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# Solitons on tori and soliton crystals

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## Abstract

Necessary conditions for a soliton on a torus  $M = \mathbb{R}^m/\Lambda$  to be a soliton crystal, that is, a spatially periodic array of topological solitons in stable equilibrium, are derived. The stress tensor of the soliton must be  $L^2$  orthogonal to  $\mathbb{E}$ , the space of parallel symmetric bilinear forms on  $TM$ , and, further, a certain symmetric bilinear form on  $\mathbb{E}$ , called the hessian, must be positive. It is shown that, for baby Skyrme models, the first condition actually implies the second. It is also shown that, for any choice of period lattice  $\Lambda$ , there is a baby Skyrme model which supports a soliton crystal of periodicity  $\Lambda$ . For the three-dimensional Skyrme model, it is shown that any soliton solution on a cubic lattice which satisfies a virial constraint and is equivariant with respect to (a subgroup of) the lattice symmetries automatically satisfies both tests. This verifies in particular that the celebrated Skyrme crystal of Castillejo *et al.*, and Kugler and Shtrikman, passes both tests.

## 1 Introduction

There are many studies in the mathematical physics literature in which a nonlinear field theory known to support topological solitons is studied not on Euclidean space  $\mathbb{R}^m$ , but on a torus  $T^m = \mathbb{R}^m/\Lambda$ . The energy minimizers found are then usually interpreted as soliton *crystals*, that is, spatially periodic arrays of solitons held in stable equilibrium. However, once we place the model on a compact domain, every homotopy class of fields will generically have an energy minimizer. This is true whatever period lattice  $\Lambda$  we choose, no matter how crazy. Clearly, for a torus such as the one depicted in figure 1, the energy minimizers cannot meaningfully be interpreted as soliton crystals: they are an artifact of the choice of boundary conditions<sup>1</sup>. The choice of a cubic period lattice certainly *looks* more plausible. But since we were bound to find an energy minimizer, by compactness, why should we assume that minimizers found on cubic lattices are not also artifacts of the boundary conditions? To be sure, we should vary the energy not just with respect to the field, but also with respect to the period lattice  $\Lambda$ . In

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<sup>1</sup>Actually, we will see that in the case of baby Skyrme models, *every* two-torus, no matter how bizarre, does support a soliton crystal for an appropriate choice of potential.



Figure 1: A possible period lattice for a toric soliton. Clearly a soliton on such a torus is an artifact of the choice of boundary conditions. Or is it?

most numerical studies, this latter variation (over tori) is only partially performed: the torus is fixed to be cubic, but its side length is varied. Of course, there is a good reason for this omission: it is numerically rather intricate to study field theories on non-rectangular tori. We are aware of only one numerical study which systematically does so: Hen and Karliner, in a study of baby skyrmions, minimized energy over all two-tori [13].

The purpose of this paper is to develop a scheme for determining whether a given energy minimizer on  $\mathbb{R}^m/\Lambda$ , assumed to minimize energy among all fields in its homotopy class on this fixed torus, also minimizes energy locally with respect to variations of the period lattice. The key idea is that varying the torus among all flat tori with Euclidean metric is equivalent to fixing the torus but varying the *metric* on the torus among all constant coefficient metrics. This manoeuvre allows us to formulate the energy variation with respect to the torus in terms of the stress tensor of the field. We compute both the first and second variation formulae to give two criteria, involving the stress tensor, for the field to be a critical point of energy with respect to variations of the lattice (the first variation) and then, further, a local minimum of energy with respect to such variations (the second variation). The first criterion is that the stress tensor should be  $L^2$  orthogonal to the  $\frac{1}{2}m(m+1)$  dimensional space  $\mathbb{E}$  of parallel symmetric  $(0,2)$  tensors on  $\mathbb{R}^m/\Lambda$ . The second criterion is that a certain symmetric bilinear form on  $\mathbb{E}$ , called the hessian, should be positive. These criteria, which can easily be checked numerically, give necessary, but not sufficient, conditions for a soliton on a torus to be a genuine physical soliton crystal. (There are two subtleties. First, varying  $\Lambda$  locally does not account for the possibility that energy could be lowered by period-increasing variations, in which the field is reinterpreted as living on a larger torus obtained by gluing neighbouring tori together. Such variations do not just jump discontinuously in the space of lattices, they also jump to another homotopy class of fields, and so are inherently inaccessible to variational calculus. Second, our scheme considers variations of the field and the lattice separately, so that cross terms in the second variation, arising from simultaneous variations of both field and lattice, are not considered.) Nonetheless, for the sake of terminological convenience, we shall say that an energy minimizer on a fixed torus is a soliton *lattice* if it satisfies the first criterion, and a soliton *crystal* if it also satisfies the second.

The paper is structured as follows. In section 2 we derive the criteria for a general scalar field theory and determine how symmetries of the field imply symmetries of its stress tensor and hessian. In section 3 we apply the criteria in the context of the baby-Skyrme model. We find that a toric baby skyrmion is a soliton lattice if it satisfies a virial constraint of Derrick type and is conformal on average, in the  $L^2$  sense. We show further that every baby skyrmion

lattice is a soliton crystal, that is, the second criterion follows immediately from the first in the baby Skyrme case. We also show that for any choice of period lattice  $\Lambda$ , there exists a potential for which the baby Skyrme model has a crystal with this periodicity. Throughout this paper our focus is on variation of the energy with respect to the period lattice, rather than the more usual problem of varying the field on a fixed domain. Indeed, the existence, on any compact domain and in any degree class, of an energy minimizer in a function space with sufficient regularity for our criteria to make rigorous sense is typically already known in the literature. An exception to this would appear to be the baby Skyrme model, whose general existence problem on compact domains does not seem to have been studied rigorously. At the end of section 3 we fill in this gap, proving an existence result on general tori for the model with target space  $S^2$  and an arbitrary potential. In section 4 we consider the usual nuclear Skyrme model in three dimensions, and some interesting variants which are of current phenomenological interest. It is shown that, on a cubic lattice, any energy minimizer which satisfies a certain virial constraint of Derrick type, and is equivariant with respect to (a certain subgroup of) the lattice symmetries is automatically a soliton crystal. This result implies that, in particular, the ‘‘Skyrme crystal’’ found numerically by Castillejo *et al.* [6] and Kugler and Shtrikman [15] is a soliton crystal according to our definition. Some concluding remarks are presented in section 5.

## 2 Varying over the space of tori

Consider a general static scalar field theory defined by some energy functional  $E(\varphi)$  for a field  $\varphi : \mathbb{R}^m \rightarrow N$ , where  $N$  is some target space. Given an energy minimizer  $\varphi : \mathbb{R}^m/\Lambda_* \rightarrow N$ , where  $\Lambda_* = \{n_1\mathbf{X}_1 + n_2\mathbf{X}_2 + \cdots + n_m\mathbf{X}_m : \mathbf{n} \in \mathbb{Z}^m\}$  is some fixed lattice in  $\mathbb{R}^m$ , when can the lifted map  $\mathbb{R}^m \rightarrow N$  be interpreted as a soliton crystal? The answer is that it should be critical, and in fact stable, with respect to variations of the lattice  $\Lambda$  around  $\Lambda_*$ , as well as variations of the field. Now, all  $m$ -tori are diffeomorphic through linear maps  $\mathbb{R}^m \rightarrow \mathbb{R}^m$ , so we can identify them all with  $M = \mathbb{R}^m/\Lambda_*$ , the torus of interest. So the manifold and  $\varphi : M \rightarrow N$  are now fixed, and varying the lattice is equivalent to varying the *metric* on  $M$  by pulling back the standard Euclidean metric on  $\mathbb{R}^k/\Lambda$  to  $M$  by the inverse of the diffeomorphism  $\mathbb{R}^k/\Lambda \rightarrow M$ . Let us denote this metric on  $M$  by  $g_\Lambda$ . Varying among all lattices, one sees that

$$g_\Lambda = \sum_{ij} g_{ij}(\Lambda) dx_i dx_j \quad (2.1)$$

where  $g_{ij}(\Lambda)$  are constant and  $g_{ij}(\Lambda_*) = \delta_{ij}$ . Consider now a curve  $g_t$  in this space of metrics on  $M$  such that  $g_0 = g_{\Lambda_*}$  the Euclidean metric, and denote by

$$\varepsilon = \partial_t|_{t=0} g_t \in \Gamma(T^*M \odot T^*M) \quad (2.2)$$

its initial tangent vector. Then  $\varepsilon$  lies in the space of allowed variations

$$\mathbb{E} = \left\{ \sum_{ij} \varepsilon_{ij} dx_i dx_j : \varepsilon_{ij} \text{ constant, } \varepsilon_{ij} = \varepsilon_{ji} \right\}. \quad (2.3)$$

This is a  $m(m+1)/2$  dimensional subspace of the space of sections of the rank  $m(m+1)/2$  vector bundle  $T^*M \odot T^*M$  (where  $\odot$  denotes symmetrized tensor product), which is canonically isomorphic to any fibre. The canonical isomorphism  $\mathbb{E} \rightarrow T_x^*M \odot T_x^*M$  is given by evaluation  $\varepsilon \mapsto \varepsilon(x)$ . Now each fibre  $T_x^*M \odot T_x^*M$  has a canonical inner product,

$$\langle \widehat{\varepsilon}, \varepsilon \rangle = \sum_{i,j} \widehat{\varepsilon}(e_i, e_j) \varepsilon(e_i, e_j) \quad (2.4)$$

where  $\{e_1, \dots, e_m\}$  is any orthonormal frame of vector fields on  $M$ . Hence, the isomorphism  $\mathbb{E} \rightarrow T_x^*M \odot T_x^*M$  equips  $\mathbb{E}$  with a canonical inner product, which we will denote  $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ . Note that this is independent of the choice of base point  $x$ . The inverse isomorphism is defined by parallel propagation, so we refer to  $\mathbb{E}$  as the space of parallel symmetric bilinear forms.

Now for *any* variation  $g_t$  of the metric,

$$\left. \frac{dE(\varphi, g_t)}{dt} \right|_{t=0} =: \langle \varepsilon, S \rangle_{L^2} \quad (2.5)$$

where  $S$ , by definition, is the *stress tensor* of the field, defined in analogy with the stress-energy-momentum tensor familiar from relativity theory (see [3] for the original derivation of  $S$  in the important case that  $E$  is the Dirichlet energy). Like  $g$  and  $\varepsilon$ ,  $S$  is a section of  $T^*M \odot T^*M$ . So  $E$  is critical for variations of the lattice  $\Lambda$  if and only if  $S \perp_{L^2} \mathbb{E}$ . We can reformulate this condition as follows. Let  $\mathbb{E}_0$  denote the orthogonal complement of  $g$  in  $\mathbb{E}$ , that is, the space of *traceless* parallel symmetric bilinear forms. Then  $S \perp_{L^2} \mathbb{E}$  if and only if

$$\langle S, g \rangle_{L^2} = 0 \quad (2.6)$$

and

$$S \perp_{L^2} \mathbb{E}_0. \quad (2.7)$$

Condition (2.6) is a virial constraint, analogous to the constraint obtained for the model on  $\mathbb{R}^m$  using the Derrick scaling argument [7]. In fact, if we replace the torus  $M$  by  $\mathbb{R}^m$  and assume that  $\varphi$  is a finite energy solution, then (2.6) coincides precisely with the Derrick virial constraint. Similarly (2.7) also coincides (under the replacement of  $M$  by  $\mathbb{R}^m$ ) with the collection of virial constraints “beyond Derrick’s theorem” obtained by Manton [19] and generalized recently by Domokos *et al.* [8]. In brief, then, in order to be critical with respect to variations of the period lattice, a toric soliton must satisfy the generalized Derrick virial constraints in each lattice cell.

