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Unwrapping the Phase Response Functions

for Nonlinear Systems

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Abstract: The higher order phase response functions contain important information about the behaviour of nonlinear systems. Unfortunately, simply adopting the linear convention of plotting the phase between ±180 produces higher order phase response functions which are virtually impossible to interpret. In the present paper two new algorithms are introduced to overcome these problems by unwrapping the multidimensional nonlinear phase. The unwrapped phase responses are shown to be much easier to interpret and details of the implementation together with examples illustrating the new techniques are included.

1. Introduction

Generalised or higher order frequency response functions are defined as the Fourier transform of the Volterra kernels and represent a natural extension of the linear frequency domain concepts to nonlinear systems. The higher order frequency response functions can be estimated either using multi-dimensional FFT and windowing based algorithms(e.g., Vinh et al, 1987; Kim and Powers, 1988) or by employing the NARMAX parametric approach(Billings and Tsang,1989a, 1989b; Storer and Tomlinson, 1989). The multidimensional frequency response functions which are obtained provide insight into the operation of complex nonlinear systems and phenomena such as harmonics, gain compression/expansion, desensitisation and intermodulation can be studied(Peyton-Jones and Billings, 1990; Zhang and Billings, 1992). Both the magnitude and phase information is provided but unfortunately the inverse tangent which is used to compute the phase angle yields the principal value of the phase between $-\pi$ and π. Because of this the multidimensional phase plots often appear to be highly complex and

discontinuous even when the corresponding gain plot exhibits clearly defined ridges and peaks. This problem arises because the phase is wrapped in the narrow range of $[-\pi,\pi]$.

In many cases it may be preferable to have a smooth continuous phase function rather than a discontinuous principal value to aid graphical interpretation and to provide an analytical phase function.

A procedure which produces a continuous phase curve from the principal value was initially used to compute the complex cepstrum and is known as phase unwrapping(Oppenheim and Schafer, 1975). Several numerical algorithms have subsequenctly been proposed to unwrap the phase spectrum for a given signal(Tribolet, 1975; McGowan and Kuc, 1982; Moura and Baggeroer, 1988). In the present paper these ideas are extended to the nonlinear case where the phase responses become multidimensional. Two phase unwrapping algorithms are developed for the multidimensional phase plots associated with the generalised frequency response functions of nonlinear systems. The unwrapped phase plots which are produced are shown to be far easier to interpret and several examples are included to illustrate the concepts and ideas.

2. Nonlinear Phase Response and the Principal Value

Consider a single input single output nonlinear system with an input $u(t)$ and output $y(t)$ which can be described by the Volterra model

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{1}$$

where $y_n(t)$, the nth-order output of the system is defined by

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \qquad n > 0 \tag{2}$$

and $h_n(\tau_1, \cdots, \tau_n)$ is the n-th order Volterra kernel. In the frequency domain the n-th order generalised frequency response function $H_n(j\omega_1, \cdots j\omega_n)$ can then be defined as the n-dimensional Fourier transform of $h_n(\tau_1, \cdots, \tau_n)$.

$$H_n(j\omega_1, \cdots j\omega_n) = \int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)} \, d\tau_1 \cdots d\tau_n \qquad (3)$$

Clearly the linear system case is given by $n=1$ where $h_1(\tau)$ is the impulse response and $H_1(j\omega)$ the frequency response function. In polar form, the multivariate complex-valued function $H_n(\cdot)$ is represented as

$$H_n(j\omega_1, \cdots j\omega_n) = \Gamma_n(\omega_1, \cdots, \omega_n)e^{j\Phi_n(\omega_1, \cdots, \omega_n)} \qquad (4)$$

where, following the convention of the linear case, $\Gamma_n(\cdot) = |H_n(j\omega_1, \cdots, j\omega_n)|$ is called the magnitude response and

$$\Phi_n(\omega_1, \cdots, \omega_n) = \angle\left[H_n(j\omega_1, \cdots j\omega_n) \right] \qquad (5)$$

is called the phase response. Strictly speaking, $\Phi_n(\cdot)$ is a multi-valued function since any integral multiple of 2π radians can be added to $\Phi_n(\cdot)$ without changing the complex value of $H_n(\cdot)$. In order to avoid ambiquilty $\Phi_n(\cdot)$ is usually restricted to lie in the range $-\pi$ to π. The value in this range is called the principal value. It has been an acceptable convention to plot the principal value of phase over $[-\pi, \pi]$. If the computed value exceeds this range a jump of $\pm 2\pi$ is required to bring the phase back into the range. This problem is also the result of using the inverse tangent function to compute the phase angle. That is, ordinarily, $\Phi_n(\cdot)$ is given as its principal value

$$\text{ARG}[H_n(j\omega_1, \cdots j\omega_n)] = \tan^{-1}\frac{\text{Im}[H_n(j\omega_1, \cdots j\omega_n)]}{\text{Re}[H_n(j\omega_1, \cdots j\omega_n)]} \qquad (6)$$

Although the use of principal value does not affect $H_n(\cdot)$ at any individual point, many artificial discontinuities of 2π are introduced and this makes it almost impossible to measure

the net change in the phase of $H_n(\cdot)$ as the frequency variables $\omega_1, \cdots, \omega_n$ go from 0 to ∞ (0 to 2π for discrete-time systems). As a consequence the classical analysis method in which the phase and magnitude response are decomposed as an addition of functional factors become difficult to implement. Graphically, plots of the phase functions are often disrupted when the principal phase value is used. For example, the phase of $e^{-j(\omega_1+\omega_2)}$ is simply an analytical function $\Phi = -(\omega_1+\omega_2)$ which should be a sloping plane in the two dimensional frequency domain. But the plot of the principal phase value illustrated in Fig.1 appears to be far more complicated and is almost impossible to interpret and recognise. Many practical nonlinear systems contain a pure delay element which will be reflected in the nonlinear transfer functions as a multi-variable exponential multiplier $e^{-j(\omega_1+ \cdots +\omega_n)}$ in each $H_n(\cdot)$. It is therefore not surprising that inspection of the phase plots associated with the generalised frequency response functions of many real systems appear to be very complex and intricate. Clearly a new set of procedures need to be developed to overcome these problems.

