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A Simple Observer Design Method For Nonlinear Systems

by

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Abstract

Based on a perturbation theorem, a method of designing an observer for nonlinear systems of the form $\dot{x} = A(x)x + B(x)u$ is presented. It has been shown that if the nonlinear perturbations satisfy some boundness condition, the unobservable states of the nonlinear system can be estimated using a linear observer designed for the linear part of the system. The method is demonstrated on a practical model of the ball and beam system.

Keywords:Observers, Nonlinear Systems, ball and beam system, Perturbation method.

1 Introduction

Although the observer design of a general nonlinear system is a problem of central importance in control theory which has received more and more attention since 1970's, most of the developed observer design methods are difficult to employ. This is because nonlinear systems are extremely hard to analyse. Therefore, in the last few years attention has been focused on designing observers for some classes of nonlinear systems with practical applications (see [5], [3] and [4]).

Both [5] and [3] have considered in their study of observer design the following widely applicable class of nonlinear systems

$$\begin{aligned} \dot{x}(t) &= A(x)x + B(x)u \\ y &= C(x)x \end{aligned} \quad (1)$$

which naturally describe many nonlinear systems, e.g. vehicle dynamics ([16], [10], [12]), ship dynamics ([17], [15]), and aircraft dynamics([13], [11]). Moreover, a nonlinear damped oscillator ([14]), any system described by Van der Pol's equation, and many pendulum problems show such behaviour. Finally, many nonlinear systems of more general classes may be put into this form by employing a Taylor series expansion (including as many terms as is feasible).

In [5] an observer design method is presented which requires transforming the nonlinear system in the form of (1) into what the authors call a nonlinear observer form (by analogy to the linear observer form) given by:

$$\begin{aligned} \dot{x}^o &= A^o(x^o)x^o + B^o(x^o)u \\ y(t) &= C^o x^o \end{aligned} \tag{2}$$

where the superscript 'o'denotes observer form, and

$$A^o(x^o) = \begin{bmatrix} a^o_{11}(x^o) & 1 & 0 & \dots & 0 \\ a^o_{21}(x^o) & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ a^o_{n1}(x^o) & 0 & \dots & \dots & 0 \end{bmatrix}$$

$$B^o(x^o) = \begin{bmatrix} b^o_1(x^o) \\ b^o_2(x^o) \\ \cdot \\ \cdot \\ \cdot \\ b^o_n(x^o) \end{bmatrix}$$

$$C^{o} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Then they design an observer as in the linear case. The major complexity with their method is the nonlinear transformation from system (1) to system (2) essentially required in the design. 'This involves solving a system of partial-differential equations and can get extremely involved, especially for higher-order systems ($n > 2$) '([5]). In [3] a generalization of linear observer design methods to nonlinear systems is presented. Using this method of design, it has been shown that the difference between observer states z (the estimated system states) and actual (unobservable) system states x converges to a 'ball 'of certain diameter provided that some boundness conditions are met. The work of [3] on the other hand does not require the nonlinear transformations of system (1) to system (2) needed by the method of [5], therefore it removes the limitations of applying the method due to this transformation. Also it reduces the computations associated with [5].

In this paper the nonlinear system is described as a linear part plus a (seperated) nonlinear perturbation. Then provided that this nonlinear perturbation satisfies some boundness condition, the unobservable states of the nonlinear system are estimated using a linear observer designed for the linear part of the system.

2 Perturbation theorem

In this section the stability of a nonlinear system, where the nonlinearities are represented as perturbation to the linear (stable) part of the system, will be investigated. This is included in the following:

Theorem 1. For the nonlinear system

$$\dot{x}(t) = A(x)x \tag{3}$$

let

$$A(x) = A_0 + A_1(x) \tag{4}$$

where A_0 is the constant stable part of $A(x)$ and $A_1(x)$ is the nonlinear perturbations of A_0. Then if

$$\|A_1(x)\| < \frac{w}{M^2} \tag{5}$$

where w and M are positive numbers determined by the matrix A_0 through the following equation

$$\left\| e^{A_0 t} \right\| \leq M e^{-wt}$$

system (3) is asymptotically stable.

