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Input-Output Maps for Nonlinear Systems, Fractional Integration and Rational Representations

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Abstract

In this paper we shall make a systematic use of the fractional integration operator to derive input-output maps in compact form for linear and bilinear systems and general autonomous nonlinear systems. This will enable us to obtain simple rational approximations to input-output maps.
Keywords:Nonlinear Systems, Fractional Integration, Rational Input-Output Maps.

1 Introduction

A great deal of effort has been directed towards obtaining input-output maps for several classes of systems. These response maps have been given, except for linear systems, as functional expansions (see [1], [2],[3]). In this paper we shall make use of the fractional integration operator ([10]) to derive input-output maps in compact forms for linear and bilinear time-varying systems and general autonomous nonlinear systems. This will lead to simple rational approximations to input-output maps which have been obtained in other more complex ways in [4].

In the case of nonlinear systems we shall use the global bilinearization of such systems introduced in [1],[2],[3], where the state space becomes a space of tensors. One could also use the approach of Brockett [7] or Sira-Ramirez [8],[9], which are again based on Carleman linearization, or the Lie series approach introduced in [4],[5].

In this paper we shall denote by $L^{1,n}[0,T]$, $0 < T < \infty$ the space of absolutely integrable functions $x : [0,T] \longrightarrow \mathbf{R}^n$, with norm

$$||x||_1 = \int_0^T ||x(\tau)||d\tau$$

where $||\cdot||$ is any norm on $\mathbf{R}^n$. For $A : [0,T] \longrightarrow \mathbf{R}^{n\times n}$, we define

$$||A(\cdot)||_\infty = \sup_{t\in[0,T]} ||A(t)||.$$

Let I^μ denote the Riemann-Liouville integration operator of order μ, where μ

is a complex number with **Re** $\mu > 0$ from $L^{1,n}[0,T]$ into itself, defined by

$$(I^{\mu}g)(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} g(\tau) d\tau$$

for almost all $t \in [0,T]$, Γ being the gamma function.

If **Re** $\mu < 0$ we define

$$I^{\mu} = (I^{-\mu})^{-1}$$

and we have, for all $\mu, \nu \in \mathbf{C}$

$$I^{\mu} I^{\nu} = I^{\nu} I^{\mu} = I^{\mu+\nu}$$

where we define

$$I^{\mu} = I^{-1} I^{1+\mu}$$

if **Re** $\mu = 0$. Note that $I^0 g = g$.

2 Linear Time-Varying Systems

Consider the time-varying linear system

$$(\Sigma) \quad \begin{cases} \frac{dx}{dt} = A(t)x + B(t)u & x(0) = 0 \\ y = C(t)x & t \in [0,T] \end{cases} \tag{2.1}$$

where u, x, y are respectively the input,state and output of the system with dimensions $m \times 1, n \times 1, r \times 1$. The input-output map of this system is well-known to be

$$\begin{aligned} y(t) &= \int_0^t h_{\Sigma}(t,\tau) u(\tau) d\tau \\ &= (H_{\Sigma} u)(t) \end{aligned} \tag{2.2}$$

where

$$h_\Sigma(t,\tau) = C(t)\phi_\Sigma(t)\phi_\Sigma^{-1}(\tau)B(\tau) \tag{2.3}$$

and ϕ_Σ is the fundamental matrix satisfying

$$\begin{cases} \frac{d\phi_\Sigma}{dt} = A(t)\phi_\Sigma \\ \phi_\Sigma(0) = I_{n\times n} \quad \text{(the identity)} \end{cases} . \tag{2.4}$$

The objective is to derive an equivalent characterization of H_Σ. For this we shall introduce linear operators $A, B, C, \mathbf{1}$ between appropriate spaces (e.g. $A : L^{1,n}[0,T] \longrightarrow L^{1,n}[0,T]$, defined by

$$\begin{cases} (Ax)(t) = A(t)x(t) \\ (Bu)(t) = B(t)u(t) \\ (Cx)(t) = C(t)x(t) \end{cases} \tag{2.5}$$

where $\mathbf{1}$ denotes the identity operator. (Using the same notation for the matrix function $A(t)$ and the operator A, etc. should not cause any confusion.) The following lemma is trivial:

Lemma 2.1 If I denotes the fractional integral operator of order 1 then

$$||I|| = 1.$$

□

We then have

Theorem 2.2 The input-output map H_Σ is given by

$$H_\Sigma = C(\mathbf{1} - IA)^{-1}IB$$

for $||A(.)||_\infty < 1$ and maps $L^{1,m}[0,T]$ into $L^{1,r}[0,T]$.

