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COMPUTER AIDED ANALYSIS OF NONLINEAR SYSTEMS

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CHAPTER 6

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I. INTRODUCTION

In general the method of analysis applied to any nonlinear system depends to a large extent on the structure of the model representation. To date little has been done to study the more qualitative aspects of the type of nonlinear models produced as a result of control system modelling and system identification. Analytical methods meet some of the requirements for certain classes of models. Such methods however, tend to be valid for only one particular model structure or type. For instance dynamical system theorists have studied the general nonlinear ODE as a dynamical system in its own right, see for example Mees (1981), where the general form of system representation is taken to be

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mu) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \qquad \mu \in \mathbf{R}^{n_\mu} \tag{1}$$

where the vector μ represents some quasi static parameter set. Fundamental results, both analytical and geometric in nature have been provided for (1) and these now form the basis for much of the work in this area, see for example Guckenheimer and Holmes (1983). On the other hand the attention of the control theorists has, to a large extent, been constrained to less general model forms. This is largely as a result of a desire to solve specific problems in system theory. Fundamental results in this area were obtained by Sandberg (1964) and Zames (1963, 1964). An algebraic theory for the representation, realisation and analysis of nonlinear control systems has since developed (Mayne and Brockett, 1973). In this work bilinear models have attracted a lot of attention largely because they are more amenable to analysis. Many concrete examples of bilinear systems do exist, see for example Mohler (1970, 1977). Subsequent work based around these systems spawned a more general class of system known as the linear analytic system. These take the form

$$\begin{aligned} \dot{\mathbf{x}}(t) &= f(\mathbf{x}(t),t) + \mathbf{u}(t)\, g(\mathbf{x}(t)) + \mathbf{u}(t) \qquad \mathbf{u} \in \mathbf{R}^{n_u} \\ \mathbf{y}(t) &= h(\mathbf{x}(t),t) \end{aligned} \tag{2}$$

Realisation theory for these systems was provided by Brockett (1972), D'Alessandro, Isidori and Ruberti (1974), Sussmann (1977), Fliess (1982) and Jakubczyk (1987). An algebraic approach to the analysis of the discrete time equivalent of (2) has been pursued by Normand-Cyrot and

Monaco (1984). Realisation theory for discrete systems was supplied by Clancy and Rugh (1978), Sontag (1979) and Schwartz and Dickinson (1986). Additionally extensions to the familiar concepts found in system theory and control have been provided by many authors, see for example Hermann and Krener (1977), Brockett (1981), Rugh (1981), Isidori (1985), Desoer and Lin (1985), Byrnes and Lindquist (1986), Fliess and Hazewinkel (1986), Vidyasagar (1986) and Isidori (1989).

The discrete equivalent of (1), sometimes referred to as a *recurrence* or *mapping*, has been shown to exhibit a richer set of characteristics than its continuous counterpart and spawned at least an equal, if not greater, number of analysis techniques. Discrete maps have been extensively studied as dynamical systems in themselves, prompted initially by the work of Li and Yorke (1975). Here the general model representation takes the form.

$$x_{k+1} = g(x_k, \mu) \qquad x \in R^{n_x} \; ; \; \mu \in R^{n_\mu} \tag{3}$$

where μ is once again some parameter vector. Extensive effort has gone into trying to classify the recurrent behaviour of (3), see for instance Bernussou (1977), Guckenheimer (1979), Iooss (1979), Gumowski and Mira (1980), Collet and Eckmann (1980), Preston (1983), Salvadori (1984) or Holmes and Whitley (1984).

From the control viewpoint the main application of the dynamical systems approach has been the analysis of chaos in feedback systems, see for instance Baillieul, Brockett and Washburn (1980), Sparrow (1980, 1981), Holmes (1983), Cook (1985), Ushio and Hirai (1983) and Salam (1985). Success in this area has not however had such widespread coverage as other areas of exploration, in particular nonlinear circuit theory, see for example Chua and Lin (1990). This may be partly due to the flourishing interest by control theorists in the algebraic and geometric approach. Notable exceptions that make direct use of the qualitative approach include Mehra (1976) and Sparrow (1980, 1981) on bifurcation free control; by Aeyels (1985) and Abed (1986, 1987) on constructing stabilising feedback control for continuous systems; by Hahn (1985), using describing functions; by Chang and Chen (1984) who considered PID control; and by Salam and Bai (1986), Ydstie and Golden (1988) and Mareels and Bitmead (1988) who have considered bifurcation in adaptive control systems.

In control studies the augmentation of the system model with one or more parameters enables the study of the systems behaviour over a specified range of parameter values. It is important to see how the behaviour of a system or model changes if the equations that make up that representation

change in some manner, if only because such models are seldom known accurately. In the general nonlinear setting this problem is placed within the framework of structural stability and bifurcation theory. Detailed knowledge of the models solution structure is then required in order to classify the complete *unfolding* of the behaviour to be expected (Golubitsky and Schaeffer, 1985 and Golubitsky, Stewart and Schaeffer, 1988). In control this problem appears under the title of robustness. The dynamical systems approach is attractive in that it provides information on the very type of qualitative behaviour the nonlinear model was constructed to emulate. In addition, the method becomes of use precisely at the point where traditional linear control theory breaks down, that is, when one or more linearised eigenvalues become degenerate.

Unfortunately these methods do have some drawbacks. First, being generally analytically based, they depend on a good deal of a priori knowledge of the solution structure of the system. In a general parameterised nonlinear model such information will not be available and indeed may be difficult to obtain. Secondly, most of the analytical approaches available require the model structure to take a particular form.

One possible solution to this problem, which we outline in this work, is to suspend the parametric models in a discrete, or *cellular*, state space. This allows the application of a particularly attractive and simple numerical algorithm which can be applied to a wide variety of model types whilst at the same time maintain the attractive qualitative aspects of the dynamical systems approach. Information on the systems stationary and periodic solution structure is provided along with both local and global stability characterisation over a predefined region of the systems state/parameter space. This information can be presented in much the same way as a traditional bifurcation diagram. Although the method itself does not attempt to identify any particular bifurcation behaviour, such characteristics can usually be identified by virtue of the changes detected in the systems solution structure.

As an added bonus, the analysis of an unknown system can proceed in an interactive manner whereby the parameter space of the problem is probed over specified parameter region for nonlinear characteristics. This probing approach has proved to be a powerful aid in the analysis of competing nonlinear model structures (Haynes and Billings, 1991).

II. GLOBAL ANALYSIS OF NONLINEAR SYSTEMS

The stability of a model, both absolute and structural, is dependent on the location and distribution of degenerate singularities on the solution manifold of that system. Analytical methods meet some of the requirements for certain classes of models. Such methods however, tend to be valid for only one particular model structure or type and tend to rely on constructing a reduced order system around a point where the *interesting* dynamics are deemed to be situated. In dynamical systems studies most analytical approaches tend to rely on a reduction method in order to reduce the dimension of the problem under consideration. Such methods, when employed, not only rely heavily on a priori knowledge of the solution structure of the problem, but even if successfully applied, are only locally valid in a small region about the degenerate eigenvalue. In discrete dynamical systems, where a profusion of periodic behaviour is all too common, constructing such a system is even more cumbersome. Obviously for application to the type of discrete models encountered in sampled data systems, system modelling and system identification a flexible approach that does not suffer these drawbacks is essential (Haynes and Billings, 1991).

Traditionally numerical analysts have used *path following* or *continuation* methods to trace out branches or arcs in the solution manifold of a system of nonlinear equations dependent upon one parameter. To illustrate the basic concept consider an autonomous system, the stationary solutions of which satisfy

$$\mathbf{f}(\mathbf{x}(t),\mu) = 0 \qquad \mathbf{x}\in\mathbf{R}^n \quad \mu\in\mathbf{R} \tag{4}$$

A smooth *branch* of solutions, $\mathbf{x}(s)$, is made up of a one parameter family of solutions to (1) or (3), where s is some arbitrary parameter. Fig. 1. shows some typical solution branches. The points μ_a and μ_b represent simple *bifurcations*. The point μ_c a *turning* point or *limit* point. At the point μ_d a multiple bifurcation occurs, where more than one eigenvalue or complex conjugate pair of eigenvalues become degenerate. The basic problem then is to compute large segments of the solution structure of (4), including the branches, as μ varies. A secondary problem also exists, that is the assignment of stability and the estimation of the corresponding influence domain within $\mathbf{R}^n$.

In the general case the stationary solutions will be smooth functions of μ. For $\mu\in\mathbf{R}$ this solution set forms a *path* in $\mathbf{R}^{n+1}$, for a two parameter family, $\mu\in\mathbf{R}^2$, a *surface*, and for $\mu\in\mathbf{R}^m$, where $m>2$, a *hypersurface* of solutions. If an initial point, $\mathbf{x}_0,\mu_0$, is known then a path following algorithm can

be used to trace out the branch (Keller, 1977). Given the general one parameter nonlinear equation

$$g(x,\mu) = 0 \qquad g: R^n \times R \rightarrow R^n \quad x \in R^n \quad \mu \in R \tag{5}$$

and a point x_0,μ_0 satisfying $g(x,\mu)=0$ consider the problem of calculating the solution set of (5) near x_0,μ_0. The mathematical basis for the path following method is the Implicit Function theorem. This ensures the existence of a smooth path of solutions $x=x(\mu)$ near x_0,μ_0, provided $g_x(x_0,\mu_0)$ is nonsingular, that is x_0,μ_0 is a regular point, where $g_x(x,\mu)$ is the usual $n \times n$ Jacobian matrix. Given a regular point x_0,μ_0 the problem is to compute a neighbouring point corresponding to $\mu=\mu_0+\Delta\mu$ where $\Delta\mu$ is a small increment in μ. A predictor corrector method is typically used. Differentiation of $g(x(\mu),\mu)$ with respect to μ gives

$$g_x \frac{\partial x}{\partial \mu} + g_\mu = 0 \tag{6}$$

where g_μ denotes the n vector with components

$$g_\mu(x,\mu) = \left[\frac{\partial g_1}{\partial \mu}, \ldots, \frac{\partial g_n}{\partial \mu} \right]^T \tag{7}$$

Rearranging and evaluating about x_0,μ_0 gives

$$\frac{\partial x}{\partial \mu}(x_0,\mu_0) = -\left[g_x^0\right]^{-1} g_\mu^0 \qquad g_x^0 = g_x(x_0,\mu_0) \quad g_\mu^0 = g_\mu(x_0,\mu_0) \tag{8}$$

A Euler predictor approximation x_p to $x(\mu_0+\Delta\mu)$ is then given by

$$x_p = x_0 + \Delta\mu \left(\frac{\partial x}{\partial \mu} \right)_0 \tag{9}$$

which is used as a starting value to solve the system.

$$g(x, \mu+\Delta\mu) = 0 \tag{10}$$

This simple Euler-Newton method works well and forms the basis of many computer codes. However, the path following method runs into trouble at points where g_x^0 is singular, in particular if the path $x(\mu)$ bends back upon itself, as in Fig. 1. at μ_c, then the method fails. Addition of an auxiliary equation to (5) is often used to circumvent this difficulty.

