of linear systems of equations has been generalized to nonlinear systems. The ideas have been based largely on Volterra series and global linearizations (see [1] and the references cited therein). However, apart from bilinear systems, there has been little success in finding a frequency domain theory for nonlinear systems. In fact, most work starts from an assumed input-output map in Volterra form and the spectrum is determined in terms of each kernel. In this paper we shall consider a nonlinear differential equation and show how to associate with it a set of natural singularities in multi-dimensional complex spaces. Two special cases will be considered–nilpotent and rational systems. In the nilpotent case it will be shown that the singularities are all 'at the origin' or, more precisely, are singular linear manifolds parallel to the complex axes. This directly generalizes the linear case where the poles are all at the origin with multiplicity the order of nilpotency of the system matrix. The rational case gives rise to an even more remarkable generalization; this is that the solutions of such a system can be written as sums of integrals of exponentials over singular varieties.

In section 2 we shall derive the formal expansion of a system using a Lie series type argument and the two special cases will be discussed in sections 3 and 4, respectively. Much work remains to be done before anything like a complete frequency domain theory of nonlinear systems can be achieved. A particularly important step would be to characterize all rational systems, a point we shall take up in a future paper.

2 Formal Solution

In order to set the scene, consider first the linear differential equation

$$\dot{x}(t) = Ax(t)\ ,\ x(0) = x_0. \tag{2.1}$$

Taking Laplace transforms, we have

$$sX(s) - x(0) = Ax(s),$$

so that

$$sX(s) = Ax(s) + x_0.$$

Inverting the Laplace transform gives

$$\dot{x}(t) = Ax(t) + x_0\delta(t)\ ,\ x(0) = 0 \tag{2.2}$$

since $\mathcal{L}(\delta) = 1$.

It follows that the system (2.1) is equivalent to the system (2.2) with zero initial condition. Of course, the solution of (2.2) is

$$\begin{aligned} x(t) &= \epsilon^{At} \cdot x(0) + \int_0^t \epsilon^{A(t-s)} x_0 \delta(s) ds \\ &= 0 + \epsilon^{At} \int_0^t \epsilon^{-As} x_0 \delta(s) ds \\ &= \epsilon^{At} x_0 \end{aligned} \tag{2.3}$$

as is expected from (2.1).

Equation (2.2) is, however, strictly speaking a distributional equation, although in this case it is easy to make the calculation in (2.3) rigorous. For, if $t \geq 0$,

$$x(t) = U(t)e^{At}x_0$$

where U is the unit step function. Hence,

$$\begin{aligned} \dot{x}(t) &= U(t)Ae^{At}x_0 + \delta(t)e^{At}x_0 \\ &= Ax(t) + \delta(t)x_0 \end{aligned}$$

since $\frac{d}{dt}U = \delta$ and $\delta(t)p(t) = \delta(t)p(0)$ for any differentiable function p. (Here, $\frac{d}{dt}$ denotes the distributional derivative.)

Now consider the analytic nonlinear system

$$\dot{x}(t) = f(x(t)) \ , \ x(0) = x_0. \tag{2.4}$$

We shall assume that this equation has a unique Laplace transformable solution. Then, as in (2.2) we see that this equation is formally identical to the equation

$$\dot{x}(t) = f(x(t)) + x_0\delta(t) \ , \tag{2.5}$$

with zero initial condition. However, it is now very questionable whether we can even make sense of (2.5) as a distributional equation. One possible route is to try to use Colombeau's theory of the multiplication of distributions [7] and to express $x(t)$ as an analytic distributional series. We shall take a more pragmatic approach and express equation (2.5) as an infinite-dimensional Volterra series using Lie series techniques [2], although it is important to note that the product of delta functions can be defined rigorously and has the expected properties.

