Article:

Palczewski, J and Schenk-Hoppe, KR (2010) Market selection of constant proportions investment strategies in continuous time. Journal of Mathematical Economics, 46 (2). 248 - 266. ISSN: 0304-4068

https://doi.org/10.1016/j.jmateco.2009.11.011

Market Selection of Constant Proportions Investment Strategies in Continuous Time*

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October 29, 2009

Abstract

This paper studies the wealth dynamics of investors holding self-financing portfolios in a continuous-time model of a financial market. Asset prices are endogenously determined by market clearing. We derive results on the asymptotic dynamics of the wealth distribution and asset prices for constant proportions investment strategies. This study is the first step towards a theory of continuous-time asset pricing that combines concepts from mathematical finance and economics by drawing on evolutionary ideas.

JEL-Classification: G11, G12.
Key words: evolutionary finance, wealth dynamics, endogenous asset prices, random dynamical systems.

1 Introduction

This paper aims at developing a theory of asset pricing that is based on the market interaction of traders in a continuous-time mathematical finance framework. While mathematical finance has offered deep insights in the dynamics of portfolio payoffs under the assumption of an exogenous price

*We are most grateful for comments by participants of the AMaMeF 2007 Midterm Conference, Vienna, and the 2008 Hausdorff workshop on Finance, Stochastics and Insurance at the University of Bonn. Jesper Lund Pedersen provided helpful comments during his visit to the University of Leeds in 2007. Financial support by the Finance Market Fund, Norway (project Stability of Financial Markets: An Evolutionary Approach) and the National Center of Competence in Research "Financial Valuation and Risk Management," Switzerland (project Behavioural and Evolutionary Finance) is gratefully acknowledged.

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process, economists prefer—in a market context—equating demand and supply through an endogenous price mechanism. This market interaction of investors plays a central role in our approach; it is modeled through the introduction of endogenous prices (driven by demand and supply) in the classical mathematical finance model. Randomness stems from exogenous asset payoff processes (dividends) and variation in traders' behavior.

Our analysis focuses on the survival and extinction of investment strategies which is defined through the asymptotic outcome of the wealth dynamics. This evolutionary view to financial markets is explored in discrete-time models by Evstigneev et al. [10, 11, 12] (see also their survey [9]). In the present paper these ideas are used to develop an evolutionary finance model in continuous time. This approach incorporates market interaction of traders into the workhorse model of mathematical finance. The continuous-time setting overcomes the problem of a priori setting a frequency of trade. It also defines a benchmark for the specification of discrete-time models in which the wealth dynamics is consistent over different trading frequencies (see Palczewski and Schenk-Hoppé [23]).

The wealth dynamics in our continuous-time evolutionary finance model is described by a random dynamical system (Arnold [2]). This system can be written as a non-linear differential equation with random coefficients that has not been studied before. Its asymptotic analysis requires the application of techniques that are new in this context. Specifically, we use the concept of a random fixed point (Schenk-Hoppé and Schmalfuss [28]), the ergodic theory for Markov processes (Anderson [1]) and the arcsine law for Markov chains (Freedman [14]).

Whilst the model is developed in the most general setting, our analysis is restricted to time-invariant investment strategies. These self-financing strategies prescribe to rebalance a portfolio so as to maintain constant proportions over time. This class is quite common in financial theory and practice, see e.g. Browne [7], Mulvey and Ziemba [21], Perold and Sharpe [26]. The assumption of time-invariance considerably reduces the level of mathematical difficulty. This allows to place the main emphasis on the novel ideas developed in this paper. Future research will aim to investigate the general, but mathematically more demanding, case of time-dependent strategies.

Our main result is the identification of a unique investment strategy λ^* that is asymptotically optimal in a market in which only time-invariant strategies are present. This strategy prescribes to divide wealth proportionally to the average relative dividend intensities of assets. We show that any other time-invariant investment strategy interacting in the market will become extinct by losing its wealth to the strategy λ^*. This finding has implications for asset pricing. If at least one investor follows the strategy λ^*, asset prices converge to a 'fundamental' value which (except for risk-free bonds) does not coincide with the usual valuation because it is based on averaging *relative* rather than absolute instantaneous dividend payments.

Our results provide the basis for a recommendation to portfolio managers targeting the well-documented success of pairs-trading strategies, see e.g. Gatev et al. [15]. Investing according to the strategy λ^* generates excess returns (in the long-term) if there are assets whose relative valuation does not coincide with the benchmark. Indeed the proposed portfolio strategy is less risky than pairs-trading as it does not involve short positions.

The modeling approach presented here provides an alternative framework for portfolio optimization under the market impact of trades. Rather than postulating an exogenous price impact function for one large trader (e.g. Bank and Baum [3]), our model provides an endogenous mechanism for the market impact of transactions (large and small). The impact increases with the size of a transaction but the precise market response depends on the wealth distribution across investors and their investment strategies.

The evolutionary approach to asset pricing presented here is related to asset pricing theories based on the notion of excess returns, e.g. Luenberger [18] and Platen [24], which can be traced back to the setting of betting markets studied by Kelly [17]. These theories, however, do not take into account market interaction of traders. Stochastic general equilibrium models (which do have endogenous prices), on the other hand, suffer from the interconnectedness of consumption and investment decisions. This feature precludes clear-cut results on the long-term dynamics of asset prices if markets are incomplete, see e.g. Blume and Easley [6].

The paper is organized in the following fashion. Section 2 introduces the model. Section 3 presents general selection results on the market dynamics. Section 4 considers the particular case of Markovian dividend intensities. Section 5 concludes. All proofs are collected in the Appendix.

2 The Model

This section derives the general evolutionary finance model in continuous time.

Consider the following description of a financial market in continuous time which is based on the standard approach in mathematical finance (e.g., Björk [5] or Pliska [25]). There are K assets (stocks) with cumulative dividend payments $D(t) = (D_1(t), ..., D_K(t))$, $t \geq 0$. Denote the price process, which will be specified later, by $S(t) = (S_1(t), ..., S_K(t))$. Each asset is in positive net supply of one. There are I investors. The portfolio (in numbers of physical units of assets) of investor i is denoted by $\theta^i(t) = (\theta^i_1(t), ..., \theta^i_K(t))$. His cumulative consumption process is given by $C^i(t)$. For a self-financing portfolio-consumption process $(\theta^i(t), C^i(t))$, the dynamics of investor i's wealth $V^i(t) = \sum_{k=1}^K \theta_k^i(t) S_k(t)$ is

$$dV^i(t) = \sum_{k=1}^K \theta_k^i(t-)\big(dS_k(t) + dD_k(t)\big) - dC^i(t) \tag{1}$$

($t-$ denotes the left-hand limit). The self-financing property of the strategy means that changes in value can be attributed either to changes in asset prices, dividend income or consumption expenditure. If there is a jump in either price or dividend, the gain (or loss) is attributed to the investor holding the asset just prior to the occurrence of this event.

Each investor's portfolio can be written as

$$\theta_k^i(t) = \frac{\lambda_k^i(t) V^i(t)}{S_k(t)} \tag{2}$$

with some real-valued process $\lambda_k^i(t)$, provided $V^i(t) \neq 0$ and $S_k(t) \neq 0$. The quantity $\lambda_k^i(t)$ can be interpreted as the trader's budget share allocated to the holding in asset k. We will refer to $\lambda^i(t) = (\lambda_1^i(t), ..., \lambda_K^i(t))$ as an *investment strategy*. It follows from the definition of $V^i(t)$ that $\lambda_1^i(t) + \ldots + \lambda_K^i(t) = 1$. Since assets are in net supply of one, market clearing and (2) imply

$$S_k(t) = \lambda_k^1(t)\, V^1(t) + \ldots + \lambda_k^I(t)\, V^I(t) = \langle \lambda_k(t), V(t) \rangle, \tag{3}$$

where $\lambda_k(t) = (\lambda_k^1(t), ..., \lambda_k^I(t))$, i.e. the market value of each asset is given by the aggregate investment in that asset. Within a framework in which economic agents maximize utility, this pricing equation can be seen as a linear approximation of the market demand function at an equilibrium. Equation (3) gives an explicit description of the marginal impact of changes in investors' strategies and wealth on the market-clearing price. The relation of this pricing equation and demand functions is explained in detail in Evstigneev et al. [11].

Combining the self-financing condition (1) with (2) and (3), one obtains

$$dV^i(t) = \sum_{k=1}^K \frac{\lambda_k^i(t-) V^i(t-)}{\langle \lambda_k(t-), V(t-) \rangle} \big(d\langle \lambda_k(t), V(t) \rangle + dD_k(t)\big) - dC^i(t) \tag{4}$$

for $i = 1, ..., I$.

The system of equations (4) provides the foundation of our approach. It describes the wealth dynamics for *given* investment strategies. This approach is opposite to the usual view in mathematical finance which derives portfolio budget shares (investment strategies) from asset holdings (portfolios, see e.g. Pliska [25, Sect. 2.5 and 5.6] and Björk [5, Sect. 6.2]). In a (very different) economic context a similar approach is promoted by Shapley and Shubik [29, Sect. III].

Suppose for each trader $i = 1, ..., I$ a process $\lambda^i(t) = (\lambda^i_1(t), ..., \lambda^i_K(t))$ is given with $\lambda^i_k(t) > 0$ for all k and $\lambda^i_1(t) + \ldots + \lambda^i_K(t) = 1$. Then (4) defines the dynamics for the vector of investors' wealth $V(t) = (V^1(t), ..., V^I(t))$ (provided it makes sense from a mathematical point of view). The main innovation is that the traders' strategies rather than the price process are taken as a primitive of the model. The evolution of investors' wealth depends on the investment strategies present in the market, the wealth distribution and the asset dividend payoffs. Asset prices, which are determined by the investment strategies and the wealth distribution, (3), are *endogenous*.