Many implementations of the above algorithm are in existence (Rheinboldt, 1986, Doedel, 1981, 1986, Brindley, Kass-Petersen and Spence, 1989). Unfortunately the available path following methods provide information on only the local stationary solutions. In addition a priori

knowledge is needed in the form of one or more points on a known solution branch in order to start tracing the solution arc. In order to detect any splitting at bifurcation points an additional branch switching algorithm must be employed. Furthermore continuation methods provide no information on the extent of the influence domains of the branches traced out. Indeed no account at all is taken of the possibility of disjoint or isolated solutions existing.

Under natural conditions on **f** the set of solutions of (5) constitutes a differentiable manifold in the product of state space and parameter space. The dimension of this manifold equals that of the parameter space. At present the standard path following computational methods require the user to construct a picture of a p-dimensional manifold from information along a 1-dimensional path, (Rheinboldt, 1988).

In a true nonlinear control design attention would center not so much on computing a few segments of this manifold, but rather in determining the form and special features of the entire solution manifold and designing control schemes or selecting system parameters to achieve some desired aim. Once a possible control had been identified robustness or parameter sensitivity, and indeed structural stability would be considered by allowing variations in the nominal parameter set. In general this is not easy to achieve and can lead to misinterpretation of the solution structure.

In this work a dual approach has been adopted specifically with this aim in mind. This combines the essentially qualitative ideas of Bifurcation Theory with a simple yet attractive numerical algorithm. The analysis of nonlinear systems using a *cell map* approximation was first carried out by Hsu, see for example Hsu and Guttalu (1980). This method has been extended to the qualitative analysis of general nonlinear parameterised models such as (1) and (3). The approach proves attractive for a number of reasons. Firstly, it provides a method of enumerating both the stationary and periodic solution structure of the system over a given parameter range. Secondly, it has been shown to detect all of the typically found bifurcation phenomena (Haynes and Billings, 1992). Thirdly, and more importantly, information of a global nature is provided on the extent of the systems stability domains. As a result both local and global behaviour can be studied with little need for a priori information.

III. CELL MAPPINGS SYSTEMS

The process of analysing a system using cell map analysis comprises of a number of steps. The first is the suspension of the nonlinear system, (1) or (3), in a *cell state space,* $\mathbf{Z}^n$. This is an n-dimensional space whose elements are n-tuples of integers. Each element is called a *cell vector,* or simply a *cell*, and is denoted by $\mathbf{z}$. There are many ways to obtain a cell structure over a given euclidean state space, (Hsu, 1987). The simplest way, which we make use of here, is to construct a cell structure consisting of rectangular parallelepipeds of uniform size (squares, cubes etc). Let x_i, $i = 1,..,n$ be the state variables and let each coordinate axis of the state variable be divided into a number N_i, of intervals of uniform size h_i. The interval z_i along the x_i axis is defined such that it covers all the x_i of interest and

$$(z_i - \frac{1}{2})h_i \le x_i \le (z_i + \frac{1}{2})h_i \qquad z_i = 1,2,..,N_i \tag{11}$$

The n-tuple z_i, $i = 1,..,n$ is then called the cell vector, denoted by $\mathbf{z}$. A point $\mathbf{x}$ belongs to a cell $\mathbf{z}$ iff x_i and z_i satisfy (11) $\forall i \in [1,n]$. Each cell $\mathbf{z}$ is considered as a *cell entity* and the entire collection of cells as the *cell state space.* Consider now the mapping between two cells $z(j)$ and $z(j+1)$, where $j = 1,2,..$ is used to denote a sequence in the same manner as the iterates of a mapping. The *cell map,* $\mathbf{C}(\mathbf{z})$, is a mapping of a set of integers $\{N+\}$ such that

$$\mathbf{z}(j+1) = \mathbf{C}(\mathbf{z}(j)) \qquad \mathbf{z}(j) \in \mathbf{Z}^n \subset \mathbf{S} \tag{12}$$

The *cell function,* $\mathbf{F}(\mathbf{z},\mathbf{C})$, is then defined as

$$\mathbf{F}(\mathbf{z},\mathbf{C}) = \mathbf{C}(\mathbf{z}) - \mathbf{z} \tag{13}$$

A *singular cell* $\mathbf{z}^*$ as a cell satisfying the relationship

$$\mathbf{F}(\mathbf{z}^*,\mathbf{C}) = 0 \qquad \text{or} \qquad \mathbf{z}^* = \mathbf{C}(\mathbf{z}^*) \tag{14}$$

A *periodic cell cycle,* given that $\mathbf{C}^0(\mathbf{z})$ denotes the identity mapping, is a sequence of K distinct cells $\mathbf{z}^*(j)$, $j = 1,..,K$, K being the minimum value which satisfies

$$\mathbf{z}^*(m+1) = \mathbf{C}^m(\mathbf{z}^*(1)) \qquad m = 1,..,K-1 \qquad \mathbf{z}^*(1) = \mathbf{C}^K(\mathbf{z}^*(1)) \tag{15}$$

Each element of the periodic cycle is a *periodic cell.* The complete cell cycle is labelled as P-K. Additionally those cells eventually mapped onto the P-K cycle by (12) are defined as within the *domain of attraction*, or DOA, of the cell cycle and labelled as the DOA-K cells.

The size of the cell state space is determined by the system itself. For most practical systems there are ranges of values of the state variable beyond which we are no longer interested. This means that there is only a finite region of the state space which is of concern. Similarly for a dynamical system governed by a cell mapping there is only a finite region of cell space of interest, and correspondingly a finite number, N_c, of cells, named *regular cells*. The *sink cell* is used to encompasses all possible cells outside the region of interest. If the mapped image of a regular cell lies outside the region of interest it is then said to be mapped into the sink cell. The regular cells are labelled by positive integers $\{1,2,..,N_c\}$. The sink cell is labelled as $\{0\}$, the zero cell. This makes the total number of cells N_c+1, such that $\mathbf{S}=\{N_c+\}$. The set, **S**, is closed under the mapping described by

$$\begin{aligned} \mathbf{z}(j+1) &= \mathbf{C}(\mathbf{z}(j)) & \mathbf{z}(j), \mathbf{z}(j+1) &\in \mathbf{Z}^n \subset \mathbf{S} \\ \mathbf{C}(0) &= 0 & \mathbf{S} &= \{N_c+\} \end{aligned} \tag{16}$$

The sink cell, **C**(0), is a P-1 cell. The set of regular cells within the influence domain of the sink cell, these being eventually mapped to **C**(0), are in the domain of attraction of the sink cell, and labelled the DOA-Sink cells.

IV. CELL MAPPING DISCRETISATION

The cell map, **C**(**z**), system may be considered as a discrete system similar to the point mapping

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \tag{17}$$

In order to make use of the qualitative ideas provided by dynamical systems theory it is necessary to extend the algorithm of Hsu to the analysis of a more general parameterised nonlinear system (Haynes and Billings, 1992).

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k, \mu) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \quad \mu \in \mathbf{R}^{n_\mu} \tag{18}$$

Applying the center point method of discretisation requires the division of $\mathbf{R}^{n_x}$ into a collection of cells according to (11) and the calculation of each cells center point $\mathbf{x}^{(d)}(j)$ such that

$$x_i^{(d)}(j) = x_i^{(l)} + h_i z_i(j) - \frac{h_i}{2} \qquad z_i(j)=1,..,N_i \quad j=1,..,N_i \quad i=1,..,n_x \tag{19}$$

where h_i is the cell size and $x_i^{(l)}$ the lower bound defining the region of interest such that $\mathbf{x}^{(l)} \le \mathbf{x} \le \mathbf{x}^{(h)}$, where $\mathbf{x}^{(l)} = (x_1^{(l)},..,x_{n_x}^{(l)})$ and $\mathbf{x}^{(h)} = (x_1^{(h)},..,x_{n_x}^{(h)})$. Similarly $\mathbf{R}^{n_\mu}$ is discretised using

$$\mu_i^{(d)}(j) = \mu_i^{(l)} + g_i\, z_i(j) - \frac{g_i}{2} \qquad z_i(j)=1,..,N_i \quad j=1,..,N_i \quad i=n_x+1,..,n_x+n_\mu \tag{20}$$

where g_i is the cell size over the region defined by $\mu^{(l)} \le \mu \le \mu^{(h)}$ where $\mu^{(l)} = (\mu_1^{(l)},..,\mu_{n_\mu}^{(l)})$ and $\mu^{(h)} = (\mu_1^{(h)},..,\mu_{n_\mu}^{(h)})$. The point mapping or image of the center point $\mathbf{x}^{(d)}(j)$ is then calculated using (18) such that

$$\mathbf{x}_{k+1}^{(d)} = \mathbf{g}(\mathbf{x}_k^{(d)} , \mu^{(d)}) \tag{21}$$

and the cell map $\mathbf{C}(\mathbf{z})$ can be constructed by determining the image of each cell within S using

$$C_i(z_i(j)) = z_i(j+1) = \mathrm{INT}\left[\frac{x_i^{(d)}(k+1) - x_i^{(l)}}{h_i} + 1 \right] \qquad i = 1,..,n_x \quad j = 1,..,N_i$$

$$C_i(z_i(j)) = z_i(j) \qquad i = n_x+1,..,n_x+n_\mu \quad j = 1,..,N_i \tag{22}$$

Note we are in effect constructing n_μ separate cell mappings of dimension n_x, each representing a *slice* within the parameter space $\mu = (\mu_1,..,\mu_{n_\mu})^T$.