Proof Consider the unperturbed linear system of (3) i.e.

$$\dot{x} = A_0 x \tag{6}$$

From the linear stability theory, the Liapunov matrix equation for the above system is:

$$A_0^T P + P A_0 = -I \tag{7}$$

and as A_0 is a stability matrix, the above Liapunov matrix equation has the solution [18]

$$P = \int_0^\infty e^{A_0^T t} e^{A_0 t} dt \tag{8}$$

taking norms of the previous equation gives

$$\begin{aligned} \|P\| &\leq \int_0^\infty \left\| e^{A_0 t} \right\|^2 dt \\ &\leq \int_0^\infty M^2 e^{-2wt} dt \\ &= \frac{1}{2w} M^2 \end{aligned} \tag{9}$$

where

$$\left\| e^{A_0 t} \right\| \leq M e^{-wt} \tag{10}$$

Now, for system (3) if we define the liapunov function

$$V = x^T P x \tag{11}$$

where P (as in equations 7 and 8) is a constant positive definite matrix defined with respect to the unperturbed system (6). Then the time derivative of V with respect to (3) is

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T (A_0^T + A_1^T(x)) P x + x^T P (A_0 + A_1(x)) x \\
&= -\|x\|^2 + 2 \left\| x^T A_1^T(x) P x \right\| \\
&\leq -\|x\|^2 + 2\|x\|^2 \|A_1(x)\| \|P\| \\
&\leq -\|x\|^2 + \frac{M^2}{w} \|x\|^2 \|A_1(x)\| \\
&= -\|x\|^2 (1 - \frac{M^2}{w} \|A_1(x)\|) \qquad (12)
\end{aligned}$$

so that for stability of nonlinear system (3) we need

$$\|A_1(x)\| < \frac{w}{M^2} \tag{13}$$

□.

3 Observer Design Method

Based on the previous perturbation theorem, an observer will be designed for the unperturbed linear system of (3). Then, and using the estimated states from this observer, stability of the closed loop system will be tested against the nonlinear perturbation.

Consider now designing an observer for system (3) i.e. for

$$\dot{x}(t) = (A_0 + A_1(x))x \tag{14}$$

Let

$$\dot{z}(t) = Fz + Gx \tag{15}$$

where F is a stability matrix and F and G are constant matricies, be an observer for the unperturbed free system of (14) i.e. for

$$\dot{x} = A_0 x$$

Lemma 1. Suppose there is a matrix T (constant) which satisfies the equation

$$FT - TA_0 + G = 0 \tag{16}$$

then

$$z = Tx + e^{Ft}((z(0) - x(0)) - \int_0^t e^{Fs} TA_1(x)x(s)ds) \tag{17}$$

Proof From (14) and (15) we have

$$\begin{aligned} \frac{d}{dt}(z - Tx) &= Fz + Gx - TA_0x - TA_1(x)x \\ &= F(z - Tx) - TA_1(x)x \end{aligned} \tag{18}$$

As the matrix F is stability matrix, the solution of (18) is

$$(z - Tx) = e^{Ft}((z(0) - x(0))) - \int_0^t e^{F(t-s)} TA_1(x)x(s)ds \tag{19}$$

and equation (17) follows directly. □.

Equation (19) is in the form

$$(z - Tx) = e^{Ft}((z(0) - x(0))) + \int_0^t e^{F(t-s)} \Pi(s)ds \tag{20}$$

where

$$\Pi(s) = -TA_1(x)x(s) \tag{21}$$

In equation (20), the eigenvalues of the matrix F are negative (chosen) and provided that (21) is bounded i.e. provided that

$$||\Pi(s)|| \leq \epsilon \tag{22}$$

where $\epsilon > 0$, then we have that

$$z \longrightarrow Tx, \qquad t \longrightarrow \infty \tag{23}$$

[Vidyasagar, 1978].

Instead of (14), if we consider the controlled nonlinear system

$$\dot{x}(t) = (A_0 + A_1(x))x + (B_0 + B_1(x))u \tag{24}$$

$$y(t) = H_0 x \tag{25}$$

then, as for the free system, the observer of the linear part of the above system is

$$\dot{z} = Fz + Gx + TB_0 u \tag{26}$$

where T satisfies (16), and equation (18) is now

$$\frac{d}{dt}(z - Tx) = F(z - Tx) - T(A_1(x) + B_1(x)u) \tag{27}$$

It is then straightforward to see that provided (20) is satisfied with

$$\Pi(s) = -T(A_1(x)x + B_1(x)u) \tag{28}$$

the convergence of (23) still holds.

Suppose that the matrix

$$\begin{bmatrix} H_0 \\ T \end{bmatrix} \tag{29}$$

is invertible. Then

$$\begin{bmatrix} H_0 \\ T \end{bmatrix}^{-1} \begin{bmatrix} H_0 \\ T \end{bmatrix} x = x \tag{30}$$

or

$$\begin{bmatrix} H_0 \\ T \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} \simeq x, \quad t \longrightarrow \infty \tag{31}$$

Equation (31) means that in the limit we can get information about system states x from the observation y and the observer output z.