Our model is closed, i.e. all markets clear, because (3) ensures market-clearing for the financial assets, while Walras' law gives market clearing for the consumption good. To verify this claim, sum up equation (4) over $i = 1, ..., I$ and denote $\bar{C}(t) = \sum_{i=1}^{I} C^i(t)$ and $\bar{D}(t) = \sum_{k=1}^{K} D_k(t)$ to obtain

$$d\bar{C}(t) = d\bar{D}(t) \tag{5}$$

because $\sum_{k=1}^{K} \lambda^i_k(t) = 1$ for all i, i.e. every trader exhausts his budget. The price of the consumption good is normalized to one and, therefore, does not appear in (1). Asset prices are therefore relative prices, i.e. expressed in terms of the consumption good price. The consumption good does not play the role of money because it cannot be stored and, thus, cannot be used for the intertemporal transfer of wealth. On the other hand, consols (financial assets with a constant payoff stream but with price risk) can be accommodated in this model.

An alternative derivation of the above model is provided in Palczewski and Schenk-Hoppé [23]. They obtain the dynamics (4) as the limit of the discrete-time evolutionary finance model (Evstigneev et al. [10, 11, 12]) as the length of the time-step tends to zero.

This paper studies the wealth dynamics (4) under the following assumptions.

(A.1) The dividend dynamics is expressed by intensities, $dD_k(t) = \delta_k(t)dt$ with non-negative processes $\delta_k(t)$, $k = 1, ..., K$. Moreover, $\bar{\delta}(t) = \sum_{k=1}^{K} \delta_k(t) > 0$ is a continuous process of finite variation.

(A.2) Investment strategies are time-invariant, $\lambda^i_k(t) = \lambda^i_k$ with $\lambda^i_k > 0$ and $\sum_{k=1}^{K} \lambda^i_k = 1$.

(A.3) The consumption rate is proportional to wealth, $dC^i(t) = c\,V^i(t)dt$ with some constant $c > 0$ which is independent of i.

Under these assumptions the wealth dynamics can be described by a system of differential equations with random coefficients. Solutions to such systems have a sample-path-wise representation, which considerably facilitates the analysis. Relaxing these assumptions to allow for diffusion processes as a description of both time-dependent strategies and the dividend process leads to several complications; even the existence and uniqueness of solutions to the wealth dynamics equation is an open problem. When restricting the set of time-dependent strategies to processes of finite variation and keeping assumption (A.1) in place still leads to the wealth dynamics that cannot be described by a differential equation. Equation (4) has to be interpreted in the integral form and contains integrals with respect to time and $\lambda_k(t)$, $k = 1, \ldots, K$. This implies that the long-term evolution of $V(t)$ can be radically changed by the short-term behavior of the investment strategies. This makes the analysis of the wealth dynamics much more involved and the results will supposedly be less clear-cut than in the case studied in this paper. The analysis of the wealth dynamics under less restrictive assumptions than those imposed here is left for future research.

The assumption of time-invariance of investment strategies implies that changes in asset prices can be attributed to the wealth dynamics: condition (A.2) yields $dS_k(t) = d\langle \lambda_k, V(t)\rangle = \langle \lambda_k, dV(t)\rangle$. Time-invariant strategies yield portfolio positions with constant investment proportions over time. This class of strategies is quite common in financial theory and practice, see e.g. Browne [7], MacLean, Thorp and Ziemba [19], Mulvey and Ziemba [21], Perold and Sharpe [26]. The particular proportions $(\lambda_1^i, ..., \lambda_K^i)$ can depend on a random event occurring at the initial time.

Postulating a common consumption rate for all the investors ensures that no trader has an advantage from a lower consumption rate (i.e. consumption per unit owned). Consider the difference in the growth rate of the wealth of two investors i, j with identical strategies. Suppose investor i has a lower consumption rate, $c^i < c^j$. Then (4) implies

$$\frac{dV^i(t)}{V^i(t-)} - \frac{dV^j(t)}{V^j(t-)} = (-c^i + c^j)dt > 0,$$

i.e. the growth rate of investor i's wealth is always higher than that of investor j. Assumption (A.3) therefore ensures a level playing-field for all the investors. It also places the emphasis on the return-based 'finance view' rather than the consumption-based 'economics view' in this line of research.

Under assumption (A.3) the total book value of the wealth $\bar{V}(t) = V^1(t) + \ldots + V^I(t)$ and the instantaneous total dividend $\bar{\delta}(t) = \delta_1(t) + \ldots + \delta_K(t)$ are related by

$$\bar{V}(t) = \bar{\delta}(t)/c \tag{6}$$

(which is immediate from (5)). The relation (6) is merely a reformulation of the market-clearing condition for the consumption good. The total amount spent on consumption (measured in the price of the consumption good) is equal to its supply: $c\bar{V}(t) = \bar{\delta}(t)$. This imposes an additional condition on the initial value $(V^1(0), ..., V^I(0))$ because the market for the consumption good only clears if $\bar{V}(0) = \bar{\delta}(0)/c$.

Under conditions (A.1)–(A.3) equation (4) can be written as

$$dV^i(t) = \sum_{k=1}^{K} \frac{\lambda_k^i V^i(t-)}{\langle \lambda_k, V(t-)\rangle} \big(\langle \lambda_k, dV(t)\rangle + \delta_k(t)dt\big) - cV^i(t)dt \tag{7}$$

with $i = 1, ..., I$.

The following lemma shows that the dynamics (7) is well defined and the solution to this integral equation generates a random dynamical system with continuous sample paths. The proof, as well as that of any other result in this paper, is given in the Appendix.

Lemma 1 *Assume (A.1)–(A.3). The dynamics (7) has a unique solution for every initial value $V(0) \in \mathbb{R}_+^I$ with $\bar{V}(0) = \bar{\delta}(0)/c$. Moreover, the solution $V(t)$ has continuous sample paths of finite variation satisfying* (6)*.*

The dynamics for feasible initial values can be described by a one-dimensional system under the following assumption.

(A.4) There are two investors in the market, i.e. $I = 2$.

Define the relative wealth $w^i(t) = V^i(t)/\bar{V}(t)$, $i = 1, 2$. Since $dw^i(t) = [dV^i(t) - w^i(t)d\bar{V}(t)]/\bar{V}(t)$, (7) yields

$$dw^1(t) = c\, w^1(t)\, \frac{\sum_{k=1}^{K} \frac{\lambda_k^1}{[\lambda_k^1-\lambda_k^2]w^1(t)+\lambda_k^2} \rho_k(t) - 1}{\sum_{k=1}^{K} \frac{\lambda_k^1\lambda_k^2}{[\lambda_k^1-\lambda_k^2]w^1(t)+\lambda_k^2}}\, dt \tag{8}$$

with $\rho_k(t) = \delta_k(t)/\bar{\delta}(t)$ denoting the relative dividend intensity. Models with just two investors are common e.g. in evolutionary game theory where an 'incumbent' competes against a 'mutant.'

Lemma 2 *Assume (A.1)–(A.4). Then one has:*

(i) The dynamics (8) *is well-defined for every initial value $w^1(0) \in [0, 1]$.*

(ii) The solutions to (7) *and* (8) *are related as follows:*

Suppose $(V^1(t), V^2(t)) \geq 0$ is a solution to (7)*. If the initial value satisfies $\bar{V}(0) = \bar{\delta}(0)/c$ then $w^1(t) = V^1(t)/\bar{V}(t)$ solves* (8)*.*

Suppose $w^1(t)$ is a solution to (8)*. Then $(V^1(t), V^2(t)) = (\bar{\delta}(t)/c)(w^1(t), 1 - w^1(t))$ solves* (7)*.*

The dynamics (8) possesses the steady states, $w^1(t) = 0$ and $w^1(t) = 1$. Each steady state corresponds to a situation in which one investment strategy holds the entire wealth. Therefore, the (relative) asset prices are given by that strategy. For instance, if $w^1(t) = 0$,

$$\frac{S_k(t)}{S_n(t)} = \frac{\lambda_k^2\, V^2(t)}{\lambda_n^2\, V^2(t)} = \frac{\lambda_k^2}{\lambda_n^2}, \qquad k, n = 1, \ldots, K.$$

The remainder of the paper is concerned with the convergence of the wealth dynamics to the steady state $w^1(t) = 0$.

3 Selection Dynamics

The main results on the long-term behavior of the wealth dynamics are presented in this section. We identify a particular investment strategy which, under certain conditions, gathers all wealth regardless of the initial state. This constant proportions strategy is selected by the market through its global dynamics.

Throughout the remainder of the paper we impose assumptions (A.1)–(A.4) and

(A.5) There are two assets, i.e. $K = 2$.

Under condition (A.5) each trading strategy can be represented by a real-valued random variable λ^i with $\lambda^i_1 = \lambda^i$ and $\lambda^i_2 = 1 - \lambda^i$. We further write $\rho(t)$ for $\rho_1(t) = \delta_1(t)/[\delta_1(t) + \delta_2(t)]$ (with $\rho_2(t) = 1 - \rho(t)$).

If both investors' strategies are identical, $\lambda^1 = \lambda^2$, their relative wealth is constant ((8) simplifies to $dw^1(t) = 0\,dt$) and the ratio of asset prices is time-invariant because (3) implies

$$\frac{S_1(t)}{S_2(t)} = \frac{\lambda^1 \bar{V}(t)}{(1-\lambda^1)\bar{V}(t)} = \frac{\lambda^1}{1-\lambda^1}.$$

Assuming that $\lambda^1 \neq \lambda^2$, (8) can be factorized as

$$dw^1(t) = c\frac{-w^1(t)\big(1-w^1(t)\big)\big((\lambda^2-\lambda^1)^2 w^1(t) + (\lambda^2-\lambda^1)(\rho(t)-\lambda^2)\big)}{\big(\lambda^2(\lambda^2-1)-\lambda^1(\lambda^1-1)\big)w^1(t) + \lambda^2(1-\lambda^2)}\,dt. \tag{9}$$

Our aim is to study the dynamics of (9) with the goal of finding a unique investment strategy that is selected against any other strategy when both interact in the asset market.

Let us assume existence of the limit

$$\bar{\rho} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \rho(s)\,ds \tag{10}$$

with $0 < \bar{\rho} < 1$. In general the sample mean $\bar{\rho}$ is a random variable. In many applications, however, $\bar{\rho}$ is constant; for instance if $\rho(t)$ is a positive recurrent Markov process or a stationary ergodic process.

Define the time-invariant strategy

$$\lambda^* = \bar{\rho}. \tag{11}$$

The construction of this investment strategy only uses 'fundamental' data: the investment proportions correspond to the time-average of the relative dividend intensities of the assets. This strategy has a game-theoretic interpretation, see Section 4.