A similar approach to the construction of the cell map system may be taken for any continuous nonlinear model that can be written in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t) , \mu) \qquad \mathbf{x} \in \mathbf{R}^n \quad \mu \in \mathbf{R}^m \tag{23}$$

It is straight forward to approximate the image of a cells center point for (23). Take a point $\mathbf{x}_k^{(d)}$, $\mu^{(d)}$ within $\mathbf{x} \in \mathbf{R}^n$, fix values for the parameters $\mu^{(d)} \in \mathbf{R}^m$, choose a time interval τ and calculate the resulting trajectory for $t = 0-\tau$ to give $\mathbf{x}_{k+1}^{(d)}$, $\mu^{(d)}$. All this requires is a suitable integration scheme to solve (21) given (23)

$$\mathbf{x}_{k+1}^{(d)} = \mathbf{x}(\tau)^{(d)} = \mathbf{x}(0)^{(d)} + \int_0^\tau \mathbf{f}(\mathbf{x}^{(d)}(t) , \mu^{(d)})\, dt \tag{24}$$

The only additional information needed is the time step τ and an integration step size. T is chosen as would be normal for the problem, bearing in mind the choice of integration method. The time step τ is dependent on the cell size and should be chosen such that image of the center point of a cell lies, on average, within a neighbouring cell. This obviously depends upon the strength of the local vector field and the cell size chosen. If the interior of the cell contains an attracting equilibria, the trajectory described by (21) will remain within that cell. An interactive approach to the choice of τ is best adopted. Generally choosing τ to be 3-15 times the integration interval T gives good results. Repeating the above calculations for each cell within S and applying (22-24) approximates the cell map $\mathbf{C}(\mathbf{z})$.

V. CELL MAP ANALYSIS

Having constructed $\mathbf{C(z)}$ the classification of all cells within $\mathbf{S}$ is carried out using a modified version of the *unravelling algorithm* (Hsu and Guttalu, 1980). The algorithm involves calling up each cell in turn and processing it in order to determine its global characteristics. Each cell is then classified accordingly.

A cell may be a singular cell, P-1, or a periodic cell cycle, P-K, satisfying (5) or (6). The set of all such cells make up the invariant orbits within $\mathbf{Z}^n$. When $K=1$ a fixed point has been located, when $K>1$ a periodic solution or limit cycle has been detected. Alternatively a cell may simply be a regular cell in the DOA of a P-K cell. Each such cell is then said to be in the same *group* and have the same *periodicity* as that cell cycle and is labelled accordingly. Finally a cell may be mapped by (21) outside the region of interest into the Sink cell. Such a cell is then said to be in the domain of attraction of the sink cell, the DOA-Sink, and is labelled accordingly. For more detail on the unravelling algorithm and is variants see Hsu (1987) or Haynes and Billings (1992).

The advantage of the parameterised cell map approach outlined above is that it combines the qualitative aspects of dynamical systems theory with the global aspect afforded by the unravelling algorithm. In implementing the above algorithm many variations are possible. The version developed in this work focuses on the problems of enumerating the global characteristics of a wide variety of parameterised nonlinear systems. One aim throughout this work has been to maintain an interactive aspect to the algorithm. This enables the analyst to probe the dynamics of a problem by varying both the extent and form of the cell state space. In order to facilitate this the algorithms outlined here have been packaged into a prototype CAD tool that enables the probing and analysis of a wide variety of nonlinear systems.

VI. ANALYSIS OF A NONLINEAR FEEDBACK SYSTEM

In a nonlinear system the type of bifurcation characteristic obtained can often depend on the bifurcational parameter selected. In real problems, where there may be a number of parameters of interest the selection of the primary parameter may be critical. Two or more parameters may interact strongly, requiring greater powers of analysis to be applied to the problem. In the following example a parameterised cell map system is constructed for the nonlinear feedback system depicted in Fig. 2. described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{F}(\mathbf{x}(t)) + \mathbf{B}\,\mathbf{u}(t) \qquad x\in\mathbf{R}^{n_x}\ \ u\in\mathbf{R}^{n_u}\ \ y\in\mathbf{R}^{n_y}\ \ \mathbf{F}:\mathbf{R}^{n_x}\rightarrow\mathbf{R}^{n_x}$$

$$\mathbf{y}(t) = \mathbf{C}\,\mathbf{x}(t) \qquad \mathbf{A}: n_x\times n_x\ \ \mathbf{B}: n_x\times n_u\ \ \mathbf{C}: n_y\times n_x$$

$$\mathbf{u}(t) = \mathbf{r} - \mathbf{G}\,\mathbf{x}(t) \qquad r\in\mathbf{R}^{n_u}\ \ \mathbf{G}: n_u\times n_x \tag{25}$$

Choosing $n_x=2$, $n_u=1$ and $n_y=1$ gives a SISO system with second order dynamics for the plant element. Note in this example we are dealing with a linear plant albeit as part of a nonlinear feedback loop. It should be noted that we could just as well be considering a nonlinear plant characteristic and are by no means limited to the model structure depicted in Fig. 2.

In this problem the main focus of our attention is on determining what influence the control matrix G and the set point vector **r** have on the overall system characteristic. In order to simplify this example further assume that the feedback matrix comprises just a single parameter and the nonlinear element is a simple cubic polynomial

$$\mathbf{G} = \begin{bmatrix} k & -k \end{bmatrix} \qquad F(\mathbf{x}) = \begin{bmatrix} 2(x_1 - x_2)^3 \\ 0 \end{bmatrix} \tag{26}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{27}$$

so that system set point, r, and feedback gain, k, become the chosen system parameters.

The characteristic of this system is dominated by the system gain k. This is easily demonstrated by performing a set of simple step tests. Large gain k, or large set point values r, result in an oscillatory response. As it stands the system can be analysed directly, both Hopf Bifurcation theory (Marsden and MaCracken, 1976) and Describing Function methods (Hahn, 1985) have been applied with some success.

A. Cell Map Analysis $\mu=k,\ r=0$

Consider the cell map analysis of the system, writing (25) in the form of (23) gives

$$\dot{\mathbf{x}}(t) = F(\mathbf{x},\mathbf{r},\mathbf{k}) \qquad \mathbf{x}\in\mathbf{R}^2 \quad \mathbf{r}\in\mathbf{R} \quad \mathbf{k}\in\mathbf{R} \tag{28}$$

Setting $r=0$ reduces the dimension of the parameter vector to one, further simplifying the problem, and leaving the primary parameter, $\mu=k$, the system gain. Thus

$$\dot{\mathbf{x}}(t) = F(\mathbf{x},\mu) \qquad \mu = k \quad r = 0.0 \quad \mathbf{x}\in\mathbf{R}^2 \quad \mu\in\mathbf{R} \tag{29}$$

It is now necessary to define the region of interest, that is the range over which to construct the cell map (12). Choosing $x_1 \in [-1,1]$, $x_2 \in [-1,1]$, $\mu \in [-0,2]$ and the number of cells to be 175x175x30 cells allows the application of (19-24) to give **C(z)**. Subsequent application of the unravelling algorithm gives rise to the cell entities depicted in Fig. 3 This diagram provides almost complete characterisation of the systems behaviour over the parameter range $0.0 \leq k \leq 2.0$ for $r = 0.0$. Within this range the shaded areas represent the stable DOA's of a complex sequence of equilibria indicated by the dotted curves. These curves are made up of either P-1 singular cells (fixed points) or P-K cell cycles (limit cycles) that have been detected by the unravelling algorithm. The entire DOA comprises the set of all cells that are ultimately attracted to either a stable P-1 or P-K asymptote within this region.

For $0.0 \leq k \leq 1.0$ a number of P-1 cells are detected along the x-axis of the diagram. This implies a stable attractor exists over this parameter range which exerts an influence domain, the size of which is defined by the extent of the shaded DOA. The DOA forms a cone around the x-axis about which a number of longer period P-K cycles can be detected. These unstable cell cycles mark the separatrix between the stable DOA and the unstable sink cell, there form being more clearly depicted in the lower plots of Fig. 3.

At $k = 1$ a split in the solution structure is evident leading to the formation of two new stable asymptotes located symmetrically either side of the x-axis with a further, unstable, asymptote existing along the x-axis for $1.0 \leq k \leq 1.5$. This qualitative change in the systems characteristic implies a pitchfork bifurcation has occurred around $k = 1.0$ (Guckenheimer and Holmes, 1983). At $k = 1.5$ a further split in the solution structure occurs resulting in the formation of a stable P-K limit cycle for $1.5 \leq k \leq 2.0$. This qualitative change implies a Hopf type bifurcation has occurred. The lower diagrams of Fig. 3. emphasise this essentially 3-D characteristic which we christen the *feedback induced* behaviour for the system.

A number of points are worth emphasising when considering this analysis. First, very little a priori knowledge of the system characteristic is necessary in order to carry out the analysis. Yet a broad and complete picture of the systems characteristic can be quickly obtained. Secondly, it is quite feasible to alter the coarse cell size chosen so that more quantitative information can be

extracted from the cell diagrams at any particular point.

B. Cell Map Analysis $\mu = r$, $k = 0.5, 1.0, 1.25$

In order to determine the influence of the set point r on the overall system characteristic now choose the primary parameter $\mu = r$ and fix the value of feedback gain to $k = 0.5, 1.0$ and 1.25. The newly parameterised problem is suspended in the cell map framework such that

$$\dot{\mathbf{x}}(t) = F(\mathbf{x},\mu) \qquad \mu = r \quad k = 0.5-1.25 \quad \mathbf{x}\in\mathbf{R}^2 \quad \mu\in\mathbf{R} \tag{30}$$

Firstly fixing $k = 0.5$ and defining the cell state space over $x_1\in[-1,1]$, $x_2\in[-1,1]$, $\mu\in[-0.2,0.2]$, for 175x175x30 cells, allows the construction of a new cell map $C(\mathbf{z})$. Subsequent application of the unravelling algorithm results in the pattern of cell entities in Fig. 4.

Once again the upper plots show the distribution of P-K cycles but this time the shaded area represents the DOA of just the P-1 cycles, the DOA P-1. That is just those cells that are asymptotic to a fixed point. These vary with the input set point over the range $-0.2\leq r\leq 0.2$. About the point $r\approx\pm 0.2$, a sequence of periodic, Hopf type, cycles appear marking the transition to instability at the edge of the DOA. The lower phase plane plots show this *input induced* behaviour more clearly. For $k = 0.5$, and a small input, the system is attracted to the stable equilibrium. For larger input set points, unstable limit cycles surrounding the origin induce a more oscillatory response.