A convenient reformulation of (2.7) arises as follows. Given any point  $x \in M$  and any pair of tangent vectors  $X, Y \in T_xM$ , these have unique parallel propagations over  $M$ , which we also denote  $X, Y$ . Associated to the map  $\varphi$ , we can define a symmetric bilinear form

$$\Delta : T_xM \times T_xM \rightarrow \mathbb{R}, \quad \Delta(X, Y) = -2 \int_M S_0(X, Y) \text{vol}_g \quad (2.8)$$

where  $S_0$  denotes the trace-free part of  $S$  (the factor of  $-2$  is for later convenience). Using the canonical identification of  $\mathbb{E}$  with any fibre  $T_x^*M \odot T_x^*M$ , we can identify  $\Delta$  with an element

of  $\mathbb{E}$ . Then  $S \perp_{L^2} \mathbb{E}_0$  if and only if  $S_0 \perp_{L^2} \mathbb{E}_0$ , which holds if and only if  $\Delta$  is orthogonal to  $\mathbb{E}_0$  with respect to  $\langle \cdot, \cdot \rangle_{\mathbb{E}}$ , and hence, if and only if

$$\Delta = \lambda g, \quad (2.9)$$

for some constant  $\lambda \in \mathbb{R}$  which, if required, can be found by evaluating  $\Delta$  on any unit vector.

Assume that the pair  $(\varphi, g)$  satisfies these conditions, (2.6), (2.7). Then we can define the second variation of  $E$  with respect to the metric in the affine space  $g + \mathbb{E}$ . Let  $g_{s,t}$  be a two-parameter family of metrics in  $g + \mathbb{E}$  with  $g_{0,0} = g$ , and let  $\widehat{\varepsilon} = \partial_s g_{s,t}|_{(0,0)}, \varepsilon = \partial_t g_{s,t}|_{(0,0)} \in \mathbb{E}$ . Then the *hessian* of  $E : g + \mathbb{E} \rightarrow \mathbb{R}$  at  $g$  is, by definition, the symmetric bilinear form  $\text{Hess} \in \mathbb{E}^* \odot \mathbb{E}^*$  such that

$$\text{Hess}(\widehat{\varepsilon}, \varepsilon) = \left. \frac{\partial^2 E(\varphi, g_{s,t})}{\partial s \partial t} \right|_{s=t=0}. \quad (2.10)$$

The pair  $(\varphi, g)$  is a local minimum of  $E$ , with respect to variations of the lattice, if this bilinear form is positive definite. This leads us to the following:

**Definition 1** An  $E$  minimizer  $\varphi : \mathbb{R}^m / \Lambda \rightarrow N$  is a *soliton lattice* if its stress tensor  $S(\varphi)$  is  $L^2$  orthogonal to  $\mathbb{E}$ , the space of parallel symmetric bilinear forms. A soliton lattice is a *soliton crystal* if, in addition, its hessian is positive definite.

The detailed structure of  $S$ , and hence of  $\text{Hess}$ , depend on the details of the field theory, We can, however, find a semi-explicit formula for  $\text{Hess}$  which turns out to be rather useful for our purposes. To state it, we need to define the natural contraction for pairs of bilinear forms on  $M$ . So, let  $A, B$  be bilinear forms (i.e.  $(0, 2)$  tensors) on  $M$  and  $e_1, \dots, e_m$  be any local orthonormal frame of vector fields on  $M$ . Then we define  $A \cdot B$  to be the bilinear form

$$(A \cdot B)(X, Y) = \sum_i A(X, e_i) B(e_i, Y). \quad (2.11)$$

If we identify a bilinear form with its matrix of components relative to the frame  $\{e_i\}$  then this contraction coincides with matrix multiplication.

**Proposition 2** Let  $\varphi$  be a soliton lattice, and  $\text{Hess}$  be its hessian. Then, for all  $\widehat{\varepsilon}, \varepsilon \in \mathbb{E}$ ,

$$\text{Hess}(\widehat{\varepsilon}, \varepsilon) = \langle \dot{S}, \varepsilon \rangle_{L^2} - 2 \langle \widehat{\varepsilon}, S \cdot \varepsilon \rangle_{L^2},$$

where  $\dot{S} = \partial_s|_{s=0} S(\varphi, g_s) \in \Gamma(T^*M \odot T^*M)$  for any generating curve  $g_s$  for  $\widehat{\varepsilon}$ . In particular, if  $\widehat{\varepsilon} = \varepsilon$ , then

$$\text{Hess}(\varepsilon, \varepsilon) = \langle \dot{S}, \varepsilon \rangle_{L^2}.$$

*Proof:* Let  $g_{s,t}$  be any two-parameter variation of  $g = g_{0,0}$  in  $g + \mathbb{E}$ , and  $\widehat{\varepsilon} = \partial_s|_{s=0} g_{s,0}$ ,  $\varepsilon_s = \partial_t|_{t=0} g_{s,t}$ ,  $\varepsilon = \varepsilon_0$ . Let  $g_s = g_{s,0}$ . Then

$$\begin{aligned} \text{Hess}(\widehat{\varepsilon}, \varepsilon) &= \left. \frac{d}{ds} \right|_{s=0} \left. \frac{\partial E(\varphi, g_{s,t})}{\partial t} \right|_{t=0} = \left. \frac{d}{ds} \right|_{s=0} \langle S(\varphi, g_s), \varepsilon_s \rangle_{L^2, g_s} \\ &= \left. \frac{d}{ds} \right|_{s=0} \int_M \langle S(\varphi, g_s), \varepsilon_s \rangle_{g_s} \text{vol}_{g_s} \\ &= \int_M \left\{ \langle \dot{S}, \varepsilon \rangle \text{vol}_g + \langle S, \dot{\varepsilon} \rangle \text{vol}_g + \left. \frac{d}{ds} \right|_{s=0} \langle S, \varepsilon \rangle_{g_s} \text{vol}_g + \langle S, \varepsilon \rangle_g \left. \frac{d}{ds} \right|_{s=0} \text{vol}_{g_s} \right\} \end{aligned} \quad (2.12)$$

Now  $\hat{\varepsilon} = \partial_s|_{s=0}\varepsilon_s \in \mathbb{E}$  and  $S \perp_{L^2} \mathbb{E}$  by assumption ( $\varphi$  is assumed to be a lattice), so the second term vanishes, upon integration over  $M$ .

For any fixed pair  $A, B$  of symmetric bilinear forms,

$$\frac{d}{ds}\Big|_{s=0} \langle A, B \rangle_{g_s} = -2\langle A \cdot B, \hat{\varepsilon} \rangle_g. \quad (2.13)$$

To check this, we can work in a local coordinate chart and use the Einstein summation convention. Since  $g_{ij}(s)g^{jk}(s) = \delta_i^k$  for all  $s$ , we deduce that  $\dot{g}^{ij} = dg^{ij}(s)/ds|_{s=0} = -g^{ip}g^{jq}g^{pq}$ . Hence

$$\begin{aligned} \frac{d}{ds}\Big|_{s=0} \langle A, B \rangle_{g_s} &= \frac{d}{ds}\Big|_{s=0} A_{ij}g^{jk}(s)B_{kl}g^{li}(s) \\ &= -A_{ij}g^{jp}\varepsilon_{pq}g^{qk}B_{kl}g^{li} - A_{ij}g^{jk}B_{kl}g^{lp}\varepsilon_{pq}g^{pi} \\ &= -2(A \cdot B)_{il}g^{lp}\varepsilon_{pq}g^{qi} = -2\langle A \cdot B, \varepsilon \rangle_g. \end{aligned} \quad (2.14)$$

Hence the third term in (2.12) reproduces the second term in the formula claimed.

The variation of the volume form is known to be [4, p. 82]

$$\frac{d}{ds}\Big|_{s=0} \text{vol}_{g_s} = \frac{1}{2}\langle \hat{\varepsilon}, g \rangle \text{vol}_g. \quad (2.15)$$

Note that, since  $\hat{\varepsilon}, g \in \mathbb{E}$ ,  $\langle \hat{\varepsilon}, g \rangle = \langle \hat{\varepsilon}, g \rangle_{\mathbb{E}}$ , which is constant. Hence, the last term reduces to  $\langle \hat{\varepsilon}, g \rangle_{\mathbb{E}} \langle S, \varepsilon \rangle_{L^2}$  which vanishes since  $\varphi$  is a lattice. This completes the proof of the first formula.

Now assume  $\hat{\varepsilon} = \varepsilon$ . For any triple of symmetric bilinear forms  $A, B, C$ ,  $\langle A, B \cdot C \rangle_g = \langle C, A \cdot B \rangle_g$  and so  $\langle \varepsilon, S \cdot \varepsilon \rangle_{L^2} = \langle S, \varepsilon \cdot \varepsilon \rangle_{L^2} = 0$  since, for all  $\varepsilon \in \mathbb{E}$ ,  $\varepsilon \cdot \varepsilon \in \mathbb{E}$ , and  $S \perp_{L^2} \mathbb{E}$ .  $\square$

We will see that, in the case of three-dimensional Skyrme models, the task of checking that a given  $E$ -minimizer  $\varphi$  satisfies Definition 1 can be greatly simplified if  $\varphi$  is equivariant with respect to (some subgroup of) the symmetries of the period lattice  $\Lambda$ . To exploit equivariance we first need to extract its consequences for  $S$  and  $\text{Hess}$ . The following basic symmetry properties may prove useful in contexts other than toric solitons, so we formulate them in some generality.

Let  $K$  be any group acting isometrically on the left on the riemannian manifold  $(M, g_0)$ , and let  $E(\varphi, g)$  be a geometrically natural energy functional on the space of smooth maps  $\varphi : M \rightarrow N$  and metrics on  $M$ . By geometrically natural, we mean that, for any diffeomorphism  $\psi : M \rightarrow M$ ,

$$E(\varphi \circ \psi, g) = E(\varphi, (\psi^{-1})^*g) \quad (2.16)$$

for all  $\varphi$  and  $g$ . In local coordinates,  $\psi$  can be thought of as a passive transformation, that is, a change of coordinate, in which case, the condition above simply means that  $E$  is independent of the choice of coordinates. Note that  $E(\varphi, g_0)$  (with fixed metric) is automatically invariant under the  $K$  action on  $M$ , since  $E(\varphi \circ k^{-1}, g_0) = E(\varphi, k^*g_0) = E(\varphi, g_0)$  as  $K$  acts by isometries. Let  $K$  also act on  $N$  on the left, in such a way that  $E$  is invariant under this  $K$  action, that is,

$$E(k \circ \varphi, g) = E(\varphi, g) \quad (2.17)$$

for all  $k \in K$ ,  $\varphi$  and  $g$ . Such a functional will be called  $K$ -invariant. We say that a field  $\varphi : M \rightarrow N$  is  $K$ -equivariant if  $\varphi \circ k = k \circ \varphi$  for all  $k \in K$ .