To do this we define the functions g_i inductively by $g_1(x) = x$ and

$$g_i(x) = \begin{cases} \frac{\partial g_{i/2}(x)}{\partial x} \cdot f(x) & \text{if } i \text{ is even} \\ \frac{\partial g_{(i-1)/2}(x)}{\partial x} \cdot x_0 & \text{if } i \text{ is odd} \end{cases}$$

Then,

$$\begin{aligned} \dot{g}_i(x) &= \frac{\partial g_i(x)}{\partial x}\dot{x} \\ &= \frac{\partial g_i(x)}{\partial x} \cdot f(x) + \delta(t)\frac{\partial g_i(x)}{\partial x} \cdot x_0 \\ &= g_{2i}(x) + \delta(t) g_{2i+1}(x). \end{aligned}$$

Hence we obtain the linear system

$$\dot{G} = AG + \delta(t)BG \ , \ G(0) = G_0 \tag{2.6}$$

where

$$G = (g_1, g_2, g_3, \cdots)^T \ ,$$

$A = (a_{ij}), B = (b_{ij})$ are infinite-dimensional matrices defined by

$$\begin{aligned} a_{ij} &= \delta_{2i,j} \\ b_{ij} &= \delta_{2i+1,j} \end{aligned} \tag{2.7}$$

and G_0 is given by

$$G_0 = (0, f(0), x_0, ((\partial f)f)(0), ((\partial f)x_0)(0), \cdots)^T,$$

where $\hat{\partial} = \partial/\partial x$.

We can obtain a more explicit expression for G_0 in terms of Lie derivatives. To do this we define the vector fields $v(i)$ by

$$v(i) = \begin{cases} f & \text{if } i \text{ is even} \\ x_0 & \text{if } i \text{ is odd} \end{cases}$$

(i.e. $v(i)$ is a constant vector field if i is odd). We also require the two functions

$$\ell(i) = \begin{cases} i/2 & \text{if } i \text{ is even} \\ (i-1)/2 & \text{if } i \text{ is odd} \end{cases}$$

and

$$\mu(i) = \text{number of bits in the binary expression of } i \text{ minus 1.}$$

(i.e.

$$i = b_{\mu(i)} b_{\mu(i)-1} \cdots b_1 b_0 \tag{2.8}$$

where

$$b_k = 0 \text{ or } 1 \text{ for } 0 \leq k \leq \mu(i)$$

and

$$b_{\mu(i)} = 1).$$

Then we have

Lemma 2.1 $G_0 = (g_{10}, g_{20}, g_{30}, \cdots)$ where

$$g_{i0} = L_{v(i)}(L_{v(\ell(i))}(L_{v(\ell(\ell(i)))}(\cdots L_{v(\ell^{\mu(i)-1}(i))} g_{\ell^{\mu(i)}(i)})))|_{x_0}$$

Lemma 2.2 Let

$$\gamma_k(t_1, t_2, \cdots, t_k) = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_k} e^{A\rho_1}Be^{A(\rho_2-\rho_1)}B\cdots Be^{A(\rho_{k-1}-\rho_{k-2})}B$$

$$e^{A(\rho_k-\rho_{k-1})}BG_0u(t_1-\rho_1)\cdots u(t_k-\rho_k)d\rho_1\cdots d\rho_k, \tag{2.11}$$

where, again, $u = \delta$. Then

$$\xi_k(t) = \gamma_k(t, t, \cdots, t).$$

Proof First note that, since $u(\tau_k) = \delta(\tau_k)$, (2.10) implies that

$$\begin{aligned}\xi_k(t) &= \int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}} e^{A(t-\tau_1)}Be^{A(\tau_1-\tau_2)}B\cdots Be^{A(\tau_{k-1}-\tau_k)}BG_0\\ &\quad\times u(\tau_1)\cdots u(\tau_k)d\tau_1\cdots d\tau_k.\end{aligned} \tag{2.12}$$

since $u = \delta$. Now put

$$\rho_i = t - \tau_i \quad , \quad 1 \le i \le k.$$

We have,

$$\xi_k(t_1, t_2, \cdots, t_k) = \int_0^t\int_0^{t-\rho_1}\cdots\int_0^{t-\rho_{k-1}} e^{A\rho_1}Be^{A(\rho_2-\rho_1)}B\cdots Be^{A(\rho_{k-1}-\rho_{k-2})}$$

$$Be^{A(\rho_k-\rho_{k-1})}BG_0u(t_1-\rho_1)\cdots u(t_k-\rho_k)d\rho_1\cdots d\rho_k.$$