By considering the problem as essentially two dimensional the analysis shows periodic behaviour to occur at lower values of gain k than expected when $r = 0.0$, that is we have detected limit cycles for $k = 0.5$. In essence of course we have a two parameter problem in which the limit cycle characteristic is both feedback and input induced.

Taking $k = 1.0$ and repeating the above analysis, results in the diagram Fig. 5. A similar scenario exists, the stable origin undergoes a pitchfork type bifurcation. Notice the system now has a reduced domain of stability with respect to the set point, roughly $|r| < 0.1$, but increased internal stability with respect to the state variable x (the DOA in the lower plots being increased in size). For values outside this range a sequence of periodic bifurcations again leads to instability.

Taking $k = 1.25$ results in the diagram Fig. 6. which further confirms this trend. Oscillatory behaviour increases further as the equilibria move closer to the unstable limit cycles marking the division between the stable and unstable domains.

VII. ANALYSIS OF A NONLINEAR SAMPLED DATA SYSTEM

To further illustrate the utility of our approach consider now the analysis of the sampled data system depicted in Fig. 7. If the feedforward element or plant model is linear then

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{31}$$

and, given digital control with sampling period T and an implied ZOH element, then within the $(k+1)^{th}$ sampling interval (31) becomes

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}_k \qquad \mathbf{u}(t_k) = \mathbf{u}_k$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \qquad t \in [t_k, t_{k+1}[\tag{32}$$

and under the usual assumption that the plant is time invariant

$$\mathbf{x}_{k+1} = e^{AT}\mathbf{x}_k + \int_0^T e^{A(t-\tau)}\,B\,\mathbf{u}_k\,d\tau \tag{33}$$

Thus an exact recursive relationship has been obtained in the form

$$\mathbf{x}_{k+1} = \Phi\,\mathbf{x}_k + \Delta\,\mathbf{u}_k$$

$$\mathbf{y}_k = C\,\mathbf{x}_k \tag{34}$$

where Φ and Δ are constants dependent upon T. Assume now that feedback is applied such that

$$\mathbf{u}_k = \mathbf{r}_k - \mathbf{g}(\mathbf{y}_k) \tag{35}$$

where g(.) is either a linear or nonlinear function of the output. If g(.) is linear the system (32-35) can be analysed using well known techniques. If g(.) is nonlinear (32) can be rewritten as

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k, \mu) \qquad \mathbf{x} \in \mathbf{R}^{n_x}\ ;\ \mu \in \mathbf{R}^{n_\mu} \tag{36}$$

where the parameter μ is say the set point $\mu = \mathbf{r}_k$. If the plant is instead a general nonlinear system the model (31) is no longer valid and the description

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \qquad \mathbf{f}:\mathbf{R}^{n_x}\times\mathbf{R}^{n_u}\rightarrow\mathbf{R}^{n_x} \tag{37}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t)) \qquad \mathbf{h}:\mathbf{R}^{n_x}\rightarrow\mathbf{R}^{n_y}$$

must be used but in this case an exact discrete model of the plant cannot be formulated. However, all that is necessary for the system to be analysed using a parameterised cell state space is that the feedforward and feedback elements can be combined into the form

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \mu) \tag{38}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mu) \tag{39}$$

In the first case the plant has been transformed directly into the nonlinear map F(). Normally this requires that the feedforward element in Fig. 7. is linear. The construction of the cell state space system, C(z), then proceeds using (18-22). In the second case, the continuous plant exhibits a nonlinear continuous characteristic and use must be made of (24).

A. Cell Map Analysis

Consider now the effect of say the sampling interval on the qualitative behaviour of a nonlinear sampled data system. Assume in this case the plant is a linear system and the feedback g(.) nonlinear. Setting $n=2$, $l=1$, $m=1$ and A, B, C and $g(x)$ such that

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_1 \end{bmatrix} \quad g(x) = x^3 \tag{40}$$

and taking $\lambda_1 = 1$, $\lambda_2 = -1$, $b_1 = b_2 = 1$ and $c_1 = c_2 = 1$, completes the system definition.

The adoption of this particular block structure may seem restrictive, bearing in mind the flexibility of the Cell Map approach. However this system was originally considered by Ushio and Hirai (1983) in order to demonstrate chaos using an analytical approach. There the linear form of the feed forward element played an important role in the analysis. The same structure is adopted here purely as a basis for comparison.

Suspending the system in the Cell Map framework the input set point r is selected as the primary parameter. Next define the cell state space such that $x_1 \in [-2,2]$, $x_2 \in [-2,2]$ and $\mu \in [-2,2]$ over a mesh of 150×150×30 cells. Then applying (18-22) to construct C(z), over the region of interest for the three separate cases; $T=0.1$, $T=0.2$ and $T=0.3$. Application of the unravelling algorithm this results in the cell diagrams Fig. 8

The diagrams produced exhibit a number of interesting features. For $T=0.1$ the system exhibits an asymptotically stable fixed point over the range of input set point considered, $\mu = r \in [-2,2]$. This in indicated by the curve of P-1 periodic cells on the graph. For $T=0.2$ a stable P-2 cycle is produced at $r = \pm 0.4$ which is in agreement with the behaviour predicted by Ushio and Hirai

(1983). For values of r outside this range a sequence of periodic P-K cycles are detected which grow in amplitude as r increases. Note that the coarse cell size used in this analysis and the corresponding plots leads to some gaps that distort the cone like characteristic. This situation can easily be improved by adjusting the number of cells used, however, we choose to leave the results in this first-cut form, preferring to emphasise the ease of analysis of the method as a quick route to obtaining a broad global picture.

For $T=0.3$ this trend increases with P-K cycles evident over the full range of set point values considered. Indeed this periodic behaviour is so pronounced that it deserves further attention.

Notice that the Cell diagrams show the origin, at $r=0$, as unstable for this system. This is due to two unstable manifolds emanating from the *fold* bifurcations located asymmetrically either side of the origin.

Fig. 9. shows the response for a step input $r=0.4$ applied to the system for the cases $T=0.1$, $T=0.2$ and $T=0.3$. The resulting outputs confirm the P-1, P-2 and P-K behaviour predicted above. A similar pattern of cyclic behaviour is displayed if the sampling interval is fixed and the set point r is varied. In the lower plot in Fig. 9. the behaviour for $T=0.3$ and at $r=4$, appears as P-K cycles of significantly larger period. Fig. 10. shows the phase plane plot of this response plotted over 2000 points. This plot displays the characteristic typical of an aperiodic or chaotic system, the outline of the invariant strange attractor being clearly visible in this graph. Of course due to the finite number of cells used in this analysis, the unravelling algorithm can not detect aperiodic chaotic phenomena. However, it is usually a straight forward matter to decide whether periodic, P-K, behaviour with K large, is actually periodic or indeed aperiodic/chaotic. Typically either a phase plane, frequency domain or Poincare Map based analysis can be used to ascertain this. As it stands the analysis of this simple nonlinear sampled data system has served again to emphasise the utility of the probing approach made possible by carrying out a parameterised cell map analysis within the framework of a CAD analysis / design tool.

VIII. CONCLUSIONS

The cell-to-cell mapping provides a convenient framework for the analysis of both continuous and discrete systems. Development of the suspended cell state space system, enabling the analysis of systems of parameterised bifurcation problems, has proven successful. The preceding examples,

although simple, succeed in illustrating the concepts behind this approach. The combination of qualitative and numerical techniques has provided a useful, and more importantly flexible, methodology for the analysis of nonlinear systems. Providing both local and global characterisation of a wide range of systems models at a level of detail appropriate to the problem.

From the theoretical point of view the methods are still in the early stages of development. The effectiveness of the algorithms shows itself in the concrete global results which are difficult to obtain by previous methods. This work has gone some way to achieving the aim of developing an integrated tool for the analysis and probing of nonlinear dynamics within different representations of a nonlinear system. At the moment a bifurcation is detected, or inferred to exist, simply by evidence of change in the solution structure at some point within the cell state space. Further work is required to enable a more detailed, possibly analytical, classification at these points. The examples in this section have been chosen for their illustrative properties. Another productive area of application of the approach is in the analysis, and qualitative validation of nonlinear models constructed using system identification techniques (Haynes and Billings, 1992).

It has been shown how a qualitative approach based around using the parameterised cell map can provide useful information on periodic, and to some extent aperiodic, characteristics within a system. Furthermore this study has served to illustrate just how important parameters such as sampling interval, set point and input excitation can be in determining the qualitative characteristics of nonlinear systems. Both stability characteristics, periodic behaviour and to some extent aperiodic behaviour can all be revealed in the analysis. The qualitative approach allows both a broad based coarse probing of nonlinear dynamics as well as more detailed or focused analysis around any particular points of interest.

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References

ABED E.A., 1986, Local feedback stabilisation and bifurcation control I: Hopf Bifurcation, *Syst. Control Lett.*, **7**, 11-17.

ABED E.A., 1987, Local feedback stabilisation and bifurcation control II : Stationary Bifurcation, *Syst. Control Lett.*, **8**, 467-73.

AEYELS D., 1985, Stabilisation of a class of nonlinear systems by a smooth feedback control, *Syst. Control. Lett.*, **5**, 289-94.

BAILLIEUL J., R.W. BROCKETT, R.B. WASHBURN, 1980, Chaotic motion in nonlinear feedback systems, *IEEE Trans. Circ. Syst.*, **TCS-27**, 990-7.

BERNUSSOU J., 1977, Point Mapping Stability, (Pergamon Press:New York).

BRINDLEY J., C. KASS-PETERSEN, A. SPENCE, 1989, Path following methods in bifurcation problems, *Physica D*, **34**, 456-61.

BROCKETT R.W., 1972, On the algebraic structure of bilinear systems, in, *Theory and Application of Variable Structure Systems*, [Ed] R.R. MOHLER AND A. RUBERTI, (Academic Press:London New York).

BROCKETT R.W., 1981, Differential Geometric Control Theory, (Birkhauser).

BYRNES C.I., A. LINDQUIST, 1986, Theory and Application of Nonlinear Control Systems, (North Holland:New York).

CHANG H.C., L.H. CHEN, 1984, Bifurcation characteristics of nonlinear systems under conventional PID control, *Chem. Eng. Sci.*, **39**, 1127-42.

