Monograph:
Peyton-Jones, J.C. and Billings, S.A. (1992) Mean Levels in Nonlinear Analysis and Identification. Research Report. ACSE Research Report 454 . Department of Automatic Control and Systems Engineering

Mean Levels in Nonlinear Analysis and Identification

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Research Report No 454

June 1992

June 11, 1992

MEAN LEVELS IN NONLINEAR ANALYSIS AND IDENTIFICATION

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Abstract: The effects of data pretreatment and mean levels upon nonlinear model structures is investigated. Techniques which are commonly used to prefilter data in linear system identification are shown to alter the model structure in the nonlinear case. The effects of mean levels are considered in detail and a new unravelling algorithm is derived to recover the underlying system model when the offsets are external the the system. A new mapping from the time domain to the frequency domain is also introduced for the case where offsets can be considered as an implicit part of the system.

1. Introduction

The treatment of d.c or constant terms in linear system analysis and identification is well established [Astrom 1980], [Isermann 1980], and the temptation is perhaps to apply the same techniques to the nonlinear case. Unfortunately such an approach may give misleading results, and new methods must be developed for the identification of nonlinear systems subjected to constant output additive disturbances, or to enable the analysis in the generalised frequency domain of nonlinear systems incorporating d.c. components.

In linear system identification for example it is common practice to remove mean levels from the data, partly because the static behaviour of the system might not be of interest, and partly because unknown measurement offsets (which cannot be distinguished from the system d.c. gain) may introduce bias into the estimates. Indeed mean removal may be regarded as a special case of frequency weighting of estimates through the use of prefilters.

Nonlinear systems however often exhibit harmonics or complex intermodulations which transfer energy between frequencies, giving an output at some quite different frequency to that of the input. Simple mean removal may therefore remove significant dynamic information contained in those intermodulations which produce d.c. output components, and hence bias the resulting nonlinear model. In the same way, linear prefiltering techniques may also be inappropriate in a nonlinear context.

Another approach employed in linear system identification is to estimate a constant term explicitly. The term may then be discarded if it is regarded as an additive disturbance to the real system dynamics of interest, or incorporated without difficulty into subsequent system analysis if it is regarded (by virtue of its unchanging nature) as part of the system itself.

In the nonlinear case however, where superposition does not apply, neither of these two perspectives is straightforward. If for example the estimated constant is regarded as being caused by a pure additive disturbance at the output, then this same additive disturbance will have become crossmultiplied with any nonlinear autoregressive terms. Hence the system dynamics cannot be obtained just by discarding the estimated constant, without also removing this crossmultiplicative bias from the system parameters.

Alternatively if the constant term is to be regarded as part of the system itself, then for similar reasons it may have a significant effect on system behaviour and needs to be incorporated in subsequent system analysis. Previous work for example has shown that higher order frequency domain analysis of nonlinear systems can provide a useful insight into system behaviour, but these analysis methods are currently restricted to models without constant components.

The aim of the present paper is to examine the effects of mean levels in nonlinear frequency response analysis and system identification, and to address the problems outlined above. The paper begins in Section 2 with an introduction to the classes of nonlinear models which will be used in the development. In Section 3 the effects of applying simple linear prefiltering techniques to nonlinear systems is investigated, and it is shown that this fundamentally alters the nonlinear model structure.

Algebraic expressions which reveal the precise form of the changes introduced to the system parameters of an input/output model through the presence of a purely additive constant offset, are developed in Section 4. These expressions are used as the basis of a method outlined in Section 5 whereby such effects may be removed from an identified model, and the parameters of the original system recovered.

Although the work in previous sections is geared towards removing an unwanted constant disturbance at the output, it is sometimes the case that the constant is regarded not as a disturbance but as part of the system itself. In Section 6 therefore methods for analysing nonlinear systems in the generalised frequency domain are extended to accomodate models with constant terms, and it is shown that such terms may have a significant effect on system behaviour.

2. Model Representations

2.1. The Volterra model

Most classical descriptions of nonlinear systems have been based on the Volterra series, [Schetzen 1980; Volterra 1959],

$$y(t) = \sum_{n=0}^{N} y_n(t) \tag{1}$$

where $y_n(t)$ the 'n-th order output' is a homogenous functional of degree n,

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i \tag{2}$$

Note that the Volterra model yields the familiar linear convolution integral for the case $N = 1$, and simply extends the linear system description to higher orders by introducing additional dimensions. Consequently $h_n(\cdot)$ is termed the 'n-th order impulse response function'.

Equation (1) may also be expressed in operator form, using the notation developed by Brilliant and George,

$$y(t) = \mathbf{H}[u(t)] = \sum_{n=0}^{N} \mathbf{H_n}[u(t)] \tag{3}$$

where $\mathbf{H_n}[]$ indicates the operator form of equation (2), i.e. $\mathbf{H_n}[u] = y_n$. The operator notation will be used in the more axiomatic proofs, while the functional notation will be used in the development of more algebraic relationships.

The Volterra model is important not only because it encompasses a wide class of nonlinear systems, but because it forms the basis for analysing such systems in the frequency domain. For example, by analogy with linear systems, an n-th order transfer function $H_n(\cdot)$ can be defined as the Fourier pair of the n-th order impulse response function of equation (2) giving,

$$h_n(\tau_1, \cdots, \tau_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots, j\omega_n)\, e^{j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\omega_1 \cdots d\omega_n \tag{4}$$

Substituting (4) into (2) and carrying out the multiple integration on $\tau_1, \cdots, \tau_n$ gives,

$$y_n(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} H_n(j\omega_1, \cdots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i)\, e^{j(\omega_1 + \cdots + \omega_n)t}\, d\omega_i \tag{5}$$

where $U(j\omega_i)$ represents the input spectrum.

It is also useful to "symmeterise" the n-th order transfer function so that its values are independent of the order of the individual frequency arguments. This property is obtained by summing an asymmetric function over all possible permutations of its arguments and dividing by their number, according to,

$$H_n^{sym}(j\omega_1, \cdots, j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \text{of } \omega_1 \cdots \omega_n}} H_n^{asym}(j\omega_1, \cdots, j\omega_n) \tag{6}$$

The generalised frequency response functions described above, though multi-dimensional in form, can be interpreted to provide a revealing description of nonlinear input/output behaviour [Peyton Jones, Billings 1990]. Although the description is based on the Volterra model, algorithms have been developed which enable the higher order transfer functions of other (perhaps more commonly used) model forms to be computed [Peyton Jones, Billings 1989]. Unfortunately such algorithms do not accomodate models which incorporate constant terms, and this issue will be addressed in Section 6.

2.2. The NARX Model

The Volterra system representation of the previous section, provides a black box description for a wide class of nonlinear systems. Unfortunately direct estimation of the system parameters is both computationally burdensome, and results in inordinately large system descriptions [Billings 1980].

