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Abstract:

Polynomial and rational expansions of nonlinear stochastic dynamic models are compared. The structure, approximation properties, and stability are discussed and a unified identification algorithm is introduced.

1 Introduction

System characterization and identification are fundamental problems in systems theory. The problem of characterization is related to the analytical modelling of a system and the problem of identification is related to the process of constructing or learning a mathematical model of a system from observations of the input and output signals. Determining the model structure is an important part of system identification because an appropriate model selection is both more likely to provide an accurate approximation and saves effort in the computation and mathematical manipulation. Ideally the final model will be accurate, concise, and easy to manipulate. While it may not always be possible to achieve such an ideal good model and structure selection will increase the probability of achieving these objectives in model based identification of both linear and nonlinear systems.

Order determination and term selection are quite straightforward for linear models and the well known parametric ARMAX model provides an excellent representation for the class of real linear systems. Nonlinear model selection however is very complicated because there is no unique model which can represent all nonlinear systems and the number of combinations of possible terms becomes very large. Usually a special model form is adopted to describe the specific nonlinear phenomenon under investigation. Linearization for example provides a linear approximation of the system and avoids the nonlinear aspects of the problem, but often at the expense of very poor approximation. The Volterra series (Schetzen 1980) is a widely studied model but with the disadvantage that this description requires thousands of parameters to specify the kernels even for simple nonlinear systems.

In the early 1980's, the NARMAX model was introduced (Leontaritis and Billings 1985) as an expansion of past inputs, outputs, and prediction error terms. The main advantage of this representation is that the dynamic information encoded by the output terms permits very concise model representations. It can be shown (Chen and Billings 1989) that the NARMAX model provides a very general representation for a large class of nonlinear systems and various forms of the model have been studied. These range from linear in the parameters expansions such as the polynomial NARMAX to nonlinear in the parameters expansions such as the nonlinear dynamic rational model.

The polynomial model has been studied in detail and the model characteristics, identification, and properties are now well known. In contrast the rational model is much more difficult to estimate and has only recently been considered. But both models are closely related and the present study is an attempt to compare the two representations. This will involve a study of the structure of the models, the approximation properties, a discussion of stability and the introduction of a unified identification procedure. The ideas are illustrated throughout by simple examples.

2 Models

The method of representing nonlinear dynamic systems using nonlinear differential or difference equations is well established in systems theory. But if such models are to be used as a basis for identification then it is important to consider the approximation, stability, and identifiability properties of this class of models.

2.1 Structure

The NARMAX model is defined as

$$y(k) = F(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r),$$
$$e(k-1), \cdots, e(k-r)) + e(k) \tag{2.1.1}$$

where $k = 1, 2, \cdots$ is a discrete time index, $u(k)$ and $y(k)$ are the input and output respectively at time k, $e(k)$ is an unobservable independent and identically distributed (iid) noise with zero mean and finite variance σ_e^2, and $F(.)$ is some nonlinear function for example a polynomial or rational function.

The polynomial NARMAX model can be expressed as

$$y(k) = \sum_{j=1}^{m} p_j(k)\theta_j \tag{2.1.2}$$

where $p_j(k) = p_j(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$ is defined as a term which is a function of the variables $(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$. The rational NARMAX model is of the form

$$y(k) = \frac{a(k)}{b(k)} = \frac{\sum_{j=1}^{num} p_{nj}(k)\theta_{nj}}{\sum_{j=1}^{den} p_{dj}(k)\theta_{dj}} \tag{2.1.3}$$

where the numerator term $p_{nj}(k) = p_{nj}(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$ and the denominator term $p_{dj}(k) = p_{dj}(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$ are typically polynomials.

The NARMAX model has the following characteristics:

Remark 1:

If the nonlinear function $F(.)$ in eqn (2.1.1) is continuous, which may include finite singular points, then it can always be arbitrarily well approximated by polynomial models (Billings and Chen 1989b), this follows from the famous Stone-Weierstrass theorem (Haaser and Sullivan 1971).

Remark 2:

Usually the NARMAX model has a finite realization in terms of lagged inputs and outputs with a constant nonlinear degree. For example an infinite power series may be simply expressed as a rational model (Zhu and Billings 1992).

Remark 3:

The NARMAX model is a natural representation for many nonlinear models and systems.

Remark 4:

The NARMAX model stability is dependent on both the model structure and the input amplitude.

Remark 5:

The NARMAX model can be identified using special least squares algorithms. This will be presented in section 4.

Chen and Billings(1989) have proved that the NARMAX model is a general and natural representation of nonlinear systems and contains, as a special case several existing nonlinear models such as the Hammerstien, bilinear, output-affine and rational models. Using the following definitions the NARMAX model can be classified into several subsets. Consider eqn (2.1.2),

(i) A model is linear in the parameters if the model is linear in θ_j, $j = 1, \cdots, m$.

(ii) A model is linear in the output variable if the model is linear in $y(k-j)$, $j = 1, \cdots, r$.

(iii) A model is linear in the input variable if the model is linear in $u(k-j)$, $j = 1, \cdots, r$.

Converse definitions of nonlinearity in the parameters, output variables and input variables can also be defined in an obvious manner.

These concepts are easy to be illustrated with examples

Example 1: The polynomial NARMAX model (Chen and Billings 1989)

$$y(k) = a_1 y(k-1) + a_2 y^2(k-1)u(k) + u^2(k-1) \tag{2.1.4}$$

is classified as linear in the parameters and nonlinear in the input and output variables.

Example 2: The output-affine model (Chen and Billings 1989)

$$y(k) = \frac{1}{1 + u(k-1)}\{[u(k-1) + u^2(k-2)]y(k-1) + u(k-2)y(k-2) + u(k)\} \tag{2.1.5}$$

is classified as linear in the output variable, but nonlinear in the parameters and input variable.