CHUA L.O., G. LIN, 1990, Canonical Realization of Chua's Circuit Theory, *IEEE Trans. Circ. Syst.*, **TCS-37**, 885-902.

CLANCY S.J., W.J. RUGH, 1978, On the realisation problem for stationary homogeneous discrete time systems, *Automatica*, **14**, 357-66.

COLLET P., J.P. ECKMANN, 1980, Universal properties of maps on the interval, *Comm. Math. Phys.*, **76**, 211-54.

COOK P.A., 1985, Simple feedback systems with chaotic behaviour, *Syst. Control Lett.*, **6**, 223-7.

D'ALESSANDRO P., A. ISIDORI, A. RUBERTI, 1974, Realisations and structure theory of bilinear dynamical systems, *SIAM J. Control Optim.*, **12**, 517-35.

DESOER C.A., C. LIN, 1985, Tracking and disturbance rejection of MIMO nonlinear systems with PI controllers, *IEEE Trans. Auto. Control*, **30**, 860-67.

DOEDEL E.J., 1981, AUTO - A program for the automatic bifurcation analysis of autonomous

systems, *Congress. Numer.*, **30**, 265-84.

DOEDEL E.J., 1986, AUTO - Software for continuation and bifurcation problems in ordinary differential equations, (Calif. Inst. Tech.:USA).

FLIESS M., 1982, Local realisation of linear and nonlinear time varying systems, in, *Proc. IEEE Conference on Decision and Control, Orlando, FL*, 733-8.

FLIESS M., M. HAZEWINKEL [ED], 1986, Algebraic and Geometric Methods in Nonlinear Control Theory, 55-75, (Reidel:Dordrecht Holland).

GOLUBITSKY M., I. STEWART, P.G. SCHAEFFER, 1988, Singularities and Groups in Bifurcation Theory, Vol. II, *Appl. Math. Sci.*, **69**, (Springer:Berlin Heidelberg New York).

GOLUBITSKY M., P.G. SCHAEFFER, 1985, Singularities and Groups in Bifurcation Theory, Vol. I, *Appl. Math. Sci.*, **51**, (Springer:Berlin Heidelberg New York).

GUCKENHEIMER J., 1979, Sensitive dependence on initial conditions for one dimensional maps, *Comm. Math. Phys.*, **70**, 133-60.

GUCKENHEIMER J., P.J. HOLMES, 1983, Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields, *Appl. Math. Sci.*, **42**, (Springer:Berlin Heidelberg New York).

GUMOWSKI I., C. MIRA, 1980, Recurrences and Discrete Dynamical Systems, *Lect. Notes Math.*, **809**, (Springer:Berlin Heidelberg New York).

HAHN H., 1985, Computation of branching solutions for a nonlinear control system via dual input describing function and root locus techniques, *Int. J. Control*, **42**, 21-31.

HAYNES B.R., S.A. BILLINGS, 1991, Global Analysis and Model Validation in Nonlinear System Identification, (Submitted for publication).

HAYNES B.R., S.A. BILLINGS, 1992, A Method for the Global Analysis of Nonlinear Systems, *Int. J. Control*, (Accepted for publication).

HERMANN R., A.J. KRENER, 1977, Nonlinear controllability and observability, *IEEE Trans. Auto. Control*, **TAC-22**, 728-40.

HOLMES P.J., 1983, Bifurcation and Chaos in simple feedback systems, in, *Proc. IEEE Conf. Decision and Control*, 365-70.

HOLMES P.J., D.C. WHITLEY, 1984, Bifurcations in one and two-dimensional maps (Erratum vol. 312, pp. 601-2), *Phil. Trans. R. Soc. London*, **A311**, 43-102 .

HSU C.S., 1987, Cell-to-Cell Mapping: A method for the global analysis of nonlinear systems, *Appl. Math. Sci.*, **64**, (Springer:Berlin Heidelberg New York).

HSU C.S., R.S. GUTTALU, 1980, An unravelling algorithm for the global analysis of dynamical systems. An application of cell-to-cell mappings, *Trans. ASME J. Appl. Mech.*, **47**, 940-48.

IOOSS G., 1979, Bifurcation of Maps and Applications, *Math. Studies*, **36**, (North-Holland:Amsterdam).

ISIDORI A., 1985, Nonlinear Control Systems: An Introduction, *Lect. Notes Control Info. Sci.*, **72**, (Springer:Berlin Heidelberg New York).

ISIDORI A., 1989, Nonlinear Control Systems: An Introduction, 2nd Ed., (Springer:Berlin Heidelberg New York).

JAKUBCZYK B., 1987, Feedback linearisation of discrete time systems, *Syst. Control Letters*, **9**, 411-16.

KELLER H.B., 1977, Numerical solution of bifurcation and nonlinear eigenvalue problems, in, *Application of Bifurcation Theory, Proceedings Seminar, Math. Res. Center, University Wisconsin*, (Academic Press:London New York).

LI T.Y., J.A. YORKE, 1975, Period-3 implies chaos, *Amer. Math. Monthly*, **82**, 985-92.

MAREELS I.M.Y., R.R. BITMEAD, 1988, Bifurcation effects in robust adaptive control, *IEEE Trans. Circ. Syst.*, **TCS-35**, 835-41.

MARSDEN J., M. MACRACKEN, 1976, The Hopf Bifurcation and its Applications, *Appl. Math. Sci.*, (Springer:Berlin Heidelberg New York).

MAYNE D.O., B.W. BROCKETT, 1973, Geometric Methods in System Theory, (Reidel:Boston MA).

MEES A.I., 1981, Dynamics of Feedback Systems, (Wiley:New York).

MEHRA R.K., 1976, Catastrophe theory, nonlinear system identification and bifurcation control , in, *Proc. Amer. Joint Auto. Control Conf.*, 823-31.

MOHLER R.R., 1970, Natural bilinear control processes, *IEEE Trans. SSC*, **SCC-6**, 192-77.

MOHLER R.R., 1977, Bilinear control Processes, (Academic Press:London New York).

NORMAND-CYROT D., S. MONACO, 1984a, On the realisation of nonlinear sampled data systems, *Syst. Control Lett.*, **5**, 145-52.

PRESTON C., 1983, Iterates of maps on a Interval, *lect. Notes. Math.*, **999**, (Spinger-Verlag:Berlin Heidelberg New York).

RHEINBOLDT W.C., 1986, Numerical Analysis of Parameterised Nonlinear Equations, (Wiley:New York).

RHEINBOLDT W.C., 1988, On the computation of multidimensional solution manifolds of parameterised equations, *Numer. Math.*, **53**, 165-81.

RUGH W.J., 1981, Nonlinear Systems Theory, (The John Hopkins University Press:Baltimore MD).

SALAM F.M.A., 1985, Feedback stabilisation, stability and chaotic dynamics, in, *Proc. 24th IEEE Decision and Control Conference*, **1**, 467-72.

SALAM F.M.A., S. BAI, 1986, Disturbance generated bifurcation in simple adaptive system: Simulation evidence, *Syst. Control Lett.*, **7**, 269-80.

SALVADORI L. [ED], 1984, Bifurcation of maps, in, *Bifurcation Theory and Applications. CIME, Montecatine, Italy, Lect. Notes Math.*, **1057**, (Springer:Berlin Heidelberg New York).

SANDBERG I.W., 1964, A frequency-domain condition for the stability of feedback systems containing a single time-varying nonlinear element, *Bell System Tech. Journal*, **43**, 1601-8.

SCHWARTZ C.A., B.W. DICKINSON, 1986, Some finite dimensional realisation theory for nonlinear systems, in, *Theory and Application of Nonlinear Control Systems*, [Ed] C.I. BYRNES AND A. LINDQUIST, (Reidel:Dordrecht Holland).

SONTAG E.D., 1979, Realisation theory of discrete-time nonlinear systems: Part I: The bounded case, *IEEE Trans. Circ. Syst.*, **TCS-26**, 342-356.

SPARROW C.T., 1980, Bifurcation and chaotic behaviour of simple feedback systems, *J. Theor. Biol.*, **83**, 93-105.

SPARROW C.T., 1981, Chaotic behaviour in a 3-D feedback system with piecewise linear feedback function, *J. Math. Anal. Appl.*, **83**, 275-91.

SUSSMANN H.J., 1977, Existence and uniqueness of numerical realisations of nonlinear systems, *Math. Syst. Theory.*, **10**, 263-84.

USHIO T., K. HIRAI, 1983, Bifurcation and chaos in sampled data systems with nonlinear continuously differentiable elements, *Elect. Comm., Japan*, **66**, 1-9.

VIDYASAGAR M., 1986, New directions of research in nonlinear system theory, *Proc. IEEE*, **74**, 1060-91..

YDSTIE B.E., M.P. GOLDEN, 1988, Bifurcation in adaptive control, *Syst. Control Letters*, **11**, 413-30.

ZAMES G., 1963, Functional analysis applied to nonlinear feedback systems, *IEEE Trans. Circ. Theory*, **CT-10**, 392-404.

ZAMES G., 1964, Realisation conditions for nonlinear feedback systems, *IEEE Trans. Circ. Theory*, **CT-12**, 186-94.

FIGURES

FIG. 1 Typical solution branch structure.