3. Phase Unwrapping for Linear Systems

Phase unwrapping will be introduced for linear systems initially and then extended to the nonlinear case. For linear systems the frequency response functions are one dimensional that is $n=1$ in eqn.(1) and (3). Given a linear frequency response function $H(j\omega)$ the phase response at some frequency ω can be written as

$$\arg[\, H(j\omega)] \;=\; \tan^{-1}\frac{\mathrm{Im}[H(j\omega)]}{\mathrm{Re}[H(j\omega)]} + 2\pi l \tag{7}$$

where the integer l allows the phase to be continuous function of ω. Determining the value of l defines the phase unwrapping process and this can be achieved in two ways. One approach is to compute the principal value of the phase response using the inverse tangent of $H(j\omega)$ and

then add or substract 2π at the point where a discontinuity is encountered. This is a very simple routine for the one dimensional linear case. The algorithm relies on the the detection of discountinuiuties which is done by computing the difference between the principal values of the phase at two adjacent frequencies ω_{k-1} and ω_k along the one dimensional frequency axes. Whenever this difference is greater than a given threshold, a 2π shift is used. In order to distinguish this method from the analytical approach to be described below, this algorithm will be called the "2π amending algorithm". This simple approach will yield the correct unwrapped phase providing the frequency sampling is fine enough and the phase function is relatively smooth. However, in many cases the sorting between natural variations of the phase and the discontinuity induced by the modulo 2π operation becomes dubious.

Another less popular approach is to re-define the phase response analytically in terms of integration of the phase derivative. For a given linear frequency response function $H(j\omega)$, although the argument of $H(j\omega)$ is a multi-valued function, its first derivative, denoted as $\phi'(\cdot)$, is still well-defined. For notational convenience define

$$H_R(\omega) \stackrel{\Delta}{=} \mathrm{Re}[H(j\omega)] \qquad H_I(\omega) \stackrel{\Delta}{=} \mathrm{Im}[H(j\omega)] \tag{8}$$

then, by formally computing the first derivative of the inverse tangent function on the right hand side of eqn.(7), the phase derivative is obtained as

$$\phi'(\omega) = \frac{1}{1+\left[\dfrac{H_I(\omega)}{H_R(\omega)}\right]^2} \frac{d}{d\omega}\left[\frac{H_I(\omega)}{H_R(\omega)}\right]$$

$$= \frac{H_R^2(\omega)}{|H(j\omega)|^2} \frac{d}{d\omega}\left[\frac{H_I(\omega)}{H_R(\omega)}\right] = \frac{H_R(\omega)H'_I(\omega) - H_I(\omega)H'_R(\omega)}{|H(j\omega)|^2} \tag{9}$$

based on the standard expression for the derivative of an inverse tangent function

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$(\arctan v)' = (\dfrac{v'}{1+v^2})$. If $H_R(\omega)$ and $H_I(\omega)$ are continuous functions and $|H(j\omega)| \neq 0$, the phase

derivative $\phi'(\cdot)$ will be a continuous function of ω. Thus an unwrapped phase $\phi(\omega)$ at a particular frequency ω can be unambiquiously defined as the integral of the derivative

$$\phi(\omega) = \phi(\omega_0) + \int\limits_{\omega_0}^{\omega} \phi'(\eta)\, d\eta \qquad (10)$$

where $\phi(\omega_0)$ is the principal value obtained from the inverse tangent. For linear systems $H(j\omega)$ is a rational polynomial so the derivative of $\phi(\omega)$ and consequently $\phi(\omega)$ itself will be continuous as long as $|H(j\omega)| \neq 0$. Therefore if the frequency response function is given, either in the form of data or as an analytical expression the unwrapped phase can be computed easily from eqn.(9) and (10).

In signal processing phase unwrapping is mainly used for finding the continuous phase spectrum for a given signal. The signal is normally in the form of a discrete sequence and the phase spectrum is extracted from the FFT data on a set of uniformly spaced frequencies. Most existing algorithms therefore are based on the calulation of the phase derivative and then integration from FFT data. But for the phase response functions estimated by parametric methods (eg NARMAX), the problem is much easier because an analytic form of the system frequency response function is given at the relevant frequencies and hence both the derivative and the integration can be accurately evaluated. The following example illustrates the the computation involved.

Consider a linear system described by the transfer function

$$G(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)} \qquad (11)$$

The frequency response function $G(j\omega)$ is given by replacing s with $j\omega$. Consider the

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frequency range from 0.1 to 100 Hz. Two derivatives, $H'_I(\cdot)$ and $H'_R(\cdot)$, are obtained by perturbing the frequency variable ω with a small deviation, that is

$$H'_R(\omega) = \frac{H_R(\omega+\Delta\omega)-H_R(\omega)}{\Delta\omega} \qquad H'_I(\omega) = \frac{H_I(\omega+\Delta\omega)-H_I(\omega)}{\Delta\omega}$$

where $\Delta\omega=0.001$ was used in the present analysis. All the values of $H_R(\cdot)$ and $H_I(\cdot)$ are directly extracted from the complex values of $H(j\omega)$. The phase derivative can now be computed using eqn.(9) and is shown in Fig.2. The unwrapped phase can then be evaluated at any given frequency ω by integrating the derivative. An adaptive recursive Simpson's rule was used for the numerical integration(Forsy et al,1977). To obtain the unwrapped phase curve on a given sequence of frequency values ω_k, $k=1,2...,N$, along the frequency axis, the unwrapped value at each given ω_k can be evaluated seperately, all from the initial point ω_0. However the computation can be speeded up by using the following recursive integration

$$\phi(\omega_k) = \phi(\omega_{k-1}) + \int_{\omega_{k-1}}^{\omega_k} \phi'(\eta)d\eta \qquad k=1,2,\cdots \qquad (12)$$

with

$$\phi(\omega_0) = \tan^{-1}\left[\frac{H_I(\omega)}{H_R(\omega)}\right]$$

Notice that although the frequencies may be spaced very sparsely, the integration can still be undertaken accurately between two adjacent frequencies, ω_{k-1} and ω_k, by means of numerical routines. Therefore the accuracy of the results does not depend on the frequency distribution like many existing algorithms which are based on FFT derived estimates. The continuous phase response unwrapped by the above precedure is given in Fig.3(a). It is seen that the phase decreases monotonically from -75 to -270 degrees $(-3\pi/2)$. In comparison, the principal value computed by inverse tangent is given in Fig.3(b), where a 2π jump is observed at 1.15 Hz to maintain the phase in the range $[-\pi,\pi]$.