The following theorem provides a result on the asymptotic dynamics of a λ^*-investor's relative wealth.

Theorem 1 *Let $\lambda^2 = \lambda^*$ and $\lambda^1 \neq \lambda^2$. Then, for each initial value $w^1(0) \in (0,1)$,*

(i) $\lim_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\,ds = 0$*; and*

(ii) $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{[0,\varepsilon]}\big(w^1(s)\big)\,ds = 1$ *for all* $\varepsilon > 0$.

The first assertion of Theorem 1 states that, for every sample path, the relative wealth of the λ^*-investor converges to 1 in the sense of Cesàro, while that of the other investor converges to 0. The second result shows that the relative wealth of the λ^*-investor asymptotically stays arbitrarily close to 1. In this sense the market selects the λ^*-investor when competing with an investor who uses any other time-invariant strategy.

The convergence result in Theorem 1 cannot be strengthened without additional assumptions on the relative dividend process. The asymmetry of the dynamics of $w^1(t)$ allows for the construction of a process $\rho(t)$ such that $w^1(t)$ converges to 0 in the Cesàro sense but reaches a deterministic level $l > 0$ on every time interval $[t,\infty)$. The process $w^1(t)$ therefore does not converge to 0 in the usual sense. The estimates and results used in the following example are derived in detail in the Appendix.

Counterexample Let $\lambda^1 = 1/4$, $\lambda^2 = 1/2$ and $c = 1$. The dynamics of $w^1(t)$, (9), simplifies to

$$dw^1(t) = -\frac{w^1(t)(1-w^1(t))\big(w^1(t) + 4(\rho(t)-\frac{1}{2})\big)}{4 - w^1(t)}\,dt. \tag{12}$$

The following estimates hold: If $\rho(u) \leq \frac{1}{4}$ and $w^1(u) \leq \frac{1}{2}$ for all $u \in [s,t]$, $0 \leq s < t$, then

$$w^1(t) \geq w^1(s)\exp\Big(\frac{t-s}{16}\Big). \tag{13}$$

If $\rho(u) \leq \frac{1}{2}$ and $w^1(u) \leq \frac{1}{2}$ for all $u \in [s,t]$, $0 \leq s < t$, then

$$w^1(t) \geq \frac{1}{\frac{2}{7}(t-s) + \frac{1}{w^1(s)}}. \tag{14}$$

Fix any $l \in (0,1/2)$. If $w^1(0) \geq l$, let $a_0 = b_0 = 0$. Otherwise, let $a_0 = 0$ and $b_0 = 16\log\big(l/w^1(0)\big)$. Further, define for $n \geq 1$

$$a_n = \frac{10^n}{l},$$
$$b_n = 16\,n\log(10),$$

and for $t \geq 0$

$$\rho(t) = \begin{cases} \frac{1}{2} \text{ if, for some } k,\ t \in \Big[\sum_{n=0}^{k}(a_n+b_n), \sum_{n=0}^{k}(a_n+b_n)+a_{k+1}\Big), \\ \frac{1}{4} \text{ if, for some } k,\ t \in \Big[\sum_{n=0}^{k}(a_n+b_n)+a_{k+1}, \sum_{n=0}^{k+1}(a_n+b_n)\Big). \end{cases} \tag{15}$$

Lemma 3 *The model with $\rho(t)$ defined by* (15) *has the following properties:*

(i) $\bar{\rho} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \rho(u)du = \frac{1}{2}$*; and*

(ii) $w^1\left(\sum_{n=0}^{k}(a_n+b_n)\right) \geq l$ *for all* $k \geq 1$.

The process $\rho(t)$ fluctuates between the two values $1/4$ and $1/2$. The length of time during which it is equal to $1/4$ grows linearly in n, while the time it is equal to $1/2$ grows exponentially in n. This implies $\rho(t)$ has a long-term mean of $1/2$ which is equal to λ^2 and, therefore, investor 2 follows the strategy λ^*. Lemma 3(i) and Theorem 1 ensure convergence of $w^1(t)$ to 0 in the Cesàro mean. The asymmetry in the dynamics of $w^1(t)$, however, precludes that $\lim_{t\to\infty} w^1(t) = 0$. If the relative dividend intensity $\rho(t) = 1/4$, investor 1's share of the total wealth increases exponentially fast. In contrast, if $\rho(t) = 1/2$, the relative wealth of investor 1 decreases linearly. This asymmetry holds as long as the relative wealth of investor 1 is small enough, i.e. $w^1(t) \leq 1/2$. Lemma 3(ii) quantifies these findings: the relative wealth of investor 1, $w^1(t)$, reaches the level $l > 0$ at a sequence of times t converging to infinity, i.e. $\limsup_{t\to\infty} w^1(t) \geq l$. □

The following Theorem presents conditions on the relative dividend intensity ensuring convergence of $w^1(t)$.

Theorem 2 *Let* $\lambda^2 = \lambda^*$ *and* $\lambda^1 \neq \lambda^2$*. Fix a sample path of the dividend intensity. Suppose*

(B) there is a real number z such that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{(-\infty,z)}\Big(\operatorname{sgn}(\bar{\rho}-\lambda^1)\int_s^t (\rho(u)-\bar{\rho})\,du\Big)\,ds > 0.$$

Then, for every initial value $w^1(0) \in (0,1)$*, the relative wealth of investor 1 converges to* 0 *(while the relative wealth of investor 2 converges to* 1*), i.e.*

$$\lim_{t\to\infty} w^1(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} w^2(t) = 1.$$

Theorem 2 strengthens the convergence result of Theorem 1 under a condition on the degree of asymmetry of $\rho(t)$ with respect to its time-average $\bar{\rho}$. The condition is trivially satisfied if $\rho(t)$ is time-invariant, i.e. $\rho(t) = \rho(0)$ for all t. The formulation of this result uses the sample-path wise interpretation of solution. If a realization of the process $\rho(t)$ satisfies condition (B), then convergence $w^1(t) \to 0$ occurs for this particular sample path.

Under condition (B), if one investor divides the wealth in proportions $(\lambda^*, 1-\lambda^*)$ while the other investor pursues a different time-invariant strategy, the first will overtake the latter: the relative wealth of the λ^*-investor tends to one. The result holds globally, i.e. regardless of the initial wealth distribution across the investors. Interpreting the dynamics as a fight between an incumbent and a mutant strategy, the result says an incumbent pursuing the strategy λ^* cannot be driven out of the market by any investor with some other time-invariant strategy. In this sense, a λ^*-market is stable.

A benchmark for the asset prices can be derived from Theorem 2. If the strategy λ^* is present on the market, the asset prices have the following asymptotic property:

$$\lim_{t\to\infty} \frac{S_1(t)}{S_2(t)} = \frac{\bar{\rho}}{1-\bar{\rho}} = \frac{\lim_{t\to\infty} \frac{1}{t}\int_0^t \delta_1(s)/(\delta_1(s)+\delta_2(s))\,ds}{\lim_{t\to\infty} \frac{1}{t}\int_0^t \delta_2(s)/(\delta_1(s)+\delta_2(s))\,ds}.$$

The ratio of the asset prices, in the long-term, is equal to that of the time-averages of the relative dividend intensities. This valuation is fundamental as it only uses dividend data. For consols it coincides with the usual concept of the fundamental value.

The portfolio positions of the λ^*-investor are given by

$$\frac{\theta_1^2(t)}{\theta_2^2(t)} = \frac{\lambda^*}{1-\lambda^*}\,\frac{S_2(t)}{S_1(t)}.$$

If the relative prices coincide with the benchmark $(\bar{\rho}, 1-\bar{\rho})$, the λ^*-investor holds the same number of units of each asset. Otherwise, he purchases a portfolio that is geared towards the undervalued asset. This asset allocation implies that the λ^*-investor's relative wealth grows asymptotically (at the expense of the competitor). In this sense the λ^*-investor experiences excess growth even though the asset prices are endogenous and change their statistical behavior (which is non-stationary) over time.

The speed of convergence of $w^1(t) \to 0$ in Theorem 2 is not exponentially fast which is at odds with the corresponding models in discrete time, [10, 16]. This observation follows from an analysis of the linearization at the steady state $w^1(t) = 0$. The variational equation at $w^1(t) = 0$ of the dynamics (9) is given by

$$dv(t) = c\,\frac{\lambda^1-\lambda^2}{\lambda^2(1-\lambda^2)}\big(\rho(t)-\lambda^2\big)\,v(t)\,dt,$$

which shows that the exponential growth rate of $v(t)$ is

$$c\,\frac{\lambda^1-\lambda^2}{\lambda^2(1-\lambda^2)}\big(\bar{\rho}-\lambda^2\big).$$

If $\lambda^2=\lambda^*$, the exponential growth rate is equal to zero for *every* investment strategy λ^1. For any time-invariant investment strategy $\lambda^2\neq\lambda^*$, however, there is an investment strategy λ^1 such that the growth rate is strictly positive, i.e. $v(t)$ diverges from 0 exponentially fast. If $\lambda^2<\lambda^*$, take any $\lambda^1\in(\lambda^2,1)$; otherwise take $\lambda^1\in(0,\lambda^2)$.

4 Markovian Dividend Intensities

This section studies the condition imposed in the main convergence result, Theorem 2. The condition (B) is quite technical and does not reveal the true nature of the assumption imposed on the relative dividend intensity process $\rho(t)$. The main obstacle is that, in general, different sample paths of the process $\rho(t)$ do not necessarily share properties beyond the required ergodicity.

This section provides sufficient conditions for (B) under a particular assumption on the process $\rho(t)$. The framework is that of time-homogeneous Markov processes, which is at the heart of many models in financial mathematics and puts at our disposal a rich toolbox of results. The main finding is the identification of statistical regularities that are needed to obtain the convergence result in Theorem 2.

Throughout this section the dividend intensity process $\rho(t)$ is a positively recurrent Markov process, i.e. the time of transition between any two states is a finite random variable (which depends on a realization of the process). This assumption ensures that there are no transient states and the state space contains one communicating set.[1] For simplicity (and without restriction on practical applications), the state space E of the process $\rho(t)$ is assumed to be countable. We assume that E contains at least two states. Otherwise condition (B) is trivially satisfied.