An alternative, and more compact parametric model form, is given by the Nonlinear AutoRegressive with eXogeneous inputs (NARX) representation, [Leontaritis,Billings 1985]. In this case the system is described by some discrete time nonlinear operator

F[·] of lagged input signals $u(t-k_u)$, and outputs $y(t-k_y)$, with t used to enumerate the sampling intervals, and k the lags:

$$y(t) = \mathbf{F}[u(t-k_i), y(t-k_i)] \qquad i = 1,2,.... \tag{7}$$

In the present study a polynomial form will be assumed for the operator F[·], and the NARX model can then be expressed as,

$$y(t) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{k_1, k_{p+q}=1}^{K} c_{p,q}(k_1, \cdots, k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \tag{8}$$

Each term is seen to contain a p-th order factor in $y(t-k_i)$ and a q-th order factor in $u(t-k_i)$ (such that $p+q=m$), and each is multiplied by a coefficient $c_{p,q}(k_1, \cdots, k_{p+q})$. The multiple summation over the k_i, ($k_i = 1, \cdots, K$), generates all the possible permutations of lags which might appear in these terms.

In the development below it will be notationally convenient to specify more explicitly which lags are associated with the input, and which with the output. Equation (8) will therefore be rewritten as,

$$y(t) = \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{k_{y_1}, k_{y_p}=1}^{K} \sum_{k_{u_1}, k_{u_q}=1}^{K} c_{p,q}(k_{y_1}, ..., k_{y_p}; k_{u_1}, ..., k_{u_q}) \prod_{i=1}^{p} y(t-k_{y_i}) \prod_{i=1}^{q} u(t-k_{u_i}) \tag{9}$$

These equations represent the general algebraic form of the NARX model, and encompass all possible terms. In practice however a physical system can generally be described using only a few NARX parameters. Thus for example a specific instance of the NARX model such as,

$$y(t) = a_0 y(t-1) + a_1 u(t-1) + a_2 y(t-2) u(t-1) + a_3 u(t-1)^2 \tag{10}$$

may be obtained from the general form (9) with

$$c_{1,0}(1) = a_0; \qquad c_{0,1}(1) = a_1; \tag{11}$$

$$c_{1,1}(2;1) = a_2; \qquad c_{0,2}(1,1) = a_3; \qquad else \;\; c_{p,q}(\cdot) = 0; \tag{11}$$

Practical NARX models have this comparatively modest parameter set because they encode information from past outputs as well as past inputs, and it is this recursive property which fundamentally distinguishes the NARX from the Volterra model.

2.3. The NARMAX model

The NARMAX model extends the NARX system description to include a characterisation of the disturbance process. Unlike linear systems however the disturbance terms are often multiplicative with the system inputs and outputs, and it is this fact that can introduce bias in the system parameters if the disturbance contains constant components (c.f. Section 4).

Consider for example the inclusion of a disturbance process $v(t)$ in the NARX equation,

$$y(t) = \mathrm{F}[u(t-k_i), y(t-k_i)] + v(t) \qquad i = 1,2,... \qquad (12)$$

At first sight the additional terms seem to be explicit, and independent of the nonlinear model. On closer inspection however, it is seen that the disturbance process appears also in all the recursive variables $y(t-k_i)$ of the system model, and the two are therefore closely inter-related. Indeed a less deceptive form of equation (12) is given by the the NARMAX representation,

$$y(t) = \mathrm{F}[u(t-k_i), y(t-k_i), e(t-k_i)] \qquad i = 1,2,... \qquad (13)$$

Expanding (13) to its polynomial form gives,

$$y(t) = \sum_{m=0}^{M} \sum_{p=0}^{m} \sum_{q=0}^{m-p} \sum_{k_{y_1},k_{y_p}=1}^{K} \sum_{k_{u_1},k_{u_q}=1}^{K} \sum_{k_{e_1},k_{e_r}=1}^{K} c_{p,q,r}(k_{y_1},...,k_{y_p};\, k_{u_1},...,k_{u_q};\, k_{e_1},...,k_{e_r}) \qquad (14)$$

$$\times \prod_{i=1}^{p} y(t-k_{y_i}) \prod_{i=1}^{q} u(t-k_{u_i}) \prod_{i=1}^{r} e(t-k_{e_i})$$

where r is used to denote the degree of nonlinearity in the noise signal $e(t)$, such that $p+q+r = m$.

Notice also that an initial constant $c_{0,0,0}$ is included in the model (14), corresponding to the case $m=0$ in the first summation. This additional term is straightforward to estimate, but can have a considerable effect on the system behaviour because it enters internally and appears in all the recursive variables $y(t-k_i)$. It might be expected therefore that constant parameters or mean levels may have a significant effect in nonlinear system identification or frequency response analysis, and indeed this will be shown to be the case.

3. Mean Levels and Prefiltering of Nonlinear systems

The characteristics of input/output data used for system identification, fundamentally determines the final model obtained. It is common practice therefore to pretreat the data or weight the identification algorithm, so as to reflect in some way the intended model application and structural form.

In many cases this predisposition is expressed in the frequency domain, [Wahlberg,Ljung, 1986], though amplitude criteria may also be introduced [Huber 1981].

Such methods however, being designed for linear system identification, do not necessarily carry over to the nonlinear case.

A particularly common special case of frequency weighting of estimates lies in the treatment of d.c. levels in input/output data, where the d.c. component is often removed entirely, either by subtracting mean levels from the data, or by methods equivalent to differencing (high pass filtering) the data, [Isermann 1980].

Consider for example a system described by the Volterra model (3), acted on by an input $cu(t)$. The (constant) dummy variable c has been included to keep track of the various order terms. This yields,

$$y(t) = \sum_{n=1}^{N} \mathbf{H_n}[cu(t)] = \sum_{n=1}^{N} c^n \, \mathbf{H_n}[u(t)] \tag{15}$$

where the second equality follows from the fact that $\mathbf{H_n}[\cdot]$ is homogeneous of degree n.