4. Phase Unwrapping for Nonlinear Systems

As stated earlier the frequency response functions for nonlinear systems are multi-dimensional in nature and therefore so are the phase response functions. For many practical systems, with moderate nonlinearities, the first two or three higher order frequency response functions in the series of eqn.(2) and (3) will be sufficient to characterise the major features of the system. Following the basic ideas introduced in last section, two phase unwrapping algorithms are derived for nonlinear systems by initially considering the second order generalised phase response and then extending these results to the higher order generalised frequency response functions.

Algorithm 1: Integration Approach

Consider the second order generalised frequency response function $H(j\omega_1, j\omega_2)$ in eqn.(3) and (4). To simplify the notation denote this as $H(j\omega_1, j\omega_2) = H_R(\omega_1, \omega_2) + j H_I(\omega_1, \omega_2)$ so that the principal value of the phase response function can be computed as

$$\text{ARG}\left[H(j\omega_1, j\omega_2)\right] = \tan^{-1}\left[\frac{H_I(\omega_1, \omega_2)}{H_R(\omega_1, \omega_2)}\right] \tag{13}$$

which is in the range $[-\pi, \pi]$. Analagous to the linear case a continuous phase response function can be re-defined in terms of the integration of the partial derivatives

$$\Phi(\omega_1, \omega_2) = \Phi(\omega_{10}, \omega_{20}) + \int_{\omega_{20}}^{\omega_2} \frac{\partial \Phi(\omega_{10}, v)}{\partial v} dv + \int_{\omega_{10}}^{\omega_1} \frac{\partial \Phi(v, \omega_2)}{\partial v} dv \tag{14}$$

where the partial derivatives are given by

$$\frac{\partial \Phi(\omega_{10}, v)}{\partial v} = \frac{H_R(\omega_{10}, v)\dfrac{\partial H_I(\omega_{10}, v)}{\partial v} - H_I(\omega_{10}, v)\dfrac{\partial H_R(\omega_{10}, v)}{\partial v}}{|H(j\omega_{10}, jv)|^2} \tag{15}$$

and

$$\frac{\partial \Phi(v, \omega_2)}{\partial v} = \frac{H_R(v, \omega_2)\dfrac{\partial H_I(v, \omega_2)}{\partial v} - H_I(v, \omega_2)\dfrac{\partial H_R(v, \omega_2)}{\partial v}}{|H(jv, j\omega_2)|^2} \tag{16}$$

The initial value $\Phi(\omega_{10},\omega_{20})$ is computed from the inverse tangent. For a wide class of non-linear systems, it has been shown (Peyton-Jones and Billings, 1989) that the higher order frequency response functions are rational polynomials of the frequency variables $\omega_1, \cdots ,\omega_n$ ($e^{-j\omega_1}, \cdots ,e^{-j\omega_n}$ for nonlinear discrete-time systems). Therefore the partial derivatives should be well-defined and consequently so should the unwrapped phase response as long as $|H_n(\cdot)|\neq0$.

The effects of eqn.(14) can be illustrated using a simple complex function $e^{-j(\omega_1+\omega_2)}$ which is a basic delay element in the second order transfer function. As noted earlier the phase delay induced by this element is a linear function given by the two input frequency sum $(\omega_1+\omega_2)$. However, this property can hardly be recognised from the plot of the principal value given in Fig.1. Now use the new definition:

$$e^{-j(\omega_1+\omega_2)} = \cos(\omega_1+\omega_2) - j\,\sin(\omega_1+\omega_2) \tag{17}$$

Let $\omega_{10}=\omega_{20}=0$. The partial derivatives in eqn.(15) and (16) are calculated as

$$\frac{\partial\Phi(v,\omega_2,)}{\partial v} = -\cos(v+\omega_2)\cos(v+\omega_2) - \sin(v+\omega_2)\sin(v+\omega_2) = -1 \tag{18}$$

and similarly

$$\frac{\partial\Phi(0,v)}{\partial v} = -1 \tag{19}$$

Using eqn.(14) yields

$$\arg\left[e^{-j(\omega_1+\omega_2)}\right] = \Phi(0,0)+[-\omega_2+0]+[0-\omega_1]=-(\omega_1+\omega_2) \tag{20}$$

which is the desired result.

Implementation of the integration method for the nonlinear case is similar to the linear one. No

matter how complicated the transfer function is the partial derivatives which are used in eqn.(15) and (16) can always be obtained by perturbation using a small deviation (e.g. $\Delta\omega=0.001$) on the relevant frequency variable while keeping the other frequencies fixed. The numerical integrations in eqn.(14) can then be undertaken to give the unwrapped phase response value $\Phi(\omega_1,\omega_2)$ at any particular point (ω_1,ω_2). The initial point can be taken as the origin (0,0) or chosen arbitrarily as $(\omega_{10},\omega_{20})$. The selection of the initial point may affect the integration time and probably the relative position of the resulting phase surface. The results obtained with different initial points may differ from each other by 2π but will have the same shape.