Proof This follows directly from the Neumann series. □

Example 2.3 In the linear time-invariant case, we have

$$\begin{aligned}(H_\Sigma u)(t) &= \left[\sum_{k\geq 0} C(IA)^k IBu\right](t) \\ &= \sum_{k\geq 0} CA^k(I^{k+1}Bu)(t) \\ &= \sum_{k\geq 0} CA^k \int_0^t \frac{(t-\tau)^k}{t!} Bu(\tau)d\tau \\ &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau.\end{aligned}$$

In general we also have the error estimate:

Proposition 2.4

$$||H_\Sigma - \sum_{k=0}^{p} C(IA)^k IB|| \leq \frac{||C(\cdot)||_\infty \cdot ||B(\cdot)||_\infty}{1-||A(\cdot)||_\infty} \cdot ||A(\cdot)||_\infty^{p+1}$$

assuming $||A(\cdot)||_\infty < 1$. □

3 Bilinear Time-Varying Systems

Consider now the bilinear time-varying system

$$(\Sigma') \begin{cases} \frac{dx}{dt} = A(t)x + \sum_{i=1}^m u_i D_i(t)x \\ y = C(t)x \ , \ x(0) = x_0 \end{cases} \tag{3.1}$$

for $t \in [0,T]$. Then we have

Theorem 3.1 The input-output map $H_{\Sigma'}$ associated with Σ' is given by

$$H_{\Sigma'}u = C\left\{\mathbf{1} - IA - \sum_{i=1}^{m} Iu_i D_i\right\}^{-1} x_0$$

where $||A(\cdot)||_\infty < 1$, and maps the ball of radius ρ in $L^{1,m}[0,T]$ into $L^{1,r}[0,T]$ for

$$0 < \rho < \frac{1 - ||A(\cdot)||_\infty}{\sum_{i=1}^m ||D_i(\cdot)||_\infty}.$$

Proof Indeed,

$$H_{\Sigma'} = C(\mathbf{1} - IA)^{-1} \left\{ \mathbf{1} - (\mathbf{1} - IA)^{-1} \sum_{i=1}^m I u_i D_i \right\}^{-1} x_0. \tag{3.2}$$

However,

$$||(\mathbf{1} - IA)^{-1}|| \le \frac{1}{1 - ||A(\cdot)||_\infty}$$

since $||A(\cdot)||_\infty < 1$, and

$$\begin{aligned} \left\| (\mathbf{1} - IA)^{-1} \sum_{i=1}^m I u_i D_i \right\| &\le ||(\mathbf{1} - IA)^{-1}|| \cdot \sum_{i=1}^m ||I u_i D_i|| \\ &\le \frac{1}{1 - ||A(\cdot)||_\infty} \cdot \sum_{i=1}^m ||D_i(\cdot)||_\infty \cdot ||u|| \\ &< 1 \end{aligned}$$

since

$$||u|| \le \rho < \frac{||A(\cdot)||_\infty}{\sum_{i=1}^m ||D_i(\cdot)||_\infty}.$$

Therefore,

$$\begin{aligned} ||H_{\Sigma'} u|| &\le \frac{||C(\cdot)||_\infty \cdot ||x_0||}{1 - ||A(\cdot)||_\infty - \rho \sum_{i=1}^m ||D_i(\cdot)||_\infty} \\ &< \infty. \end{aligned}$$