FIG. 2 Nonlinear Feedback System Block Diagram

FIG. 3 CELL Diagram System $\mu = k\,r = 0.0$

FIG. 4 CELL Diagram System $\mu = r\,k = 0.5$

FIG. 5 CELL Diagram System $\mu = r\,k = 1.0$

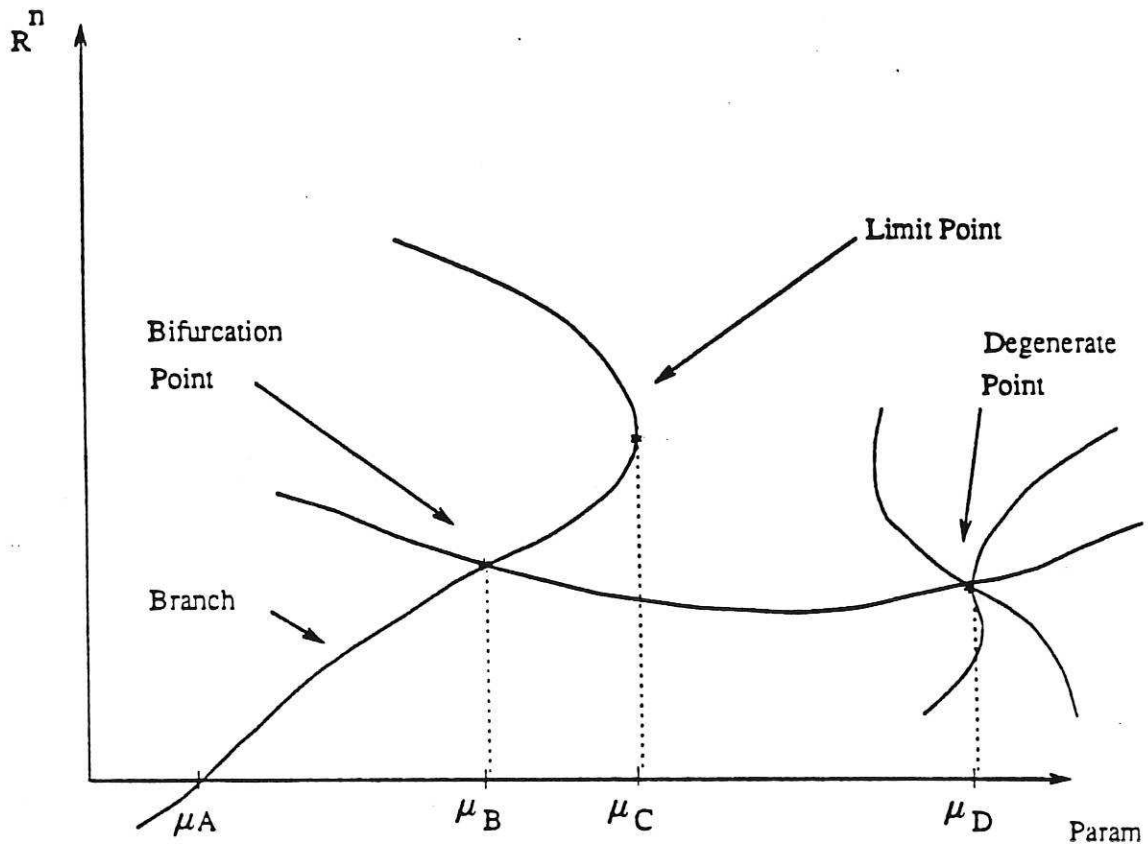
FIG. 6 CELL Diagram System $\mu = r\,k = 1.25$

FIG. 7 Nonlinear Sampled Data System

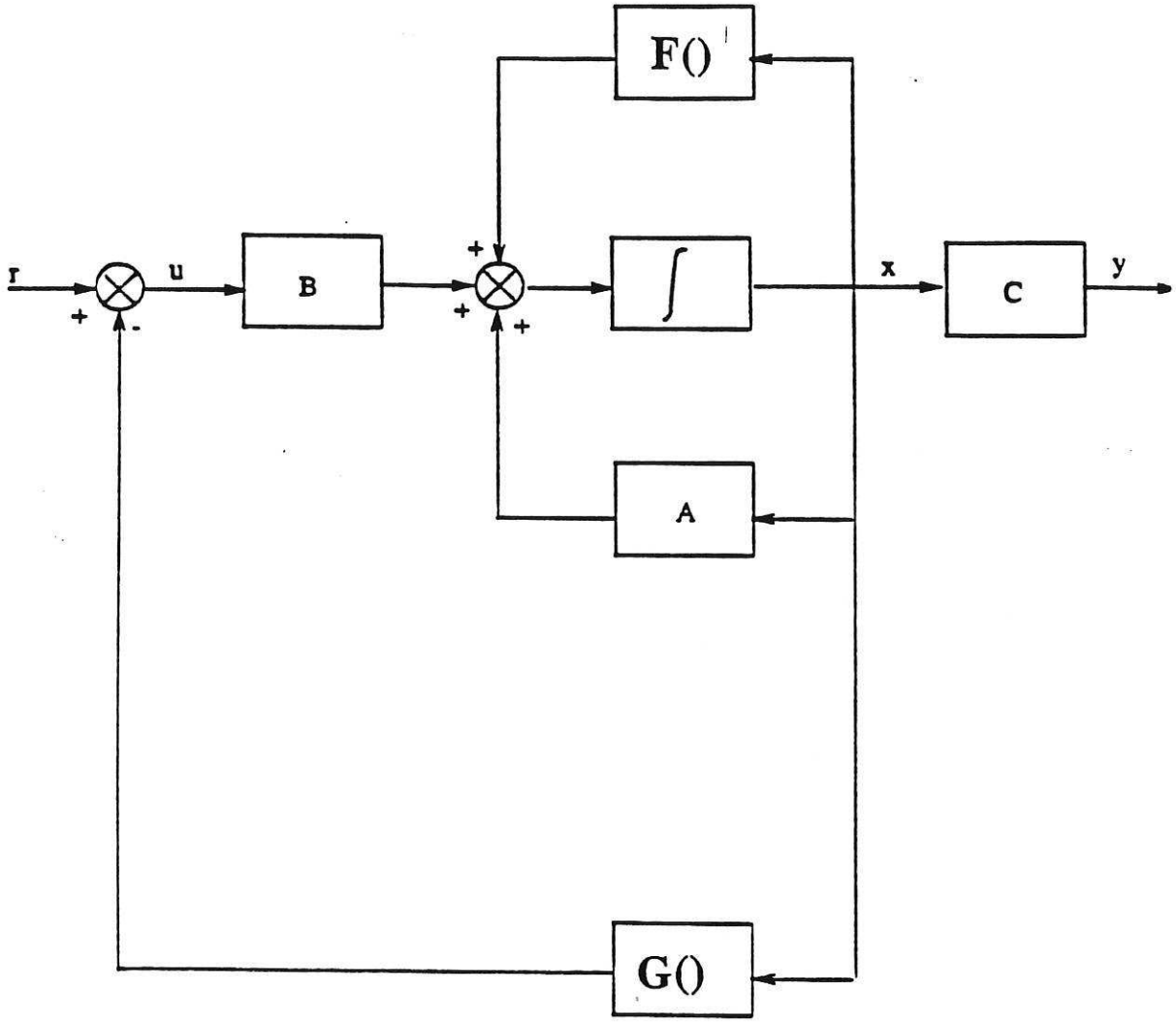
FIG. 8 CELL Diagram System

FIG. 9 Step Responses, $T = 0.1$, $T = 0.2$ and $T = 0.3$, $r = 0.4$

FIG. 10 Chaos Phase Plane Plot $T = 0.3$

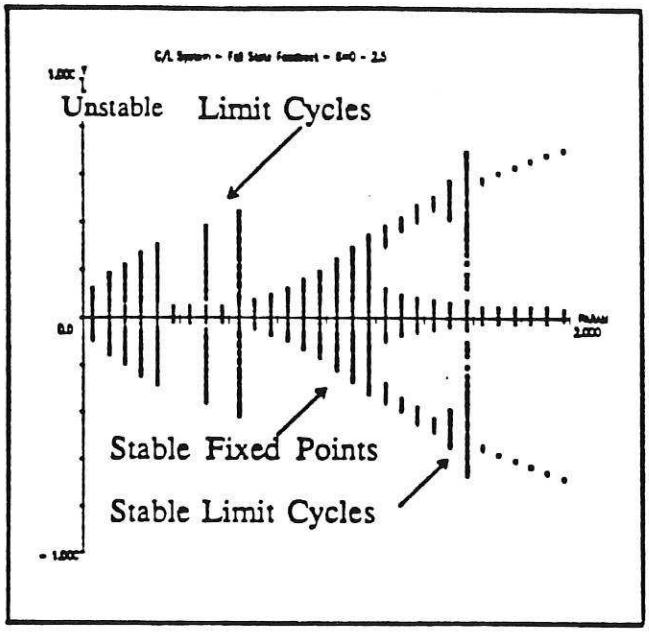


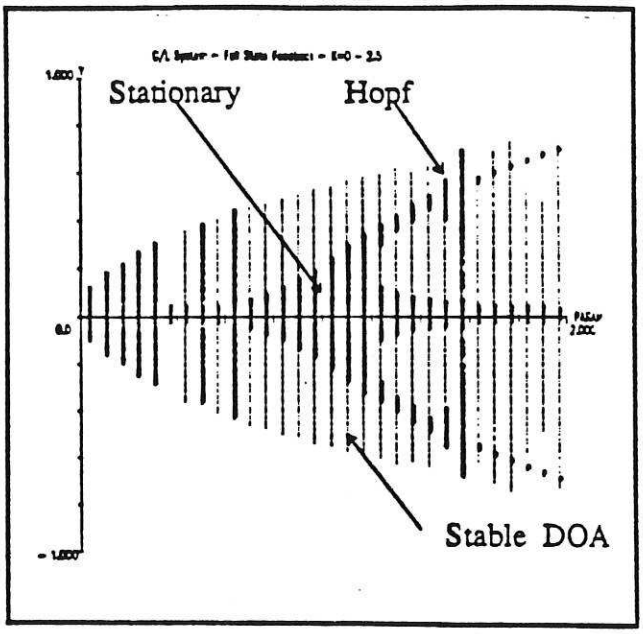
Typical solution branch structure

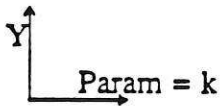


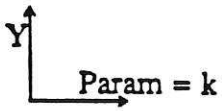
Hahn Nonlinear Feedback System Block Diagram