If a linear weighting filter $L(\cdot)$ were to be applied to both input and output, then a similar expression could be used to describe the relationship between the filtered quantities $u^*(t), y^*(t)$, namely,

$$y^*(t) = \sum_{n=1}^{N} \mathbf{H_n^*}[u^*(t)] \tag{16}$$

where,

$$u^*(t) = L(cu(t)) = c\ L(u(t)) \qquad\qquad y^*(t) = L(y(t)) \tag{17}$$

Notice the filtered system is described by the operators $\mathbf{H_n^*}[\cdot]$, where the star is used to indicate that these may be different from the $\mathbf{H_n}[\cdot]$ of the original system (15). Indeed substituting for $u^*(t), y^*(t)$ from (17) in (16) and rearranging slightly gives,

$$y(t) = L^{-1} \sum_{n=1}^{N} \mathbf{H}_n^*[c\ L(u(t))] = L^{-1} \sum_{n=1}^{N} c^n\ \mathbf{H}_n^*[L(u(t))] \tag{18}$$

Hence by equating coefficients of c across equations (15) and (18), the relationship between $\mathbf{H}_n^*[\cdot]$ and $\mathbf{H}_n[\cdot]$ is seen to be,

$$\mathbf{H}_n[\cdot] = L^{-1}\ \mathbf{H}_n^*[L(\cdot)] \qquad or \qquad \mathbf{H}_n^*[\cdot] = L\ \mathbf{H}_n[L^{-1}(\cdot)] \tag{19}$$

It is clear therefore that the process of prefiltering the data leaves the transfer function estimate unchanged only in the linear case, where by the special commutativity property of linear convolution (18) reduces to,

$$\mathbf{H}_1^*[\cdot] = \mathbf{H}_1[\cdot] \tag{20}$$

For the nonlinear case the very act of filtering will introduce bias into the estimates even in the ideal noise free case considered here.

3.1. Example: difference prefilter

It is important to emphasise that the continuous time Volterra series model is chosen as a convenient representation for a wide class of nonlinear systems. The fact that the derivation above is based on this model does not constrain the applicability of the result to Volterra models only. Consider for example the NARX model,

$$y(t) = 0.7y(t-1) + 0.3u(t-1) + 0.2u(t-1)^2 \tag{21}$$

Applying a simple difference filter $L(q) = 1 - q^{-1}$ to both input and output, gives the filtered quantities $u^*(t), y^*(t)$ as,

$$u^*(t) = (1 - q^{-1})u(t); \qquad y^*(t) = (1 - q^{-1})y(t) \tag{22}$$

where q is used to represent the backwards shift operator. The new relationship between the filtered quantities can be found by eliminating $u(t), y(t)$ between(21),(22) and rearranging to give,

$$y^*(t) = 0.7y^*(t-1) + 0.3u^*(t-1) + \frac{0.2}{(1 - q^{-1})} u^*(t-1)^2 \tag{23}$$

Notice that the *linear* structure of this model is identical to that of (21), and is unaffected by the filtering process. Conversely the *nonlinear* term has been changed quite considerably, and parameter estimates based on the filtered data will therefore yield quite different results to those obtained without applying a prefilter.

4. Mean Level Induced Bias in Nonlinear Parameter Estimation

In linear system identification an alternative approach to the treatment of mean levels and unknown measurement offsets in data, is to include a constant term explicitly in the estimation process. In this way bias is avoided, and the additional (constant) disturbance term can then be discarded to leave the system parameters of interest.

This approach however, is only valid if the constant term is purely additive in the system description. Physically such an assumption is reasonable since disturbances often enter the system through the measurement process itself, giving a system description of the form,

$$z(t) = \mathrm{F}[u(t-k_i),\, z(t-k_i)] \qquad i=1,2,... \tag{24}$$

$$y(t) = z(t) + v(t); \tag{25}$$

Here $z(t)$ is used to denote the disturbance free, (unmeasured), ouput from the system. Typically the disturbance $v(t)$ is a linear moving average process, but this may often contain a d.c. component as for example when mean signal levels are deliberately adjusted to enable the maximum range of an A/D converter to be utilised.

Unfortunately the additive disturbance, (and the mean level in particular), cannot be estimated directly. Instead equations (24),(25) must first be reduced to a difference equation in terms of the measured input/output signals $u(t)$,$y(t)$ alone, i.e. a model in the NARMAX form. The task then is to see whether identification applied to a model in this form still yields correct system parameter estimates for the model in its output additive form, - or whether the latter are biased by utilising the NARMAX structure during the identification phase.

Eliminating $z(t)$ between (24) and (25) yields the input/output relation,

$$y(t) = v(t) + \mathrm{F}[u(t-k_i),\, (y(t-k_i)-v(t-k_i))] \qquad i=1,2,... \tag{26}$$

with the corresponding polynomial expansion,

$$y(t) = v(t) + \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{k_{y_1},k_{y_p}=1}^{K} \sum_{k_{u_1},k_{u_q}=1}^{K} c'_{p,q}(k_{y_1},...,k_{y_p};\, k_{u_1},...,k_{u_q}) \times \prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{y_i})) \prod_{i=1}^{q} u(t-k_{u_i}) \tag{27}$$

Notice a prime appears on the coefficients $c'_{p,q}(\cdot)$ which describe the *physical* (output additive) system, since these are not necessarily identical to the *estimated* parameters $c_{p,q}(\cdot)$ of the NARMAX model. Indeed if the NARMAX estimation techniques are to be applied to data from systems with output additive disturbances, it is important to determine exactly how the estimated NARMAX parameters are related to those of the original physical system.

To this end, consider the expansion of the first product in (27) into a sum of nonlinear product terms,

$$\prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{y_i})) = \sum_{r'=0}^{p} \sum_{\substack{\text{all combinations} \\ \text{of } \{k_{y_1},\ldots,k_{y_q}\} \\ \text{taken } p-r' \text{ at a time}}} \prod_{i=1}^{p-r'} y(t-k_{y_i}) \prod_{i=p-r'+1}^{p} v(t-k_{p_i}) \tag{28}$$

An immediate result follows by extracting the first term of (28) which has $r'=0$ (i.e. a pure output nonlinearity with no disturbance factors). Providing the disturbance $v(t)$ contains no constant/d.c component, this gives,

$$\prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{y_i})) = \prod_{i=1}^{p} y(t-k_{y_i}) + \left\{ \begin{matrix} \textit{sum of product terms containing} \\ \textit{at least one factor in } v(t-k_{y_i}) \end{matrix} \right\} \tag{29}$$

Substituting (29) in (27) and comparing with (14) shows by inspection that,

$$c_{p,q,0}(k_1,\ldots,k_{p+q}) = c'_{p,q}(k_1,\ldots,k_{p+q}) \tag{30}$$

This is an important result for it shows that the physical *system* parameters are identical to those estimated using the NARMAX model structure, (providing there is no constant/d.c component in the disturbance).

Although it is possible to derive a complete expansion of equation (29) assuming $v(t)$ to be some zero-mean moving average process, the only additional information this reveals is the structure of the noise model. If however the disturbance $v(t)$ contains a constant or d.c. component, the same conclusion does not necessarily hold. Unlike linear systems where such a disturbance would have no effect on the system parameters, the system parameters in the nonlinear case are likely to be influenced by the presence of the output additive constant component.

Consider for example the case where the additive disturbance consists of a constant offset v_0 alone. Such restrictions will not limit the generality of the result, since it has already been observed that the dynamic noise terms do not affect the identified system parameters. Equation (28) then becomes,

$$\prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{y_i})) = \sum_{r'=0}^{p} \sum_{\substack{\text{all combinations} \\ \text{of } \{k_{y_1}, \ldots, k_{y_{p'+r'}}\} \\ \text{taken } p' \text{ at a time}}} \left[-v_0\right]^{r'} \prod_{i=1}^{p'} y(t-k_{y_i}) \tag{31}$$

where $p' = p - r'$ has been introduced to represent the degree of nonlinearity of the output factor.