□

4 Nonlinear Systems

In this section we shall consider an analytic system of the form

$$\begin{cases} \dot{x} = f(x,u) & x(0) = x_0 \\ y = g(x) & t \in [0,T] \end{cases} \tag{4.1}$$

where $x_0 \in \mathbf{R}^n$, $u(t) \in \mathbf{R}$ and $f : \mathbf{R}^n \times \mathbf{R} \longrightarrow \mathbf{R}^n$ and $g : \mathbf{R}^n \longrightarrow \mathbf{R}$ are analytic functions. (We are considering the single-input single-output case for simplicity; the general case follows in a similar way.) It is well known that this nonlinear system can be transformed into a bilinear system in an appropriate space of tensors if one uses the technique of Carelman linearization [11]. Hence, if $\mathbf{i}$ denotes the multi-index $(i_1, \cdots, i_n)$ and $\phi_{i_1,\cdots,i_n}(x) \triangleq x^{\mathbf{i}}$ then

$$(\Sigma'') \begin{cases} \frac{d\phi}{dt} = A_0\phi + \sum_{j\geq 1} u^j A_j \phi \\ y = G\phi \quad , \quad \phi(0) = \phi^0 \end{cases}$$

where ϕ is the tensor with components $\phi_{i_1,\cdots,i_n}(x)$ and the A_j's with $j \geq 0$ are tensor operators given in detail in [3] . The A_j's may be shown to be bounded in an appropriate space ([5]) and so the next result follows just as in theorem 2:

Theorem 4.1 The input-output map $H_{\Sigma''}$ of Σ'' is given by

$$H_{\Sigma''}u = G(\mathbf{1} - IA_0)^{-1}\left\{\mathbf{1} - (\mathbf{1} - IA_0)^{-1}\sum_{j\geq 1} u^j A_j\right\}^{-1} \phi^0$$

for $||A_0||_\infty < 1$ and

$$\sum_{j\geq 1} ||u||^j \cdot ||A_j||_\infty < 1.$$

□

5 Rational Representations

In this section we shall show that the formulae developed above can be used to derive rational approximations to input-output maps in a simple way. We shall consider the case of time-varying bilinear systems as in section 3, but the ideas generalize easily to other systems. Hence, consider the system (Σ') in (3.1), i.e.

$$(\Sigma') \begin{cases} \frac{dx}{dt} = A(t)x + \sum_{i=1}^{m} u_i D_i(t)x \\ y = C(t)x \ , \ x(0) = x_0 \end{cases} \tag{5.1}$$

where $A(\cdot), D_i(\cdot)$ and $C(\cdot)$ are analytic matrix-valued functions. In order to do this we consider the algebra $\mathbf{A}$ of operators on $C^{\omega,n}[0,T] \subseteq L^{1,n}[0,T]$ generated by I and the multiplication operators $M_f : C^{\omega,n}[0,T] \longrightarrow C^{\omega,n}[0,T]$ given by

$$M_f(g) = fg \quad , \quad f, g \in C^{\omega,n}[0,T]$$

where $C^{\omega,n}[0,T] \subseteq L^{1,n}[0,T]$ is the linear space of analytic functions on $[0,T]$. We shall find a matrix representation for $\mathbf{A}$ in the following way. First consider the case $n = 1$ for simplicity.

Lemma 5.1 The integration operator I has the matrix representation given by

$$I \sim \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{2} & 0 & \cdot & \cdot \\ 0 & 0 & \frac{1}{3} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \triangleq \mathcal{I}$$

with respect to the Hamel basis $\{t^i\}_{i\geq 0}$ of $C^{\omega,1}[0,T]$.

Proof Let $f = \sum_{i\geq 0} f_i t^i \in C^{\omega,1}[0,T]$. Then,

$$If = \sum_{i\geq 0} f_i \frac{t^{i+1}}{i+1}$$

and $\mathcal{I}(f_0, f_1, \cdots)^T = (0, f_0, f_1/2, f_2/3, \cdots)^T$. □

Lemma 5.2 The multiplication operator M_f has matrix representation

$$M_f \sim \begin{pmatrix} f_0 & 0 & 0 & \cdot & \cdot \\ f_1 & f_0 & 0 & \cdot & \cdot \\ f_2 & f_1 & f_0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \triangleq \mathcal{M}_f.$$