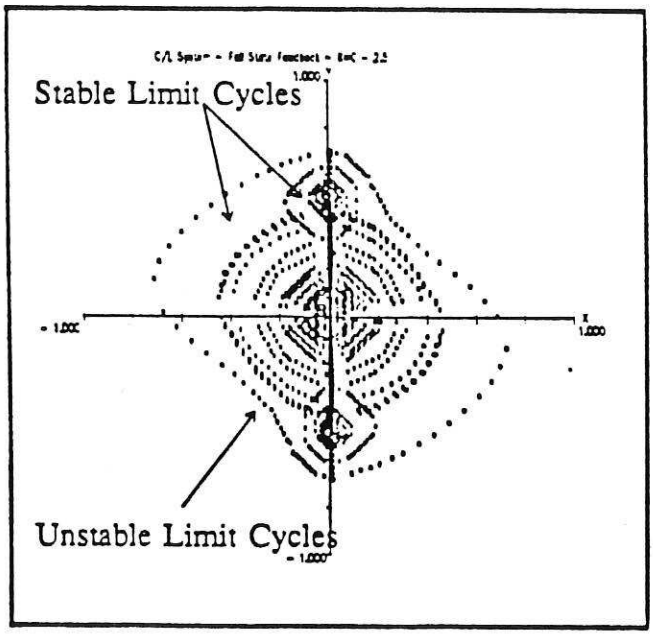
Fig. 2

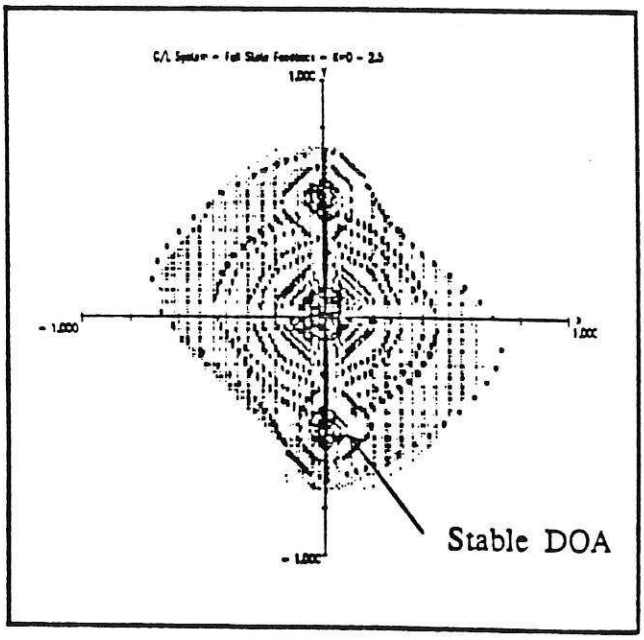


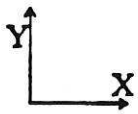


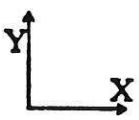






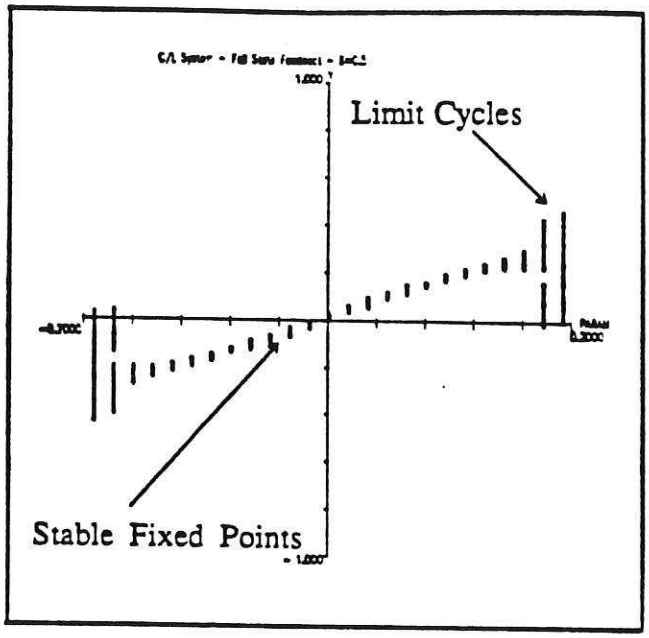


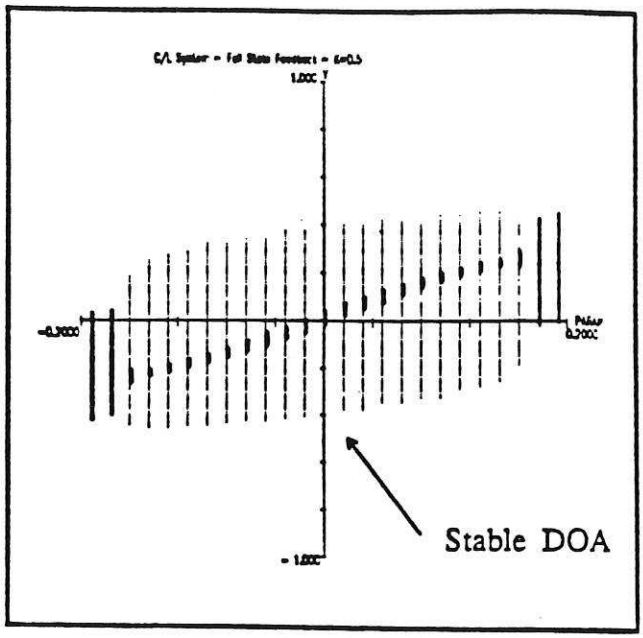


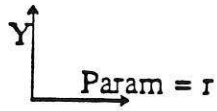


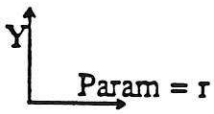
CELL Diagram, $\mu = k\,r = 0.0$

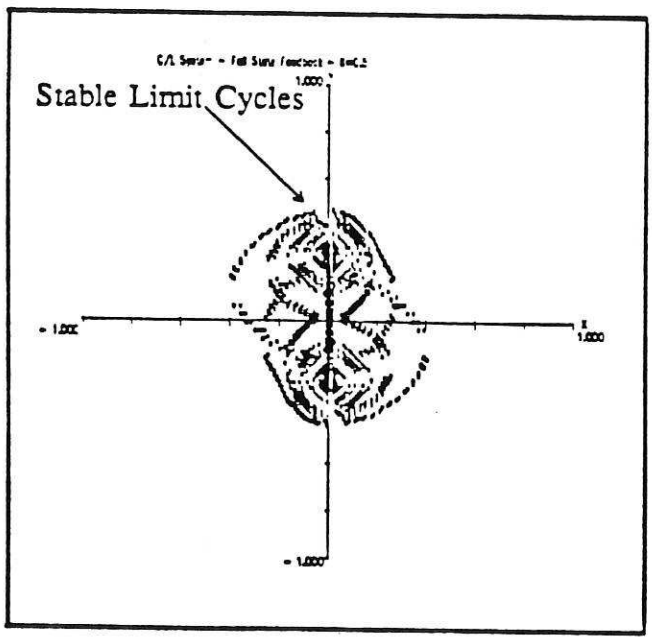
Fig. 5

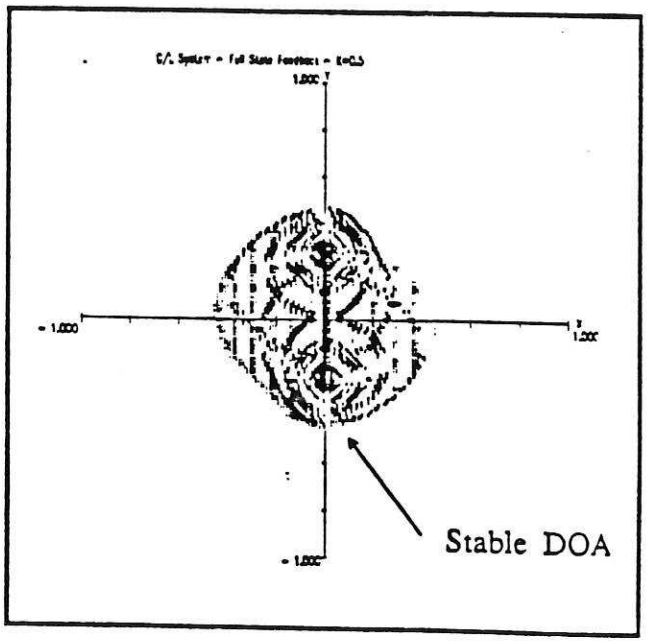


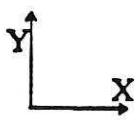


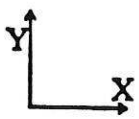






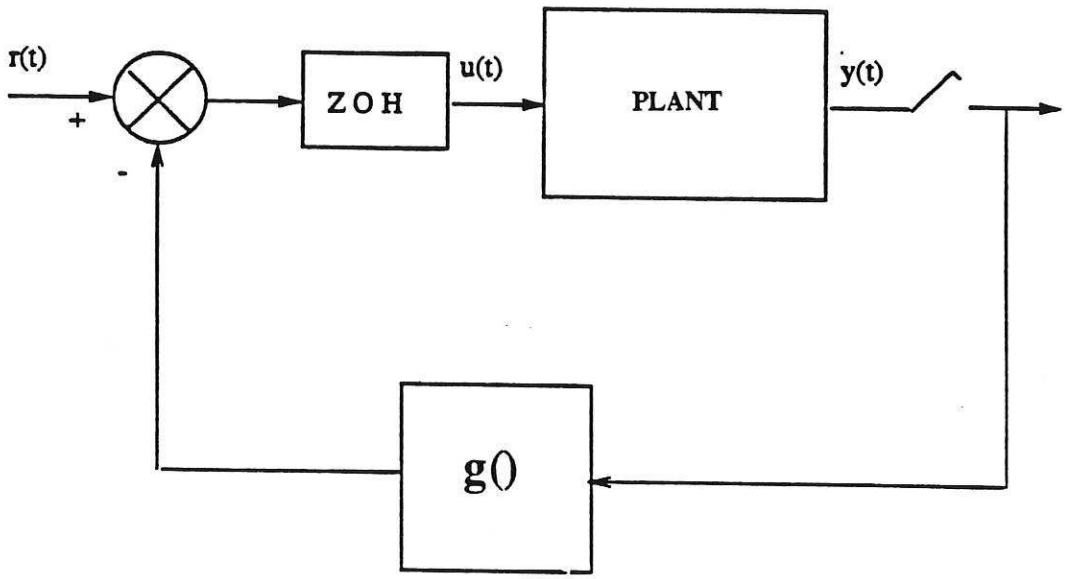






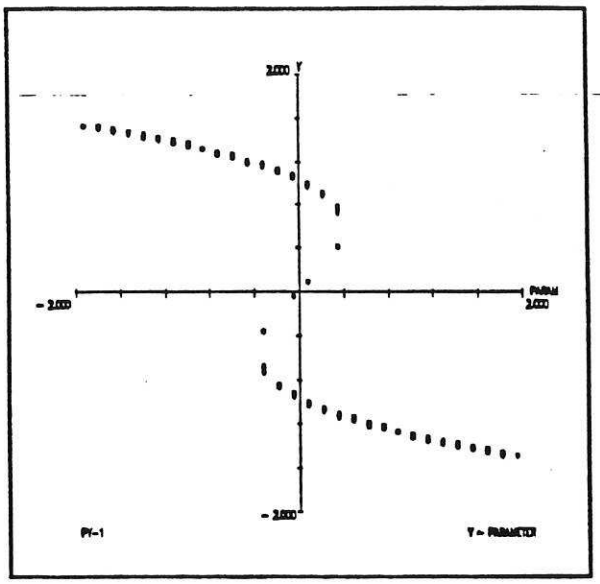
CELL Diagram, $\mu = r$, $k = 0.5$

Fig 6.