Notice that cross-products between the input and the *constant* disturbance v_0, are now structurally indistinguishable from a pure output nonlinearity multiplied by some factor. Indeed the constant disturbance, acting on a p-th order nonlinearity in the output, contributes to all the lower p'-th order input terms of the NARMAX system model.

Notice also that the second summation of equation (31) generates many terms which share the same degree of nonlinearity, but which are associated with different lags. However the lags in the set $\{k_{y_1}, \ldots, k_{y_{p'+r'}}\}$ are not necessarily all unique, and consequently the summation could contain a number of repeated terms. Such terms may be collected together, thereby introducing a factor $\mathbf{C}_{p',r'}(\cdot)$ given by,

$$\mathbf{C}_{p',r'}(k_{y_1}, \ldots, k_{y_{p'+r'}}) = \begin{cases} \prod_{i=1}^{n_c} \begin{bmatrix} N_i \\ n_i \end{bmatrix} & p' \geq 1 \\ 1 & p' = 0 \end{cases} \tag{32}$$

where the term in large brackets represents the binomial coefficient, and the other quantities are given by,

N_i = *number of repetitions of the i-th distinct lag in* $\{k_{y_1}, \ldots, k_{y_{p'+r'}}\}$

n_i = *number of repetitions of the i-th distinct lag in* $\{k_{y_1}, \ldots, k_{y_{p'}}\}$

n_c = *number of distinct lags in* $\{k_{y_1}, \ldots, k_{y_{p'}}\}$

Equation (31) can therefore be rewritten as a sum of unique terms, each multiplied by the appropriate factor $\mathbf{C}_{p',r'}(\cdot)$, so that,

$$\prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{y_i})) = \sum_{r'=0}^{p} \sum_{\substack{\text{all unique combs} \\ \text{of } \{k_{y_1}, \ldots, k_{y_{p'+r'}}\} \\ \text{taken } p' \text{ at a time}}} \mathbf{C}_{p',r'}(k_{y_1}, \ldots, k_{y_{p'+r'}}) \left[-v_0\right]^{r'} \prod_{i=1}^{p'} y(t-k_{y_i}) \tag{33}$$

Hence a direct expression for the expansion of any given term of the true system description into NARMAX form is given by,

$$c'_{p,q}(k_{y_1},\ldots,k_{y_p};\ k_{u_1},\ldots,k_{u_q}) \prod_{i=1}^{p}(y(t-k_{y_i}) - v(t-k_{u_i})) \prod_{i=1}^{q} u(t-k_{u_i}) = \quad (34)$$

$$\sum_{r'=0}^{p} \sum_{\substack{\text{all unique combs} \\ \text{of } \{k_{y_1},\ldots,k_{y_{p'+r'}}\} \\ \text{taken } p' \text{ at a time}}} C_{p',r'}(k_{y_1},\ldots,k_{y_{p'+r'}})\, c'_{p'+r',q}(k_{y_1},\ldots,k_{y_{p'+r'}};\ k_{u_1},\ldots,k_{u_q}) \quad (34)$$

$$\times \left[-v_0\right]^{r'} \prod_{i=1}^{p'} y(t-k_{y_i}) \prod_{i=1}^{q} u(t-k_{u_i}) \quad (34)$$

Equation (34) means that a single term of the true (output additive) system description, containing a p-th degree nonlinearity in the input, when expanded to the NARMAX form, will contribute to all similar but lower order terms of the NARMAX model having only a p'-th degree output nonlinearity.

By the same token a given NARMAX parameter will receive contributions from a number of different terms in the original system description. Hence system parameters estimated using the NARMAX model are *no longer* identical to those of the true system, but are modified by the action of the additive constant. The precise relationship between the two is therefore of considerable interest so that this bias could be removed where appropriate. Substituting then from (34) gives the NARMAX form of equation (27) as,

$$y(t) = v_0 + \sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{r'=0}^{m-p} \sum_{k_{y_1},k_{y_{p'}}=1}^{K} \sum_{k_{y_{p'+1}},k_{y_{p'+r'}}=1}^{K} \sum_{k_{u_1},k_{u_q}=1}^{K} c'_{p'+r',q}(k_{y_1},\ldots,k_{y_{p'+r'}};\ k_{u_1},\ldots,k_{u_q})$$

$$\times \sum_{\substack{\text{all unique combs} \\ \text{of } \{k_{y_1},\ldots,k_{y_{p'+r'}}\} \\ \text{taken } p' \text{ at a time}}} C_{p',r'}(k_{y_1},\ldots,k_{y_{p'+r'}}) \left[-v_0\right]^{r'} \prod_{i=1}^{p'} y(t-k_{y_i}) \prod_{i=1}^{q} u(t-k_{u_i}) \quad (35)$$

where the multiple summation over the lags k_{y_i} in (35) has been broken into two parts, to facilitate direct comparison with the NARMAX expression (14). Indeed for the special case disturbance under consideration, equation (14) reduces to the NARX-like subset,

$$y(t) = \sum_{m=0}^{M} \sum_{p=0}^{m} \sum_{k_{y_1},k_{y_p}=1}^{K} \sum_{k_{u_1},k_{u_q}=1}^{K} c_{p,q,0}(k_{y_1},\ldots,k_{y_p};\ k_{u_1},\ldots,k_{u_q}) \prod_{i=1}^{p} y(t-k_{y_i}) \prod_{i=1}^{q} u(t-k_{u_i}) \quad (36)$$

Equating now the coefficients of like terms across these two equations yields (for

$p,q \neq 0$) the relation between the coefficients $c'(\cdot)$, and the NARMAX parameters $c(\cdot)$,

$$c_{p,q,0}(k_{y_1},\ldots,k_{y_p};\, k_{u_1},\ldots,k_{u_q}) =$$
$$\sum_{r=0}^{M-q} \left[-v_0\right]^r \sum_{k_{y_{p+1}},k_{y_{p+r}}=1}^{K} \mathrm{C}_{p,r}(k_{y_1},\ldots,k_{y_{p+r}})\, c'_{p+r,q}(k_{y_1},\ldots,k_{y_{p+r}};\, k_{u_1},\ldots,k_{u_q}) \tag{37}$$

Notice however the special case of the estimated NARMAX constant term $c_{0,0,0}$ which receives a contribution from the initial term v_0 of (35), in addition to those from all the higher order $c'_{p,q}(\cdot)$ coefficients. In this case also the factor $\mathrm{C}_{p,r}(\cdot)$ reduces to unity by definition. Hence the NARMAX constant term is given by,

$$c_{0,0,0} = v_0 + \sum_{r=0}^{M} \left[-v_0\right]^r \sum_{k_1,k_r=1}^{K} c'_{r,0}(k_1,\ldots,k_r) \tag{38}$$

From these relationships it can be seen that the system parameters of the identified NARMAX model are no longer identical to those of the physical plant, but are related through the action of the output additive constant. However equations (34) and (37) reveal the form of this mapping, in which case it may be possible to recover the original parameters. This approach is adopted in the following section.