**Proposition 3** *Let  $\varphi : (M, g_0) \rightarrow N$  be a  $K$ -equivariant field, and  $S$  be its stress tensor with respect to a geometrically natural  $K$ -invariant energy functional. Then  $k^*S = S$  for all  $k \in K$ .*

*Proof:* Let  $g_t$  be an arbitrary variation of  $g_0$ , and  $\varepsilon = \partial|_{t=0}g_t$ . Since  $K$  acts isometrically on  $(M, g_0)$ , each  $k^*$  is  $L^2$  self-adjoint on  $\Gamma(T^*M \odot T^*M)$ . Hence

$$\begin{aligned}
\langle k^*S, \varepsilon \rangle_{L^2} &= \langle S, k^*\varepsilon \rangle_{L^2} = \left. \frac{d}{dt} \right|_{t=0} E(\varphi, k^*g_t) \\
&= \left. \frac{d}{dt} \right|_{t=0} E(\varphi \circ k^{-1}, g_t) \quad (E \text{ is geometrically natural}) \\
&= \left. \frac{d}{dt} \right|_{t=0} E(k^{-1} \circ \varphi, g_t) \quad (\varphi \text{ is } K\text{-equivariant}) \\
&= \left. \frac{d}{dt} \right|_{t=0} E(\varphi, g_t) \quad (E \text{ is } K\text{-invariant}) \\
&= \langle S, \varepsilon \rangle_{L^2}.
\end{aligned} \tag{2.18}$$

Since this holds for all variations, and all  $k$ , the result follows.  $\square$

A similar argument shows that the hessian is also  $K$ -invariant.

**Proposition 4** *Let  $\varphi : (M, g_0) \rightarrow N$  be a  $K$ -equivariant field, and  $S$  be its stress tensor with respect to a geometrically natural  $K$ -invariant energy functional  $E$ . Assume that  $S$  is  $L^2$  orthogonal to some  $K$ -invariant linear subspace  $\mathbb{E} \subset \Gamma(T^*M \odot T^*M)$ , and let  $\mathbf{Hess} : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$  be the hessian of  $E$  with respect to variations in the affine space  $g_0 + \mathbb{E}$ . Then  $(k^*)^*\mathbf{Hess} = \mathbf{Hess}$  for all  $k \in K$ .*

*Proof:* Let  $g_{s,t}$  be an arbitrary two-parameter variation of  $g_0 = g_{0,0}$ ,  $\widehat{\varepsilon} = \partial_s|_{s=0}g_{s,0}$ ,  $\varepsilon = \partial_t|_{t=0}g_{0,t}$ , and  $k \in K$ . Then

$$\begin{aligned}
(k^*)^*\mathbf{Hess}(\widehat{\varepsilon}, \varepsilon) &= \mathbf{Hess}(k^*\widehat{\varepsilon}, k^*\varepsilon) = \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} E(\varphi, k^*g_{s,t}) \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} E(\varphi \circ k^{-1}, g_{s,t}) \quad (E \text{ is geometrically natural}) \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} E(k^{-1} \circ \varphi, g_{s,t}) \quad (\varphi \text{ is } K\text{-equivariant}) \\
&= \left. \frac{\partial^2}{\partial s \partial t} \right|_{s=t=0} E(\varphi, g_{s,t}) \quad (E \text{ is } K\text{-invariant}) \\
&= \mathbf{Hess}(\widehat{\varepsilon}, \varepsilon).
\end{aligned} \tag{2.19}$$

Since this holds for all variations, and all  $k$ , the result follows.  $\square$

To conclude this section, we return to the setting of interest:  $M = \mathbb{R}^m/\Lambda$ , a torus with the euclidean metric  $g$ ,  $\mathbb{E}$  is the space of parallel symmetric bilinear forms on  $M$ , and  $K$  is

some subgroup of  $O(m)$  which preserves  $\Lambda$  (and hence acts isometrically on  $M$ ). Note that  $\mathbb{E}$  is indeed  $K$ -invariant. Let  $\varphi$  be  $K$ -equivariant, and  $\Delta \in \mathbb{E}$  be the bilinear form defined in (2.8). Then  $\Delta$  is itself  $K$ -invariant.

**Proposition 5** *If  $\varphi : M \rightarrow N$  is  $K$ -equivariant, then  $k^*\Delta = \Delta$  for all  $k \in K$ .*

*Proof:* Clearly

$$\Delta(X, Y) = \lambda g(X, Y) - 2 \int_M S(X, Y) \text{vol}_g =: \lambda g(X, Y) + \Delta'(X, Y) \quad (2.20)$$

for some constant  $\lambda$ , and  $k^*g = g$  for all  $k \in K$  since  $K$  acts isometrically. Hence, it suffices to show that  $k^*\Delta' = \Delta'$ . But this follows immediately from Proposition 3.  $\square$

### 3 Baby skyrmion crystals

The baby Skyrme model concerns a single scalar field  $\varphi : M \rightarrow N$  where  $(M, g)$  is an oriented riemannian two-manifold (in our case, a torus  $\mathbb{R}^2/\Lambda$ ) and  $(N, h, \omega)$  is a compact kähler manifold (usually chosen to be  $S^2$ ) with metric  $h$  and kähler form  $\omega$ . The energy is

$$E(\varphi) = \frac{1}{2} \int_M \{ |d\varphi|^2 + |\varphi^*\omega|^2 + U(\varphi)^2 \} \text{vol}_g \quad (3.1)$$

where we have found it convenient to write the potential as  $\frac{1}{2}U^2$  where  $U : N \rightarrow \mathbb{R}$  is some function. It is conventional to label the three terms in  $E$  as  $E_2$ ,  $E_4$  and  $E_0$  respectively. The subscript specifies the degree of the integrand thought of as a polynomial in spatial partial derivatives.

Let  $\varphi : M = T^2/\Lambda \rightarrow N$  minimize  $E$  among maps (on this fixed torus) in its homotopy class. Under what circumstances is  $\varphi$  a soliton lattice, as defined in section 2? The stress tensor of  $\varphi$  is easily computed [10],

$$S(\varphi, g) = \frac{1}{4} (|d\varphi|_g^2 - |\varphi^*\omega|_g^2 + U(\varphi)^2) g - \frac{1}{2} \varphi^*h. \quad (3.2)$$

Now  $\varphi$  is a lattice if and only if  $S(\varphi, g)$  is  $L^2$  orthogonal to both  $g$  and  $\mathbb{E}_0$ , the space of traceless parallel symmetric bilinear forms, yielding conditions (2.6) and (2.7). Now  $\langle g, g \rangle = 2$  and

$$\langle g, \varphi^*h \rangle = \sum_{i,j} g(e_i, e_j) \varphi^*h(e_i, e_j) = \sum_i \varphi^*h(e_i, e_i) = |d\varphi|^2 \quad (3.3)$$

so

$$\langle S, g \rangle_{L^2} = \frac{1}{2} (E_0 - E_4). \quad (3.4)$$

Hence (2.6) becomes the familiar virial constraint

$$E_0 = E_4. \quad (3.5)$$

This is the condition which is enforced in numerical studies of baby skyrmion crystals by minimization of  $E$  over the sidelength of the torus.

For all  $\varepsilon \in \mathbb{E}_0$ ,  $\langle S, \varepsilon \rangle_{L^2} = -\frac{1}{2} \langle \varphi^* h, \varepsilon \rangle_{L^2}$  since  $\varepsilon$  is pointwise orthogonal to  $g$ . In two dimensions,  $\mathbb{E}_0$  is spanned by

$$\varepsilon_1 = dx_1^2 - dx_2^2, \quad \varepsilon_2 = 2dx_1 dx_2. \quad (3.6)$$

So (2.7) is equivalent to

$$\langle \varphi^* h, \varepsilon_1 \rangle_{L^2} = \int_M \left( \left| d\varphi \frac{\partial}{\partial x_1} \right|^2 - \left| d\varphi \frac{\partial}{\partial x_2} \right|^2 \right) \text{vol}_g = 0 \quad (3.7)$$

$$\langle \varphi^* h, \varepsilon_2 \rangle_{L^2} = \int_M h(d\varphi \frac{\partial}{\partial x_1}, d\varphi \frac{\partial}{\partial x_2}) \text{vol}_g = 0. \quad (3.8)$$

Note that these conditions, like the virial constraint, are easy to check numerically. If the latter two conditions hold pointwise, rather than just as integrals, this means precisely that the mapping  $\varphi$  is (weakly) conformal. So a natural way to describe the condition  $S \perp_{L^2} \mathbb{E}_0$  is that the mapping  $\varphi$  must be *conformal on average* (in the  $L^2$  sense). We can reformulate this condition using the symmetric bilinear form  $\Delta$ , defined in (2.8). In this case,  $-2S_0 = \varphi^* h$ , so  $\varphi$  is conformal on average if and only if

$$\Delta = \int_M \varphi^* h \text{vol}_g = \lambda g \quad (3.9)$$

where  $\lambda$  is some constant. Taking the trace of both sides, one sees that  $\lambda = E_2$ . In abbreviated notation, then, we may express the conditions for  $\varphi$  to be a soliton lattice as

$$E_0 = E_4, \quad \Delta = \int_M \varphi^* h \text{vol}_g = E_2 g. \quad (3.10)$$

This result was already derived, via a slightly different argument, in [10].

So, given an  $E$  minimizer  $\varphi$  which is conformal on average and satisfies  $E_0 = E_4$ , we know that  $E$  is critical with respect to variations of the period lattice. To check if, in addition, it is *stable* with respect to variations of the period lattice, we need to show that its hessian is positive definite. In fact this second step turns out to be redundant, as we now show.

**Proposition 6** *Let  $\varphi$  be a baby skyrmion lattice. Then its hessian is positive definite. Hence every baby skyrmion lattice is a soliton crystal.*

*Proof:* We are given that  $S(\varphi)$  is  $L^2$  orthogonal to  $\mathbb{E}$ . Let  $g_{s,t}$  be a two-parameter variation of  $g = g_{0,0}$  in  $g + \mathbb{E}$ , and  $\widehat{\varepsilon}, \varepsilon, g_s, \dot{S}$  be as defined as in Proposition 2. From (3.2) we see that

$$\dot{S} = \lambda g + \frac{1}{4} (|d\varphi|^2 - |\varphi^* \omega|^2 + U(\varphi)^2) \widehat{\varepsilon}. \quad (3.11)$$

Hence, if  $\widehat{\varepsilon} = g$  and  $\varepsilon \in \mathbb{E}_0$  then  $\dot{S} = \lambda g$ , so

$$\text{Hess}(g, \varepsilon) = \langle \lambda g, \varepsilon \rangle_{L^2} - 2 \langle g, S \cdot \varepsilon \rangle_{L^2} = 0 - 2 \langle S \cdot g, \varepsilon \rangle_{L^2} = -2 \langle S, \varepsilon \rangle_{L^2} = 0 \quad (3.12)$$

since  $\varphi$  is a soliton lattice. Hence it suffices to show that  $\mathbf{Hess}(g, g) > 0$  and  $\mathbf{Hess}(\varepsilon, \varepsilon) > 0$  for all  $\varepsilon \in \mathbb{E}_0 \setminus \{0\}$ .