The assumption of positive recurrence implies that there exists a unique invariant probability measure μ for the process $\rho(t)$. This measure has the property that $\mu(\{x\})>0$ for all $x\in E$. If μ is taken as the initial distribution, the process $\rho(t)$ is stationary. The interpretation of such a model is that the dividend-paying firms operate under stable, though random, conditions.

[1]Transient states (which are the states that are not visited after a finite time) can be ignored because the analysis focuses on the long-run behavior of the market. The assumption that there is one communicating set is not restrictive. If the state space consists of several communicating sets, every such set can be considered separately, and the investment strategy λ^* depends on the state of the dividend intensity process at time 0.

Let $\mathbb{P}^x$ denote the probability measure under which the distribution of $\rho(0)$ is concentrated in the state x. $\mathbb{P}^\mu$ is the probability measure corresponding to $\rho(0)$ being distributed as μ. Since the state space E is countable, a relation between probability distributions $\mathbb{P}^x$ and $\mathbb{P}^\mu$ has the following form:

$$\mathbb{P}^\mu = \sum_{x\in E} \mu\big(\{x\}\big)\, \mathbb{P}^x. \tag{16}$$

We collect some properties on the ergodicity of the positively recurrent process $\rho(t)$ which will be needed in the following.

Lemma 4 *If the initial distribution of $\rho(t)$ is given by the invariant measure μ, then*

$$\mathbb{P}^\mu\Big(\lim_{t\to\infty} \frac{1}{t}\int_0^t \rho(s)\, ds = \bar\rho\Big) = 1,$$

where

$$\bar\rho = \mathbb{E}^\mu \rho(0)$$

and $\mathbb{E}^\mu$ is the expectation with respect to measure $\mathbb{P}^\mu$. Moreover, if $x \in E$ is the initial state of $\rho(t)$, then

$$\mathbb{P}^x\Big(\lim_{t\to\infty} \frac{1}{t}\int_0^t \rho(s)\, ds = \bar\rho\Big) = 1.$$

This result ensures the time-invariant strategy $\lambda^* = \bar\rho$ is well-defined and independent of the initial distribution. If the dividend-paying firms operate under stable economic conditions, which prevail when μ is the initial distribution, the strategy λ^* states that one should invest according to expected relative dividend payments. It turns out that this strategy is selected by the market dynamics even if the relative dividend process $\rho(t)$ starts from an arbitrary state.

The strategy λ^* has the following game-theoretic interpretation. For the incumbent's constant strategy λ, the gross return on holding asset 1 over the time $[t, T]$ is

$$R_{t,T} = \frac{\lambda + c\int_t^T \rho(s)ds}{\lambda}$$

and, therefore, the instantaneous (marginal) return at time t is

$$\frac{\partial R_{t,T}}{\partial T}\,\Big|_{T=t} = \frac{c\rho(t)}{\lambda}.$$

Similarly, the instantaneous (marginal) return at time t on asset 2 is

$$\frac{c(1-\rho(t))}{1-\lambda}.$$

Define the concave function

$$\lambda \mapsto \arg\max_{\eta} \mathbb{E}^{\mu} \log\left(\frac{c\rho(t)}{\lambda}\eta + \frac{c(1-\rho(t))}{1-\lambda}(1-\eta)\right)$$

mapping an incumbent's strategy to the set of "best responses," i.e. all constant strategies maximizing the expected logarithmic growth rate at prices given by the incumbent's strategy λ. The first-order condition for a maximum is

$$\mathbb{E}^{\mu}\left[\frac{\frac{\rho(t)}{\lambda} - \frac{1-\rho(t)}{1-\lambda}}{\frac{\rho(t)}{\lambda}\eta + \frac{1-\rho(t)}{1-\lambda}(1-\eta)}\right] = 0.$$

The incumbent's strategy is the best response to itself (i.e. at the prices determined by this strategy) if $\eta = \lambda$. In this case the above first-order condition is equivalent to

$$\mathbb{E}^{\mu}\left[\frac{\rho(t)}{\lambda} - \frac{1-\rho(t)}{1-\lambda}\right] = 0,$$

which implies $\lambda = \mathbb{E}^{\mu}\rho(t) = \mathbb{E}^{\mu}\rho(0) = \lambda^*$. A comprehensive discussion of this line of results in evolutionary finance can be found in Evstigneev, Hens and Schenk-Hoppé [13].

We now turn to the main result of this section which relates the dynamics of the sample mean of the process $\rho(t)$ and the condition (B) of Theorem 2. This theorem asserts the sample-path-wise convergence to 1 of the λ^*-investor's relative wealth provided the other investor follows a different strategy.

Assume there exists a state $x \in E$ such that if the process $\rho(t)$ starts at x, the proportions of time its sample mean is strictly above, resp. below, the mean $\bar{\rho}$ are both positive:

$$\text{(C)} \qquad \begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}^x\left(\frac{1}{s}\int_0^s \rho(u)du < \bar{\rho}\right) ds > 0, \\ &\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}^x\left(\frac{1}{s}\int_0^s \rho(u)du > \bar{\rho}\right) ds > 0. \end{aligned}$$

The condition (C) implies the analogous condition for the process $\rho(t)$ with the initial distribution μ:

$$\text{(C')} \qquad \begin{aligned} &\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}^{\mu}\left(\frac{1}{s}\int_0^s \rho(u)du < \bar{\rho}\right) ds > 0, \\ &\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}^{\mu}\left(\frac{1}{s}\int_0^s \rho(u)du > \bar{\rho}\right) ds > 0. \end{aligned}$$

Indeed, for every $x \in E$, the relation (16) and the fact that $\mu(\{x\}) > 0$ imply the following inequalities:

$$\mathbb{P}^{\mu}\left(\frac{1}{s}\int_0^s \rho(u)du < \bar{\rho}\right) \geq \mu(\{x\})\ \mathbb{P}^{x}\left(\frac{1}{s}\int_0^s \rho(u)du < \bar{\rho}\right),$$

and

$$\mathbb{P}^{\mu}\Big(\frac{1}{s}\int_0^s \rho(u)du > \bar{\rho}\Big) \geq \mu\big(\{x\}\big)\ \mathbb{P}^{x}\Big(\frac{1}{s}\int_0^s \rho(u)du > \bar{\rho}\Big).$$

Conditions (C) and (C') are satisfied by a large class of processes. Examples are presented below.

Theorem 3 *Suppose the dividend intensity process $\rho(t)$ is a positively recurrent Markov process on a countable state space E satisfying assumption (C'). Then, for each initial state $y \in E$ and for every real-valued $\xi \neq 0$,*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{(-\infty,0)}\Big(\xi\int_s^t (\rho(u)-\bar{\rho})\,du\Big)\,ds > 0 \quad \mathbb{P}^y - a.s.$$

Therefore, if $\lambda^2 = \lambda^*$ and $\lambda^1 \neq \lambda^*$, then $w^1(t) \to 0$ (and $w^2(t) \to 1$) $\mathbb{P}^y$-a.s. for every $w^1(0) \in (0,1)$.

The condition in Theorem 3, which implies assumption (B), is written in statistical terms and, moreover, it is easily verifiable in applications.

Conditions (C) and (C') exclude the asymmetry of the process $\rho(t)$ that was exploited in the construction of the counterexample in Section 3. These conditions impose fluctuations of the sample mean of the relative dividend intensity process $\rho(t)$ around its mean. In this realistic setting, the market dynamics selects the λ^*-investor for almost every realization of the dividend process. This selection result holds independently of the initial distribution of the process $\rho(t)$.

Two examples are discussed in the following.

Symmetric Markov processes. Assume that the process $\rho(t)$ has initial distribution μ and is symmetric around its expected value $\bar{\rho} = \mathbb{E}^{\mu}\rho(0)$, i.e. all finite-dimensional distributions of $(\rho(t)-\bar{\rho})$ and $(\bar{\rho}-\rho(t))$ under $\mathbb{P}^{\mu}$ are identical. Then, for every $s \geq 0$,

$$\mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du < 0\Big) = \mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du > 0\Big)$$

and, consequently,

$$\mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du < 0\Big) = \frac{1}{2}\Big(1 - \mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du = 0\Big)\Big).$$

Lemma 6, see the Appendix, implies

$$\lim_{s\to\infty} \mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du < 0\Big) = \frac{1}{2}$$

and, by symmetry,

$$\lim_{s\to\infty} \mathbb{P}^{\mu}\Big(\int_0^s \big(\rho(u)-\bar{\rho}\big)du > 0\Big) = \frac{1}{2}.$$

Thus condition (C') is satisfied.

Exponentially ergodic Markov processes. Denote by τ_x the time of the first return of the process $\rho(t)$ to the state x:

$$\tau_x = \inf\{t \geq \sigma_x : \rho(t) = x\},$$

where σ_x is the first time of leaving x,

$$\sigma_x = \inf\{t \geq 0 : \rho(t) \neq x\}.$$

Both random variables are well-defined by right-continuity of the process $\rho(t)$.

Theorem 4 *If*

$$\mathbb{E}^x(\tau_x)^2 < \infty$$

for some state $x \in E$, then condition (C) is satisfied.

The assumption of Theorem 4 ensures applicability of the Central Limit Theorem for Markov processes. While (by positive recurrence of $\rho(t)$) the expectation $\mathbb{E}^x\tau_x$ is finite for each initial state x, finiteness of the second moment (variance) of τ_x is an additional condition. It is satisfied e.g. for exponentially ergodic Markov processes (as their transition probabilities converge to the invariant distribution exponentially fast); cf. Anderson [1, Lemma 6.3].

5 Conclusion

This paper develops a model of the market interaction of heterogenous investment strategies within a continuous-time financial mathematics framework. The wealth dynamics is described by a random dynamical system which is driven by an exogenous dividend process. We analyze the asymptotic behavior of this dynamics for constant proportions investment strategies. Our results lead to the identification of a unique investment strategy that outcompetes any other time-invariant strategy. This finding has implications for the long-term evolution of asset prices by providing an asymptotic benchmark.

Future research on evolutionary finance models in continuous time will aim at the study of adapted, time-variant investment strategies, more investors and more assets. Another line of inquiry is concerned with the corresponding diffusion-type model which requires the use of stochastic analysis.