5. An Unravelling Algorithm for Systems with Constant Output Disturbances

Although equations (34) and (37) show exactly how system parameters estimated using a NARMAX structure are altered from those of a given "output additive" system, the inverse relation whereby the original system parameters may be recovered from the (identified) system estimates is not immediately apparent.

Consider then the expansion of (37), which represents a set of simultaneous equations between the parameters of the two model forms. The estimated system parameters for example are expressed by equations of the form, (39)

$$c_{p,q,0}(\cdot) = c'_{p,q}(\cdot) - v_0 \sum_{k_{y_{p+1}}}^{K} \mathrm{C}_{p,1}(\cdot)c'_{p+1,q}(\cdot) + v_0^2 \sum_{k_{y_{p+2}}}^{K} \mathrm{C}_{p,2}(\cdot)c'_{p+2,q}(\cdot) - v_0^3 \sum_{k_{y_{p+3}}}^{K} \mathrm{C}_{p,3}(\cdot)c'_{p+3,q}(\cdot) + \cdots$$

$$c_{p+1,q,0}(\cdot) = c'_{p+1,q}(\cdot) - v_0 \sum_{k_{y_{p+2}}}^{K} \mathrm{C}_{p+1,1}(\cdot)c'_{p+2,q}(\cdot) + v_0^2 \sum_{k_{y_{p+3}}}^{K} \mathrm{C}_{p+1,2}(\cdot)c'_{p+3,q}(\cdot) - \cdots$$

$$c_{p+2,q,0}(\cdot) = c'_{p+2,q}(\cdot) - v_0 \sum_{k_{y_{p+3}}}^{K} \mathrm{C}_{p+2,1}(\cdot)c'_{p+3,q}(\cdot) + \cdots$$

The original system parameters however have to be recovered by solving such equations for the coefficients $c'_{p,q}(\cdot)$. Using the equations (39) for example the first few terms in the solution for $c'_{p,q}(\cdot)$ are given by,

$$c_{p,q,0}(\cdot) + v_0 \sum_{k_{y_{p+1}}}^{K} C_{p,1}(\cdot)\, c_{p+1,q,0}(\cdot) - v_0^2 \sum_{k_{y_{p+2}}}^{K} \left[C_{p,2}(\cdot) - \sum_{k_{y_{p+1}}}^{K} C_{p,1}(\cdot)\, C_{p+1,1}(\cdot) \right] c_{p+2,q,0}(\cdot) +$$

$$v_0^3 \sum_{k_{y_{p+3}}}^{K} \begin{bmatrix} [C_{p,3}(\cdot) - \sum_{k_{y_{p+1}}}^{K} C_{p,1}(\cdot)\, C_{p+1,2}(\cdot)] - \\ \sum_{k_{y_{p+2}}}^{K} [C_{p,2}(\cdot) - \sum_{k_{y_{p+1}}}^{K} C_{p,1}(\cdot)\, C_{p+1,1}(\cdot)] C_{p+2,1} \end{bmatrix} c_{p+3,q,0}(\cdot) - \cdots = c'_{p,q}(\cdot) \qquad (40)$$

In general $c'_{p,q}(\cdot)$ is a function of all the higher degree estimated parameters $c_{p+i,q,0}(\cdot)$ cross-multiplied by $-v_0^i$. Hence the desired system coefficients can be expressed more concisely by, (41)

$$c'_{p,q}(k_{y_1},\ldots,k_{y_p}) = c_{p,q,0}(k_{y_1},\ldots,k_{y_p}) - \sum_{i=1}^{M-q} \left[-v_0\right]^i \sum_{k_{y_1},\ldots,k_{y_{p+i}}=1}^{K} \alpha_{p,i}(k_{y_1},\ldots,k_{y_{p+i}})\, c_{p+i,q,0}(k_{y_1},\ldots,k_{y_{p+i}})$$

where by inspection of (40) the $\alpha_{p,i}(\cdot)$ are cast in recursive form according to,

$$\alpha_{p,i}(k_{y_1},\ldots,k_{y_{p+i}}) = C_{p,i}(k_{y_1},\ldots,k_{y_{p+i}}) - \sum_{j=1}^{i-1} C_{p+j,i-j}(k_{y_1},\ldots,k_{y_{p+i}})\, \alpha_{p,j}(k_{y_1},\ldots,k_{y_{p+j}}) \qquad (42)$$

Equation (41) however still contains the unknown constant v_0 which must first be determined. The additional relation required is obtained by including equation (38) in the set of simultaneous equations (39), and solving to give,

$$v_0 = c_{0,0,0} - \sum_{i=1}^{M-q} \left[-v_0\right]^i \sum_{k_{y_1},\ldots,k_{y_i}=1}^{K} \alpha_{0,i}(k_{y_1},\ldots,k_{y_i})\, c_{i,0,0}(k_{y_1},\ldots,k_{y_i}) \qquad (43)$$

Thus equations (41),(43), together with the recursive relation (42), define an algorithm for recovering the original parameters of an "output additive" model from a set of estimated parameters in difference equation form. The algorithm is perhaps best demonstrated by considering a simple example.

5.1. Example: heat exchanger system

Consider then the model of a real heat exchanger identified by Billings and Fadzil,

$$y(t) = 2.072 + 0.9158\, y(t-1) + 0.4788\, u(t-1) - 0.01572\, y^2(t-1) \qquad (44)$$

$$- 0.01133\, u^2(t-1) - 0.002244\, y^2(t-1)u(t-1) - 0.002239\, u^3(t-1) \qquad (44)$$

so that the estimated parameters are given by the set,

$$
\begin{array}{llll}
c_{0,0,0} = 2.072 & & c_{0,1}(;1) = 0.4788 & \quad (45) \\
c_{1,0,0}(1) = 0.9158 & & c_{0,2}(;1,1) = -0.01133 & \\
c_{2,0}(1,1) = -0.01572 & & c_{0,3}(;1,1,1) = -0.002239 & \\
c_{2,1}(1,1;1) = -0.002244 & & &
\end{array}
$$

Although in this case the model may be perfectly adequate for the purpose, suppose the physical system was known to have a d.c. additive disturbance at the output. This would have altered the estimated NARMAX parameters away from the correct physical values. Hence it may be desirable to use the relationships developed above to cast the identified NARMAX model back into the form of equations (24),(25).