Example 3: The Bilinear model (Chen and Billings 1989)

$$y(k) = a_1 y(k-1) + b_0 u(k) + c_1 y(k-2)u(k-1) \tag{2.1.6}$$

is classified as linear in the parameters, input and output variables.

Example 4: The Hammerstien model (Narendra and Gallmann 1966)

$$y(k) = a_1 y(k-1) + b_0 + b_1\, u(k) + b_2\, u^2(k) \tag{2.1.7}$$

is classified as linear in the parameters and output variable, but nonlinear in the input variable.

Example 5: The rational NARMAX model (Chen and Billings 1989)

$$y(k) = \frac{a_1 y(k-1) + a_2 y^2(k-1)u(k-1)}{1 + b_1 y^2(k-1)} \tag{2.1.8}$$

is classified as nonlinear in the parameters, output and input variable.

2.2 Representation

Mathematical models can be used for both function approximation and data fitting. In the former case for example a complex function such as a hyperbolic function can be expanded and approximated by a Taylor series model say, whereas in the later case the model is estimated as a best fit to a selected data set usually in the presence of noise. This is normally called system identification and it often involves both structure detection and parameter estimation.

2.2.1 Function approximation

In function approximation the expansions are usually static. The polynomial model takes the form

$$y = 1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n \tag{2.2.1}$$

and the rational model is expressed as

$$y = \frac{1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n}{1 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_m x^m} \tag{2.2.2}$$

This is sometimes called Pade function (Braess 1986).

The polynomial model is used mainly to represent the class of continuous functions. The rational model however can also be used to describe certain types of discontinuous functions.

In function approximation the major concerns are usually approximation error analysis and error convergence speed. There is no need to detect function structure or to estimate parameters because both the structure of the approximated and approximating functions are exactly known a priori. Table 2.1 illustrates the order of error of approximation for both polynomial and rational models, in the table n may be thought of as the number of evaluations of a given function required to obtain the approximation and c denotes a positive constant.

Model	analytic	analytic with singularity
Polynomial	$\exp(-cn)$	n^{-c}
Rational	$\exp(-cn)$	$\exp(-cn^{1/2})$

Table 2.1 Order of error of approximation using polynomial and rational models

The table shows that the polynomial and the rational model both have the same rate of convergence $exp(-cn)$ of error of approximation for an analytic functions without singularities. But the rational model provides a much better convergence property compared to the polynomial model in function approximation with singularities. In this latter case the polynomial model will exhibit a very slow convergence to zero error (Stenger 1980).

There exists a simple transformation between the polynomial model of eqn (2.2.1) and the rational model of eqn (2.2.2) using the famous Pade table (Braess 1986). In theory, the polynomial model can be obtained by expanding the rational model into a power series.

2.2.2 Data approximation

Many dynamic phenomena can be described by nonlinear differential equations or difference equations. Usually data collected from practical environments are modelled based on these equations and the approximation properties are discussed below.

Sampled differential equation

The polynomial and rational models are the natural representatives for the sampled continuous time systems described by differential equations. Consider a nonlinear differential equation

$$y^{(1)}(t) - u^2(t)y(t) = u(t) \tag{2.2.3}$$

where t is a continuous time and $y^{(1)}$ denotes the first derivative of y with respect to t. Assuming the sampling period is sufficiently small, the forward difference scheme can be used to produce

$$\frac{y(k+1) - y(k)}{h} - u^2(k)y(k) = u(k) \tag{2.2.4}$$

where k=1, 2, $\cdots$ is a discrete time index with equal sampling period h. Simple algebra operation gives

$$y(k+1) = y(k) + hu^2(k)y(k) + hu(k) \tag{2.2.5}$$

which is a typical polynomial model.

Sampling the same differential equation but using the backward difference scheme produces

$$\frac{y(k) - y(k-1)}{h} - u^2(k)y(k) = u(k) \tag{2.2.6}$$

which gives a rational model or an output affine model

$$y(k) = \frac{y(k-1) + h\,u(k)}{1 - hu^2(k)} \tag{2.2.7}$$

Noticed that both the forward and the backward difference schemes deliver a linear difference equation structure when a linear differential equation is sampled. For example

$$y^{(1)}(t) - ay(t) = u(t) \tag{2.2.8}$$

where a is a constant, becomes using the forward difference scheme

$$y(k+1) = (h + a)y(k) + hu(k) \tag{2.2.9}$$

The backward difference scheme produces

$$y(k) = \frac{1}{1 - ha}y(k-1) + \frac{h}{1 - ha}u(k) \tag{2.2.10}$$

and both are linear difference models.

Chaos

The discovery of new types of dynamic behaviour in engineering systems over the last decade has introduced new analytic and experimental techniques in dynamics. The principal amongst these new discoveries is chaos. Chaotic behaviour has been observed in many areas including solid mechanics, fluid mechanics, thermo fluid phenomena, electromagnetic systems, acoustical systems, and general adaptive control problems. Usually the chaotic motion is described by some complicated models (Genesio and Tesi 1991), for example (Ozaki 1985)

$$y(k) = (\ 1 - 18\exp(-y^2(k-1))\ )y(k-1) - (\ 0.25 - 72.5\exp(-y^2(k-1))\ )y(k-2) \tag{2.2.11}$$

which is a typical nonlinear time series model.

The polynomial model can be easily used to model the chaotic behaviour in eqn (2.2.11) by using the Taylor series expansion

$$\exp(-y^2(k)) = 1 - y^2(k) + y^4(k) - \cdots \tag{2.2.12}$$

to replace the exponential terms of eqn (2.2.11). Alternatively a rational model could be employed.