Let

$$\begin{bmatrix} H_0 \\ T \end{bmatrix} = [M_1, M_2] \tag{32}$$

then

$$\begin{aligned} u(t) &= KM_1y + KM_2z \\ &= KM_1H_0x + KM_2z \end{aligned} \tag{33}$$

If we define

$$\zeta = z - Tx \tag{34}$$

as the error between the observation z and Tx, then from (24) and using (33), we have

$$\begin{aligned} \dot{x} &= A_0 x + B_0 u + A_1(x)x + B_1(x)u \\ &= (A_0 + B_0 K)x + B_0 K M_2 \zeta \\ &+ (A_1(x) + B_1(x)K)x + B_1(x) K M_2 \zeta \end{aligned} \tag{35}$$

Also from equation (34),

$$\begin{aligned} \dot{\zeta} &= \dot{z} - T(x)\dot{x} \\ &= F\zeta - T((A_1(x) + B_1(x)K)x + B_1(x) K M_2 \zeta \end{aligned} \tag{36}$$

and the composite system is

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} &= \begin{bmatrix} A_0 + B_0 K & B_0 K M_2 \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \\ &+ \begin{bmatrix} (A_1(x) + B_1(x)K) & B_1(x) K M_2 \\ -T(A_1(x) + B_1(x)K) & -T B_1(x) K M_2 \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \end{aligned} \tag{37}$$

If we set

$$\Psi = \begin{bmatrix} x & \zeta \end{bmatrix}^T \tag{38}$$

then the above system (37) can be written as

$$\dot{\Psi} = \Gamma_0 \Psi + \Gamma_1(x)\Psi \tag{39}$$

where Γ_0 and $\Gamma_1(x)$ are as in (37). For system (39), the eigenvalues of the matrix Γ_0 are those of $(A_0 + B_0 K)$ and F which can be chosen arbitrarily. Then for stability of (39) the matrix $\Gamma_1(x)$ should satisfy the following: (see theorem 1 in section 2)

$$\|\Gamma_1(x)\| < \frac{w_0}{M_0^2} \tag{40}$$

where the positive numbers w_0 and M_0 are defined in

$$\left\| e^{\Gamma_0 t} \right\| \leq M_0 e^{-w_0 t} \tag{41}$$

Both (40) and (41) means that the nonlinearities allowed in (39) will depend on the stability of Γ_0 (note that the eigenvalues of Γ_0 are to be chosen arbitrarily).

4 Example

As an application of the result which has been proved in this paper, we shall consider the example of the Ball and Beam system shown in Figure 1. (This system has been also considered in [3]).

In this model the beam is symmetric and is made to rotate in a vertical plane by applying a torque at the point of rotation (the centre). The ball is restricted to frictionless sliding along the beam (as a bead along a wire). This allows for complete rotations and arbitrary angular accelerations of the beam without the ball losing contact with the beam. We shall be interested in controlling the position of the ball along the beam i.e. we would like the ball to track an arbitrary trajectory.

Let the moment of inertia of the beam be J, the mass of the ball be M, and the acceleration due to gravity be G. If we choose the angle ϕ of the beam and the position r of the ball as a generalised coordinates for this system, then the Lagrangian equations of the motion are given by

$$\begin{aligned} 0 &= \dot{r} + G sin\phi - r\dot{\phi}^2 \\ t_0 &= (Mr^2 + J)\dot{\phi} + 2Mr\dot{r}\dot{\phi} + MGrcos\phi \end{aligned} \tag{42}$$

where t_0 is the torque applied to the beam and there is no force applied to the ball. Using the invertible transformation

$$t_0 = 2Mr\dot{r}\dot{\phi} + MGrcos\phi + (Mr^2 + J)u \tag{43}$$

to define a new input u the system can be written in state space form as

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1x_4^2 - Gsinx_3 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{44}$$

$$y = x_1$$

where $x = (x_1, x_2, x_3, x_4)^T =: (r, \dot{r}, \phi, \dot{\phi})^T$ is the state and $y = h(x) := r$ is the output of the system (i.e. the variable that we want to control). System (44) is in the form