Now

$$\mathbf{Hess}(g, g) = \left. \frac{d^2}{dt^2} \right|_{t=1} E(\varphi, g_t) \quad (3.13)$$

where  $g_t = tg$ . Clearly

$$E_0(\varphi, g_t) = tE_0(\varphi, g), \quad E_2(\varphi, g_t) = E_2(\varphi, g), \quad E_4(\varphi, g_t) = t^{-1}E_4(\varphi, g) \quad (3.14)$$

so

$$\mathbf{Hess}(g, g) = 2E_4 > 0. \quad (3.15)$$

Finally, if  $\widehat{\varepsilon} = \varepsilon \in \mathbb{E}_0 \setminus \{0\}$  then, by Proposition 2,

$$\begin{aligned} \mathbf{Hess}(\varepsilon, \varepsilon) &= \langle \dot{S}, \varepsilon \rangle_{L^2} = \langle \lambda g + \frac{1}{4}(|d\varphi|^2 - |\varphi^*\omega|^2 + U(\varphi)^2)\varepsilon, \varepsilon \rangle_{L^2} \\ &= \frac{1}{4} \langle \varepsilon, \varepsilon \rangle_{\mathbb{E}} \int_M (|d\varphi|^2 - |\varphi^*\omega|^2 + U(\varphi)^2) \text{vol}_g \\ &= \frac{1}{2} E_2 \langle \widehat{\varepsilon}, \varepsilon \rangle_{\mathbb{E}} > 0 \end{aligned} \quad (3.16)$$

since  $E_0 = E_4$ . □

Hence, rather remarkably, for baby Skyrme models every soliton lattice is a soliton crystal. It follows that the baby skyrmion lattices found in [10], for example, for which the conditions (3.5), (3.8) were checked numerically, are soliton crystals according to our definition. These were defined on equianharmonic tori, that is tori with  $\Lambda$  generated by  $L$  and  $Le^{i\pi/3}$ . Another remarkable feature of baby Skyrme models is that *every* torus, no matter how bizarre its period lattice, supports a soliton crystal for an appropriate choice of potential function  $U$ . We next outline the construction of this special potential.

We specialize to the case where  $N = S^2$ , the unit sphere with its usual metric and complex structure. There is a well-known topological energy bound on  $E_2(\varphi)$  due to Lichnerowicz [17]

$$E_2(\varphi) \geq 2\pi \deg(\varphi) \quad (3.17)$$

where  $\deg(\varphi)$  denotes the degree of the map  $\varphi : M \rightarrow S^2$ . Equality holds if and only if  $\varphi$  is holomorphic. Less well-known is that there is also a topological lower energy bound on  $E_0 + E_4$  [23]

$$E_0(\varphi) + E_4(\varphi) \geq 4\pi \langle U \rangle \deg(\varphi) \quad (3.18)$$

where  $\langle U \rangle$  denotes the average value of the function  $U : S^2 \rightarrow \mathbb{R}$ , with equality if and only if

$$\varphi^*\omega = *U \circ \varphi. \quad (3.19)$$

Let us choose and fix a torus  $M = \mathbb{R}^2/\Lambda$  and use a stereographic coordinate

$$W = \frac{\varphi_1 + i\varphi_2}{1 - \varphi_3} \quad (3.20)$$

on  $S^2$  and a complex coordinate  $z = x_1 + ix_2$  on  $M$ . Then there is a degree 2 holomorphic map  $\varphi_\wp : M \rightarrow S^2$  defined by taking  $W(z) = \wp(z)$ , the Weierstrass p-function corresponding to lattice  $\Lambda$ . Then  $W(z)$  satisfies the ordinary differential equation

$$W'(z)^2 = 4W(z)^3 - c_2W(z) - c_3 \quad (3.21)$$

where  $c_2, c_3$  are constants, depending on  $\Lambda$ , which are more conventionally denoted  $g_2, g_3$  [16, p. 159]. Since  $\varphi_\wp$  is holomorphic,

$$\varphi_\wp^* \omega = \frac{4|W'(z)|^2}{(1 + |W|^2)} \frac{i}{2} dz \wedge d\bar{z}. \quad (3.22)$$

Hence, if we choose the potential function so that

$$U(W) = \frac{4|4W^3 - c_2W - c_3|}{(1 + |W|^2)^2} \quad (3.23)$$

then the field  $\varphi_\wp$  is holomorphic and satisfies (3.19) and so simultaneously minimizes both  $E_2$  and  $E_0 + E_4$  among all degree 2 maps  $M \rightarrow S^2$ . Hence, this field is an  $E$  minimizer. It follows immediately from (3.19) that  $E_0 = E_4$ , and since  $\varphi_\wp$  is holomorphic, it is weakly conformal, and hence certainly conformal on average. Hence  $\varphi_\wp$  is a soliton lattice and further, by Proposition 6, a soliton crystal. Note that this construction, which generalizes in obvious fashion an observation of Ward [24] in the case of a square lattice, works no matter what choice of period lattice  $\Lambda$  we start with, although, of course, the potential function  $U$  depends on  $\Lambda$ . In all cases, the potential  $V = \frac{1}{2}U^2$  has four vacua, located at the critical values of the p-function associated with  $\Lambda$ . One of these is always the North pole (corresponding to  $W = \infty$ ) but the other three, located at roots of the polynomial  $4W^3 - c_2W - c_3$  move around as  $\Lambda$  is varied. Conversely, given any choice of four distinct points on  $S^2$ , one can rotate them so that one lies at  $(0, 0, 1)$ , then construct a degree two elliptic function with critical values at precisely those (rotated) points. This function determines a period lattice  $\Lambda$ , and a potential function  $U$  such that the field  $\varphi_\wp$  is a soliton crystal for the model with potential  $U$ . Of course, these potentials are specially constructed to support exact soliton crystals, but there are other examples of four-vacuum potentials outside this class which are known numerically, in light of Proposition 6, to support soliton crystals with topological charge per unit cell equal to two. In all known cases, the period lattice has the same geometry (up to scale) as that predicted by the p-function with critical values at the vacua (though only exceptionally symmetric cases where the corresponding p-function has square or equianharmonic period lattice have been studied). It would be interesting to see whether this is a general phenomenon, by conducting a thorough numerical analysis along the lines of Karliner and Hen's [13].

As we have seen, for a given fixed period lattice  $\Lambda$ , we can reverse engineer a potential  $V(\varphi) = \frac{1}{2}U(\varphi)^2$  so that the baby Skyrme model with target space  $S^2$  has a smooth (in fact, holomorphic) energy minimizer of degree 2 on the torus  $M = \mathbb{R}^2/\Lambda$ . But what about degree classes  $\deg(\varphi) \neq 2$ , or potentials other than this very special choice? Existence of minimizers of the baby Skyrme energy on compact domains has not been rigorously studied previously, and is not entirely trivial; for example, it is known that no degree  $\pm 1$  minimizer exists on any torus for the pure sigma model (with the potential and Skyrme terms absent). Existence of

minimizers on  $\mathbb{R}^2$  with degree  $\pm 1$  for the potential  $V(\varphi) = \lambda(1 - \varphi_3)^2$  with  $\lambda > 0$  sufficiently small has been established via the concentration-compactness method by Lin and Yang in [18]. Their analysis suggests the essential estimate required for our purposes.

Choose and fix a period lattice  $\Lambda$  and let  $M = \mathbb{R}^2/\Lambda$ . Denote by  $L^2$  the space of square integrable functions  $M \rightarrow \mathbb{R}^3$  and by  $H^1$  the subspace of  $L^2$  consisting of maps whose first partial derivatives are also in  $L^2$ . These are Hilbert spaces with respect to the usual inner products

$$\langle \varphi, \psi \rangle_{L^2} = \int_M \varphi \cdot \psi, \quad \langle \varphi, \psi \rangle_{H^1} = \langle \varphi, \psi \rangle_{L^2} + \langle \varphi_x, \psi_x \rangle_{L^2} + \langle \varphi_y, \psi_y \rangle_{L^2}. \quad (3.24)$$

For each fixed  $k \in \mathbb{Z}$  we define

$$X_k = \{\varphi \in H^1 : |\varphi| = 1 \text{ almost everywhere, } \deg(\varphi) = k\} \quad (3.25)$$

where

$$\deg(\varphi) = \frac{1}{4\pi} \int_M \varphi \cdot (\varphi_x \times \varphi_y). \quad (3.26)$$

The result below rests on three standard theorems of functional analysis: Alaoglu's Theorem [21, p. 125] (every bounded sequence in a reflexive Banach space, for example, a Hilbert space, has a weakly convergent subsequence), Rellich's Lemma [2, p. 144] (the inclusion  $H^1(M) \hookrightarrow L^2(M)$  is compact for compact  $M$ ) and Tonelli's Theorem [9, p. 22] ( $f \mapsto \int_M F(f)$  is sequentially weakly lower semicontinuous on  $L^2$  if  $F : \mathbb{R}^m \rightarrow \mathbb{R}$  is convex).

**Theorem 7** *Let  $V : S^2 \rightarrow \mathbb{R}$  be any function admitting a convex extension to  $\mathbb{R}^3$ . Then, for any  $k \in \mathbb{Z}$  the functional*

$$E : X_k \rightarrow [0, \infty], \quad E(\varphi) = \int_M \left( \frac{1}{2} |\varphi_x|^2 + \frac{1}{2} |\varphi_y|^2 + \frac{1}{2} |\varphi_x \times \varphi_y|^2 + V(\varphi) \right),$$

*attains a minimum.*

*Proof:* Let  $\varphi_n \in X_k$  be a minimizing sequence for  $E$ , that is,  $E(\varphi_n) \rightarrow \inf_{\psi \in X_k} E(\psi)$ . We will repeatedly extract nested subsequences from  $\varphi_n$ , still denoted  $\varphi_n$ , with various convergence properties. We will denote strong convergence by  $\rightarrow$  and weak convergence by  $\rightharpoonup$ , the space concerned being explicitly specified.