Bifurcations

Chaotic phenomena are often preceded by a series of bifurcations including subharmonic and limit cycle or Hopf bifurcations. Iooss (1988) presented two typical bifurcations, the first one is called a saddle node bifurcation can be described by

$$\alpha_1 \, [\, \tan(y(t))]^{(2)} + \alpha_2 \, [\, \tan(y(t))]^{(1)} + \alpha_3 \, [\, \cos(y(0)) - \cos(y(t)) \,] \, \sin(y(t)) + \alpha_4 = 0 \tag{2.2.13}$$

where $f^{(1)}$ and $f^{(2)}$ denote the first and second derivative of f with respect to t respectively. A second example known as the pitckfork bifurcation can be expressed by

$$y^{(2)}(t) + \alpha_1 \, y^{(1)}(t) + \alpha_2 \sin(y(t)) + \alpha_3 \sin(y(t))\cos(y(t)) = 0 \tag{2.2.14}$$

The two differential equations can be approximated by either a polynomial model or a rational model. Using the Taylor series expansions of

$$\begin{aligned}
\tan(y(t)) &= y(t) + \frac{1}{3}y^3(t) + \frac{2}{15}y^5(t) + \cdots \\
\sin(y(t)) &= y(t) - \frac{y^3(t)}{3!} + \frac{y^5(t)}{5!} - \cdots \\
\cos(y(t)) &= 1 - \frac{y^2(t)}{2!} + \frac{y^4(t)}{4!} - \cdots
\end{aligned} \tag{2.2.15}$$

to replace the trigonometrical terms in equations (2.2.13) and (2.2.14), and then applying forward difference scheme

$$y^{(1)}(t) = \frac{y(k+1) - y(k)}{h} \tag{2.2.16}$$

gives polynomial approximations to the saddle node bifurcation and the Hopf bifurcation. Alternatively the extended model set ideas of Billings and Chen (1989b) can be used with the advantage that the trigonometrical terms can be represented without approximation.

When the backward difference scheme

$$y^{(1)}(t) = \frac{y(k) - y(k-1)}{h} \tag{2.2.17}$$

is applied to equations (2.2.13) and (2.2.14) rational approximations to the bifurcations are obtained.

Fractals

Fractal phenomena like coastlines, galaxies, and frost patterns on windows have been observed over many years. Barnsley and Demko (1980) used both the static polynomial and rational model to approximate fractal phenomena. Two such models are (Barnsley and Demko 1980)

$$Y = \frac{(z^2 - 81)}{9}$$

$$Y = \frac{3}{4} z + \frac{1}{4 z^3} \tag{2.2.18}$$

where z is a complex variable.

The polynomial model has also been used to in the identification of a variety of nonlinear systems including a heat exchanger (Liu, Korenberg, Billings, and Fadzil 1987), a diesel generator (Billings and Fadzil 1988), an automotive diesel engine (Billings and Chen 1989), nonlinear fluid loading systems (Worden, Stansby, Tomlinson, and Billings 1991) and so on. Application of the rational model to these types of problems is however only just beginning.

3 Stability

The stability of a class of NARMAX models which are characterised as linear in the output variable is analysed below and a linear equivalent stability criterion is proposed and illustrated with several examples.

3.1 Linear equivalent principle of stability

The stability analysis of the class of nonlinear systems, which are linear in the output, a subset of the NARMAX model, can be investigated using linear methods.

When the input is treated as a time-variant parameter the model can be expressed as

$$y(k) = \alpha_1(u)y(k-1) + \cdots + \alpha_r(u)y(k-r) + U \tag{3.1.1}$$

where $\alpha_1(u)$ to $\alpha_r(u)$ can be functions of the input and original parameters, and U represents possibly nonliner terms in $u(k-j)$, $j = 1, \cdots, r$. Eqn (3.1.1) has the same structure as the linear ARMAX model, except that some of the parameters are time variant and therefore the stability may be analysed with linear stability criteria.

Remark 1:

The stability of the class of systems considered is input-dependent, this is obvious by noticing that the parameters $\alpha_1(u)$ to $\alpha_r(u)$ in eqn (3.1.1) are functions of the input. This confirms the well-known result that the stability of most of nonlinear systems depends on the input amplitude as well as the system structure and parameters.

Remark 2:

A hypersurface $S(u(k), \cdots, u(k-r))$ distinguishes the stable and unstable spaces in a hyperspace spanned by $(u(k), \cdots, u(k-r))$. This can be proved from the stability analysis which results in a set of equations by root-solving. Consider the Z transform of the characteristic equation of eqn (3.1.1)

$$z^r - \alpha_1(u)z^{r-1} - \cdots - \alpha_r(u) = 0 \tag{3.1.2}$$

Let $R_i(\alpha_1(u), \cdots, \alpha_r(u))$ be one of the roots of the characteristic equation. Then for a stable system the following set of inequalities must be satisfied.

$$|R_1(\alpha_1(u), \cdots, \alpha_r(u))| < 1$$

$$|R_2(\alpha_1(u), \cdots, \alpha_r(u))| < 1$$

$$\vdots$$

$$|R_r(\alpha_1(u), \cdots, \alpha_r(u))| < 1 \tag{3.1.3}$$

Let $S(u(k), \cdots, u(k-r))$ be the solution of eqn (3.1.3), therefore this corresponds, in the hyperspace, to the stable solution space which is delimited by $|R_1(.)| < 1 \;\cdots\; |R_r(.)| < 1$ in a geometrical sense.

Remark 3:

The stable space delimited by $S(u(k), \cdots, u(k-r))$ maps into the inside of the unit circle in the Z plane, the unstable space is outside the unit circle. This follows from linear stability theory.

Remark 4:

The Hammerstien model has the same stability property as the corresponding linear part of the model because in this case $\alpha_1(u), \cdots, \alpha_r(u)$ in eqn (3.1.1) are input-independent.

3.2 Stochastic stability

It is very possible that the system described by eqn(3.1.1) switches between stable and unstable regions due to the parameters $\alpha_1(u), \cdots, \alpha_r(u)$ varying with the input $u(k)$. This naturally produces a stochastic stability problem.