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 & 0 & f(x_3) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{45}$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where

$$f(x_3) = -G(1 - \frac{x_3^2}{3!} + \frac{x_3^4}{5!} - +...)$$

and its clear that for this system only x_1 is available for measurement at the output. In order to apply the results of this paper, we write system (44) in the form of equation (24), i.e. The above system can be written as

$$\begin{aligned} \dot{x}(t) &= A_0 x + B_0 u + A_1(x)x \\ y(t) &= H_0 x \end{aligned} \tag{46}$$

where the linear parts are

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -9.8 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

$$B_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \; ; \; H_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

and the nonlinearities are

$$A_1(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x_4^2 & 0 & g(x_3) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

with

$$g(x_3) = 9.8(\frac{x_3^2}{3!} + \frac{x_3^4}{5!} - +...)$$

For the linear part of system (46), the pair (A_0, B_0) is stabilizable and so using a standard optimal control technique with $Q = I$ and $R = 1$ (see [1]) we can get the control

$$\begin{aligned} u &= -R^{-1}B^T P x \\ &= -kx \end{aligned} \tag{50}$$

where

$$k = [-1 \quad -1.9383 \quad 13.5088 \quad 5.2932] \tag{51}$$

The above control will place the eigenvalues of the closed loop system (i.e. $[A_0 - B_0 k]$) at

$$-1.0849 + j1.8508$$

$$-1.0849 - j1.8508$$

$$-2.1182$$

$$-1.0053$$

And as we have only x_1 at the output, a reduced order observer will be designed for the rest of the states of system (46). Following [9], we write:

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} + \begin{bmatrix} B_1(x) \\ B_2(x) \end{bmatrix} u \qquad (52)$$

with

$$A_{11} = [0]$$

$$A_{12} = [1 \ \ 0 \ \ 0]$$

$$A_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 0 & -9.8 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = [0]$$

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let the observer of the above system be

$$\dot{z} = Fz + Gx + TBu \tag{53}$$

where

$$F = A_{22} - LA_{12} \tag{54}$$

$$G = ((A_{22} - LA_{12})L + (A_{21} - LA_{11}, 0) \tag{55}$$

$$T = (-L, I_{n-r}) \tag{56}$$

We then follow standard procedures in chosing L so that F is a stability matrix with suitable eigenvalues. Let these eigenvalues be equal to $\lambda_1 =$

$\lambda_2 = \lambda_3 = -2$, then we have

$$L = \begin{bmatrix} 6 \\ -1.2245 \\ -0.8163 \end{bmatrix} \tag{57}$$

Using this value for L in equations (54), (55) and (56) we get

$$F = \begin{bmatrix} -6 & -9.8 & 0 \\ 1.2245 & 0 & 1 \\ 0.8163 & 0 & 0 \end{bmatrix} \tag{58}$$

$$G = \begin{bmatrix} -24 & 0 & 0 & 0 \\ 6.5307 & 0 & 0 & 0 \\ 4.8978 & 0 & 0 & 0 \end{bmatrix} \tag{59}$$

and

$$T = \begin{bmatrix} -6 & 1 & 0 & 0 \\ 1.2245 & 0 & 1 & 0 \\ 0.8163 & 0 & 0 & 1 \end{bmatrix} \tag{60}$$

With the matrix T given in (60), we can write the matrix

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ 1.2245 & 0 & 1 & 0 \\ 0.8163 & 0 & 0 & 1 \end{bmatrix} \tag{61}$$

and then find its inverse

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ -1.2245 & 0 & 1 & 0 \\ -0.8163 & 0 & 0 & 1 \end{bmatrix} \tag{62}$$

which enables us in turn to write

$$M_1 = \begin{bmatrix} 1 \\ 6 \\ -1.2245 \\ -0.8163 \end{bmatrix} \tag{63}$$

and

$$M_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (64)$$

For this system, asw e have stated at the beginning of this example, we would like the ball to track a specific rt ajectory. We, therefore, recall that for a tracking problem of this kind the control is given by (see [1])

$$u = kM_1Hx + kM_2z - R^{-1}B^Ts_l \quad (65)$$

where k, M_1 and M_2 are as defined in equations (50), (63) and (64), s_l is given by

$$s = [(A_0 - B_0k)^T]^{-1}Qr \quad (66)$$

and r is the set point.

Using the calculations obtained so far we can now write down the composite original (nonlinear) plant-observer system as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_4^2 & 0 & f(x_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 & -6 & -9.8 & 0 \\ 6.5307 & 0 & 0 & 0 & 1.2245 & 0 & 1 \\ 4.8978 & 0 & 0 & 0 & 0.8163 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (67)$$

The above system has been simulated and the results are shown in Fig.2. The output of the observer i.e. the estimated unobservable states of the original system are shown in Fig.2b. Using these estimated states in the feedback of equation (65), the system provide good tracking for the trajectory $5 * cos(pi * t/30)$ as its clear in Fig.2a.

5 Conclusion.

In this paper we have obtained a simple method for designing observers for a wide class of nonlinear systems based on a perturbation theorem. The method has been shown to give good results in the case of a ball and beam.

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