Appendix

Proof of Lemma 1. Let us assume $V^i(0) > 0$ for all i. The case of initial states with one or more coordinates being equal to zero can be treated analogously as discussed later. Equation (6) yields

$$V^I(t) = \frac{\bar{\delta}(t)}{c} - \sum_{i=1}^{I-1} V^i(t) \tag{17}$$

and, therefore,

$$dV^I(t) = \frac{1}{c} d\bar{\delta}(t) - \sum_{i=1}^{I-1} dV^i(t). \tag{18}$$

Define the matrix $\hat{\Theta}(t) \in \mathbb{R}^{(I-1)\times K}$ by $\hat{\Theta}_{ik}(t) = \lambda_k^i V^i(t)/\langle \lambda_k, V(t)\rangle$ with $V^I(t)$ given by (17). Further, let the matrix $\hat{\Lambda} \in \mathbb{R}^{K\times(I-1)}$ be given by $\hat{\Lambda}_{ki} = \lambda_k^i$, $k = 1, ..., K$, $i = 1, ..., I-1$, and define

$$A = \hat{\Lambda} - \Lambda^I,$$

where $\Lambda^I \in \mathbb{R}^{K\times(I-1)}$ has $I-1$ identical columns, each being equal to the vector λ^I. Then (7) is equivalent to the system of equations in $\hat{V}(t) := (V^1(t), ..., V^{I-1}(t))$ which is given by

$$d\hat{V}(t) = \hat{\Theta}(t-)\Big(A\, d\hat{V}(t) + \frac{\lambda^I}{c} d\bar{\delta}(t) + \delta(t)dt\Big) - c\hat{V}(t)dt$$

with $\delta(t) = (\delta_1(t), ..., \delta_K(t))^T$. Equivalently, one can write

$$[\mathrm{Id} - \hat{\Theta}(t-)\, A] d\hat{V}(t) = \hat{\Theta}(t-)\Big(\frac{\lambda^I}{c} d\bar{\delta}(t) + \delta(t)dt\Big) - c\hat{V}(t)dt. \tag{19}$$

The matrix $[\mathrm{Id} - \hat{\Theta}(t-)\, A]$ is invertible, which is shown at the end of this proof. Equation (19) can be written in the explicit form

$$d\hat{V}(t) = [\mathrm{Id} - \hat{\Theta}(t-)\, A]^{-1}\Big[\hat{\Theta}(t-)\Big(\frac{\lambda^I}{c} d\bar{\delta}(t) + \delta(t)dt\Big) - c\hat{V}(t)dt\Big]. \tag{20}$$

Its canonical representation is

$$d\hat{V}(t) = F\big(\delta(t), \hat{V}(t-)\big) dZ(t), \tag{21}$$

where $Z(t) = (t, \bar{\delta}(t))^T$ and

$$F(\delta, \hat{v}) = [\mathrm{Id} - \hat{\Theta}(\bar{\delta}, \hat{v})\, A]^{-1}\Big(\hat{\Theta}(\bar{\delta}, \hat{v})\delta - c\hat{v},\ \hat{\Theta}(\bar{\delta}, \hat{v})\frac{\lambda^I}{c}\Big)$$

with

$$\bar{\delta} = \sum_{k=1}^{K} \delta_k \quad \text{and} \quad \hat{\Theta}_{ik}(\bar{\delta}, \hat{v}) = \frac{\lambda_k^i \hat{v}^i}{\sum_{i=1}^{I-1} (\lambda_k^i - \lambda_k^I)\hat{v}^i + \lambda_k^I \bar{\delta}/c}. \tag{22}$$

Define

$$\mathcal{D} = \Big\{ (\delta, \hat{v}) \in [0,\infty)^K \times (0,\infty)^{I-1} : \quad \sum_{i=1}^{I-1} \hat{v}^i < \frac{1}{c} \sum_{k=1}^{K} \delta_k \Big\}.$$

Continuous differentiability of F on $\mathcal{D}$ with respect to $\hat{v}$ implies F satisfies the following uniform local Lipschitz condition: for any $(\delta, \hat{v}) \in \mathcal{D}$ there exist neighborhoods O_δ of δ and $O_{\hat{v}}$ of $\hat{v}$ such that

$$\|F(\delta', \hat{v}') - F(\delta', \hat{v}'')\| \le K \|\hat{v}' - \hat{v}''\|, \qquad \hat{v}', \hat{v}'' \in O_{\hat{v}}, \quad \delta' \in O_\delta$$

for some $K > 0$ depending on O_δ and $O_{\hat{v}}$.

A left-continuous version of equation (21) is given by

$$d\hat{V}(t) = F\big(\delta(t-), \hat{V}(t-)\big) dZ(t). \tag{23}$$

Theorem 6 in [27, Chapter V] ensures that (23) has a unique local solution for every initial condition in $(\delta(0), \hat{V}(0)) \in \mathcal{D}$. For every ω, the function $t \mapsto \delta(t)(\omega)$ has finite variation. Continuity of F implies that $F(\delta(t), V(t-))(\omega)$ can differ from $F(\delta(t-), V(t-))(\omega)$ at most in a countable number of points. Continuity of $Z(t)$ implies that the signed measure induced by the function $Z(t)(\omega)$ is atomless, and, therefore,

$$\int_0^t \mathbf{1}_{F(\delta(t-),V(t-)) \neq F(\delta(t),V(t-))} dZ(t) = 0.$$

The sets of solutions to equations (21) and (23) are identical, which ensures local existence and uniqueness of solutions to (21). The continuity of $\hat{V}(t)$ follows from the absence of atoms in the measure induced by the function $Z(t)$.

Since the set $\mathcal{D}$ is invariant, this solution is global, i.e. well-defined for all $t \in [0, \infty)$. Indeed, if $V^i(0) = 0$ for one or more investors (but $V^j(0) > 0$ for at least one investor) then the absence of short-selling, (A.2), and (7) imply $V^i(t) = 0$ for all $t \ge 0$. The relation (6) together with assumption (A.1) ensures that a solution corresponding to a non-zero initial state cannot become identical to zero. The non-trivial (lower-dimensional) dynamics is defined for all investors with strictly positive initial wealth. Existence and uniqueness of the solution follows from the above.

Proof of invertibility of $[\mathrm{Id} - \hat{\Theta}(t-)\, A]$**.** Fix any $(\delta, \hat{v}) \in \mathcal{D}$ and let $\hat{\Theta} = \hat{\Theta}(\bar{\delta}, \hat{v})$, using the notation introduced in (22). We want to show invertibility of $[\mathrm{Id} - \hat{\Theta}\, (\hat{\Lambda} - \Lambda^I)]$.

The matrix $C = \mathrm{Id} - \hat{\Theta}\, \hat{\Lambda}$ has a column-dominant diagonal. Each diagonal entry strictly dominates the sum of absolute values of the remaining entries in the corresponding column:

$$C_{ii} > \sum_{j=1, j \neq i}^{I-1} |C_{ji}|, \qquad i = 1, \dots, I-1. \tag{24}$$

Indeed, the (i, j) entry of the matrix C is given by

$$\mathbf{1}_{i=j} - \sum_{k=1}^{K} \hat{\Theta}_{ik} \lambda_j^k.$$

Since all off-diagonal entries are negative, the condition (24) is equivalent to

$$\sum_{i=1}^{I-1} \sum_{k=1}^{K} \hat{\Theta}_{ik} \lambda_j^k < 1.$$

The following computation proves this inequality:

$$\sum_{i=1}^{I-1} \sum_{k=1}^{K} \hat{\Theta}_{ik} \lambda_j^k = \sum_{k=1}^{K} \Big(\sum_{i=1}^{I-1} \hat{\Theta}_{ik} \Big) \lambda_j^k < \sum_{k=1}^{K} \lambda_j^k = 1.$$

One has $\sum_{i=1}^{I-1} \hat{\Theta}_{ik} < 1$ because the investment of investor I in asset k is strictly positive (see assumption (A.2)) and assets are in net supply of 1. The above property of the matrix C implies that it is invertible and C^{-1} maps the non-negative orthant into itself (see [22, Corollary, p. 22 and Theorem 23, p. 24]).

Invertibility of $[\mathrm{Id} - \hat{\Theta}\,(\hat{\Lambda} - \Lambda^I)]$ is equivalent to invertibility of

$$[\mathrm{Id} - \hat{\Theta}\,(\hat{\Lambda} - \Lambda^I)]C^{-1} = \mathrm{Id} + \hat{\Theta}\,\Lambda^I C^{-1}.$$

It suffices to prove that $x = 0$ is the only solution to the linear equation

$$x = -\hat{\Theta}\,\Lambda^I C^{-1} x. \tag{25}$$

For any $y \in \mathbb{R}^{I-1}$, the particular form of the matrix Λ^I implies $\hat{\Theta}\,\Lambda^I y = b\bar{y}$, where

$$b = \Big[\sum_{k=1}^{K} \hat{\Theta}_{1k} \lambda_k^I, \ldots, \sum_{k=1}^{K} \hat{\Theta}_{(I-1)k} \lambda_k^I \Big]^T \quad \text{and} \quad \bar{y} = \sum_{i=1}^{I-1} y_i.$$

The linear equation (25) therefore can only have solutions of the form $x = \alpha b$ with $\alpha \in \mathbb{R}$. All coordinates of b are non-negative because short-sales are not allowed in the model. Assume that $x = \alpha b$ is a solution to (25), with $\alpha \neq 0$ and $b^i > 0$ for at least one $i = 1, ..., I-1$. The condition $\alpha \neq 0$ implies that $b = -b\,\overline{C^{-1}b}$, which further yields $\overline{C^{-1}b} = -1$. Since the matrix C^{-1} maps the non-negative orthant into itself and $b \geq 0$, all coordinates of $C^{-1}b$ are non-negative and $\overline{C^{-1}b} = \sum_{i=1}^{I-1} (C^{-1}b)_i \geq 0$—a contradiction. Hence, the only solution to (25) is $x = 0$, which proves the invertibility of the matrix $[\mathrm{Id} - \hat{\Theta}(t-)\,A]$. □

Proof of Lemma 2. Part (i). Let $I = 2$. Elementary calculations suffice to derive (8) from the relation $dw^1(t) = [dV^1(t) - w^1(t) d\bar{V}(t)]/\bar{V}(t)$ and

(7). Existence of a local solution follows as in Lemma 1. Note that all denominators are strictly positive by assumption (A.2). $w^1(t) = 0$ and $w^1(t) = 1$ are fixed points. Therefore the solution is global.