Since  $M$  is compact and  $|\varphi_n| = 1$  almost everywhere,  $\|\varphi_n\|_{H^1}^2 \leq 2E(\varphi_n) + \text{Vol}(M)$ , is bounded. Hence, by Alaoglu's Theorem, there is a subsequence  $\varphi_n$  and  $\varphi \in H^1$  such that  $\varphi_n \xrightarrow{H^1} \varphi$ . Again, since  $M$  is compact, the inclusion  $\iota : H^1 \hookrightarrow L^2$  is compact by Rellich's Lemma, so  $\varphi_n$ , a bounded sequence in  $H^1$ , has a subsequence converging strongly in  $L^2$ , and hence weakly in  $L^2$ . By uniqueness of (weak) limits, its limit must be  $\varphi$ , that is  $\varphi_n \xrightarrow{L^2} \varphi$ . Now  $|\varphi_n| = 1$  almost everywhere, so on any open set  $\Omega \subset M$ ,  $\int_\Omega (1 - |\varphi|^2) = \lim \int_\Omega (1 - |\varphi_n|^2) = 0$ , so  $|\varphi| = 1$  almost everywhere also. Hence, we can replace  $V$  in the formula for  $E$  by its convex extension to  $\mathbb{R}^3$ . Then every term in the integrand of  $E$  is convex, so, by Tonelli's Theorem,  $E : H^1 \rightarrow [0, \infty]$  is a sum of sequentially weakly lower semicontinuous functionals, and hence is itself sequentially weakly lower semicontinuous. Hence  $E(\varphi) \leq \lim E(\varphi_n) = \inf_{\psi \in X_k} E(\psi)$ . It remains to show that  $\varphi \in X_k$ , that is,  $\deg(\varphi) = k$ .

Both  $\varphi$  and (since  $E(\varphi)$  is finite)  $\varphi_x \times \varphi_y$  are in  $L^2$ , so  $\deg(\varphi)$  exists. Furthermore,

$$\begin{aligned} 4\pi|\deg(\varphi) - k| &= 4\pi|\deg(\varphi) - \deg(\varphi_n)| = \left| \int_M \varphi \cdot (\varphi_x \times \varphi_y) - \varphi_n \cdot (\partial_x \varphi_n \times \partial_y \varphi_n) \right| \\ &\leq |\langle \varphi, \varphi_x \times \varphi_y - \partial_x \varphi_n \times \partial_y \varphi_n \rangle_{L^2}| + |\langle \varphi - \varphi_n, \partial_x \varphi_n \times \partial_y \varphi_n \rangle_{L^2}| \\ &\leq |\langle \varphi, \varphi_x \times \varphi_y - \partial_x \varphi_n \times \partial_y \varphi_n \rangle_{L^2}| + \|\varphi - \varphi_n\|_{L^2} \|\partial_x \varphi_n \times \partial_y \varphi_n\|_{L^2} \end{aligned} \quad (3.27)$$

Now  $\|\partial_x \varphi_n \times \partial_y \varphi_n\|_{L^2}$  is bounded (by  $2E(\varphi)+1$ , for example) and hence, by Alaoglu's Theorem,  $\partial_x \varphi_n \times \partial_y \varphi_n$  has a subsequence converging weakly in  $L^2$ . By uniqueness of weak limits, its limit must be  $\varphi_x \times \varphi_y$ , so  $\varphi_x \times \varphi_y - \partial_x \varphi_n \times \partial_y \varphi_n \xrightarrow{L^2} 0$ , whence  $\langle \varphi, \varphi_x \times \varphi_y - \partial_x \varphi_n \times \partial_y \varphi_n \rangle_{L^2} \rightarrow 0$ . Further,  $\varphi_n \xrightarrow{L^2} \varphi$  and  $\|\partial_x \varphi_n \times \partial_y \varphi_n\|_{L^2}$  is bounded, so  $\|\varphi - \varphi_n\|_{L^2} \|\partial_x \varphi_n \times \partial_y \varphi_n\|_{L^2} \rightarrow 0$  also. Hence  $\deg(\varphi) = k$ .  $\square$

The requirement that the potential  $V : S^2 \rightarrow \mathbb{R}$  have a convex extension to  $\mathbb{R}^3$  looks, at first sight, annoyingly restrictive. In fact, *every* function  $S^2 \rightarrow \mathbb{R}$  has a convex extension to  $\mathbb{R}^3$ , provided it is sufficiently smooth, so this is practically no restriction at all:

**Proposition 8** *Let  $f : S^n \rightarrow \mathbb{R}$  be twice continuously differentiable. Then  $f$  has a convex extension  $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ .*

*Proof:* Since  $f$  is  $C^2$  and  $S^n$  is compact,

$$C_* = \sup\{|(f \circ \gamma)(t)|, |(f \circ \gamma)'(t)|, |(f \circ \gamma)''(t)|\} \quad (3.28)$$

is finite, where the supremum is over all unit speed geodesics  $\gamma : [0, 2\pi] \rightarrow S^n$  and all  $t \in [0, 2\pi]$ . Choose any  $C > 2C_*$  and consider the function

$$F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}, \quad F(x) = |x|^2(f(x/|x|) + C) - C, \quad (3.29)$$

which clearly extends  $f$ . We claim  $F$  is convex. To show this it suffices to show that its restriction to any arc-length parametrized straight line in  $\mathbb{R}^{n+1}$  is convex. This is clear for all straight lines through 0, and for the straight line  $\alpha(t) = a + tb$ ,  $a \neq 0$ ,  $|b| = 1$ ,  $a \cdot b = 0$ , one sees, by radially projecting  $\alpha$  to a geodesic arc on  $S^n$ ,

$$\gamma(\theta(t)) = \frac{\alpha(t)}{|\alpha(t)|}, \quad \theta(t) = \tan^{-1} \frac{t}{|a|} \quad (3.30)$$

that

$$\begin{aligned} (F \circ \alpha)''(t) &= 2(f \circ \gamma(\theta(t)) + C) + 4t(f \circ \gamma)'(\theta(t))\theta'(t) \\ &\quad + (t^2 + |a|^2)(f \circ \gamma)''(\theta(t))\theta'(t)^2 + (t^2 + |a|^2)(f \circ \gamma)'(\theta(t))\theta''(t) \\ &\geq \frac{1}{t^2 + |a|^2} \{2(C - C_*)(t^2 + |a|^2) - 2t|a|C_* - |a|^2C_*\} \\ &\geq \frac{C_*}{t^2 + |a|^2} \{(t - |a|)^2 + t^2\} \geq 0. \end{aligned} \quad (3.31)$$

□

Combining Theorem 7 and Proposition 8, we see that for any  $C^2$  potential  $V : S^2 \rightarrow \mathbb{R}$  (including  $V = 0$ ), any degree  $k$ , and any period lattice  $\Lambda$ , there is a minimizer of  $E$  among all  $H^1$  maps  $\mathbb{R}^2/\Lambda \rightarrow S^2$  of degree  $k$ . This minimizer is sufficiently regular for all the integral criteria developed in this section for criticality, and stability, with respect to variations of  $\Lambda$  for this fixed minimizer  $\varphi$  (that is, the virial constraint, and conformality on average) to be rigorously well-defined. There is no reason to expect  $H^1$  regularity of the minimizer  $\varphi$  to be optimal: one would hope, for smooth  $V$ , that elliptic regularity methods would, with some effort, yield smoothness of  $\varphi$ . Since variation of  $\varphi$  is outside the main focus of the present paper, and enhanced regularity is not necessary for our purposes, we do not pursue this further here.

## 4 Skyrme crystals

In this section we consider the three dimensional Skyrme model. This has a single scalar field  $\varphi : M \rightarrow G$ , where  $(M, g)$  is an oriented riemannian three manifold (in our case,  $M = \mathbb{R}^3/\Lambda$ , a torus), and  $G$  is a compact simple Lie group (usually taken to be  $SU(2)$ ) whose Lie algebra we denote  $\mathfrak{g}$ . The energy functional is conventionally<sup>2</sup> the sum of two terms

$$E_2 = \frac{1}{2} \int_M |d\varphi|^2 \text{vol}_g, \quad E_4 = \frac{1}{2} \int_M |\varphi^* \Omega|^2 \text{vol}_g \quad (4.1)$$

where  $\Omega$  is a  $\mathfrak{g}$ -valued two-form on  $G$  defined as follows. Let  $\mu \in \Omega^1(G) \otimes \mathfrak{g}$  be the left Maurer-Cartan form, that is, the  $\mathfrak{g}$ -valued one-form on  $G$  which associates to any  $X \in T_x G$  the value at the identity element  $e \in G$  of the left invariant vector field on  $G$  whose value at  $x$  is  $X$ . Then, for any  $X, Y \in T_x G$ ,

$$\Omega(X, Y) := [\mu(X), \mu(Y)]. \quad (4.2)$$

So  $\varphi^* \Omega \in \Omega^2(M) \otimes \mathfrak{g}$ , and its norm in the expression for  $E_4$  is taken with respect to  $g$  and some natural choice of  $Ad(G)$  invariant inner product on  $\mathfrak{g}$  (for example  $\langle X, Y \rangle_{\mathfrak{g}} = \frac{1}{2} \text{tr}(X^\dagger Y)$  in the case of most interest,  $G = SU(2)$ , giving  $G$  the metric of the unit 3-sphere). To be explicit, given any local orthonormal frame  $e_1, e_2, e_3$  of vector fields on  $M$ , then

$$|\varphi^* \Omega|^2 = |\varphi^* \Omega(e_1, e_2)|_{\mathfrak{g}}^2 + |\varphi^* \Omega(e_2, e_3)|_{\mathfrak{g}}^2 + |\varphi^* \Omega(e_3, e_1)|_{\mathfrak{g}}^2. \quad (4.3)$$

One can also allow for the presence of potential and sextic terms

$$E_0 = \frac{1}{2} \int_M U(\varphi)^2 \text{vol}_g, \quad E_6 = \frac{1}{2} \int_M |\varphi^* \Xi|^2 \text{vol}_g \quad (4.4)$$

where  $U : G \rightarrow \mathbb{R}$  is some potential function and  $\Xi \in \Omega^3(G)$  is some natural three-form on  $G$ , for example,

$$\Xi(X, Y, Z) = \langle \mu(X), \Omega(Y, Z) \rangle_{\mathfrak{g}} \quad (4.5)$$

---

<sup>2</sup>Actually, Manton and Sutcliffe take the Skyrme energy to be  $E' = \frac{1}{12\pi^2}(2E_2 + \frac{1}{2}E_4)$  but this can be reduced to  $E_2 + E_4$  by rescaling length and energy units [20, p. 350].

which coincides, in the case  $G = SU(2)$ , with the volume form on  $G$ . Such terms have aroused considerable interest recently because they offer the hope of constructing so-called “near-BPS” Skyrme models with drastically reduced nuclear binding energies, which addresses a fundamental phenomenological problem with the usual Skyrme model [1]. We shall begin our analysis of the model with all these terms present

$$E = E_0 + E_2 + E_4 + E_6 \quad (4.6)$$

before restricting to the usual case by choosing  $U = 0$ ,  $\Xi = 0$ . Existence of  $H^1$  minimizers in every degree class on an arbitrary compact domain for  $E = E_2 + E_4$ ,  $G = SU(2)$ , was established by Kapitanski [12]. A similar result with  $E_6$  included (with or without  $E_0$ ) follows from Proposition 8 and the obvious modification of Theorem 7 (the proof of which made no essential use of the dimension of  $M$ ). Once again, the established regularity is not thought to be optimal, but is sufficient to make the integral criteria below rigorously well-defined.