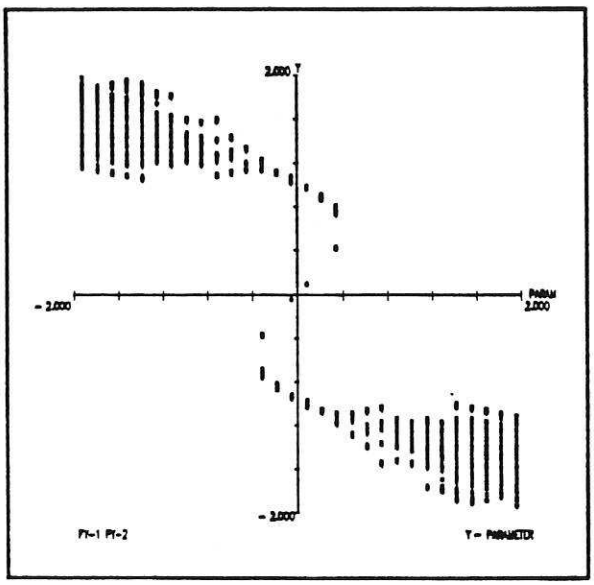
Nonlinear Sampled Data System

Fig. 7



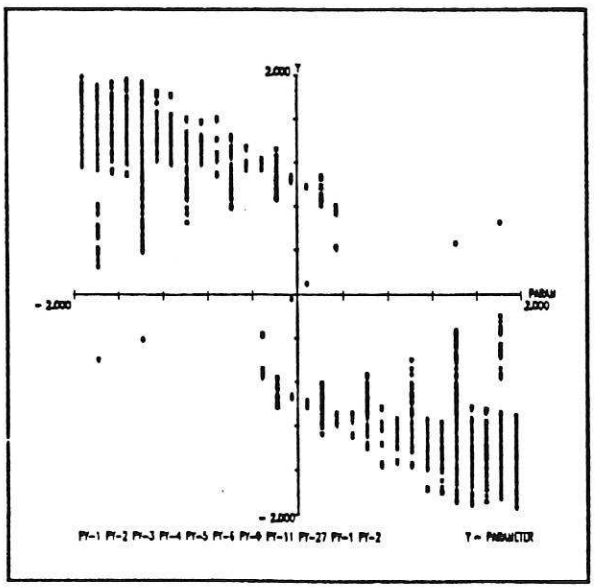
Non-Periodic

T = 0.1



Periodic

T = 0.2

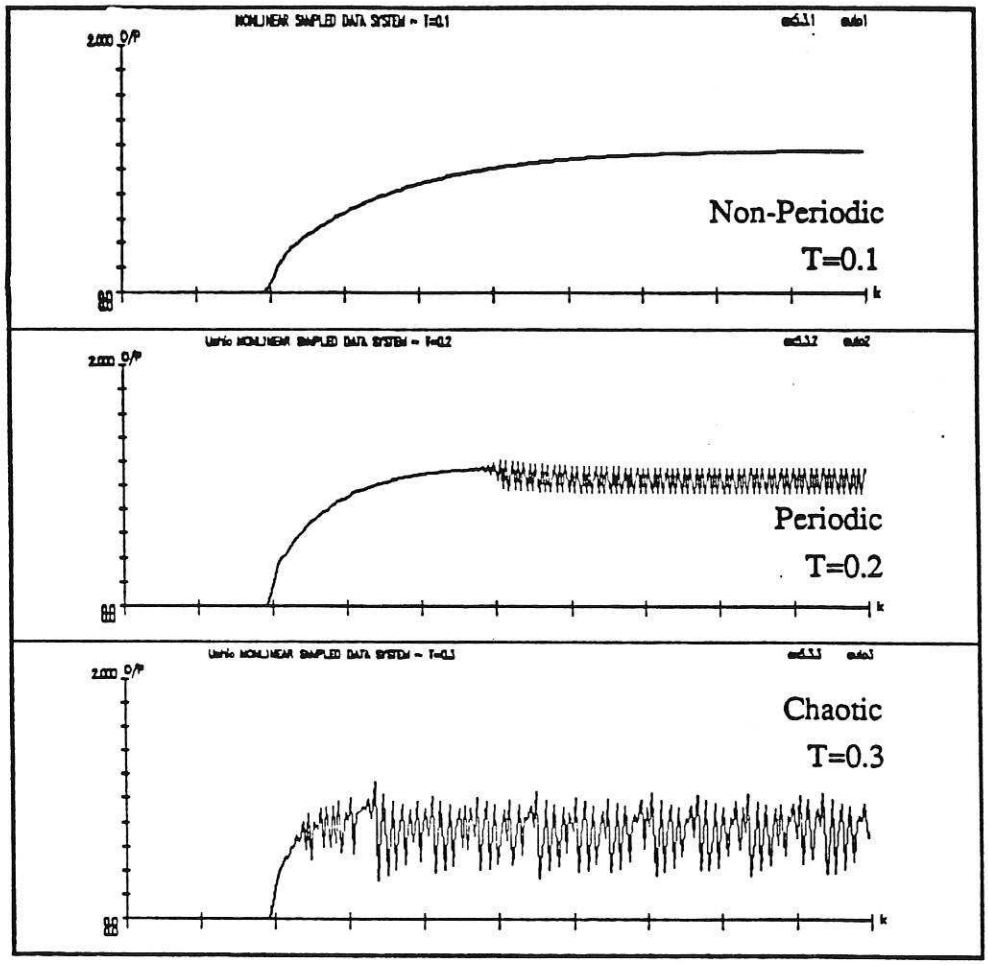


Chaotic

T = 0.3

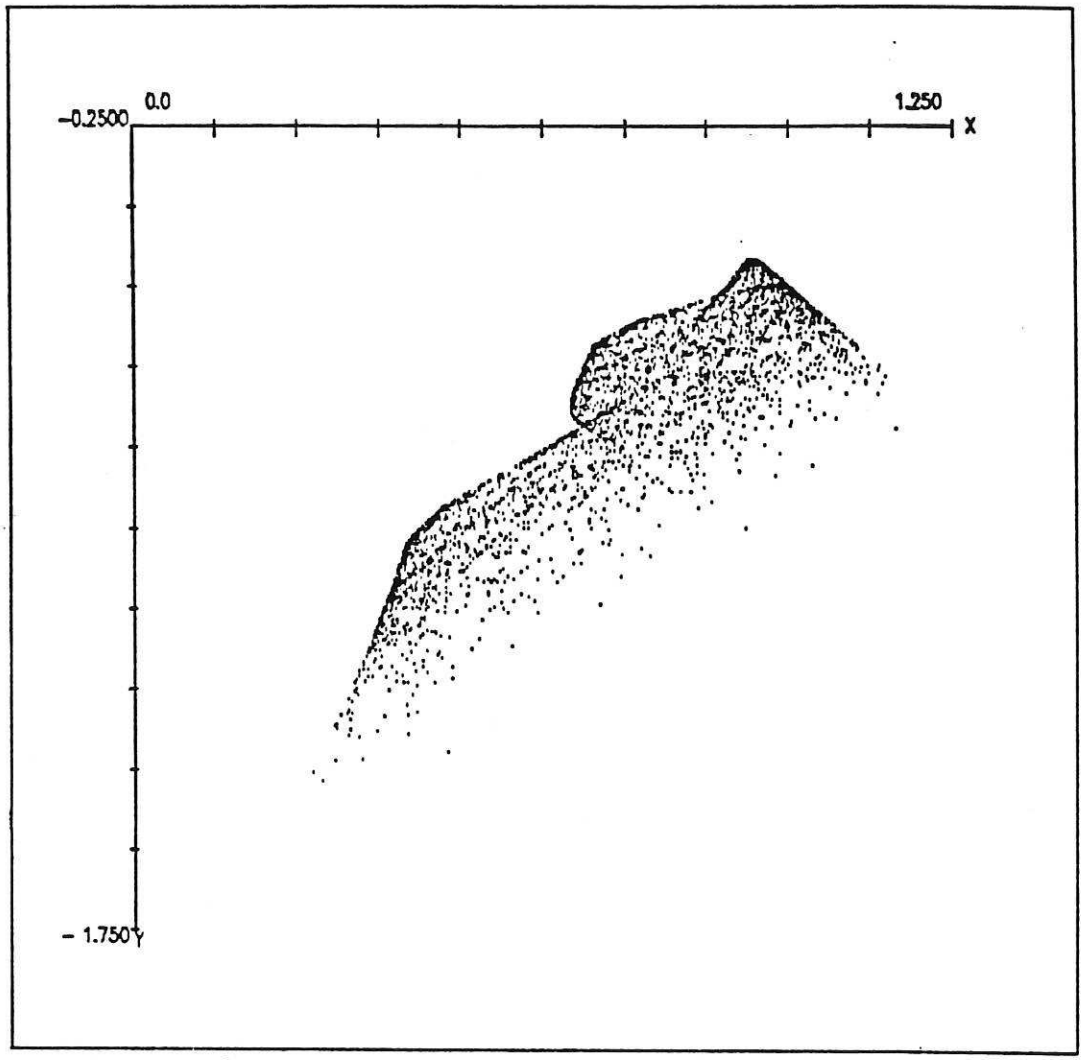
CELL Diagram System - $\mu = r$ for $T = 0.1$, $T = 0.2$ and $T = 0.3$

Fig 7.



Step Responses - $r = 0.4$ for $T = 0.1$, $T = 0.2$ and $T = 0.3$,

Fig 9



Chaos in Nonlinear Sampled Data System, Strange Attractor

Phase Plane Plot of Strange Attractor - $r = 0.4$ for $T = 0.3$

Fig 10