A large portion of publications on this topic have been devoted to the Lyapunov stability concept. We first introduce the definition of Lyapunov stability in the m'th mean, secondly we present theorem 2 to show the condition for the input-output stability in the second mean, and then give remarks for the interpretation of the theorem.

We state here, for reference, the concept of Lyapunov stability for deterministic systems. We shall always refer to the equilibrium or null solution, $x = 0$, as the solution whose stability properties are being tested, x_0 will denote the initial state at the initial time k_0. We will denote the solution with initial state x_0 at time k_0, by $x(k; x_0, k_0)$, which is assumed to be an n-vector. Finally, $\| x \|$ will denote $\sum_{i=1}^{n} |x_i|$, the simple absolute value norm.

Definition of Lyapunov stability in the m'th mean:

The equilibrium solution is stable in the m'th mean if the mth moments of the solution vector exist and given $\varepsilon > 0$, there exists $\delta(\varepsilon, k_0)$ such that $\| x_0 \|_m < \delta$ implies

$$E[\ sup\ \|x(k; x_0, k_0\|_m^m\ ] < \varepsilon, \qquad \text{for } t \geq k_0 \tag{3.2.1}$$

where $\| x \|_m^m = \sum_{i=1}^{N} |x_i|^m$. For $m = 2$, $\| x \|_2^2 = \sum_{i=1}^{N} x_i^2$.

A system described in eqn(3.1.1) is input-output stable in the second mean if $\lim_{t\to\infty}[\ E[sup\ R_1^{2k}] \cdots E[sup\ R_r^{2k}]\ ]$ exists, where $R_1 \cdots R_r$ are the roots of the eqn(3.1.1). For SISO systems Lyapunov stability is equivalent to input-output stability.

Remark 5:

A sufficient condition for the system to be stable is

$$E[R_1^2] < 1 \cdots E[R_r^2] < 1 \tag{3.2.2}$$

This is useful as an inverse criterion for the stability analysis. When conditions given in eqn (3.2.2) are not met the system will be unstable.

In conclusion the methods developed (Zhu and Billings 1990) allow the user to check analytically in which input amplitude range an identified model is stable for the class of nonlinear systems considered. The stability analysis for the class of nonlinear systems, which are nonlinear in the output variable is much more difficult than the problem considered here. This is more general problem in nonlinear integro-differential equation stability analysis.

3.3 Examples

Consider the simple first order system

$$y(k) = \alpha_1(u)y(k-1) + U(k) \tag{3.3.1}$$

The Z transform of the characteristic equation is

$$z - \alpha_1(u) = 0 \tag{3.3.2}$$

The root R_1 is given by

$$R_1 = \alpha_1(u) \tag{3.3.3}$$

Then for a stable system we require

$$|R_1| = |\alpha_1(u)| < 1 \tag{3.3.4}$$

Example 1

$$\begin{aligned} y(k) &= 0.5y(k-1)u(k-1) + u(k) \\ &= \alpha_1(u)y(k-1) + U(k) \end{aligned} \tag{3.3.5}$$

where

$$\alpha_1(u) = 0.5u(k-1), \qquad U(k) = u(k) \tag{3.3.6}$$

Example 2

$$\begin{aligned} y(k) &= \frac{1}{1 + u(k-1)}\{0.8y(k-1)u(k-1) + u(k)\} \\ &= \alpha_1(u)y(k-1) + U(k) \end{aligned} \tag{3.3.7}$$

where

$$\alpha_1(u) = \frac{0.8u(k-1)}{1 + u(k-1)}, \qquad U(k) = \frac{u(k)}{1 + u(k-1)} \tag{3.3.8}$$

Example 3

$$y(k) = (u^2(k-1) + u^2(k-2))y(k-1) + 3u(k)$$
$$= \alpha_1(u)y(k-1) + U(k) \tag{3.3.9}$$

where

$$\alpha_1(u) = u^2(k-1) + u^2(k-2), \qquad U(k) = 3u(k) \tag{3.3.10}$$

The regions where condition of eqn (3.3.4) holds are shown as the shaded parts in Figure 1 (a) to (c). The analysis indicates that the output-affine system, shown in example two, has a wide stability range, it is possible to make the system globally stable by chosing $|a(\ \alpha_1(u)\ )| < |b(\ \alpha_1(u)\ )|$, where $a(.)$ and $b(.)$ denote the parameters in the numerator and denominator respectively. For the systems, shown in examples one and three, there is at least one unstable area.

Now consider the more complicated case

$$y(k) = \alpha_1(u)y(k-1) + \alpha_2(u)y(k-2) + U(k) \tag{3.3.11}$$

The Z transform of the characteristic equation is

$$z^2 - z\ \alpha_1(u) - \alpha_2(u) = 0 \tag{3.3.12}$$

The two roots R_1, R_2 can be calculated by

$$(R_1, R_2) = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \tag{3.3.13}$$

Then for a stable system we require

$$|R_1| = \left|\frac{\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}}{2}\right| < 1$$
$$|R_2| = \left|\frac{\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}}{2}\right| < 1 \tag{3.3.14}$$

Example 4

$$\alpha_1(u) = u(k-1), \quad \alpha_2(u) = \frac{1}{2}u(k-1)u(k-2) + \frac{1}{4}u^2(k-2) \tag{3.3.15}$$

The two roots are calculated by eqn (3.3.13)

$$.R_1 = (2u(k-1) + u(k-2))/2$$
$$R_2 = (u(k-2))/2 \tag{3.3.16}$$

The stable region is shown as the shaded part in Figure 1 (d)

Example 5

$$\alpha_1(u) = 2sin(u(k)), \quad \alpha_2(u) = \cos^2(u(k)) \tag{3.3.17}$$

The two roots are calculated by eqn (3.3.13)

$$R_1 = sin(u(k)) + 1$$

$$R_2 = sin(u(k)) - 1 \tag{3.3.18}$$

This system is unstable because no solution can be obtained from the set of inequality equations $|R_1| < 1$, $|R_2| < 1$, this is shown in Figure 1 (e) which has no shaded parts or because when $|R_1| < 1$ $(u(k) \in (2t\pi, (2t \pm 1)\pi, t = 0, 1, \cdots)$ the other root becomes $|R_2| > 1$ and, when $|R_2| < 1$ $(u(k) \in ((2t+1)\pi, (2t \pm 2)\pi, t = 0, 1, \cdots)$ the other root becomes $|R_1| > 1$.