For our purposes, the key field theoretic object is the stress tensor of a field  $\varphi$ . To write this down neatly, we need to generalize slightly the contraction map  $A \cdot B$  for bilinear forms, introduced in section 2. So let  $A, B$  be  $\mathfrak{g}$ -valued bilinear forms on  $M$  (for example,  $\varphi^*\Omega$ ) and  $e_1, e_2, e_3$  be a local orthonormal frame of vector fields on  $M$ . Then by  $A \cdot B$  we will mean the (real valued) bilinear form

$$(A \cdot B)(X, Y) = \sum_i \langle A(X, e_i), B(e_i, Y) \rangle_{\mathfrak{g}}. \quad (4.7)$$

With this convention, we have:

**Proposition 9** *The stress tensor of a Skyrme field  $\varphi : M \rightarrow G$  with respect to the energy  $E = E_0 + E_2 + E_4 + E_6$  is*

$$S(\varphi, g) = \frac{1}{4} (|d\varphi|^2 + |\varphi^*\Omega|^2 - |\varphi^*\Xi|^2 + U(\varphi)^2) g - \frac{1}{2} (\varphi^*h - \varphi^*\Omega \cdot \varphi^*\Omega).$$

*Proof:* Let  $g_t$  be a smooth variation of  $g = g_0$  and  $\varepsilon = \partial_t|_{t=0}g_t$ . The terms coming from  $E_0 + E_2 + E_6$  (that is, all but the second and last terms in the formula above) were obtained previously [10]. It remains to show that

$$\left. \frac{d}{dt} \right|_{t=0} E_4(\varphi, g_t) = \langle \varepsilon, \frac{1}{4} |\varphi^*\Omega|^2 g + \frac{1}{2} \varphi^*\Omega \cdot \varphi^*\Omega \rangle_{L^2}. \quad (4.8)$$

Let us employ the abbreviation  $\Omega_{ij} = \varphi^*\Omega(\partial/\partial x_i, \partial/\partial x_j)$ , and the Einstein summation convention. Then

$$|\varphi^*\Omega|_g^2 = \frac{1}{2} \langle \Omega_{ij}, \Omega_{kl} \rangle_{\mathfrak{g}} g^{ik} g^{jl} \quad (4.9)$$

and hence

$$\left. \frac{d}{dt} \right|_{t=0} |\varphi^*\Omega|_{g_t}^2 = \varepsilon_{pq} g^{qj} \langle \Omega_{ji}, g^{il} \Omega_{lk} \rangle_{\mathfrak{g}} g^{kp} = \langle \varepsilon, \varphi^*\Omega \cdot \varphi^*\Omega \rangle. \quad (4.10)$$

It follows that

$$\begin{aligned}
\left. \frac{d}{dt} \right|_{t=0} E_4(\varphi, g_t) &= \left. \frac{d}{dt} \right|_{t=0} \frac{1}{2} \int_M |\varphi^* \Omega|_{g_t}^2 \text{vol}_{g_t} \\
&= \frac{1}{2} \int_M \left( \langle \varepsilon, \varphi^* \Omega \cdot \varphi^* \Omega \rangle \text{vol}_g + |\varphi^* \Omega|^2 \left. \frac{d}{dt} \right|_{t=0} \text{vol}_{g_t} \right) \\
&= \frac{1}{2} \int_M \left( \langle \varepsilon, \varphi^* \Omega \cdot \varphi^* \Omega \rangle \text{vol}_g + \frac{1}{2} |\varphi^* \Omega|^2 \langle \varepsilon, g \rangle \text{vol}_g \right) \tag{4.11}
\end{aligned}$$

as required, by (2.15).  $\square$

We want to use this formula to extract explicit conditions which  $\varphi$  must satisfy if it is to be a skyrmion lattice. Recall that this means precisely that  $S$  is  $L^2$  orthogonal to the space  $\mathbb{E}$  of parallel symmetric bilinear forms on  $M$ . We expect these conditions to consist of a virial constraint, similar to (3.5), and some analogue of the ‘‘conformal on average’’ condition (3.8). To formulate the latter condition in the Skyrme context, we define  $\Delta \in \mathbb{E}$  as in (2.8). That is, we choose  $x \in M$  and define  $\Delta : T_x M \times T_x M \rightarrow \mathbb{R}$  by

$$\Delta(X, Y) = \int_M (\varphi^* h(X, Y) - (\varphi^* \Omega \cdot \varphi^* \Omega)(X, Y)) \text{vol}_g \tag{4.12}$$

where  $X, Y$  on the right are the unique parallel extensions of  $X, Y$  over  $M$ . We then identify  $\Delta$  with an element of  $\mathbb{E}$  using the canonical isomorphism  $\mathbb{E} \rightarrow T_x^* M \odot T_x^* M$  defined by evaluation.

**Proposition 10**  *$\varphi : M = \mathbb{R}^3/\Lambda \rightarrow G$  is a skyrmion lattice if and only if*

$$(E_2 - E_4) + 3(E_0 - E_6) = 0, \quad \text{and} \quad \Delta = \frac{2}{3}(E_2 + 2E_4)g.$$

*Proof:* To analyze the condition  $S \perp_{L^2} g$ , we note that, for any symmetric bilinear form  $A$ ,  $\langle A, g \rangle = \text{tr } A$  pointwise, and

$$\text{tr } \varphi^* h = |\text{d}\varphi|^2 \quad \text{and} \quad \text{tr } \varphi^* \Omega \cdot \varphi^* \Omega = -2|\varphi^* \Omega|^2. \tag{4.13}$$

Hence

$$\begin{aligned}
\langle S, g \rangle_{L^2} &= \int_M \left\{ \frac{3}{4} (|\text{d}\varphi|^2 + |\varphi^* \Omega|^2 - |\varphi^* \Xi|^2 + U(\varphi)^2) - \frac{1}{2} (|\text{d}\varphi|^2 + 2|\varphi^* \Omega|^2) \right\} \text{vol}_g \\
&= \frac{1}{2} (E_2 - E_4 - 3E_6 + 3E_0) \tag{4.14}
\end{aligned}$$

which establishes the virial constraint. We have already noted that  $S \perp_{L^2} \mathbb{E}_0$  if and only if  $\Delta = \lambda g$  for some constant  $\lambda$ . Taking the trace of both sides and using (4.13) again, one finds that  $3\lambda = 2E_2 + 4E_4$ .  $\square$

Of course, we could have deduced the virial constraint directly from a Derrick scaling argument, but it is reassuring to see that it follows from our formula for  $S$ .

It would be convenient to have the analogue of Proposition 6 for skyrmion lattices, that is, a proof that every skyrmion lattice has positive hessian. While this is certainly plausible, we have been unable to prove it because the Skyrme stress tensor lacks a fundamental simplifying property enjoyed by the baby Skyrme stress tensor. Namely, in the baby Skyrme case, the derivative of  $S$  with respect to  $g$  in the direction of  $g$  is itself parallel to  $g$ ,  $\dot{S} = \lambda g$ , from which it immediately follows that  $\mathbf{Hess}$  is block diagonal with respect to the decomposition  $\mathbb{E} = \langle g \rangle \oplus \mathbb{E}_0$ . Hence, it suffices to show that  $\mathbf{Hess}(g, g) > 0$  and  $\mathbf{Hess}(\varepsilon, \varepsilon) > 0$  for all  $\varepsilon \in \mathbb{E}_0 \setminus \{0\}$ . In the Skyrme case, however,  $\partial_t|_{t=1} S(\varphi, tg) = \lambda g - \frac{1}{2} \varphi^* \Omega \cdot \varphi^* \Omega$  (the extra term coming from the  $g$  dependence in the contraction defining  $\varphi^* \Omega \cdot \varphi^* \Omega$ ), and this is not necessarily  $L^2$  orthogonal to  $\mathbb{E}_0$ . Hence, the hessian is not block diagonal in general, and no such simplification occurs. This difficulty is absent in the special case of Skyrme models for which the quartic term is absent, that is, with energy  $E = E_0 + E_2 + E_6$ , and in this case, the analogue of Proposition 6 does hold (the proof being essentially unchanged). As far as we are aware, however, all numerical studies of Skyrme crystals have addressed the conventional Skyrme model, with energy  $E = E_2 + E_4$ , so to check whether the solutions found therein are crystals according to our definition, one must independently check both the lattice conditions (given by Proposition 10) and positivity of the hessian.

So, for the rest of this section, we specialize to the usual Skyrme model, with target space  $G = SU(2)$  and energy  $E = E_2 + E_4$  by setting both  $U$  and  $\Xi$  to 0. Further, we choose  $\langle X, Y \rangle_{\mathfrak{g}} = \frac{1}{2} \text{tr}(X^\dagger Y)$ . The problem of minimizing  $E$  on a cubic torus  $T_L^3 = \mathbb{R}^3 / LZ^3$  among all Skyrme fields of degree 4 has been studied numerically by Castillejo *et al* [6] and Kugler and Shtrikman [15]. These studies minimized  $E$  for fixed side length  $L$ , and then varied  $L$ , independently finding an energy minimum at  $L \approx 18.8$  (in our units). The toric 4-skyrmion so obtained is usually called the ‘‘Skyrme crystal’’. Assuming such a minimum exists at this value of  $L$ , it must satisfy the virial constraint  $E_2 = E_4$  (since this follows from minimality with respect to dilations of  $g$ ). The numerical studies further suggest [20, p. 383] that the Skyrme crystal is equivariant with respect to a certain subgroup  $K$  of the isometry group of  $T_L^3$ , which we now describe.

Consider the linear maps  $s_i : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$s_1(x_1, x_2, x_3) = (-x_1, x_2, x_3), \quad s_2(x_1, x_2, x_3) = (x_2, x_3, x_1), \quad s_3(x_1, x_2, x_3) = (x_1, x_3, -x_2). \quad (4.15)$$

Clearly these are isometries of  $\mathbb{R}^3$  and preserve any cubic lattice  $\Lambda = LZ^3$ , and so generate a subgroup  $K$  of  $O(3)$  acting isometrically on  $T_L^3$  on the left. This group also acts (isometrically) on  $G$  on the left, as follows. We may identify  $G = SU(2)$  with the unit sphere  $S^3 \subset \mathbb{R}^4$  by means of the correspondence

$$(y_0, y_1, y_2, y_3) \leftrightarrow \begin{pmatrix} y_0 + iy_2 & y_3 + iy_1 \\ -y_3 + iy_1 & y_0 - iy_2 \end{pmatrix}. \quad (4.16)$$

This allows us to define an isometric left action of  $O(3)$  on  $G$  by  $\mathcal{O} : (y_0, \mathbf{y}) \mapsto (y_0, \mathcal{O}\mathbf{y})$ , and hence an isometric left action of  $K$  on  $G$ . The Skyrme crystal is known to be  $K$ -equivariant with respect to these actions.<sup>3</sup> We will now show that  $K$ -equivariance and the virial constraint

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<sup>3</sup>We have used a slightly non-standard embedding  $G \hookrightarrow \mathbb{R}^4$  to define the  $K$  action on  $G$ . This is to ensure that the Skyrme crystal is  $K$ -equivariant in the usual sense. Alternatively, we could use the usual embedding, and analyze the Skyrme ‘‘anticrystal’’,  $\tilde{\varphi} = P \circ \varphi$  where  $P : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is the map  $(y_0, y_1, y_2, y_3) \mapsto (y_0, y_1, y_3, y_2)$ .

alone ensure that an  $E$  minimizer is a soliton crystal according to our definition. It follows that, assuming the Skyrme crystal exists with the symmetries claimed, it is likewise a soliton crystal according to our definition.