The result now follows. □

Next we define the k-dimensional **Laplace transform** of γ_k by

$$\Gamma_k(s_1, \cdots, s_k) = \int_0^\infty\cdots\int_0^\infty \gamma_k(t_1, \cdots, t_k)e^{-\sum_{i=1}^k s_i t_i}dt_1\cdots dt_k.$$

Lemma 2.3 If $\mathcal{L}^n\{X(t_1, \cdots, t_n)\}(s_1, \cdots, s_n)$ denotes the n-dimensional Laplace transform of $X(t_1, \cdots, t_n)$, then

$$\left[\mathcal{L}^1(e^{At_1}B)(s_1)\right]_{ij} = \sum_{n=0}^\infty \frac{1}{s^{n+1}}\delta_{2^{n+1}i,j}$$

and

$$\left[\mathcal{L}^2(e^{At_1}Be^{-At_1}e^{At_2}B)(s_1,s_2)\right]_{ij}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^m\frac{(n+m)!}{n!m!}\frac{1}{s_1^{n+m+1}s_2^{p+1}}\delta_{v(i),j}\ ,$$

where

$$v(i) = 2^{p+1}(2^m(2^{n+1}i+1))+1.$$

Proof By definition of A and B we have

$$\begin{aligned}\left(e^{At_1}B\right)_{ij} &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{t_1^n}{n!}\delta_{2^n i,k}\delta_{2k+1,j}\\ &= \sum_{n=0}^{\infty}\frac{t_1^n}{n!}\delta_{2^{n+1}i+1,j}\end{aligned}$$

and the first equality follows. For the second equality note that

$$\left(e^{At_1}Be^{-At_1}\right)_{ij} = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{t_1^n}{n!}\delta_{2^{n+1}i+1,k}\frac{(-t_1)^m}{m!}\delta_{2^m k,j}$$

from which the second equality readily follows. □

By a simple induction argument, we can calculate the n^{th} order kernel in the Laplace domain:

Lemma 2.4 We have

$$\left[\mathcal{L}^n\left(e^{At_1}Be^{A(t_2-t_1)}B\cdots Be^{A(t_n-t_{n-1})}B\right)\right]_{ij} =$$
$$\sum_{p_1=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{p_2=0}^{\infty}\sum_{q_2=0}^{\infty}\cdots\sum_{p_{n-1}=0}^{\infty}\sum_{q_{n-1}=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{q_1+\cdots+q_{n-1}}\times$$
$$\Pi_{\ell=1}^{n-1}\left\{\frac{(p_\ell+q_\ell)!}{p_\ell!q_\ell!}\frac{1}{s_\ell^{p_\ell+q_\ell+1}}\right\}\frac{1}{s_n^{r+1}}\delta_{v(i),j} \tag{2.13}$$

where

$$v(i) = 2^{r+1}w(i) + 1$$

and w is given inductively by

$$\begin{aligned} w(i) &= \mu(n-1) \\ \mu(1) &= 2^{p_1}(2^{q_1}i + 1) \\ \mu(\ell) &= 2^{p_\ell}(2^{q_\ell}\mu(\ell-1)+1). \end{aligned}$$