Using equations (43),(42) the unknown additive constant v_0 is first determined according to,

$$
\begin{aligned}
v_0 &= c_{0,0,0} + v_0 [\alpha_{0,1}(1) c_{1,0,0}(1)] - v_0^2 [\alpha_{0,2}(1,1) c_{2,0,0}(1,1)] \\
&= c_{0,0,0} + v_0 c_{1,0,0}(1) - v_0^2 c_{2,0,0}(1,1) [1 - C_{1,1}(1,1)]
\end{aligned} \quad (46)
$$

Hence,

$$
0.01572 v_0^2 + (1 - 0.9158) v_0 - 2.072 = 0 \quad \Rightarrow \quad v_0 = \begin{cases} 9.11082 \\ -14.467 \end{cases} \quad (47)
$$

Notice in this case that there are in fact two possible solutions, of which we will choose the positive one for the evaluation of the other terms.

Applying now the relations (41),(42) yields expressions for each of the unknown system parameters $c'_{p,q}(\cdot)$ in terms of the known estimated parameters $c_{p,q,0}(\cdot)$ and the additive output constant v_0 above. Thus, (48)

$$
\begin{aligned}
c'_{1,0}(1) &= c_{1,0,0}(1) + v_0 [\alpha_{1,1}(1,1) c_{2,0,0}(1,1)] &&= c_{1,0,0}(1) + 2 v_0 c_{2,0,0}(1,1) \\
c'_{1,1}(1;1) &= 0 + v_0 [\alpha_{1,1}(1,1) c_{2,1,0}(1,1;1)] &&= 2 v_0 c_{2,1,0}(1,1;1) \\
c'_{0,1}(1) &= c_{0,1,0}(1) + 0 - v_0^2 [\alpha_{0,2}(1,1) c_{2,1,0}(1,1;1)] &&= c_{0,1,0}(1) + v_0^2 c_{2,1,0}(1,1;1)
\end{aligned}
$$

$$
\begin{array}{ll}
c'_{2,1}(1,1;1) = c_{2,1,0}(1,1;1); & c'_{2,0}(1,1) = c_{2,0,0}(1,1) \\
c'_{0,2}(;1,1) = c_{0,2,0}(1,1); & c'_{0,3}(;1,1,1) = c_{0,3,0}(1,1,1)
\end{array}
$$

Substituting the known values from (45) and (47) therefore gives,

$$
v_0 = 9.1108 \qquad c'_{1,1}(1;1) = -0.0409
$$

$$
\begin{array}{llll}
c'_{1,0}(1) = 0.6294 & & c'_{0,1}(1) = 0.2925 & (49) \\
c'_{2,0}(1,1) = -0.01572 & & c'_{0,2}(;1,1) = -0.01133 & \\
c'_{2,1}(1,1;1) = -0.002244 & & c'_{0,3}(;1,1,1) = -0.002239 &
\end{array}
$$

Hence an equivalent form of the NARMAX description (39) is given by the output additive equation pair,

$$
\begin{aligned}
z(t) &= 0.2925\,u(t-1) + 0.6293\,z(t-1) - 0.01572\,z^2(t-1) - 0.0133\,u^2(t-1) \qquad (50) \\
&\quad - 0.0409\,z(t-1)u(t-1) - 0.002244z^2(t-1)u(t-1) - 0.002239\,u^3(t-1)
\end{aligned}
$$

$$
y(t) = z(t) + 9.1108 \qquad (50)
$$

6. Frequency Domain Analysis of Systems with Constant Terms

Frequency domain techniques have proved to be an invaluable tool in linear system analysis and design, so recent work has been aimed at extending this approach to the nonlinear case. In particular algorithms for deriving and interpreting the higher order transfer functions of practical representations such as the discrete time NARX model, or continuous time Nonlinear Integro-Differential Equation (NIDE), have been developed [Peyton Jones, Billings 1990].

Unfortunately the existing algorithms do not accomodate models with internal constants. One approach therefore is to remove the constant term from the model prior to analysis. Notice however, if the model is nonlinear in the output, that this is not simply a matter of discarding the constant term along with any other unwanted disturbance terms, but requires the use of the algorithms developed above.

However in some cases it may be possible to regard the constant disturbance, by virtue of its unchanging nature, as part of the system itself, i.e. to redraw the black box so as to encompass the constant within the system, rather than regarding it as some external disturbance whose effects must be removed. If this approach is to be adopted, then there is a need to extend the time to frequency domain mappings to accomodate a model complete with constant term.

6.1. The frequency response of NARX models with internal constants

Computation of higher order frequency response functions is based on the probing method [Bedrosian, Rice, 1971] whereby the output of both the Volterra model and that of some other nonlinear representation are expanded for a fictional harmonic input given by,

$$u(t) = \sum_{r=1}^{R} e^{j\omega_r t} \tag{51}$$

In the Volterra case applying this input to (2) and integrating yields the expression,

$$y(t) = \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{R} H_n(j\omega_{r_1}, \cdots, j\omega_{r_n})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_n})t} \tag{52}$$

so that for $R=n$ the n-th order transfer function appears as to the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$. The corresponding coefficient in the expansion of the other nonlinear representation must therefore represent this same n-th order Volterra transfer function only this time expressed in terms of its own parameters.

Thus to derive the higher order transfer functions of the NARX model (together with internal constant term) for example, it is first necessary to expand the output in terms of the harmonic input (51). An expression for the n-th order transfer function in terms of the NARX parameters $c_{p,q}(\cdot)$ can then be found by extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$.

The working is broadly similar to that described by Peyton Jones and Billings (1989), and indeed the contribution to the n-th order transfer function arising from pure non-linearities in the input is unchanged, and is given by,

$$H_{n_u}^{asym}(j\omega_1, \cdots, j\omega_n) = \sum_{k_1, k_n=1}^{K} c_{0,n}(k_1, \cdots, k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)} \tag{53}$$

The effect of the additional constant term becomes apparent however when nonlinearities in the output are considered. In order to expand the NARX model as a function of the harmonic input alone, it is necessary to eliminate all the lagged recursive variables in $y(t-k_{y_i})$ by substituting the corresponding Volterra expression from (52), so that,

$$y(t-k_i) = \sum_{\gamma=0}^{N} \alpha^{\gamma} \sum_{r_1, r_\gamma=1}^{R} H_\gamma(j\omega_{r_1}, \cdots, j\omega_{r_\gamma})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_\gamma})(t-k_i)} \tag{54}$$

Here γ is used to enumerate the various order terms, and each lagged variable can therefore be seen to contain the influence of the constant term when γ=0. For convenience a dummy variable α=1 has also been included to keep track of all terms homogenous to degree γ.