In order to get a 3D mesh surface for the unwrapped phase response, the two frequency axes are each sampled to form an $N \times M$ grid. The 3D graphical surface can then be obtained by computing the unwrapped phase value at every point on the frequency grid. All the integrations are evaluated seperately from each other. This is the most straightforward implementation of the algorithm but it will take a long time to complete, even for a very simple example. Fortunately a fast algorithm can be derived based on recursive computation. Assume the two dimensional frequency domain is a $N \times M$ grid and an arbitrary point on the grid is denoted as $[\omega_1(i), \omega_2(k)]$ with $i=1,...,N$ and $k=1,...,M$. Denote the two integrations in eqn.(14) as I_1 and I_2, respectively, so that they are re-expressed as

$$I_1[\omega_2(k)] = \int_{\omega_{20}}^{\omega_2(k)} \frac{\partial\Phi(\omega_{10},v)}{\partial v} dv \tag{21}$$

$$I_2[\omega_1(i),\omega_2(k)] = \int_{\omega_{10}}^{\omega_1(i)} \frac{\partial\Phi(v,\omega_2(k))}{\partial v} dv \tag{22}$$

Clearly I_1 is independent of ω_1 while I_2 depends on both ω_1 and ω_2. So I_1 can be computed recursively along the ω_2 axis by implementing a single loop

$$I_1[\omega_2(k)] = I_2[\omega_2(k-1)] + \int_{\omega_2(k-1)}^{\omega_2(k)} \frac{\partial \Phi(\omega_{10},v)}{\partial v} dv \tag{23}$$

for $k=2,...,M$ and

$$I_1[\omega_2(1)] = \int_{\omega_{20}}^{\omega_2(1)} \frac{\partial \Phi(\omega_{10},v)}{\partial v} dv \tag{24}$$

For a given $\omega_2(k)$ with k fixed, I_2 can also be recursively computed along the ω_1 axis by an inner loop within the I_1 loop. That is, for any given integer k

$$I_2[\omega_1(i),\omega_2(k)] = I_2[\omega_1(i-1),\omega_2(k)] + \int_{\omega_1(i-1)}^{\omega_1(i)} \frac{\partial \Phi(v,\omega_2(k))}{\partial v} dv \qquad i=2,...N \tag{25}$$

The first value for I_2 should be computed as the initial value before starting the inner loop, that is, for any given k

$$I_2[\omega_1(1),\omega_2(k)] = \int_{\omega_{10}}^{\omega_1(1)} \frac{\partial \Phi(v,\omega_2(k))}{\partial v} dv \tag{26}$$

After going through every point on the $N \times M$ grid, I_1 and I_2 are calculated individually through two inlaid loops. Then the unwrapped phase is calculated at every point on the frequency grid by summing the two integrations and the initial value

$$\Phi[\omega_1(i),\omega_2(k)] = \Phi[\omega_{10},\omega_{20}] + I_1[\omega_2(k)] + I_2[\omega_1(i),\omega_2(k)] \qquad i=1,...N; \quad k=1,...M \tag{27}$$

The initial point can be chosen as the first point in the input domain, i.e., let $\omega_{10} = \omega_1(1)$ and $\omega_{20} = \omega_2(1)$. Then the unwrapped phase surface will always overlap with the conventional wrapped one at the begining. By using the recursive implementation, a great deal of time is saved. The examples in the next section indicated that the recursive implementation is almost ten times faster than the basic approach.

This approach can also be extended to the more general case for $n>2$ in eqn.(1) and (2). The

derivation of the unwrapped phase for these more complex cases is given in Appendix I.

Algorithm 2: 2π amending approach

Similar to the linear case this algorithm computes the principal value of the phase response using the inverse tangent associated with $H_n(\cdot)$, and then unwraps by appropriately processing the principal value data. For the second order case, the data matrix of principal values will be an $N \times M$ array with all values within the range $[-\pi,\pi]$. The unwrapping procedure involves the application of two one dimensional phase unwrappings on the two axes followed by a two dimensional point-to-point unwrapping over the remainder of the matrix.

The algorithm proceeds as follows. First, the principal phase values along the two frequency axes, $\Phi[\omega_1(k),\omega_{20}]$ and $\Phi[\omega_{10},\omega_2(i)]$ with $k=1,...,M$ and $i=1,...,N$ are unwrapped by simply indexing k and i, and adding or substracting 2π whenever a phase discontinuity is encountered. Following these one dimensional unwrappings on the axes, the remainder of the phase matrix is scanned in the manner indicated in Fig.4. Suppose for example that the "present point" is at (n,m) (boxed lattice point). The differences between the "present" phase value and the three "previously" unwrapped neighbouring phase values at $(n-1,m-1)$, $(n-1,m)$ and $(n,m-1)$, respectively (lattice points marked with 'x' in Fig.4) are each calculated. If a discontinuity is encountered, the multiple of 2π required to remove the discontinuity at that point is added (or substracted) to the phase. Then the operation moves to the next point, until the whole phase array is processed. Notice that the first row and first column (in the shadowed box) have been unwrapped by one dimensional algorithm beforehand.

This algorithm can also be used to unwrap the higher order ($n\geq3$) phase responses provided the corresponding principal value data is available for processing. The procedure still consists

of n one dimensional phase unwrappings on the n frequency axes and an n-dimensional point checking over the remainder by comparing the 'present point' with the 'previous' points. In the nth order case, the number of 'previous points' will be $\Sigma n = \dfrac{n(n+1)}{2}$.

An obvious advantage of the 2π amending algorithm is that it is simple, fast and numerically reliable since there is not much numerical computation involved. In comparison, the integration method takes much longer to complete (even with recursive implementation) and may fail for some functions due to purely numerical problems. However the integration method gives an independent evaluation of the re-defined continuous phase for a given complex-valued function, regardless of the selection of the frequency sampling, function characteristcs etc and these are significant advantage because the 2π amending algorithm requires that the natural variations between the adjacent points are relatively small to ensure the discontinuities are sorted out une-quivacally.

5. Examples

The first example is a NARMAX model given by

$$y(k) = 0.5y(k-1) + u(k-2) + 0.1u^2(k-1) \tag{28}$$

The first and second order transfer functions can be derived as(Peyton-Jone and Billings,1989)

$$H_1(j\omega) = \frac{e^{-2j\omega}}{1-0.5e^{-j\omega}} \quad \text{and} \quad H_2(j\omega_1,j\omega_2) = \frac{0.1e^{-j(\omega_1+\omega_2)}}{1-0.5e^{-j(\omega_1+\omega_2)}} \tag{29}$$

The second order phase response based on principal values is shown in Fig.5(a). In comparison, the unwrapped phase response computed by the integration method is given in Fig.5(b) which shows that the jump from $-\pi$ to π along $f_1+f_2=0$ has been united to form a

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smooth surface. The same result was also obtained by using the 2π amending algorithm.