For the general case given in eqn (3.1.2), that is the Z transform of the characteristic equation of eqn (3.1.1), some well known stability analysis methods such as Jury's criterion for linear discrete time systems can be used to check the stability.

4 Identification

In this section the identification of the polynomial model and the rational model are unified and compared.

4.1 The generalized stochastic NARMAX model

In a practical environment some uncertain behaviours or stochastic phenomena are often encounted and model fitting based on stochastic data will be necessary. The generalization or extension of the stochastic NARMAX model can be considered in three stages, the first and fundamental sub-model set is the polynomial formulation (Leontaritis and Billings 1985), the second extension is the model set including exponential, absolute value, logarithmic, and trigonometrical terms the extended model set (Billings and Chen 1989a), and the third extension is the introduction of the rational model (Billings and Chen 1989b, Billings and Zhu 1991). The first two sub-model sets are naturally characterised as linear in the parameters.

Consider the extension of the NARMAX model of eqn (2.1.1) to the following NARMAX model of the form

$$y(k) = F(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r),$$
$$e(k-1), \cdots, e(k-r), e_1(k), \cdots, e_{n_l}(k)) + e(k) \tag{4.1.1}$$

Note unlike previous studies current noise terms (eg $e_j(k)$ etc) which may be induced by different noise effects on different variables or regressors have been included on the right hand side. To understand the distinctive noise characteristics of eqn (4.1.1) compared to the previous model two block diagrams are shown in Figure 2. The model of eqn (2.1.1) shown in Figure 2(a) assumes all the current noise sources can be lumped together but the enhanced model of eqn (4.1.1), shown in Figure 2(b), allows for different noise sequences on different variables. The equivalent linear in the parameters expression of eqn (4.1.1) is

$$Y(k) = \sum_{j=1}^{m} \phi_j(k)\theta_j + e(k) \tag{4.1.2}$$

where

$$\phi_j(k) = p_j(k)(v_j(k) + e_j(k)) \tag{4.1.3}$$

$p_j(k) = p_j(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$ and $v_j(k) = v_j(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))$. Both these latter expressions may contain exponential, absolute value, logarithmic, trigonometrical, or other functions, $e_j(k)$ is the current independent noise with zero mean and finite variance σ_{ej}^2. Eqn (4.1.2) can be expressed as

$$Y(k) = \sum_{j=1}^{m} (p_j(k)\ v_j(k) + p_j(k)\ e_j(k))\theta_j + e(k) \tag{4.1.4}$$

In the sense of system identification, eqn (4.1.4) represents the problem of detecting model structure and estimating the unknown parameters when the measurements of different variables as well as the output are noise contaminated. The inclusion of current noise terms in both input and output is frequently realistic and these noises could be induced by sampling errors, human errors, modelling errors or instrument errors. These problems arise in a broad class of scientific disciplines such as signal processing, automatic control, system identification and in general engineering, statistics, physics, economics, biology, and medicine.

4.2 A unified least squares algorithm

The NARMAX model identification consists of the following steps

(i) Model based term selection

(ii) Parameter estimation

(iii) Model validation

The first two steps can be based on a least squares type algorithm. The model term selection can be obtained using the orthogonal estimation algorithm (Billings and Chen 1989b) which selects the significant terms according to the contribution that each makes to the estimated noise variance. The parameter estimation is normally achieved by either a conventional least squares estimator or the orthogonal estimator but correlated noise must be accommodated if bias is to be avoided. Model validation tests the results obtained from the algorithms. By considering the extended model of eqn (4.1.4) a unified least squares algorithm can be derived which gives a structure detection and parameter estimator for all the different NARMAX model sets. Writting eqn (4.1.4) in vector notation

$$\vec{Y} = (PV + PE)\Theta + \vec{e} \tag{4.2.1}$$

where

$$\vec{Y} = [Y(1) \cdots Y(N)]^T \tag{4.2.2}$$

N is the data length, and

$$PV = \begin{bmatrix} p_1(1)v_1(1) & \ldots & p_{m}(1)v_{m}(1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ p_1(N)v_1(N) & \ldots & p_{m}(N)v_{m}(N) \end{bmatrix}$$

$$PE = \begin{bmatrix} p_1(1)e_1(1) & \ldots & p_{m}(1)e_{m}(1) \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ p_1(N)e_1(N) & \ldots & p_{m}(N)e_{m}(N) \end{bmatrix}$$

$$\Theta = [\theta_1 \cdots \theta_{m}]^T$$

$$\vec{e} = [e(1) \cdots e(N)]^T \tag{4.2.3}$$

Let

$$\Phi = PV + PE \tag{4.2.4}$$

Then the formal least squares parameter estimator is

$$\hat{\Theta} = [\Phi^T\Phi]^{-1}\Phi^T\vec{Y}$$

$$= [[PV]^T PV + [PE]^T PE]^{-1} \ [[PV]^T\vec{Y} + [PE]^T\vec{Y}] \tag{4.2.5}$$

where, by the probability limit property (Wilks 1962),

$$\frac{1}{N}\,[PV]^T PE \approx Plim[\,\frac{1}{N}\,[PV]^T PE\,] = 0 \tag{4.2.6}$$

because $e_j(k)$ in PE is an independent zero mean noise and $Plim[f]$ denotes the probability limit of f. Define

$$Bias_1 = Plim[\,\frac{1}{N}\,[PE]^T PE\,] \approx \frac{1}{N}\,[PE]^T PE$$

$$Bias_2 = Plim[\,\frac{1}{N}\,[PE]^T \vec{Y}\,] \approx \frac{1}{N}\,[PE]^T \vec{Y} \tag{4.2.7}$$

which can be thought of as the auto-correlation of the terms and errors, and the cross-correlation errors between the output and terms.