Pure output nonlinearities may therefore be expressed as,

$$\prod_{i=1}^{p} y(t-k_i) = \prod_{i=1}^{p} \sum_{\gamma=0}^{N} \alpha^{\gamma} \sum_{r_1,r_\gamma=1}^{R} H_\gamma(j\omega_{r_1}, \cdots, j\omega_{r_\gamma})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_\gamma})(t-k_i)} \tag{55}$$

$$= \sum_{\gamma_1,\gamma_p=0}^{N} \alpha^{\gamma_1+\cdots+\gamma_p} \prod_{i=1}^{p} \sum_{r_1,r_{\gamma_i}=1}^{R} H_{\gamma_i}(j\omega_{r_1}, \cdots, j\omega_{r_{\gamma_i}})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_{\gamma_i}})(t-k_i)}$$

By inspecting the power of the dummy variable α, (55) is seen to contain terms from order 0 up to Np. This is in contrast to the result obtained if the constant component (γ=0) is not considered, in which case the terms run from order p up to Np. The difference may be interpreted as follows. In the absence of a constant term a p-th order nonlinearity in the output contributes only to the Volterra transfer functions of order p and above. With the addition of a constant term however, the output nonlinearity contributes to all Volterra transfer functions including those of order less than p. This consequence may be seen more clearly by dividing the leftmost summation of equation (55) into terms of like order n, giving,

$$\prod_{i=1}^{p} y(t-k_i) = \sum_{n=0}^{Np} \alpha^{n} \sum_{\substack{\gamma_1,\gamma_p=0 \\ \mid \sum\gamma_i=n}}^{n} \prod_{i=1}^{p} \sum_{r_1,r_{\gamma_i}=1}^{R} H_{\gamma_i}(j\omega_{r_1}, \cdots, j\omega_{r_{\gamma_i}})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_{\gamma_i}})(t-k_i)} \tag{56}$$

where the constraint that $\sum\gamma_i=n$ also lowers the limit N to n.

Extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_n)t}$ should now yield the contribution to the n-th order frequency response function that is generated by a p-th degree of nonlinearity in $y(t)$. Thus,

$$H_{n,p}^{asym}(\cdot) = \sum_{\substack{\gamma_1,\gamma_p=0 \\ \mid \sum\gamma_i=n}}^{n} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots, j\omega_{r_{X+\gamma_i}})\, e^{-j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+\gamma_i}})k_i} \tag{57}$$

where the device $X = \sum\gamma_x$, $x = 1..i-1$ has been used to ensure that $H_{n,p}^{asym}(\cdot)$ is a function of the n *different* frequencies $\{\omega_1 \cdots \omega_n\}$.

Equation (57) may be rewritten in recursive form by expanding the last term of the product, and using a more convenient set of subscripts to give,

$$H_{n,p}^{asym}(\cdot) \;=\; \sum_{i=0}^{n} H_i^{asym}(j\omega_1, \cdots, j\omega_i) H_{n-i,p-1}(j\omega_{i+1}, \cdots, j\omega_n)\, e^{-j(\omega_1 + \cdots + \omega_i)k_p} \tag{58}$$

Note that the recursion finishes with $p=1$, and that $H_{n,1}(j\omega_1, \cdots, j\omega_n)$ has the property,

$$H_{n,1}(j\omega_1, \cdots, j\omega_n) = H_n(j\omega_1, \cdots, j\omega_n)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1} \tag{59}$$

Equation (58) however gives the contribution to the n-th order frequency response function generated by a single p-th nonlinearity in $y(t)$, whereas the actual output is composed of many such terms. Their combined contribution to the n-th order frequency response function is therefore,

$$H_{n_y}^{asym}(j\omega_1, \cdots, j\omega_n) = \sum_{p=1}^{M} \sum_{k_1, k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\; H_{n,p}(j\omega_1, \cdots, j\omega_n) \tag{60}$$

Notice that the uppermost limit on the first summation includes all pure output terms of whatever order, whereas the corresponding result for systems with no constant component only includes those of order less than or equal to n. This merely reiterates that a p-th nonlinearity in $y(t)$ in the presence of a constant term contributes to all Volterra kernels, and does not exclude those of order less than p.

In this context notice also that the right hand side of (60) may contain contributions to the n-th order kernel $H_{n_y}(\cdot)$ itself, and these should be collected and brought over to the left side of the equation. The terms concerned are most easily extracted by expanding the uppermost limit of the summation in equation (58) to give,

$$H_{n,p}^{asym}(\cdot) \;=\; \overset{*\,asym}{H_{n,p}}(\cdot) + H_0^{p-1}\; H_n(j\omega_1, \cdots, j\omega_n) \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i} \tag{61}$$

where,

$$\overset{*\,asym}{H_{n,p}}(\cdot) \;=\; \sum_{i=0}^{n-1} H_i^{asym}(j\omega_1, \cdots, j\omega_i) \overset{*}{H}_{n-i,p-1}(j\omega_{i+1}, \cdots, j\omega_n)\, e^{-j(\omega_1 + \cdots + \omega_i)k_p} \tag{62}$$

Thus (60) may be rewritten as,

$$\left[1 - \sum_{p=1}^{M} \sum_{k_1, k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\, H_0^{p-1} \left(H_n(j\omega_1, \cdots, j\omega_n)\right) \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i}\right] H_{n_y}^{asym}(\cdot)$$

$$= \sum_{p=1}^{n} \sum_{k_1, k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\, \overset{*}{H}_{n,p}(j\omega_1, \cdots, j\omega_n) \tag{63}$$

Having obtained expressions for the contributions to the n-th order frequency response function generated by pure input and pure output nonlinearities, there remains the set of pure input/output cross product terms to be considered.

The multiplicative structure of these terms suggests that the n-th order response of a single cross product term would be,

$$e^{-j(\omega_{n-q+1}k_{n-q+1} + \cdots + \omega_{p+q}k_{p+q})} H_{n-q,p}(j\omega_1, \cdots ,j\omega_{n-q}) \tag{64}$$

where the exponential factor relates to the input part of the nonlinearity, and the recursive factor $H_{n-q,p}(\cdot)$ to the output part, (see equations (53),(08) respectively).

The NARX model however contains many cross product terms, so the response (64) must be combined with the appropriate coefficient, and summed over appropriate limits to cover all combinations of p,q for which (64) is non zero, giving,

$$H_{n_{uy}}^{asym}(j\omega_1, \cdots ,j\omega_n) = \tag{65}$$

$$\sum_{q=1}^{n} \sum_{p=1}^{M-1} \sum_{k_1,k_n=1}^{K} c_{p,q}(k_1, \cdots ,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{n-q+1} + \cdots + \omega_{p+q}k_{p+q})} H_{n-q,p}(j\omega_1, \cdots ,j\omega_{n-q})$$

This somewhat heuristic approach may be validated by applying the probing method directly as in the case of the pure output nonlinear terms.