If the output time data generated from eqn.(28) is now shifted back three steps with respect to the input data and a new model is estimated a model with a pure delay factor will be obtained and the second order transfer function will then become

$$H_2^d(j\omega_1,j\omega_2) = e^{-3j(\omega_1+\omega_2)}H_2(j\omega_1,j\omega_2) = \frac{0.1e^{-4j(\omega_1+\omega_2)}}{1-0.5e^{-j(\omega_1+\omega_2)}} \tag{30}$$

The wrapped second order phase response for this delayed system is given in Fig.6(a). Due to the presence of the pure delay many pleats are observed in the phase response plot which becomes very difficult to intepret. Using the integration or the 2π amending algorithm the phase response can be fully unwrapped to reveal a smoothly sloping surface as shown in Fig.6(b). A comparison with to Fig.5(b) shows the phase response of the delayed data system maintains the same basic shape but has a larger slope. This is because an extra phase lag which increases linearly with the frequency sum $(\omega_1+\omega_2)$ is added to the phase response due to the delay. The unwrapped phase response is clearly much easier to intepret

The time domain model for the second example is

$$y(k) = 0.8y(k-1) + 0.5u(k-1) + 0.2u(k-1)y(k-1) \tag{31}$$

from which the first and second order phase response functions are derived as

$$H_1(j\omega) = \frac{0.5e^{-j\omega}}{1-0.8e^{-j\omega}} \quad \text{and} \quad H_2(j\omega_1,j\omega_2) = \frac{0.2[H_1(j\omega_1)+H_1(j\omega_2)]e^{-j(\omega_1+\omega_2)}}{1-0.8e^{-j(\omega_1+\omega_2)}} \tag{32}$$

The second order phase response is shown by principal value in Fig.7. Although the time domain model is superficially no more complicated than the previous example, eqn.(28), the phase response is very different. In Fig.7, several cuts can be observed and it is difficult to tell which is caused by the 2π modulo effect and which are inherent variations of the function

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itself. The unwrapped phase is computed using the integration algorithm and is shown in Fig.8. A comparison with the wrapped phase response in Fig.7 shows that the unwrapped phase is neat and easy to interpret. In fact, Fig.8 can be decomposed as an addition of the phases of three elemental factors: $H_1(j\omega_1) + H_1(j\omega_2)$, $e^{-4j(\omega_1 + \omega_2)}$ and the denominator $\dfrac{1}{1 - 0.5e^{-j(\omega_1 + \omega_2)}}$ which are illustrated in Figs.9(a), (b) and (c), respectively. The basic pattern of the phase response plot is contributed by $H_1(j\omega_1) + H_1(j\omega_2)$ while the inclination of the whole surface indicates the presence of an exponential factor $e^{-4j(\omega_1 + \omega_2)}$. The effect of the denominator is relatively tiny, this just bends the graph slightly. The difference between the unwrapped phase response and the original principal value is illustrated in Fig.10. It is seen that a number of 2π values are used in some area to amend the wrapped phase response while the physical significance is not affected.

6. Conclusions

The phase unwrapping problem for nonlinear systems has been studied and two algorithms have been developed to unwrap the higher order generalised phase response functions. Algorithms based on integrating the partial derivatives of the phase and using multiples of 2π to amend the principal value were derived. The implementation of the algorithms has been illustrated by examples and it has been shown that the unwrapped phase responses are much easier to interpret. The use of unwrapped phase also allows the entire phase response to be calculated by superposing the phase of the various factors and this together with the parametric estimation approach for the generalised frequency response functions provides a very powerful toolkit for the analysis of nonlinear systems in the frequency domain.

7. Acknowledgements

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8. Appdendix I: Unwrapped Phase For Higher Order Frequency Response Functions

For general case, given a nth order frequency response function

$$H_n(j\omega_1,...j\omega_n) = H_R(\omega_1,...,\omega_n) + j\,H_I(\omega_1,...,\omega_n) \tag{A-1}$$

denote the unwrapped phase as $\Phi(\omega_1,...,\omega_n)$. Then

$$\int_0^{\omega_1} \frac{\partial\Phi(\nu_1,\omega_2,...,\omega_n)}{\partial\nu_1} d\nu_1 = \Phi(\omega_1,\omega_2,...,\omega_n) - \Phi(0,\omega_2,...,\omega_n)$$

$$\int_0^{\omega_2} \frac{\partial\Phi(0,\nu_2,...,\omega_n)}{\partial\nu_2} d\nu_2 = \Phi(0,\omega_2,...,\omega_n) - \Phi(0,0,...,\omega_n)$$

$$\int_0^{\omega_k} \frac{\partial\Phi(0,...,0,\nu_k,\omega_{k+1},...,\omega_n)}{\partial\nu_k} d\nu_k = \Phi(0,...,0,\omega_k,\omega_{k+1},...,\omega_n) - \Phi(0,...,0,0,\omega_{k+1},...,\omega_n)$$

$$\int_0^{\omega_n} \frac{\partial\Phi(0,...,0,\nu_n)}{\partial\nu_n} d\nu_n = \Phi(0,...,0,\omega_n) - \Phi(0,...,0,0)$$

Therefore the nth order unwrapped phase response can be defined as

$$\Phi(\omega_1,...,\omega_n) = \Phi(0,...,0) + \int_0^{\omega_n} \frac{\partial\Phi(0,...,0,\nu_n)}{\partial\nu_n} d\nu_n + ...$$

$$+ \int_0^{\omega_k} \frac{\partial\Phi(0,...,0,\nu_k,\omega_{k+1},...,\omega_n)}{\partial\nu_k} d\nu_k + ... + \int_0^{\omega_1} \frac{\partial\Phi(\nu_1,\omega_2,...,\omega_n)}{\partial\nu_1} d\nu_1 \tag{A-2}$$

The partial derivative is given by

$$\frac{\partial\Phi(\omega_1,...,\nu_k,...,\omega_n)}{\partial\nu_k} =$$

$$\frac{H_R(\omega_1,...,\nu_k,...,\omega_n)H'_I(\omega_1,...,\nu_k,...,\omega_n) - H_I(\omega_1,...,\nu_k,...,\omega_n)H'_R(\omega_1,...,\nu_k,...,\omega_n)}{|H(j\omega_1,...,j\nu_k,...,j\omega_n)|^2} \tag{A-3}$$

where $H'_R(\cdot)$ and $H'_I(\cdot)$ can always be obtained by perturbation on the relevant frequency variable. Notice that for the above during derivation the initial point has been chosen as $(0,...,0)$ for notational simplicity. This can be replaced by an arbitrary point $(\omega_{10},...,\omega_{n0})$ if required.