According to the analysis above the unbiased least squares estimate of the parameters for the model of eqn (4.1.4) is

$$\hat{\Theta} = [\Phi^T\Phi - [PE]^T PE]^{-1}\,[\Phi^T\vec{Y} - [PE]^T\vec{Y}] \tag{4.2.8}$$

The famous orthogonal least squares algorithm can also be applied to such an expression by transforming $\Phi^T\Phi$ into an orthogonal normal matrix with appropriate corrections to the normal matrix and correlation vector (Zhu and Billings 1992).

4.3 Polynomial and rational model identification

For the polynomial model eqn (4.2.8)

$$\hat{\Theta} = [\Phi^T\Phi - [PE]^T PE]^{-1}\,[\Phi^T\vec{Y} - [PE]^T\vec{Y}]$$

reduces to

$$\hat{\Theta} = [\Phi^T\Phi]^{-1}\,\Phi^T\vec{Y} \tag{4.3.1}$$

which is a straightforward unbiased estimator. This follows because the matrix $PE = 0$ since which the polynomial model only includes current noise at the output such that the normal matrix is current noise free. The covariance of the estimator is given by

$$Cov\hat{\Theta} = \sigma_e^2\,[\Phi^T\Phi]^{-1} \tag{4.3.2}$$

This algorithm and various alternatives to it have been extensively studied (Billings and Chen 1989a, Chen and Billings 1989).

For the rational model, the algorithm reduces to the rational model estimator

$$\begin{aligned}\hat{\Theta} &= [\Phi^T\Phi - [PE]^T PE]^{-1}\,[\Phi^T\vec{Y} - [PE]^T\vec{Y}]\\ &= [\Phi^T\Phi - \sigma_e^2\,\Psi]^{-1}\,[\Phi^T\vec{Y} - \sigma_e^2\,\psi]\end{aligned} \tag{4.3.3}$$

where $[PE]^T PE = \sigma_e^2\,\Psi$, $[PE]^T \vec{Y} = \sigma_e^2\,\psi$, and σ_e^2 is the noise variance. This algorithm was introduced as a new method of identification for the rational model (Billings and Zhu 1991, 1992, Zhu and Billings 1991, 1992). The covariance matrix of the algorithm was derived by Zhu and Billings (1991)

$$Cov\hat{\Theta} = \sigma_e^2\,\sigma_b^2\,[\Phi^T\Phi - \sigma_e^2\,\Psi]^{-1}|_{\sigma_e^2<1} \tag{4.3.4}$$

where σ_b^2 is the denominator variance of the rational model.

Consider a simple rational model to illustrate the algorithm

$$y(k) = \frac{a(k)}{b(k)} = \frac{a_1\,u(k-1)e(k-1)}{1 + b_1\,y^2(k-1)} + e(k) \tag{4.3.5}$$

The linear in the parameters expression is given by multiplying $b(k)$ on both sides of eqn (4.3.5) and then moving all the terms except $y(k)$ to the right hand side

$$Y(k) = a_1\,u(k-1)e(k-1) - b_1\,y^2(k-1)y(k) + b(k)e(k) \tag{4.3.6}$$

where

$$\hat{\Theta} = [\ \hat{a}_1,\ \hat{b}_1\ ]^T$$

$$\Phi = \begin{bmatrix} u(0)e(0) & y^2(0)y(1) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ u(N-1)e(N-1) & y^2(N-1)y(N) \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{k=1}^{N} y^4(k-1) \end{bmatrix}$$

$$\psi = \begin{bmatrix} 0 \\ -\sum_{k=1}^{N} y^2(k-1) \end{bmatrix}$$

$$\vec{Y} = [\ y(1), \cdots, y(N)\ ]^T \tag{4.3.7}$$

Unbiased parameter estimates can be obtained by substituting eqn (4.3.7) into eqn (4.3.4).

Inspection of eqn (4.3.1) and eqn (4.3.4) shows that the critical difference between the algorithms for the polynomial and rational models is the current noise terms which are induced in the later model.

5 Conclusions

A comparison of model representations, stability, and identification has shown that although the rational model often exhibits superior approximation properties it is much more difficult to identify compared with the polynomial model.