It now follows from lemma 2.2 that

$$\Gamma_k(s_1, \cdots, s_k) = K_k(s_1, \cdots, s_k)G_0$$

where

$$\{K_k(s_1, \cdots, s_k)\}_j = \sum_{p_1=0}^{\infty}\sum_{q_1=0}^{\infty}\sum_{p_2=0}^{\infty}\sum_{q_2=0}^{\infty}\cdots\sum_{p_{n-1}=0}^{\infty}\sum_{q_{n-1}=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{q_1+\cdots+q_{n-1}}$$

$$\Pi_{\ell=1}^{n-1}\left\{\frac{(p_\ell+q_\ell)!}{p_\ell!q_\ell!}\frac{1}{s_\ell^{p_\ell+q_\ell+1}}\right\}\frac{1}{s_n^{r+1}}\delta_{v(1),j} \tag{2.14}$$

for all $s_1, \cdots, s_k$ for which the power series converges. Suppose that $\Gamma_k(s_1, \cdots, s_k)$ is analytic apart from on the singular varieties $V_k \subseteq \mathbf{C}^k$, $1 \le k < \infty$.

Definition We call V_k the **poles** of the system (2.4) (or (2.5)). (**Zeros** can be defined in a similar way.) Particularly important classes of systems are the nilpotent ones in which $g_i \equiv 0$ for large i and those in which the 'transfer functions' $\Gamma_k(s_1, \cdots, s_k)$ are rational. We shall examine the nilpotent class in the next section.

3 Nilpotent Systems

Definition An equation of the form (2.4) (or (2.5)) is called **nilpotent** if

$$g_i = L_{v(i)}(L_{v(\ell(i))}(\cdots(\cdots L_{v(\ell^{\mu(i)-1}(i))}x))) \equiv 0$$

for $i \geq N$, for some $N \in \mathbf{N}$.

This is equivalent to the Lie algebra generated by the vector fields $x_0\frac{\partial}{\partial x}$ and $f(x)\frac{\partial}{\partial x}$ being nilpotent.

Theorem 3.1 A nilpotent system has a finite Volterra series.

Proof This follows easily from lemma 2.4, since if $g_i \equiv 0$ for $i \geq N$ then by construction of $v(i)$ the kernels of order $\geq n$ are zero if $2(n-1)+1 \geq N$. This follows from the fact that the factor $\delta_{v(1),j}$ in (2.12) ensures that the first nonzero term in (2.12) has index $2(n-1)+1$. □

(This should be compared with a similar result of Crouch, [6])

Recall that a linear system

$$\dot{x} = Ax \ , \ x \in \mathbf{R}^n$$

is nilpotent if and only if $A^m = 0$ for some $m \geq 0$. This is the case if and only if A has the form

$$\begin{pmatrix} 0 & * & \cdots & * \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & * \\ 0 & \cdot & \cdots & 0 \end{pmatrix}$$

with respect to some basis. Clearly, therefore, the denominator of the transfer function is s^m and so all the poles are at $s = 0$, with multiplicity m.

It is remarkable that the analogous result holds in the nonlinear case:

Theorem 3.2 The k^{th} order poles ($1 \leq k \leq L$) of a nonlinear system of the form (2.4) are given by

$$s_1 = 0, \cdots, s_k = 0,$$

where L is the highest order nonzero kernel. The multiplicity of the poles depends on the order of nilpotency of the system.

Proof This follows directly from (2.12). □

Remark The multiplicity of the poles is related to $v(1)$. In fact, if $g_\nu \equiv 0$ for $\nu \geq N$, then the multiplicity of s_ℓ ($1 \leq \ell \leq n-1$) is given by the highest order of $p_\ell + q_\ell + 1$ for which p_ℓ and q_ℓ occur in a $v(1)$ for which $g_{v(1)}(x_0) \neq 0$. Note that the order of nilpotency and order of the poles depends on x_0.