Fig.1 Principal value plot for $e^{-j(\omega_1+\omega_2)}$.

Fig.2 Derivative of phase response function for the linear system example.

Fig.3 The phase response function for the linear system example:

    (a) Unwrapped phase response;   (b) Principal value

Fig.4 Illustration of the 2π amending algorithm for the second order case.

Fig.5 Second order phase response before shifting

    (a) principal value;   (b) unwrapped

Fig.5 Second order phase response after shifting

    (a) principal value;   (b) unwrapped

Fig.7 Principal value for second nonlinear example.

Fig.8 Unwrapped phase response for second nonlinear example.

Fig.9 The phase plots of the factors:

    (a) $H_1(j\omega_1)+H_1(j\omega_2)$;

    (b) $e^{-j(\omega_1+\omega_2)}$;

    (c) $1/1-0.8e^{-j(\omega_1+\omega_2)}$

Fig.1

Fig.2 - Phase derivative

Fig.3(a) - unwrapped phase response

Fig.3(b) - principal value
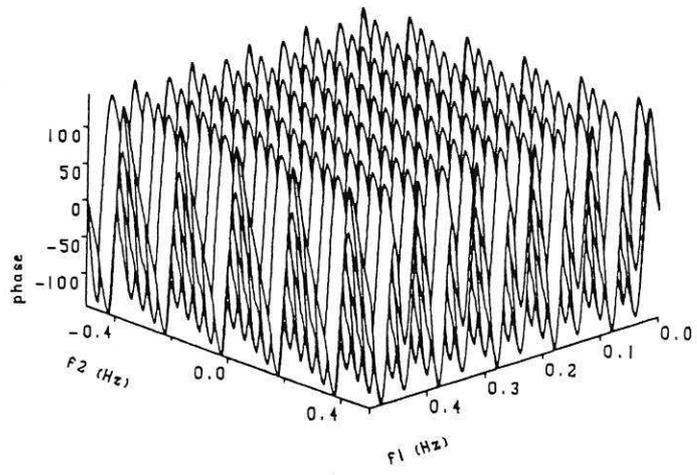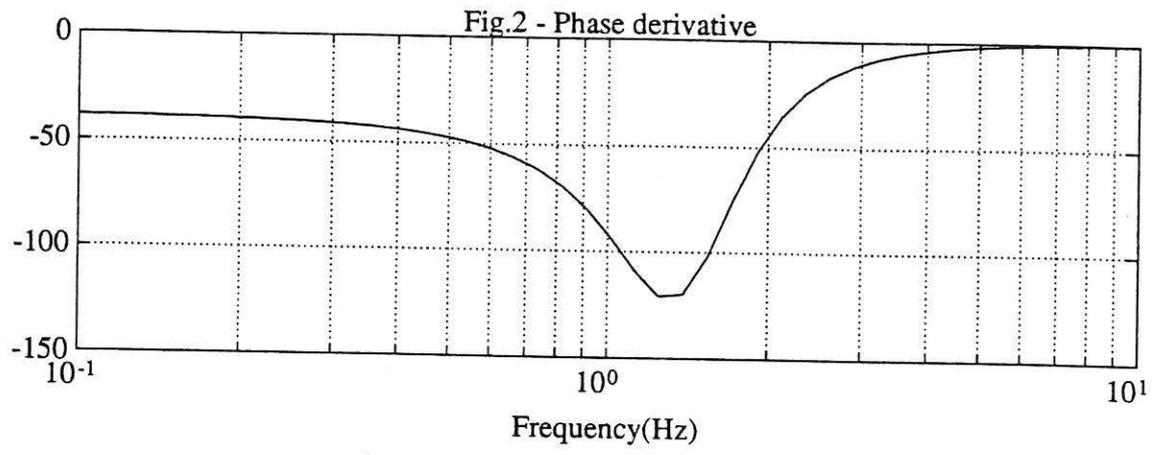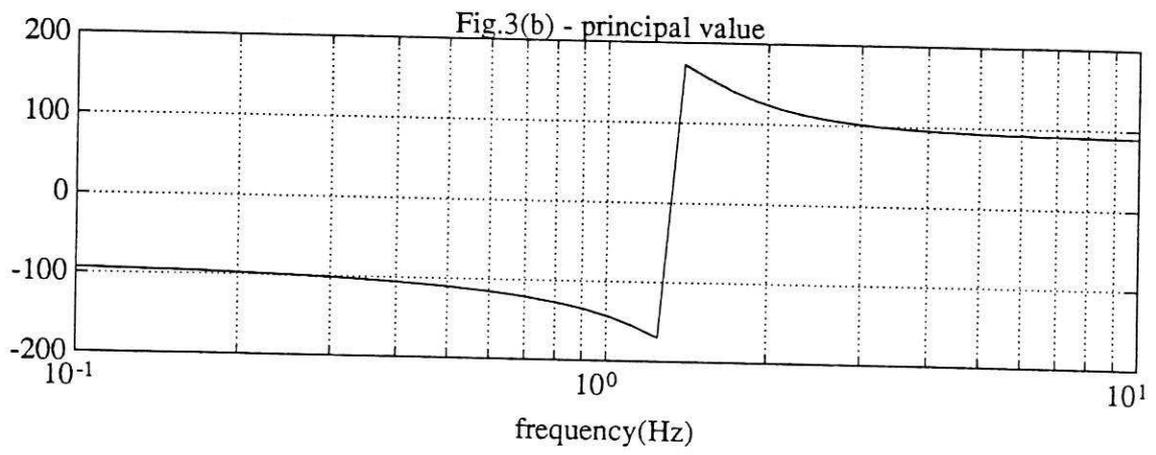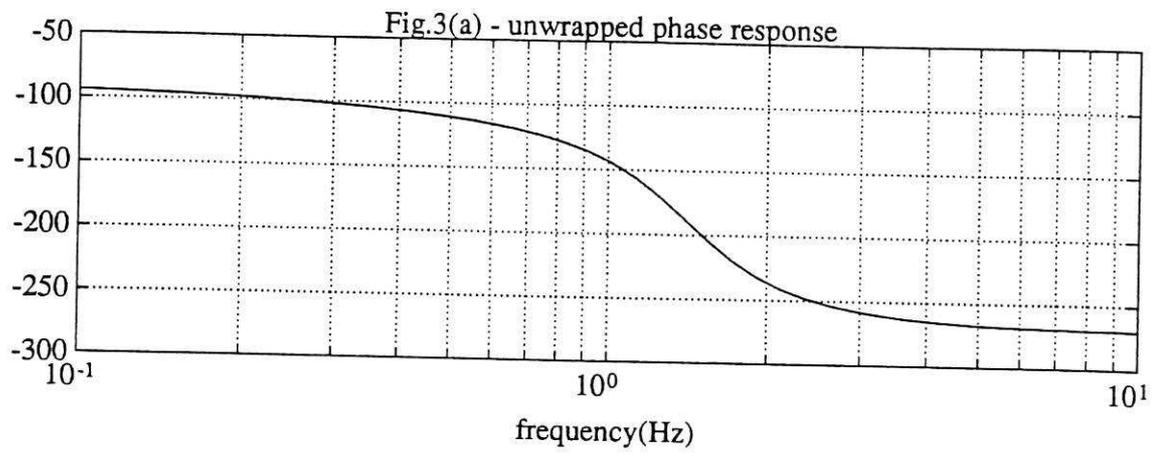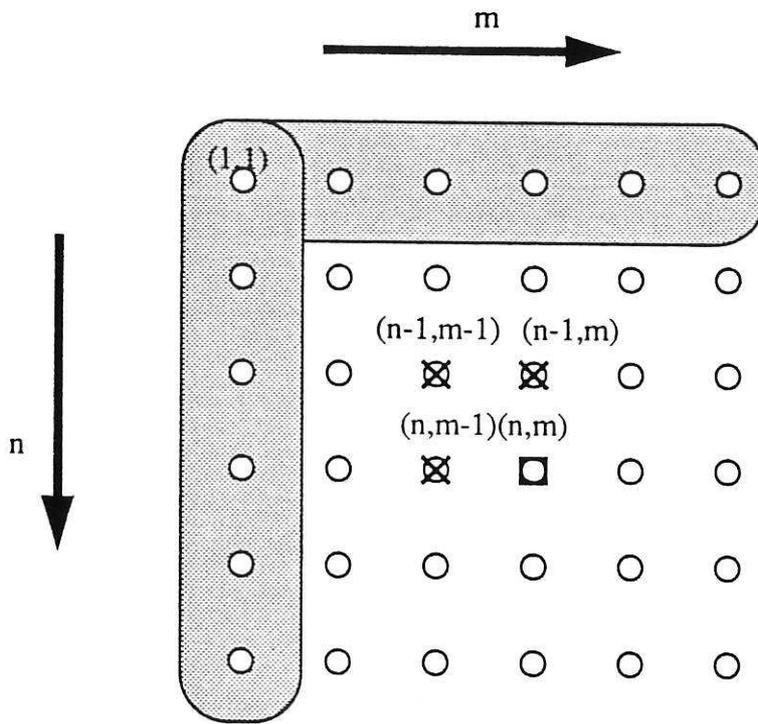