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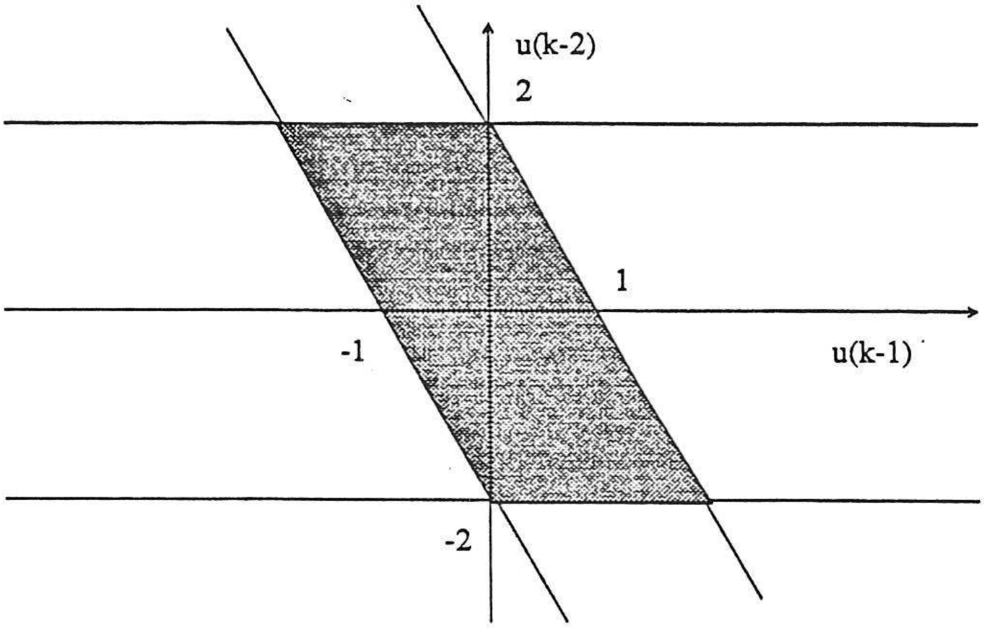
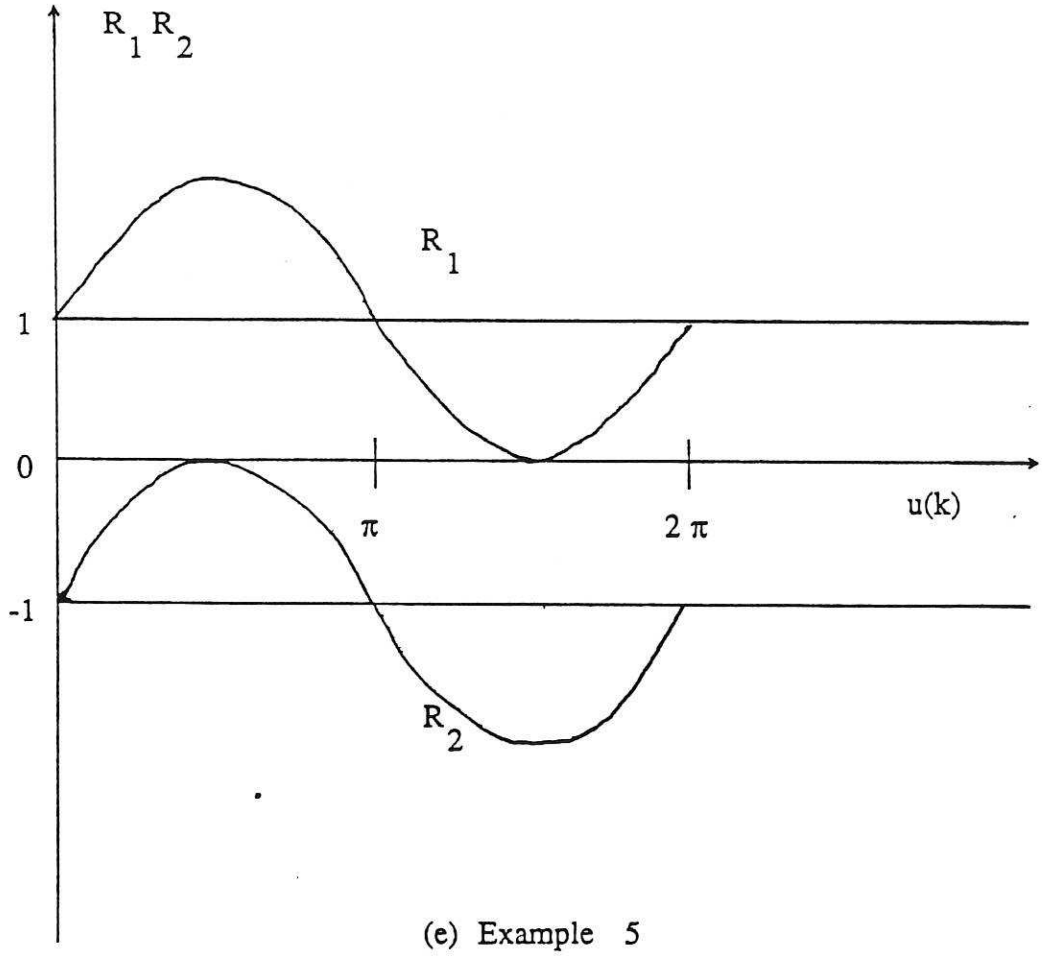
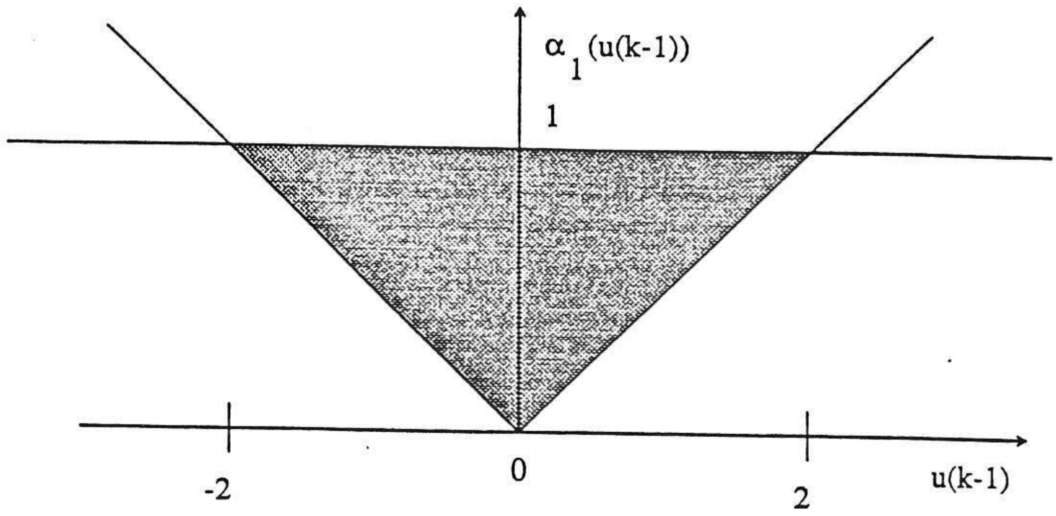