The total frequency response of the NARX model may now be found by summing the contributions from the various sub-classes,

$$H_n(j\omega_1, \cdots ,j\omega_n) = H_{n_u}(\cdot) + H_{n_{uy}}(\cdot) + H_{n_y}(\cdot) \tag{66}$$

Substituting (53),(63) and (65) in (66) gives the total frequency response (for $n > 0$) ,

$$\left[1 - \sum_{p=1}^{M} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots ,k_p)\ H_0^{p-1} \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i}\right] H_n^{asym}(j\omega_1, \cdots ,j\omega_n) =$$

$$\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1, \cdots ,k_n)\ e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+ \sum_{q=1}^{n} \sum_{p=1}^{M-1} \sum_{k_1,k_n=1}^{K} c_{p,q}(k_1, \cdots ,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{n-q+1} + \cdots + \omega_{p+q}k_{p+q})} H_{n-q,p}(j\omega_1, \cdots ,j\omega_{n-q})$$

$$+ \sum_{p=1}^{n} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots ,k_p)\ \overset{*}{H}_{n,p}(j\omega_1, \cdots ,j\omega_n) \tag{67}$$

together the recursive relations (58),(62). Note that (67) gives the asymmetric Volterra transfer function, although it is a simple matter to obtain unique symmetric values by applying (6).

6.2. The frequency response of NIDE models with internal constants

Although the treatment of mean levels is primarily an issue in system identification, it is equally important that constant terms are not ignored when a system modelling approach is adopted. Such physical analysis is generally performed when there is considerable knowledge about the components that make up the system, and results in continuous time nonlinear integro-differential equation (NIDE) models. A polynomial form for a wide class of these models may be represented as,

$$\sum_{m=1}^{M} \sum_{p=0}^{m} \sum_{l_1, l_{p+q}=-L}^{L} c_{p,q}(l_1, \cdots, l_{p+q}) \prod_{i=1}^{p} D^{l_i} y(t) \prod_{i=p+1}^{p+q} D^{l_i} u(t) + c_0 = 0 \qquad (68)$$

where $p + q = m$, and where $c_{p,q}(l_1, \cdots, l_{p+q})$ denotes the coefficient associated with a p-th order nonlinearity in $D^{l_i} y(t)$ and a q-th order nonlinearity in $D^{l_i} u(t)$. The D operator itself is defined by,

$$D^l x(t) = \begin{cases} \dfrac{d^l x(t)}{dt^l} & l \geq 0 \\ \int_{-\infty}^{t} \cdots \int_{-\infty}^{\tau_2} x(\tau_1) d\tau_1 \cdots d\tau_{|l|} & l < 0 \end{cases} \qquad (69)$$

Note that the lower limits of integration in (69) may be raised to zero for causal systems where $x(\tau)=0 \; forall \; \tau<0$.

Consider for example a specific instance of the NIDE model such as the well known Duffings equation,

$$D^2 y(t) + 2\zeta\omega_n Dy(t) + \omega_n^2 y(t) + \omega_n^2 \varepsilon y(t)^3 - u(t)=0 \qquad (70)$$

This may be obtained from the general form (68) with

$$c_{1,0}(2) = 1.0; \qquad c_{1,0}(1) = 2\zeta\omega_n; \qquad c_{1,0}(0) = \omega_n^2; \qquad (71)$$

$$c_{3,0}(0) = \omega_n^2 \varepsilon; \qquad c_{0,1}(0) = 1.0; \qquad else \;\; c_{p,q}(\cdot) = 0; \qquad (71)$$

where ζ is the damping ratio, and ω_n the natural frequency of the Duffings oscillator.

Once again the advantage of the NIDE model is its relatively small parameter set, and its widespread usage in physical system modelling.

The n-th order frequency response of NIDE models, in the absence of a constant term, has been derived previously [Billings, Peyton Jones 1989]. However the existing results may be extended to models incorporating a constant term in a manner similar to that developed for the NARX model above. This yields,

$$-\left[\sum_{p=1}^{M}\sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1,\cdots,l_p)\, H_0^{p-1} \times \sum_{i=1}^{p} (j\omega_1+\cdots+j\omega_n)^{l_i}\right] H_n^{asym}(j\omega_1,\cdots,j\omega_n) =$$

$$\sum_{l_1,l_n=-L}^{L} c_{0,n}(l_1,\cdots,l_n)\,(j\omega_1)^{l_1}\cdots(j\omega_n)^{l_n}$$

$$+\sum_{q=1}^{n}\sum_{p=1}^{M-1}\sum_{l_1,l_n=-L}^{L} c_{p,q}(l_1,\cdots,l_{p+q})\,(j\omega_{n-q+1})^{l_{n-q+1}}\cdots(j\omega_{p+q})^{l_{p+q}}\, H_{n-q,p}(j\omega_1,\cdots,j\omega_{n-q})$$

$$+\sum_{p=1}^{M}\sum_{l_1,l_p=-L}^{L} c_{p,0}(l_1,\cdots,l_p)\,\overset{*}{H}_{n,p}(j\omega_1,\cdots,j\omega_n) \qquad (72)$$

where $H_{n,p}(\cdot)$ is used to denote the contribution to the n-th order frequency response function that is generated by the p-th degree of nonlinearity in the output,

$$H_{n,p}^{asym}(\cdot) = \sum_{i=0}^{n} H_i(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\,(j\omega_1+\cdots+j\omega_i)^{l_p} \qquad (73)$$

and where $\overset{*}{H}_{n,p}(\cdot)$ is the subset of $H_{n,p}(\cdot)$ which does not contain any of the bracketed terms in equation (72). Thus,

$$\overset{*}{H}{}_{n,p}^{asym}(\cdot) = \sum_{i=0}^{n-1} H_i^{asym}(j\omega_1,\cdots,j\omega_i)\overset{*}{H}_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\,(j\omega_1+\cdots+j\omega_i)^{l_p} \qquad (74)$$

7. Conclusion

The treatment of mean levels in linear system identification and analysis is greatly eased by the superposition and commutative properties that apply in this case. For nonlinear systems however these properties do not hold, and commonly used linear techniques may give misleading results.

Data prefiltering methods for example have been shown to alter the model structure of the system under analysis, and are therefore inappropriate in a nonlinear constext.

Another approach then, is to estimate the constant component explicitly with a view to discarding it in subsequent analysis. Unfortunately the occurrence of a constant term within a nonlinear difference equation is not so straightforward, and an it cannot simply be discarded without also removing the effects its presence may have induced upon the other system parameters. Algebraic relations which reveal the form of these effects have therefore been derived, and an unravelling algorithm developed to recover the original estimates.

Alternatively an estimated constant term may be regared as part of the system itself, in which case is must be incorporated in the subsequent analysis. Previous results which enabled the frequency response of nonlinear difference and differential equations to be computed, have therefore been extended to encompass systems with constant terms. These extended mappings are substantially different from those obtained when constant terms are excluded, and therefore indicate the importance of correctly treating constant components in nonlinear system analysis.

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