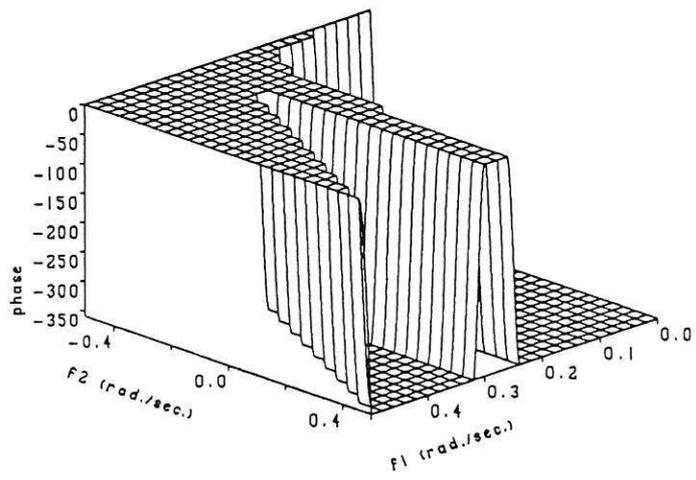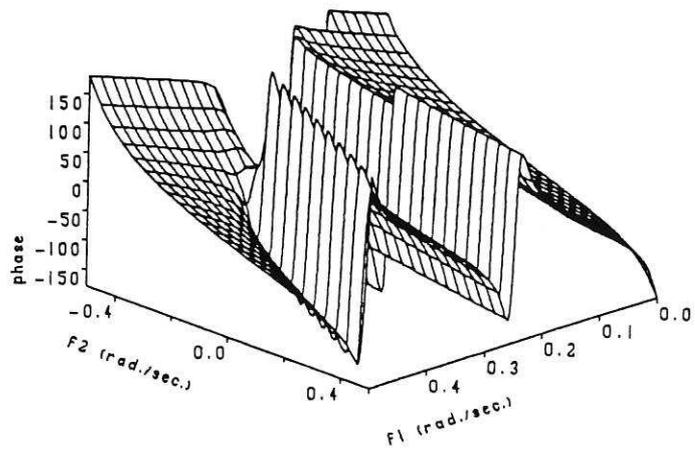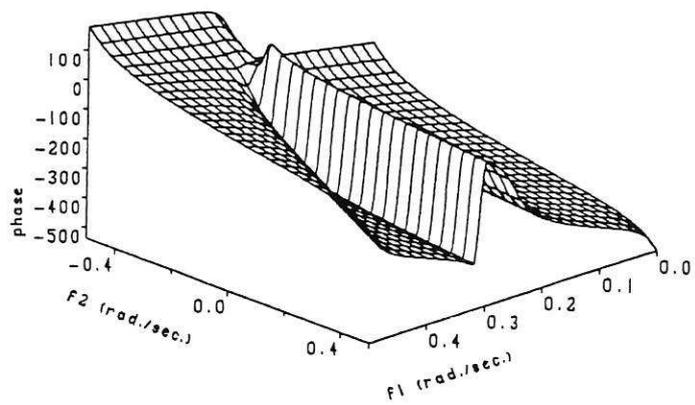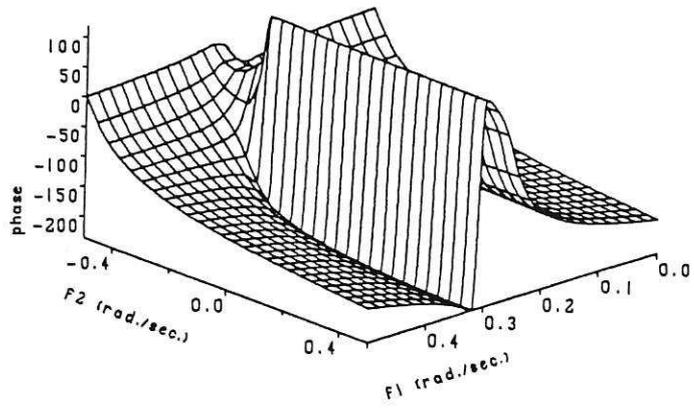
Fig.4　Illustration of the 2
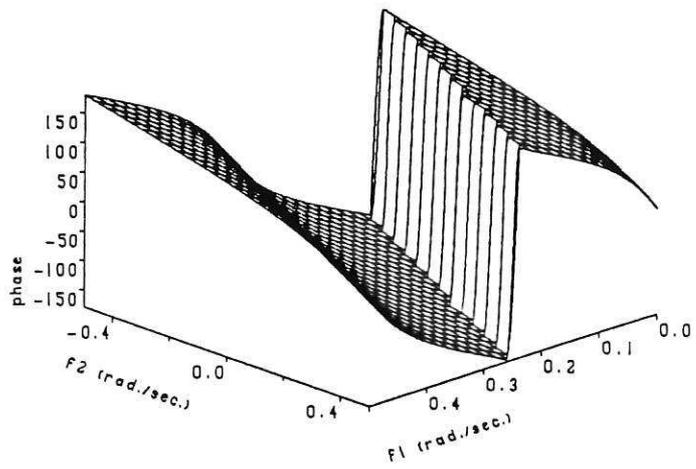
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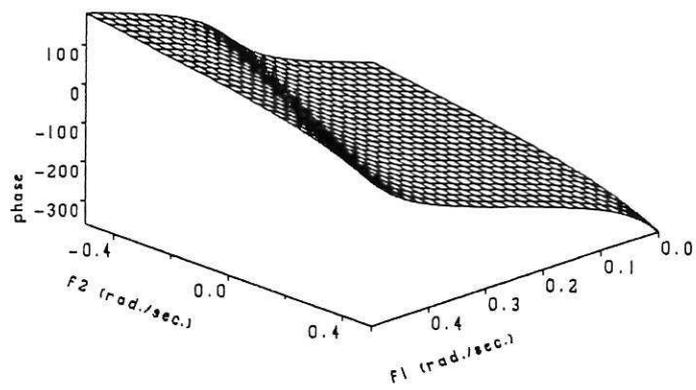
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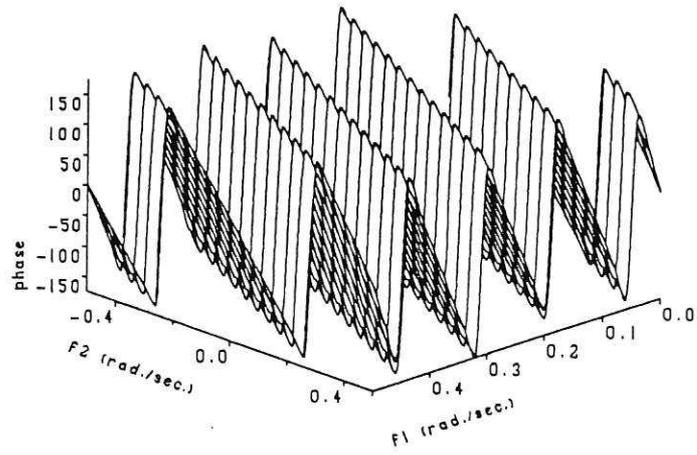
Fig.6(a)
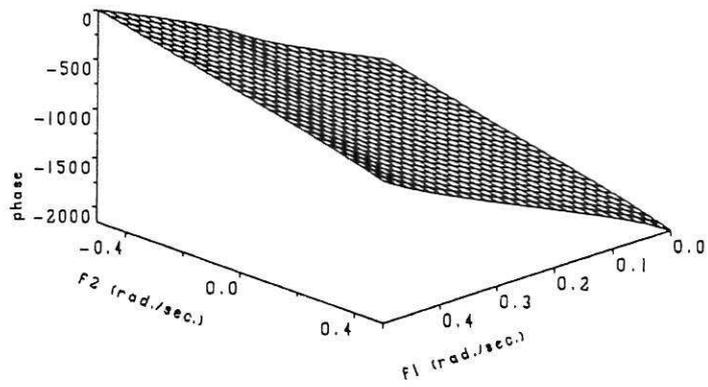
Fig.6(b)

Fig.10

Fig.7

Fig.8

Fig.9(a)

Fig.9(b)

Fig.9(c)