(d) Example 4

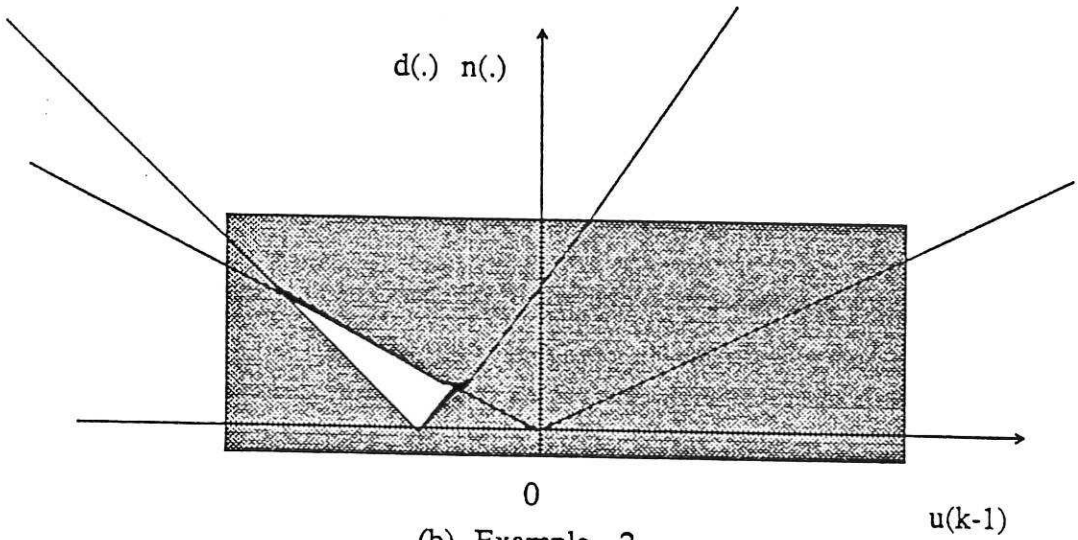


(e) Example 5

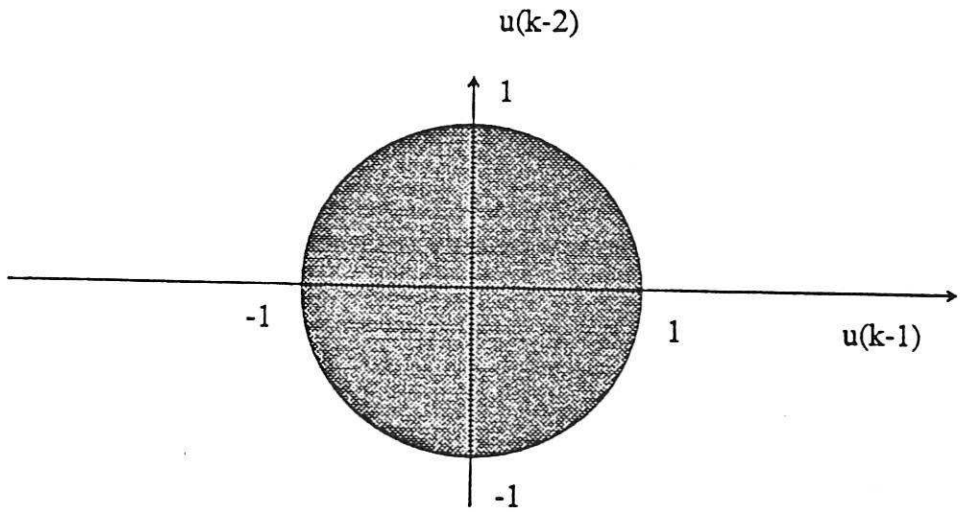
Figure 1 System stability characteristics as a function of input



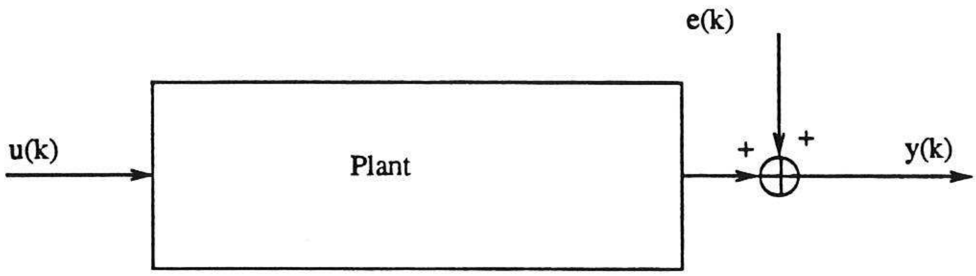
(a) Example 1



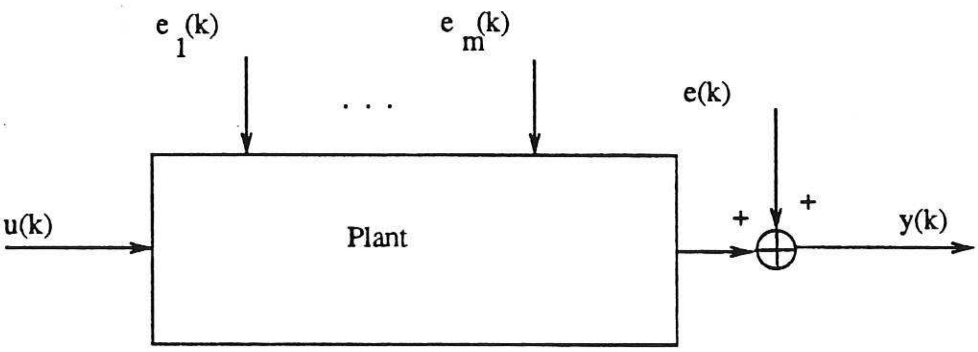
(b) Example 2



(c) Example 3



(a) Current noise at the output



(b) Current noise at the input and output

Figure 2 Block diagrams for NARMAX models