4 Rational Systems

In this section we shall consider systems for which the 'transfer functions' $\Gamma_k(s_1, \cdots, s_k)$ are all rational, and we shall prove that any solution of such a system can be written as a sum of integrals of exponentials over the singular varieties of Γ_k. This will generalize the linear case, where every solution is of the form

$$x(t) = \sum_{i=1}^{n} \alpha_i e^{\lambda_i t}$$

where the λ_i's are the eigenvalues of the 'A' matrix. Since the singular sets are now algebraic varieties in $\mathbf{C}^k$ (or $\mathbf{P}^k(\mathbf{C})$, the projective space), the sum must be replaced by an integral. This result could be proved by applying the topological ideas in [3] and Cauchy's theorem in k dimensions, but we shall give a proof based on the result of Palmadov [5] for the corresponding case of linear partial differential equations. In fact, we can derive it directly from the following result ([4]):

Theorem 4.1 If $P(D)$ is a partial differential operator in $\mathbf{R}^n$ with constant coefficients, then there is a finite set $\{A_i(\zeta, x)\}_{i=1}^T$ of polynomials in x and ζ such that if u satisfies the equation $P(D)u = 0$, then u has the representation

$$u(x) = \sum_{i=1}^{T} \int A_i(\zeta, x) e^{-i(x,\zeta)} d\mu_i(\zeta)$$

for some set $\{\mu_i\}$ of measures in $\mathbf{C}^n$. □

Remark This result is also true for systems of operators and the measures can be shown to be bounded in some sense.

Now consider the k^{th} order kernel $\Gamma_k(s_1, \cdots, s_k)$ of the nonlinear system and assume that it can be written in the form

$$\Gamma_k(s_1, \cdots, s_k) = \left\{ \frac{R_{k,i}(s_1, \cdots, s_k)}{P_{k,i}(s_1, \cdots, s_k)} \right\}_{1 \le i \le n}$$

where the maximum order of monomials in $P_{k,i}$ is greater than that in $R_{i,k}$. (Recall that Γ_k is actually a vector of rational functions with components $R_{k,i}/P_{k,i}$.) Then each rational function gives rise to a linear, constant coefficient partial differential equation

$$P_{k,i}(D_1, \cdots, D_k)u = 0$$

where the numerator term $R_{k,i}$ in Γ_k can be regarded as the boudary conditions on u. Theorem 4.1 now immediately gives our main result:

Theorem 4.2 Consider a nonlinear differential equation of the form (2.4) which has rational 'transfer functions' in the above sense. Then the solution can be written in the form

$$x(t) = \sum_{k=0}^{\infty} \sum_{i=1}^{T_k} \int A_{k,i}(\zeta, \tau) e^{-(\tau,\zeta)} d\mu_{k,i}(\zeta)$$

for some set of measures $\mu_{k,i}$ and a set of polynomials $A_{k,i}$. Here, τ is the k-dimensional vector $(t, t, \cdots, t)$. □

Remark This theorem shows that nonlinear ordinary differential equations with rational kernels have solutions with exponential representations in terms of integrals over the singular varieties. Thus we have a direct generalization of the elementary linear case, familiar in systems theory. In the nilpotent case the sum is finite and the integrals are over hyperplanes parallel to the axes.

5 Conclusions

In this paper we have generalized the notion of characteristic or singular values of a linear system of differential equations to nonlinear ordinary differential equations. Two natural classes have been identified; i.e. the nilpotent class and the rational class. (Of course, the former is a subclass of the latter.) In the nilpotent case we have seen that the 'poles' of the system generalize from the linear case in a remarkable way. Moreover, in the rational case an exponential representation of the solutions can be shown to exist.

It will be important to find conditions on $f(x)$ which gives a complete characterization of rational systems and this will be studied in a future paper. Also, it is desirable to have a better understanding of the nature of product operators when the arguments are distributions, perhaps other than the simple case of delta functions here. For this, it is necessary to use Colombeau's theory of the multiplication of distributions in which it is proved that the product of delta functions does not have a representative distribution. This is also under investigation.

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