Part (ii). Straightforward from the derivation of (8) from (7). $\square$

Proof of Theorem 1. Fix a sample path of the dividend intensity process.

Assertion (i): Equation (9) implies

$$\frac{(\lambda^1(1-\lambda^1) - \lambda^2(1-\lambda^2))w^1(t) + \lambda^2(1-\lambda^2)}{w^1(t)(1-w^1(t))}\, dw^1(t) \tag{26}$$
$$= -c\big((\lambda^2-\lambda^1)^2 w^1(t) + (\lambda^2-\lambda^1)(\rho(t)-\lambda^2)\big)\, dt,$$

which is equivalent to

$$d\log\left(\frac{w^1(t)}{(1-w^1(t))^\gamma}\right) = -c\frac{(\lambda^2-\lambda^1)^2 w^1(t) + (\lambda^2-\lambda^1)(\rho(t)-\lambda^2)}{\lambda^2(1-\lambda^2)}\, dt \tag{27}$$

where $\gamma = 1 + (\lambda^1(1-\lambda^1) - \lambda^2(1-\lambda^2))/(\lambda^2(1-\lambda^2)) = \lambda^1(1-\lambda^1)/(\lambda^2(1-\lambda^2)) > 0$. Dividing the integral form of this equation by t, one obtains

$$\frac{1}{t}\log\left(\frac{w^1(t)}{(1-w^1(t))^\gamma}\right) - \frac{1}{t}\log\left(\frac{w^1(0)}{(1-w^1(0))^\gamma}\right) \tag{28}$$
$$= -c\frac{(\lambda^2-\lambda^1)^2}{\lambda^2(1-\lambda^2)}\,\frac{1}{t}\int_0^t w^1(s)\, ds - c\frac{\lambda^2-\lambda^1}{\lambda^2(1-\lambda^2)}\,\frac{1}{t}\int_0^t \big(\rho(s)-\lambda^2\big)\, ds.$$

Equation (10) implies that $\frac{1}{t}\int_0^t \big(\rho(s)-\lambda^2\big)\, ds$ converges to zero. Since $0 < w^1(t) < 1$,

$$0 < \frac{1}{t}\int_0^t w^1(s)\, ds < 1.$$

Assume that $\limsup_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\, ds = 1$ over the fixed trajectory of the dividend intensity. There exists a sequence (s_n) such that $\lim_{n\to\infty} w^1(s_n) = 1$ and $\lim_{n\to\infty} \frac{1}{s_n}\int_0^{s_n} w^1(s)\, ds = 1$. (The construction of the sequence (s_n) is provided at the end of the proof.) From equation (28) it follows that

$$\lim_{t\to\infty} \frac{\gamma}{s_n}\log\big(1-w^1(s_n)\big) = c\frac{(\lambda^2-\lambda^1)^2}{\lambda^2(1-\lambda^2)} > 0.$$

This contradicts $\log\big(1-w^1(s_n)\big) < 0$ for all n.

Assume now that $\eta = \limsup_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\, ds$ is strictly positive, $\eta > 0$. Clearly, $\eta < 1$. There exists a sequence (t_n) (see below for its construction) such that

$$\lim_{n\to\infty} w^1(t_n) = \eta \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} w^1(s)\, ds = \eta.$$

As n tends to infinity, the left-hand side of (28) converges to zero whereas the right-hand side converges to $-\eta c(\lambda^2 - \lambda^1)^2/(\lambda^2(1-\lambda^2)) < 0$. This is a contradiction.

The conclusion is that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\,ds = 0.$$

Since $w^1(t) \geq 0$, this implies (i).

Assertion (ii): Fix any $\varepsilon > 0$. Part (a) and $w^1(t) \geq 0$ yields

$$\begin{aligned} 0 &= \lim_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\,ds \geq \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{(\varepsilon,\infty)}(w^1(s))w^1(s)\,ds \\ &\geq \lim_{t\to\infty} \frac{\varepsilon}{t}\int_0^t \mathbf{1}_{(\varepsilon,\infty)}(w^1(s))\,ds \geq 0. \end{aligned}$$

Therefore, $\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{1}_{(\varepsilon,\infty)}(w^1(s))\,ds = 0$, which implies (ii).

Construction of the sequence (s_n): Define

$$\begin{aligned} h_n &= \inf\Big\{t \geq n:\ w^1(t) \geq 1 - 1/n\Big\}, \\ s_n &= \inf\Big\{t \geq h_n:\ \frac{1}{t}\int_0^t w^1(s)\,ds \geq 1 - 1/n\Big\}. \end{aligned}$$

The sequences (h_n) and (s_n) are non-decreasing and converge to infinity because $w^1(t) < 1$ and, by assumption, $\limsup_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\,ds = 1$. It suffices to show that $w^1(s_n) \geq 1 - 1/n$ to ensure that the sequence (s_n) has the desired properties. If $h_n = s_n$, this relation holds. Otherwise one has $s_n > h_n$. Then there exists a $z > 0$ such that the function $t \mapsto \frac{1}{t}\int_0^t w^1(s)\,ds$ is non-decreasing for $t \in [s_n - z, s_n]$. This implies non-negativity of the derivative at $t = s_n$:

$$\frac{w^1(s_n)}{s_n} - \frac{1}{s_n^2}\int_0^{s_n} w^1(s)\,ds \geq 0.$$

Hence, we have

$$w^1(s_n) \geq \frac{1}{s_n}\int_0^{s_n} w^1(s)\,ds = 1 - 1/n.$$

Construction of the sequence (t_n): Define

$$\begin{aligned} h_n^1 &= \inf\Big\{h \geq n:\ \sup\Big\{\frac{1}{t}\int_0^t w^1(s)ds : t \geq h\Big\} \leq \eta + \frac{1}{n}\Big\}, \\ h_n^2 &= \inf\Big\{t \geq h_n^1:\ w^1(t) \geq \eta - 1/n\Big\}. \end{aligned}$$

The sequences (h_n^1) and (h_n^2) are non-decreasing and converge to infinity because $\eta = \limsup_{t\to\infty} \frac{1}{t}\int_0^t w^1(s)\,ds$. Let

$$t_n = \inf\Big\{t \ge h_n^2 : \ \frac{1}{t}\int_0^t w^1(s)\,ds \ge \eta - 1/n\Big\}.$$

The sequence (t_n) has the desired properties if

(a) (t_n) is non-decreasing and converges to infinity,

(b) $w^1(t_n) \ge \eta - 1/n$,

(c) $\displaystyle\frac{1}{t_n}\int_0^{t_n} w^1(s)\,ds \ge \eta - 1/n.$

Part (a) and (c) follow directly from the definitions of h_n^2 and t_n, respectively. If $h_n^2 = t_n$ then (b) holds. Otherwise one has $t_n > h_n^2$. Then there exists a $z > 0$ such that the function $t \mapsto \frac{1}{t}\int_0^t w^1(s)\,ds$ is non-decreasing on $[t_n - z, t_n]$. Therefore, its derivative is non-negative at $t = t_n$:

$$\frac{w^1(t_n)}{t_n} - \frac{1}{t_n^2}\int_0^{t_n} w^1(s)\,ds \ge 0.$$

This inequality, together with (c), implies (b). $\square$

Proof of Lemma 3 and details of the counterexample. Fix $\lambda^1 = 1/4$, $\lambda^2 = 1/2$ and $c = 1$. The dynamics of $w^1(t)$ (see (12)) simplifies to

$$dw^1(t) = -\frac{w^1(t)(1-w^1(t))\big(w^1(t) + 4(\rho(t) - \frac{1}{2})\big)}{4 - w^1(t)}\,dt. \tag{29}$$

If $\rho(u) \le \frac{1}{4}$ and $w^1(u) \le \frac{1}{2}$ for all $u \in [s,t]$, $0 \le s < t$, the following estimate holds

$$w^1(t) \ge w^1(s)\exp\Big(\frac{1}{16}\int_s^t \big(3 - 8\rho(u)\big)du\Big) \ge w^1(s)\exp\Big(\frac{t-s}{16}\Big). \tag{30}$$

Proof: Equation (29) is equivalent to

$$dw^1(t) = \frac{w^1(t)(1-w^1(t))\big(-w^1(t) - 4(\rho(t) - \frac{1}{2})\big)}{4 - w^1(t)}\,dt.$$

If $w(t) \le \frac{1}{2}$ and $\rho(t) \le \frac{1}{4}$, the right-hand side is nonnegative. An upper bound for the denominator is 4. A lower bound for the numerator is derived using the estimates $(1 - w^1(t)) \le \frac{1}{2}$ and

$$-w^1(t) - 4(\rho(t) - \frac{1}{2}) \le -\frac{1}{2} - 4(\rho(t) - \frac{1}{2}) = \frac{3}{2} - 4\rho(t).$$

Therefore,

$$dw^1(t) \geq \frac{w^1(t)\bigl(3 - 8\rho(t)\bigr)}{16}\, dt. \tag{31}$$

Integration of (31) yields (30). □

If $\rho(u) \leq \frac{1}{2}$ and $w^1(u) \leq \frac{1}{2}$ for all $u \in [s,t]$, $0 \leq s < t$, then

$$w^1(t) \geq \frac{1}{\frac{2}{7}(t-s) + \frac{1}{w^1(s)}}. \tag{32}$$

Proof: If $w(t) \leq \frac{1}{2}$, $\rho(t) \leq \frac{1}{2}$, and $w^1(t) + 4(\rho(t) - \frac{1}{2}) \geq 0$, then

$$dw^1(t) \geq -\frac{w^1(t)\bigl(w^1(t) + 4(\rho(t) - \frac{1}{2})\bigr)}{7/2}\, dt \geq -\frac{2}{7}\bigl(w^1(t)\bigr)^2\, dt.$$

The above inequality is clearly satisfied if $w^1(t) + 4(\rho(t) - \frac{1}{2}) < 0$, because this condition implies that the right-hand side of (29) is positive. Hence, the lower bound (32) for $w^1(t)$ is obtained by integration of the inequality $dw^1(t) \geq -\frac{2}{7}\bigl(w^1(t)\bigr)^2\, dt$. □

The construction of the process $\rho(t)$ is based on these two estimates. Fix any $l \in (0, 1/2)$. If $w^1(s) \geq l$ then $w^1(t) \geq l10^{-n}$ for

$$t \leq s + \frac{10^n}{l^2} \leq s + \Bigl(\frac{10^n}{l} - 1\Bigr)\frac{7}{2l}$$

provided that $\rho(u) \leq 1/2$ for $u \in [s,t]$. If $w^1(t) > 1/2$ for some $t \geq s$ then the condition $w^1(t) \geq l10^{-n}$ is trivially satisfied. Otherwise, the estimate (32) implies the result.