Proof If $f = \sum_{i\geq 0} f_i t^i$, $g = \sum_{j\geq 0} g_j t^j$, then

$$fg = \sum_{k\geq 0} \sum_{i+j=k} f_i g_j t^k$$

and this is equal, term by term, to $\mathcal{M}_f(g_0, g_1, \cdots)^T$. □

In a similar way, for the general case, we have a matrix representation for **A** generated by the infinite matrices $\mathcal{I} \otimes I_{n,n}$, $\mathcal{M}_f$ where $I_{n,n}$ is the $n \times n$ unit matrix and $f : [0,T] \longrightarrow \mathbf{R}^{n\times n}$. Now let $\mathcal{I}(k)$ and $\mathcal{M}_f(k)$ denote the truncation of $\mathcal{I}$ and $\mathcal{M}_f$, respectively, to $k \times k$ matrices where

$$\mathcal{I}(k) = P_k \mathcal{I} P_k \quad , \quad \mathcal{M}_f(k) = P_k \mathcal{M}_f P_k$$

and $P_k(f_0, f_1, \cdots)^T = (f_0, f_1, \cdots, f_{k-1})^T$, $f_i \in \mathbf{R}^{n\times n}$. We can therefore associate the following approximate matrices with the corresponding operators in

(3.2):

$$
\begin{aligned}
\mathbf{1} &\sim \mathbf{1}(k) = P_k \begin{pmatrix} I_{n,n} & & & & \\ & I_{n,n} & & & \\ & & \ddots & & \\ & & & I_{n,n} & \\ & & & & \ddots \end{pmatrix} P_k \\
I &\sim \mathcal{I}(k) \\
A &\sim \mathcal{A}(k) = P_k \begin{pmatrix} A_0 & & & \\ A_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ . & . & . & . \end{pmatrix} P_k \\
u_i &\sim \mathcal{U}_i(k) = P_k \left\{ \begin{pmatrix} u_{i0} & & & \\ u_{i1} & u_{i0} & & \\ u_{i2} & u_{i1} & u_{i0} & \\ . & . & . & . \end{pmatrix} \otimes I_{n,n} \right\} P_k \\
D_i &\sim \mathcal{D}_i(k) = P_k \begin{pmatrix} D_{i0} & & & \\ D_{i1} & D_{i0} & & \\ D_{i2} & D_{i1} & D_{i0} & \\ . & . & . & . \end{pmatrix} P_k \\
C &\sim \mathcal{C}(k) = (C_0 \; C_1 \; \cdots \; C_{k-1})
\end{aligned}
$$

where $A(t) = \sum_{i\geq 0} A_i t^i$, $D_i(t) = \sum_{j\geq 0} D_{ij} t^j$, $C(t) = \sum_{i\geq 0} C_i t^i$, $u_i(t) = \sum_{j\geq 0} u_{ij} t^j$. Hence we have proved

Theorem 5.3 The input-output map (3.2) of the bilinear system (Σ') in (3.1) can be approximated by the rational function of u_{ij} , $0 \leq j \leq k-1$:

$$(\overline{H}_{\Sigma} u)(t) = \left(\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} C_i S_{ij} t^j \right) x_0$$

where S_{ij} is the ij^{th} block submatrix (of order $n \times n$) of the $(k \times n) \times (k \times n)$ matrix

$$(\mathbf{1}(k) - \mathcal{I}(k)\mathcal{A}(k))^{-1}\{\mathbf{1}(k) - \mathbf{1}(k) - \mathcal{I}(k)\mathcal{A}(k))^{-1} \sum_{i=1}^{m} \mathcal{I}(k)\mathcal{U}_i(k)\mathcal{D}_i(k)\}^{-1}.$$

□

6 Conclusions

In this paper we have derived compact forms for the input-output maps of very general classes of bilinear and nonlinear dynamical systems. These are based on the fractional integration operator and Carleman linearization in the general nonlinear case. We have seen that these representations can be used to derive rational approximations to the input-output maps of systems in a way which is not possible from the standard Volterra series.

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