**Proposition 11** *Let  $\varphi : T_L^3 \rightarrow SU(2)$  be a  $K$ -equivariant minimizer of the Skyrme energy  $E = E_2 + E_4$  which satisfies the virial constraint  $E_2 = E_4$ . Then  $\varphi$  is a soliton crystal.*

*Proof:* We are given that  $\varphi$  satisfies the virial constraint, so it suffices to show that  $\Delta = \lambda g$ , for some constant  $\lambda$  (where  $\Delta \in \mathbb{E}$  is defined in (4.12)), and that  $\mathbf{Hess}$  is positive. It is clear that  $E(\varphi, g)$  is geometrically natural and  $K$ -invariant. Hence, by Proposition 5,  $\Delta$  is an element of

$$\mathbb{E}^K = \{\varepsilon \in \mathbb{E} : \forall k \in K, k^* \varepsilon = \varepsilon\}, \quad (4.17)$$

the fixed-point space of  $\mathbb{E}$  under the action of  $K$ , and by Proposition 4,  $\mathbf{Hess}$  is an element of

$$(\mathbb{E}^* \odot \mathbb{E}^*)^K = \{H \in \mathbb{E}^* \odot \mathbb{E}^* : \forall k \in K, (k^*)^* H = H\} \quad (4.18)$$

the fixed-point space of  $\mathbb{E}^* \odot \mathbb{E}^*$  under the action of  $K$ . A representation theoretic argument, presented in the appendix, shows that  $\mathbb{E}^K$  has dimension 1 and  $(\mathbb{E}^* \odot \mathbb{E}^*)^K$  has dimension 3. Clearly  $g \in \mathbb{E}$  is  $K$ -invariant, so  $\mathbb{E}^K$  is spanned by  $g$ , and it follows that  $\Delta = \lambda g$  for some  $\lambda$ . Hence  $\varphi$  is a soliton lattice. It remains to show that  $\mathbf{Hess}$  is positive, and for this we introduce a basis for  $(\mathbb{E}^* \odot \mathbb{E}^*)^K$  as follows.

First note that the  $K$  action on  $\mathbb{E}$  leaves the three-dimensional subspace of  $\mathbb{E}$  consisting of diagonal symmetric bilinear forms

$$\mathbb{D} = \{a_1 dx_1^2 + a_2 dx_2^2 + a_3 dx_3^2 : a_1, a_2, a_3 \in \mathbb{R}^3\} \quad (4.19)$$

invariant. It also leaves the line spanned by  $g$ , a subspace of  $\mathbb{D}$ , invariant, and preserves the inner product on  $\mathbb{E}$ , and hence leaves the orthogonal complement of  $g$  in  $\mathbb{D}$

$$\mathbb{D}_0 = \{a_1 dx_1^2 + a_2 dx_2^2 + a_3 dx_3^2 : a_1, a_2, a_3 \in \mathbb{R}^3, a_1 + a_2 + a_3 = 0\} \quad (4.20)$$

and the orthogonal complement of  $\mathbb{D}$  in  $\mathbb{E}$

$$\mathbb{D}^\perp = \{2a_1 dx_1 dx_2 + 2a_2 dx_1 dx_3 + 2a_3 dx_2 dx_3 : a_1, a_2, a_3 \in \mathbb{R}\} \quad (4.21)$$

invariant. Let us denote by  $\text{Pr}_g$ ,  $\text{Pr}_{\mathbb{D}_0}$  and  $\text{Pr}_{\mathbb{D}^\perp}$  the orthogonal projectors  $\mathbb{E} \rightarrow \langle g \rangle$ ,  $\mathbb{E} \rightarrow \mathbb{D}_0$  and  $\mathbb{E} \rightarrow \mathbb{D}^\perp$ , and by  $H_1, H_2, H_3 : \mathbb{E} \times \mathbb{E} \rightarrow \mathbb{R}$  the symmetric bilinear forms

$$\begin{aligned} H_1(\widehat{\varepsilon}, \varepsilon) &= \langle \text{Pr}_g(\widehat{\varepsilon}), \text{Pr}_g(\varepsilon) \rangle_{\mathbb{E}}, \\ H_2(\widehat{\varepsilon}, \varepsilon) &= \langle \text{Pr}_{\mathbb{D}_0}(\widehat{\varepsilon}), \text{Pr}_{\mathbb{D}_0}(\varepsilon) \rangle_{\mathbb{E}}, \\ H_3(\widehat{\varepsilon}, \varepsilon) &= \langle \text{Pr}_{\mathbb{D}^\perp}(\widehat{\varepsilon}), \text{Pr}_{\mathbb{D}^\perp}(\varepsilon) \rangle_{\mathbb{E}}. \end{aligned} \quad (4.22)$$

These are linearly independent and, by construction,  $K$ -invariant, and hence form a basis for  $(\mathbb{E}^* \odot \mathbb{E}^*)^K$ . Hence

$$\mathbf{Hess} = c_1 H_1 + c_2 H_2 + c_3 H_3 \quad (4.23)$$

for some constants  $c_1, c_2, c_3$ , and  $\mathbf{Hess}$  is positive if and only if these constants are positive.

Now

$$\begin{aligned}
3c_1 &= \text{Hess}(g, g) = \frac{d^2}{dt^2} \Big|_{t=1} E(\varphi, tg) = \frac{d^2}{dt^2} \Big|_{t=1} (t^{\frac{1}{2}} E_2(\varphi, g) + t^{-\frac{1}{2}} E_4(\varphi, g)) \\
&= \frac{1}{4} E_2 + \frac{3}{4} E_4 = E_2 > 0.
\end{aligned} \tag{4.24}$$

To compute  $c_2$  and  $c_3$ , it is convenient to use again the abbreviation  $\Omega_{ij} := \varphi^* \Omega(\partial/\partial x_i, \partial/\partial x_j)$ . Let  $g_t$  be a generating curve in  $\mathbb{E}$  for  $\varepsilon$ . Then

$$\begin{aligned}
\dot{S} &= \frac{d}{dt} \Big|_{t=0} S(\varphi, g_t) = \lambda g + \frac{1}{4} (|\text{d}\varphi|^2 + |\varphi^* \Omega|^2) \varepsilon + \frac{1}{2} \frac{d}{dt} \Big|_{t=0} (\varphi^* \Omega) \cdot_{g_t} (\varphi^* \Omega) \\
&= \lambda g + \frac{1}{4} (|\text{d}\varphi|^2 + |\varphi^* \Omega|^2) \varepsilon - \frac{1}{2} \sum_{i,j,k,l} \langle \Omega_{ik}, \varepsilon_{kl} \Omega_{lj} \rangle_{\mathfrak{g}} dx_i dx_j.
\end{aligned} \tag{4.25}$$

In the case  $\varepsilon = dx_1^2 - dx_2^2 \in \mathbb{D}_0$ , one sees that

$$\dot{S} = \frac{1}{4} (|\text{d}\varphi|^2 + |\varphi^* \Omega|^2) \varepsilon - \frac{1}{2} |\Omega_{12}|^2 \varepsilon + A \tag{4.26}$$

where  $\langle A, \varepsilon \rangle_{\mathbb{E}} = 0$ . Hence, by Proposition 2,

$$\begin{aligned}
2c_2 &= \text{Hess}(dx_1^2 - dx_2^2, dx_1^2 - dx_2^2) = \langle \dot{S}, \varepsilon \rangle_{L^2} = E_2 + E_4 - \int_M |\Omega_{12}|^2 \text{vol}_g \\
&= 2E_4 - \int_M |\Omega_{12}|^2 \text{vol}_g = \|\Omega_{23}\|_{L^2}^2 + \|\Omega_{31}\|_{L^2}^2 > 0.
\end{aligned} \tag{4.27}$$

Similarly, in the case  $\varepsilon = 2dx_1 dx_2 \in \mathbb{D}^\perp$ ,

$$\dot{S} = \frac{1}{4} (|\text{d}\varphi|^2 + |\varphi^* \Omega|^2) \varepsilon - \frac{1}{2} |\Omega_{12}|^2 \varepsilon + A' \tag{4.28}$$

where  $\langle A', \varepsilon \rangle_{\mathbb{E}} = 0$  and so, by identical reasoning,

$$2c_3 = \text{Hess}(2dx_1 dx_2, 2dx_1 dx_2) = \|\Omega_{23}\|_{L^2}^2 + \|\Omega_{31}\|_{L^2}^2 > 0. \tag{4.29}$$

□

Other periodic Skyrme solutions on cubic tori have been found numerically, and can be analyzed in similar fashion. For example, Klebanov [14] found a solution which is equivariant with respect to the subgroup  $K' < K$  generated by  $s_1, s_2$  only. Again, he minimized over period length, so that the virial constraint must hold. It turns out that  $\mathbb{E}^{K'} = \mathbb{E}^K$  and  $(\mathbb{E}^* \odot \mathbb{E}^*)^{K'} = (\mathbb{E}^* \odot \mathbb{E}^*)^K$ , so exactly the same argument given above shows that this toric soliton is also a soliton crystal.

It is slightly surprising that, in the course of the proof above, we showed that  $c_2 = c_3$ , so that the hessian actually has the simple form

$$\text{Hess}(\widehat{\varepsilon}, \varepsilon) = c_1 \langle \text{Pr}_g(\widehat{\varepsilon}), \text{Pr}_g(\varepsilon) \rangle_{\mathbb{E}} + c_2 \langle \text{Pr}_{\mathbb{E}_0}(\widehat{\varepsilon}), \text{Pr}_{\mathbb{E}_0}(\varepsilon) \rangle_{\mathbb{E}}. \tag{4.30}$$

This does not follow immediately from  $K$ -equivariance alone, but seems to rely on the detailed structure of the Skyrme energy, so should not be expected as a generic property of soliton lattices on cubic tori. Note that the symmetry  $\varphi \circ s_2 = s_2 \circ \varphi$  implies

$$\|\Omega_{12}\|_{L^2}^2 = \|\Omega_{23}\|_{L^2}^2 = \|\Omega_{31}\|_{L^2}^2 = \frac{2}{3}E_4 = \frac{2}{3}E_2 = \frac{1}{3}E, \quad (4.31)$$

so that  $K$ -equivariance (or  $K'$ -equivariance) actually implies

$$\text{Hess}(\widehat{\varepsilon}, \varepsilon) = \frac{1}{3}E \left( \langle \widehat{\varepsilon}, \varepsilon \rangle_{\mathbb{E}} - \frac{1}{6} \langle g, \widehat{\varepsilon} \rangle_{\mathbb{E}} \langle g, \varepsilon \rangle_{\mathbb{E}} \right) \quad (4.32)$$

for the standard Skyrme model. It would be interesting to see whether useful information about the vibrational modes of Skyrme crystals can be extracted from this formula.