The estimate for the case $\rho(u) \leq 1/4$ is derived similarly. If $w^1(s) \geq l10^{-n}$ then $w^1(t) \geq l$ for

$$t \geq s + 16n \log(10)$$

provided that $\rho(u) \leq 1/4$ for $u \in [s,t]$. If $w^1(t) > 1/2$ for some $t \geq s$ then the condition $w^1(t) \geq l$ is trivially satisfied. Otherwise, the estimate (30) implies the result.

The assertions of Lemma 3 follow straightforwardly from the above estimates and the construction of the process $\rho(t)$. □

Proof of Theorem 2. Define a new variable

$$v(t) = \frac{w^1(t)}{(1 - w^1(t))^\gamma}, \tag{33}$$

where $\gamma = \lambda^1(1-\lambda^1)/(\lambda^2(1-\lambda^2))$ as in the proof of Theorem 1. Transformation (33) is a diffeomorphism between the interval $(0,1)$ of values of $w^1(t)$

and $(0,\infty)$. Note that $\lim_{t\to\infty} w^1(t) = 0$ if and only if $\lim_{t\to\infty} v(t) = 0$. The latter equality will be proved.

A differential equation for $v(t)$ is obtained from (26):

$$dv(t) = -\kappa(t)v^2(t)\,dt - q(t)v(t)\,dt,$$

where

$$\kappa(t) = c\frac{(\lambda^2-\lambda^1)^2}{\lambda^2(1-\lambda^2)}(1-w^1(t))^\gamma \quad \text{and} \quad q(t) = c\frac{\lambda^2-\lambda^1}{\lambda^2(1-\lambda^2)}(\rho(t)-\lambda^2).$$

The solution to this Ricatti equation is given by

$$v(t) = \frac{\exp\Big(\int_0^t q(s)\,ds\Big)}{1/v(0) + \int_0^t \kappa(s)\exp\Big(\int_0^s q_u du\Big)ds}.$$

This is not a closed-form expression for $v(t)$ because the function $\kappa(t)$ depends on $w^1(t)$. Since $1/v(0) > 0$ one has

$$v(t) \le \frac{1}{\int_0^t \kappa(s)\exp\Big(-\int_s^t q(u)\,du\Big)\,ds}.$$

Therefore $\lim_{t\to\infty} v(t) = 0$ if

$$\lim_{t\to\infty}\int_0^t \kappa(s)\exp\Big(-\int_s^t q(u)\,du\Big)\,ds = \infty.$$

Positivity of the integrand ensures that the above equality holds if and only if

$$\limsup_{t\to\infty}\int_0^t \kappa(s)\exp\Big(-\int_s^t q(u)\,du\Big)\,ds = \infty.$$

For any $\hat\epsilon > 0$ and for any real number $\hat z$ the following sequence of estimates holds

$$\begin{aligned}
&\int_0^t \kappa(s)\exp\Big(-\int_s^t q(u)\,du\Big)\,ds\\
&\quad\ge \hat\epsilon e^{-\hat z}\int_0^t \mathbf{1}_{[\hat\epsilon,\infty)}(\kappa(s))\,\mathbf{1}_{(-\infty,\hat z)}\Big(\int_s^t q(u)\,du\Big)\,ds\\
&\quad= \hat\epsilon e^{-\hat z}\int_0^t \bigg(\mathbf{1}_{[\hat\epsilon,\infty)}(\kappa(s)) - \mathbf{1}_{[\hat\epsilon,\infty)}(\kappa(s))\mathbf{1}_{[\hat z,\infty)}\Big(\int_s^t q(u)\,du\Big)\bigg)\,ds\\
&\quad\ge \hat\epsilon e^{-\hat z}\,t\,\bigg(\frac{1}{t}\int_0^t \mathbf{1}_{[\hat\epsilon,\infty)}(\kappa(s))\,ds - \frac{1}{t}\int_0^t \mathbf{1}_{[\hat z,\infty)}\Big(\int_s^t q(u)\,du\Big)\,ds\bigg).
\end{aligned}$$

Choose

$$\hat z = \frac{c|\lambda^2-\lambda^1|}{\lambda^2(1-\lambda^2)}z \quad \text{and any} \quad \hat\epsilon \in \Big(0, \frac{c(\lambda^2-\lambda^1)^2}{\lambda^2(1-\lambda^2)}\Big).$$

Condition (B) is equivalent to

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{[\hat{z},\infty)}\Big(\int_s^t q(u)\, du\Big)\, ds < 1,$$

and Theorem 1 (b) implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{1}_{[\hat{\epsilon},\infty)}(\kappa(s))\, ds = 1.$$

Therefore

$$\limsup_{t\to\infty} \left(\frac{1}{t} \int_0^t \mathbf{1}_{[\hat{\epsilon},\infty)}(\kappa(s))\, ds - \frac{1}{t} \int_0^t \mathbf{1}_{[\hat{z},\infty)}\Big(\int_s^t q(u)\, du\Big)\, ds \right) > 0,$$

which implies $\limsup_{t\to\infty} \int_0^t \kappa(s) \exp\big(- \int_s^t q(u)\, du\big)\, ds = \infty$. □

Proof of Lemma 4. The assertions of this lemma follow from the ergodic theorem for Markov processes (see Chapter II.14 in Chung [8]). Using renewal theory, the lemma can also be proved employing Meyn and Tweedie [20, Theorem 17.0.1] which is for discrete-time models. □

Proof of Theorem 3. Assume first that the distribution of $\rho(0)$ is μ, i.e. the process $\rho(t)$ is stationary. $\rho(t)$ is defined for $t \in [0,\infty)$. Its stationarity allows the construction of a probability space and a stationary stochastic process $\tilde{\rho}(t)$, $t \in \mathbb{R}$, whose finite dimensional distributions for $t \in [0,\infty)$ are identical those of $\rho(t)$ (see Section A.3 in Arnold [2]). Define $x(t) = \tilde{\rho}(-t)$. The process $x(t)$ is a Markov process with the so-called adjoint semigroup (see Exercise 4.5.4, Stroock [30]). Assumption (C') implies

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}^{\mu}\bigg(\xi \int_0^s \big(x(u) - \bar{\rho}\big) du < 0 \bigg) ds =: \alpha \in (0,1).$$

Freedman [14, Theorem 1][2] ensures that the family of random variables

$$\frac{1}{t} \int_0^t \mathbf{1}_{(-\infty,0)}\Big(\xi \int_0^s \big(x(u) - \bar{\rho}\big) du \Big) ds$$

converges in distribution, as $t \to \infty$, to a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & \text{if } x \le 0, \\ \frac{\sin(\pi\alpha)}{\pi} \int_0^x s^{\alpha-1}(1-s)^{-\alpha} ds, & \text{if } x \in (0,1), \\ 1, & \text{if } x \ge 1. \end{cases}$$

[2]The paper studies only the discrete-time case. The regeneration approach employed there generalizes to our case; see Freedman's remark at the end of his paper [14].

Stationarity of $x(t)$ ensures that the distribution of

$$\frac{1}{t}\int_0^t \mathbf{1}_{(-\infty,0)}\Big(\xi\int_0^s \big(x(u)-\bar{\rho}\big)du\Big)ds$$

is identical to the distribution of

$$\phi(t) := \frac{1}{t}\int_0^t \mathbf{1}_{(-\infty,0)}\Big(\xi\int_s^t \big(\rho(u)-\bar{\rho}\big)du\Big)ds.$$

The sequence of random variables $\phi(t)$ therefore converges in distribution to $F(x)$ as $t\to\infty$. Since F does not have an atom in 0, Lemma 5 (see below) implies that for any sequence t_n converging to infinity

$$\mathbb{P}^{\mu}\big(\limsup_{n\to\infty}\phi(t_n)=0\big)=0.$$

Using (16), one obtains for every $y\in E$

$$\mathbb{P}^{y}\big(\limsup_{n\to\infty}\phi(t_n)=0\big)\le\frac{1}{\mu\big(\{y\}\big)}\mathbb{P}^{\mu}\big(\limsup_{n\to\infty}\phi(t_n)=0\big)=0.$$

This finding implies

$$\limsup_{t\to\infty}\phi(t)>0\quad \mathbb{P}^{y}-a.s.$$

which proves the first statement of the Theorem. $\mathbb{P}^{y}$-a.s. path-wise convergence follows from this result and Theorem 2. $\square$

Lemma 5 *Let X_n be a sequence of random variables with values in $[0,\infty)$ such that X_n converges in distribution to X. If X does not have an atom at 0, then*

$$\mathbb{P}(\limsup_{n\to\infty}X_n=0)=0.$$

Proof. Let $A=\{\omega:\limsup_{n\to\infty}X_n(\omega)=0\}$. Since $X_n\ge 0$, the set A can be written as

$$A=\{\omega:\lim_{n\to\infty}X_n(\omega)=0\}.$$

Define the function f_ϵ, for $\epsilon\in(0,1)$, by

$$f_\epsilon(x)=\max(1-x/\epsilon,0),\qquad x\ge 0.$$

f_ϵ is continuous and bounded which implies $\mathbb{E}f_\epsilon(X_n)\to\mathbb{E}f_\epsilon(X)$. Let

$$Y_n(\omega)=\begin{cases}X_n, & \omega\in A,\\ 1, & \omega\notin A.\end{cases}$$

Obviously, $Y_n \to 1_{A^c}$ a.s. Since $\epsilon < 1$, one further has

$$\mathbb{E} f_\epsilon(X_n) \geq \mathbb{E}\big(1_A f_\epsilon(X_n)\big) = \mathbb{E} f_\epsilon(Y_n).$$