## 5 Concluding remarks

We have derived necessary conditions for a soliton on a torus  $M = \mathbb{R}^m/\Lambda$  to be a soliton crystal. The stress tensor  $S$  of the soliton must be  $L^2$  orthogonal to  $\mathbb{E}$ , the space of parallel symmetric bilinear forms on  $TM$  and, further, a certain symmetric bilinear form on  $\mathbb{E}$ , called the hessian, must be positive. We have shown that, for baby Skyrme models, the first condition actually implies the second. We have also shown that, for any choice of lattice  $\Lambda$ , there is a baby Skyrme model which supports a soliton crystal of periodicity  $\Lambda$ . For the three-dimensional Skyrme model, we showed that a soliton solution on a cubic lattice which satisfies the virial constraint  $E_2 = E_4$  and is equivariant with respect to (a subgroup of) the lattice symmetries automatically satisfies both tests. This verifies in particular that the ‘‘Skyrme crystal’’ of Castillejo *et al.*, and Kugler and Shtrikman, passes both tests. Note that, although we have applied the criteria only to local minimizers of  $E(\varphi)$ , they could equally well be applied to saddle points (unstable static solutions), and there is no obvious reason why a saddle point for variations of  $\varphi$  should not be a local minimum for variations of  $\Lambda$ . Indeed, *if* saddle points of the baby Skyrme energy exist which are critical for variations of  $\Lambda$ , the proof of Proposition 6 implies that they can *only* be local minima (with respect to variations of  $\Lambda$ ). The physical significance of such saddle points, if, indeed, they exist at all, is not clear.

It would be straightforward to extend the analysis to deal with gauge theories on tori. In this context,  $\varphi$  would be a section of some vector bundle  $V$  over  $M$  with connexion  $\nabla$ . The linear diffeomorphisms used to identify all tori with  $M$  can be used to identify the bundles  $V$ , the section  $\varphi$ , and the connexion  $\nabla$ , by pullback, so that, once again, the variation over period lattices is reformulated as a variation over metrics. One expects the first and second variation formulae to be structurally identical to those presented in section 2, therefore.

A more interesting extension would be to apply the idea to soliton sheets and chains, that is, solitons on  $M = \mathbb{R}^m \times T^{m'}$ . Presumably the first variation formula will, once again, amount to the condition that the soliton satisfies all generalized Derrick constraints on  $M$ . Now, however,  $\frac{1}{2}m(m+1)$  of these conditions will be known *a priori* for free (or  $\frac{1}{2}m(m+1)+1$  if the energy has been minimized over torus volume, as is usual in numerical studies) because they involve deformations only of the  $\mathbb{R}^m$  factor, which are already accounted for in the variation of  $\varphi$ .

This leaves  $\frac{1}{2}m'(m' + 1) + mm'$  nontrivial constraints. Equivariance with respect to lattice symmetries is likely to be considerably less constraining than it is for tori so that equivariance alone is unlikely to guarantee criticality. It would be interesting to see whether the skyrmion sheets found numerically in [5] and [22] survive the analysis.

## Appendix: the symmetry group $K$ and its action on $\mathbb{E}$

Recall that the Skyrme crystal is equivariant with respect to a discrete subgroup  $K < O(3)$  generated by the matrices

$$s_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad (\text{A.1})$$

These generate a nonabelian group of order 48, consisting of all permutation matrices where each nonzero entry can be either 1 or  $-1$ . There is an induced isometric action of  $K$  on  $(\mathbb{E}, \langle \cdot, \cdot \rangle_{\mathbb{E}})$  by pullback,

$$k^* \varepsilon(X, Y) = \varepsilon(k(X), k(Y)) \quad (\text{A.2})$$

and, further, an induced action of  $K$  on  $\mathbb{E}^* \odot \mathbb{E}^*$  by pullback of the pullback,

$$(k^*)^* H(\hat{\varepsilon}, \varepsilon) = H(k^* \hat{\varepsilon}, k^* \varepsilon). \quad (\text{A.3})$$

We wish to compute the dimensions of their fixed point spaces  $\mathbb{E}^K$ ,  $(\mathbb{E}^* \odot \mathbb{E}^*)^K$ , or equivalently, to count the number of copies of the trivial representation in the decomposition of the  $K$  representations on  $\mathbb{E}$  and  $\mathbb{E}^* \odot \mathbb{E}^*$  into irreducible orthogonal representations. This we can do by using character orthogonality. Our task, therefore, boils down to the construction of character tables for these two representations.

Recall that characters are constant on conjugacy classes. There is an obvious eight-to-one “forgetful” homomorphism  $\mu : K \rightarrow S_3$ , which sends each signed permutation matrix to the permutation matrix obtained by changing all  $-1$  entries to  $+1$ . Clearly, if  $k, k' \in K$  are conjugate in  $K$ , so are  $\mu(k), \mu(k')$  in  $S_3$ , so each conjugacy class  $[k]$  in  $K$  carries a label  $[\mu(k)]$ , a conjugacy class in  $S_3$ . There are three such classes, consisting of permutations which fix 3, 1 or 0 elements, the classes of  $e$ , (23) and (132) respectively. Conjugate elements in  $K$  also have equal trace and determinant. So if  $k$  is conjugate to  $k'$ ,  $([\mu(k)], \det k, \text{tr } k) = ([\mu(k')], \det k', \text{tr } k')$  and straightforward calculation shows that the converse also holds: if  $([\mu(k)], \det k, \text{tr } k) = ([\mu(k')], \det k', \text{tr } k')$  then  $k$  is conjugate to  $k'$ . Hence, each conjugacy class is uniquely labelled by the triple  $([\mu(k)], \det k, \text{tr } k)$ . From this we deduce that  $K$  splits into 10 conjugacy classes, as specified in table 1. The final label,  $\text{tr } k$ , is also the character  $\chi^{\mathbb{R}^3}(k)$  of the fundamental representation of  $K$ . From this, we can deduce the character of the induced representation on  $\mathbb{E}$ , the space of symmetric bilinear forms on  $\mathbb{R}^3$ , using the standard formula [11]

$$\chi^{\mathbb{E}}(k) = \frac{1}{2} \left[ \chi^{\mathbb{R}^3}(k)^2 + \chi^{\mathbb{R}^3}(k^2) \right]. \quad (\text{A.4})$$

For this purpose we need to know  $[k^2]$ , the conjugacy class of  $k^2$ , for a representative  $k$  of each class. This information is recorded in column 6 of table 1, and suffices to compute  $\chi^{\mathbb{E}}(k)$  for

Representative $k$	$ [k] $	$[\mu(k)]$	$\det k$	$\operatorname{tr} k = \chi^{\mathbb{R}^3}(k)$	$[k^2]$	$\chi^{\mathbb{E}}(k)$	$\chi^{\mathbb{E}^* \odot \mathbb{E}^*}(k)$
$\mathbb{I}_3$	1	$[e]$	1	3	$[\mathbb{I}_3]$	6	21
$-\mathbb{I}_3$	1	$[e]$	-1	-3	$[\mathbb{I}_3]$	6	21
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3	$[e]$	-1	1	$[\mathbb{I}_3]$	2	5
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	3	$[e]$	1	-1	$[\mathbb{I}_3]$	2	5
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	6	$[(2, 3)]$	-1	1	$[\mathbb{I}_3]$	2	5
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$	6	$[(2, 3)]$	1	-1	$[\mathbb{I}_3]$	2	5
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	6	$[(2, 3)]$	1	1	$\left[ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$	0	1
$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$	6	$[(2, 3)]$	-1	-1		0	1
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	8	$[(1, 3, 2)]$	1	0	$\left[ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \right]$	0	0
$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$	8	$[(1, 3, 2)]$	-1	0		0	0

Table 1: Character table for the representations of the signed permutation group  $K$  on  $\mathbb{R}^3$ ,  $\mathbb{E}$  and  $\mathbb{E}^* \odot \mathbb{E}^*$ . The columns, from left to right, give a representative  $k$  for each conjugacy class, the size of the class, then the three labels ( $[\mu(k)]$ ,  $\det k$ ,  $\operatorname{tr} k$ ) which uniquely label the class. The last of these coincides with the character  $\chi^{\mathbb{R}^3}$ . The next column gives the class of  $k^2$ , which suffices to compute, inductively,  $\chi^{\mathbb{E}}$  and  $\chi^{\mathbb{E}^* \odot \mathbb{E}^*}$  using the formulae (A.4) and (A.6).

each class, yielding column 7. To compute the number of copies of the trivial representation of  $K$  in  $\chi^{\mathbb{E}}$ , we compute the character inner product between  $\chi^{\mathbb{E}}$  and  $\chi^{\operatorname{triv}}$  (where  $\chi^{\operatorname{triv}}(k) = 1$  for all  $k$ ):

$$\langle \chi^{\mathbb{E}}, \chi^{\operatorname{triv}} \rangle = \frac{1}{|K|} \sum_{k \in K} \chi^{\mathbb{E}}(k) \chi^{\operatorname{triv}}(k) = \frac{1}{|K|} \sum_{[k] \in K/\sim} |[k]| \chi^{\mathbb{E}}([k]) \times 1 = 1 \quad (\text{A.5})$$

where the second sum is over conjugacy classes. Hence,  $\mathbb{E}^K$  is one-dimensional.

Consider now the induced representation of  $K$  on  $\mathbb{E}^* \odot \mathbb{E}^*$ , the 21-dimensional space of symmetric bilinear forms on  $\mathbb{E}$ . This is just the adjoint representation associated with the orthogonal representation of  $K$  on  $\mathbb{E}$  just constructed. Hence, its character is related to  $\chi^{\mathbb{E}}$  just as in (A.4), namely

$$\chi^{\mathbb{E}^* \odot \mathbb{E}^*}(k) = \frac{1}{2} [\chi^{\mathbb{E}}(k)^2 + \chi^{\mathbb{E}}(k^2)], \quad (\text{A.6})$$

which yields column 8 of table 1. Now

$$\langle \chi^{\mathbb{E}^* \odot \mathbb{E}^*}, \chi^{\operatorname{triv}} \rangle = \frac{1}{|K|} \sum_{[k] \in K/\sim} |[k]| \chi^{\mathbb{E}^* \odot \mathbb{E}^*}([k]) = 3 \quad (\text{A.7})$$

so we deduce that  $(\mathbb{E}^* \odot \mathbb{E}^*)^K$  has dimension 3.

A similar analysis can be performed for  $K'$ , the order 24 group generated by  $s_1, s_2$  alone. One finds that the spaces  $\mathbb{E}^{K'}$  and  $(\mathbb{E}^* \odot \mathbb{E}^*)^{K'}$  again have dimension 1 and 3 respectively. Since  $K' < K$ , it follows that  $\mathbb{E}^K < \mathbb{E}^{K'}$  and  $(\mathbb{E}^* \odot \mathbb{E}^*)^K < (\mathbb{E}^* \odot \mathbb{E}^*)^{K'}$  and hence, by equality of dimensions,  $\mathbb{E}^K = \mathbb{E}^{K'}$ ,  $(\mathbb{E}^* \odot \mathbb{E}^*)^K = (\mathbb{E}^* \odot \mathbb{E}^*)^{K'}$ .

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