The dominated convergence theorem gives

$$\lim_{n\to\infty} \mathbb{E} f_\epsilon(Y_n) = \mathbb{E} f_\epsilon(1_{A^c}) = \mathbb{P}(A).$$

Hence,

$$\mathbb{P}(A) = \lim_{n\to\infty} \mathbb{E} f_\epsilon(Y_n) \leq \lim_{n\to\infty} \mathbb{E} f_\epsilon(X_n) = \mathbb{E} f_\epsilon(X).$$

Since X has no atom at zero, $\lim_{\epsilon\downarrow 0} \mathbb{E} f_\epsilon(X) = 0$. This implies $\mathbb{P}(A) = 0$. $\square$

Proof of Theorem 4. The proof is based on Central Limit Theorem for Markov processes[3] (see Theorem III.8.6 in Bhattacharya and Waymire [4]). Under the assumptions of the theorem

$$\lim_{t\to\infty} \mathbb{P}^x\left((td)^{-1/2} \int_0^t \big(\rho(s) - \bar{\rho}\big) ds \leq z \right) = \Phi(z),$$

where Φ is a cumulative distribution function of the standard normal distribution, $\bar{\rho} = \mathbb{E}^\mu \rho(0)$, and d is a positive constant depending on x. Inserting $z = 0$ yields

$$\lim_{t\to\infty} \mathbb{P}^x\left(\int_0^t \big(\rho(s) - \bar{\rho}\big) ds \leq 0 \right) = \frac{1}{2}.$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}^x\left(\int_0^s (\rho(u) - \bar{\rho}) du \leq 0 \right) ds = \frac{1}{2}.$$

This clearly implies one assertion of (C):

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}^x\left(\int_0^s (\rho(u) - \bar{\rho}) du > 0 \right) ds = \frac{1}{2}.$$

The second part of (C) follows along the same lines; one simply has to replace the process $(\rho(t))$ by $(-\rho(t))$. $\square$

Auxiliary result. Consider the Markovian framework introduced in Section 4. Let ν be a distribution of $\rho(0)$ (ν is the initial distribution of the process $\rho(t)$).

Lemma 6 *One has*

$$\lim_{t\to\infty} \mathbb{P}^\nu\left(\int_0^s \big(\rho(u) - \bar{\rho}\big) du = 0 \right) = 0,$$

and, consequently,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{P}^\nu\left(\int_0^s \big(\rho(u) - \bar{\rho}\big) du = 0 \right) ds = 0.$$

[3] One can also adapt Central Limit Theorem for discrete-time Markov chains, Meyn and Tweedie [20, Theorem 17.2.2]. Its proof is based on a regeneration technique, which easily generalizes to the class of Markov processes considered in this paper.

Proof. The proof of the first assertion of the lemma is based on the following identity:

$$\mathbb{P}^{\nu}\left(\int_0^s \big(\rho(u)-\bar\rho\big)du = 0\right) = \mathbb{P}^{\nu}\Big(\rho(u) = \bar\rho \ \text{ for all } u \le s\Big). \qquad (34)$$

For its justification it suffices to show that the probability of

$$A = \left\{\int_0^s \big(\rho(u)-\bar\rho\big)du = 0, \quad \rho(t) \neq \bar\rho \ \text{ for some } t \in [0,s]\right\}$$

is zero. This event can be decomposed into a countable number of events

$$A^n(\rho_1,\ldots,\rho_n) = \Big\{\text{there exists } 0 = t_0 < t_1 < \cdots < t_{n-1} \le t_n = s : \\ \rho(u) = \rho_i \text{ for } u \in (t_{i-1}, t_i), \sum_{i=1}^n (\rho_i - \bar\rho)(t_i - t_{i-1}) = 0\Big\},$$

where n is a non-negative integer, $\rho_1, ..., \rho_n \in E$, and $\rho_i \neq \rho_{i+1}$, $i = 1, ..., n-1$. The set $A^n(\rho_1, ..., \rho_n)$ represents the following event: there are exactly $n-1$ transitions of the process $\rho(t)$ on the interval $[0,s]$ through the states $\rho_1, ..., \rho_n$ and the integral $\int_0^s \big(\rho(u)-\bar\rho\big)du$ equals 0. To compute the probability of $A^n(\rho_1, ..., \rho_n)$, the $(n-1)$-dimensional density of transitions through the states $\rho_1, ..., \rho_n$ is integrated on the set of $(t_0, ..., t_n)$ such that

$$0 = t_0 < t_1 < \cdots < t_{n-1} \le t_n = s \quad \text{and} \quad \sum_{i=1}^n (\rho_i - \bar\rho)(t_i - t_{i-1}) = 0.$$

This set has dimension $n-2$, so the integral is equal to zero, and

$$\mathbb{P}^{\nu}\big(A^n(\rho_1,\ldots,\rho_n)\big) = 0.$$

Therefore, the probability of A is also zero because A is the union of a countable number of sets of measure zero. This completes the proof of (34).

Recall that the state space has at least two elements. If $\bar\rho$ is not an element of the state space, the first assertion of the lemma follows immediately from (34) because in this case $\mathbb{P}^{\nu}\big(\rho(u) = \bar\rho \quad \forall u \le s\big) = 0$. Otherwise, if $\bar\rho \in E$, the probability that the process persists in the state $\bar\rho$ converges to zero,

$$\lim_{s\to\infty} \mathbb{P}^{\nu}\big(\rho(u) = \bar\rho \text{ for all } u \le s\big) = 0.$$

This relation and (34) imply the first assertion of the lemma. The second assertion of the lemma is a direct consequence of the first one. $\square$

References

[1] Anderson, W. J., Continuous-Time Markov Chains: An Application-Oriented Approach, Springer, 1991.

[2] Arnold, L., Random Dynamical Systems, Springer, 1998.

[3] Bank, P. and D. Baum, Hedging and portfolio optimization in financial markets with a large trader, Mathematical Finance, 2004, **14**, 1–18.

[4] Bhattacharya, R. N. and E. C. Waymire, Stochastic Processes with Applications, Wiley Series in Probability and Mathematical Statistics, 1990.

[5] Björk, T., Arbitrage Theory in Continuous Time, Oxford University Press, 2004.

[6] Blume, L. and D. Easley, If you're so smart, why aren't you rich? Belief selection in complete and incomplete markets, Econometrica, 2006, **74**, 929–966.

[7] Browne, S., The return on investment from proportional portfolio strategies, Advances in Applied Probability, 1998, **30**, 216–238.

[8] Chung, K. L., Markov Chains with Stationary Transition Probabilities, Springer, 1960.

[9] Evstigneev, I. V., Hens, T. and K. R. Schenk-Hoppé, Evolutionary Finance, Chapter 9 in: Handbook of Financial Markets: Dynamics and Evolution (T. Hens and K. R. Schenk-Hoppé, eds.), Elsevier, 2009.

[10] Evstigneev, I. V., Hens, T. and K. R. Schenk-Hoppé, Evolutionary stable stock markets, Economic Theory, 2006, **27**, 449–468.

[11] Evstigneev, I. V., Hens, T. and K. R. Schenk-Hoppé, Globally evolutionarily stable portfolio rules, Journal of Economic Theory, 2008, **140**, 197-228.

[12] Evstigneev, I. V., Hens, T. and K. R. Schenk-Hoppé, Market selection of financial trading strategies: global stability, Mathematical Finance, 2002, **12**, 329–339.

[13] Evstigneev, I. V., Hens, T. and K. R. Schenk-Hoppé, Survival and Evolutionary Stability of the Kelly Rule, Chapter in: The Kelly Capital Growth Investment Criterion: Theory and Practice (L. C. MacLean, E. O. Thorp and W. T. Ziemba, eds.), World Scientific, forthcoming.

[14] Freedman, D. A., An arcsine law for Markov chains, Proceedings of the American Mathematical Society, 1963, **14**, 680–684.

[15] Gatev, E., Goetzmann, W. N. and K. G. Rouwenhorst, Pairs trading: performance of a relative-value arbitrage rule, Review of Financial Studies, 2006, **19**, 797–827.

[16] Hens, T. and K. R. Schenk-Hoppé, Evolutionary stability of portfolio rules in incomplete markets, Journal of Mathematical Economics, 2005, **41**, 43–66.

[17] Kelly, J. L., A new interpretation of information rate, Bell System Technical Journal, 1956, **35**, 917–926.

[18] Luenberger, D. G., A preference foundation for log mean-variance criteria in portfolio choice problems, Journal of Economic Dynamics and Control, 1993, **17**, 887–906.

[19] MacLean, L. C., Thorp, E. O. and W. T. Ziemba, (eds.), The Kelly Capital Growth Investment Criterion: Theory and Practice, World Scientific, forthcoming.

[20] Meyn, S. P. and R. L. Tweedie, Markov Chains and Stochastic Stability, Springer, 1993.

[21] Mulvey, J. M. and W. T. Ziemba (eds.), Worldwide Asset and Liability Modeling, Cambridge University Press, 1998.

[22] Murata, Y. Mathematics for Stability and Optimization of Economic Systems, Academic Press, 1977.

[23] Palczewski, J. and K. R. Schenk-Hoppé, From discrete to continuous time evolutionary finance models, Journal of Economic Dynamics and Control (forthcoming).

[24] Platen, E., A benchmark approach to finance, Mathematical Finance, 2006, **16**, 131–151.

[25] Pliska, S. R., Introduction to Mathematical Finance - Discrete Time Models, Blackwell, 1997.

[26] Perold, A. F. and W. F. Sharpe, Dynamic strategies for asset allocation, Financial Analysts Journal, 1988, **44**, 16–27.

[27] Protter, P. E., Stochastic Integration and Differential Equations, Second Edition, Springer, 2004.

[28] Schenk-Hoppé, K. R. and B. Schmalfuss, Random fixed points in a stochastic Solow growth model, Journal of Mathematical Economics, 2001, **36**, 19–30.

[29] Shapley, L. and M. Shubik, Trade using one commodity as a means of payment, Journal of Political Economy, 1977, **85**, 937–968.

[30] Stroock, D. W., An Introduction to Markov Processes, Springer, 2005.