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From Discrete to Continuous Time Evolutionary Finance Models[☆]

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Abstract

This paper aims to open a new avenue for research in continuous-time financial market models with endogenous prices and heterogenous investors. To this end we introduce a discrete-time evolutionary stock market model that accommodates time periods of arbitrary length. The dynamics is time-consistent and allows the comparison of paths with different frequency of trade. The main result in this paper is the derivation of the limit model as the length of the time period tends to zero. The resulting model in continuous time generalizes the workhorse model of mathematical finance by introducing asset prices that are driven by the market interaction of investors following self-financing trading strategies. Our approach also offers a numerical scheme for the simulation of the continuous-time model that satisfies constraints such as market clearing at every time step. An illustration is provided.

Key words: Evolutionary finance, market interaction, wealth dynamics, self-financing strategies, endogenous prices, continuous-time limit.

JEL-Classification: G11, G12.

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1. Introduction

Research in evolutionary finance models, which are employed to study the wealth dynamics driven by the market interaction of investment strategies, has seen tremendous progress in the last few years. This approach has shed light on optimal long-term investment strategies and the dynamics of asset prices in incomplete markets; see Evstigneev, Hens and Schenk-Hoppé [10] for a survey of the current state of the art in evolutionary finance.

All of the existing models in this theory are formulated in discrete time with a fixed length of the time period, cf. Evstigneev, Hens and Schenk-Hoppé [11, 12]. This approach is suitable, for instance, if investors trade only once every day, quarter or year because of institutional or market constraints. Models with myopic investors, in which the planning horizon and frequency of trade coincide, share this property; see e.g. Brock, Hommes and Wagener [5], Chiarella, Dieci and Gardini [7] and the references in the surveys by Chiarella, Dieci and He [8] and Hommes and Wagener [15].

The comparison of models with different frequency of trade and/or investors' planning horizon is potentially problematic because the dynamics might lack consistency or scalability with respect to time. The evolutionary finance model of stock markets introduced in this paper allows for time periods of arbitrary length and is time-consistent, i.e. time series with different frequency of trade are related. This model generalizes the approach presented in Evstigneev, Hens and Schenk-Hoppé [10, 11, 12] and Hens and Schenk-Hoppé [14]. Time-consistency is achieved by defining investment strategies and asset payoffs as continuous-time processes and then deriving quantities in a discrete-time setting which are meaningful from an economic point of view. Investors adjust portfolios to fit their investment strategies at the beginning of each time period only, akin to tracking a benchmark. Asset payoffs are aggregated over time to lump-sum payments which are paid to the asset holders at the end of each period.

The main result in this paper is the derivation of a continuous-time model which describes the limit of the discrete-time dynamics when the length of the time period tends to zero. The resulting dynamics has an explicit representation as a system of random differential equations. It turns out that the limit model generalizes the classical continuous-time dynamics of self-financing portfolios in mathematical finance (see, for instance, the textbooks by Björk [4] or Shreve [24]) by introducing market interaction of investors. Our findings highlight the potential of dynamic economic theory, combined with evolutionary ideas, for mathematical finance.

Our results on the existence of a continuous-time limit (and the explicit description of the limit model) are useful for practical finance because they provide a numerical scheme for the simulation of the continuous-time dynamics. We prove that the continuous-time dynamics is uniformly approximated (on finite time intervals) by the discrete-time model with small time steps. The approximation error decreases linearly with the length of the time step. This numerical method possesses an advantage over standard numerical techniques because it approximates the continuous-time wealth dynamics with a scheme that ensures clearing of the asset markets in each time period. An illustration of the convergence result is provided.

In mathematical finance discrete-time approximations of continuous-time models have a long tradition, see Prigent [23] for a comprehensive account of approximation theory for financial markets. Most continuous-time models however do not take into account the market interaction of investors. For instance, the market impact of trades (and its implications for optimal behavior of large investors) is usually studied for a single trader facing an exogenous price impact function (e.g. Bank and Baum [3]).

The existence of continuous-time limits for other discrete-time agent-based finance models with finitely many traders (see the surveys by Chiarella, Dieci and He [8] and Hommes and Wagener [15]), by-and-large, is an open problem. One exception is the paper by Buchmann and Weber [6] who derive the continuous-time limit of a betting market model à la Kelly [16]. This framework, however, rules out capital gains and therefore cannot appropriately model stock markets. There is a strand of literature studying continuous-time models inspired by locally interacting particle models in physics. These limit models are deterministic (though can display chaotic dynamics) and describe the market dynamics on an aggregate level only, see e.g. Lux [17] and the references therein. The derivation of these models usually rests on quite restrictive assumptions on the types of traders in the market. Related stochastic models in continuous time with finitely many noise traders are given in Alfarano, Lux and Wagner [1] and Lux [18].

A stochastic agent-based model with chartists and fundamentalists in continuous time is presented in Chiarella, He and Zheng [9]; but their approach relies on ad hoc assumptions on agent behavior and the sluggish adjustment of asset prices. Rheinlaender and Steinkamp [21] consider a closely related model in continuous time which is a stochastic extension of Zeeman's [25] one-dimensional deterministic model.

Section 2 presents a generalization of the discrete-time model [11, 12] with arbitrarily small time steps and time-dependent investment strategies. A heuristic derivation of its continuous-time limit is given in Section 3. Section 4 shows that this continuous-time model with market interaction of heterogenous investors leads to the classical continuous-time dynamics of self-financing portfolios with consumption. The explicit representation of both models, which are useful for numerical and analytical studies, are derived in Section 5. Section 6 contains the main result on the convergence of the dynamics of the discrete-time evolutionary stock market model to the continuous-time model. An example is discussed in Section 6.2. Section 7 shows how to extend the results to more general dividend processes. Section 8 concludes. All proofs and auxiliary results are collected in Appendix A-E.

2. Discrete-time evolutionary model

This section introduces a discrete-time evolutionary stock market model with time periods of arbitrary length. The model extends the approach presented in Evstigneev, Hens and Schenk-Hoppé [11, 12].

Given the length $\nu > 0$ of a time period, time is discrete and proceeds through the set $t_n^\nu = n\nu$, $n = 0, 1, 2, \ldots$. There are K long-lived assets (stocks) with random dividend intensity $\delta(t) = (\delta_1(t), \ldots, \delta_K(t))$, $t \geq 0$, with $\delta_k(t) \geq 0$ for $k = 1, \ldots, K$. Although the dependence on the random event $\omega \in \Omega$ is suppressed in this notation, the process $\delta(t)$ depends on two arguments and is assumed to be measurable with respect to the product σ-algebra.

Each asset is in positive net supply of one. The total dividend paid by asset k at time t_{n+1}^ν (to the investors who hold the asset over the time period $[t_n^\nu, t_{n+1}^\nu)$) is given by

$$D_{\nu,k}(t_{n+1}^\nu) = \int_{t_n^\nu}^{t_{n+1}^\nu} \delta_k(s)\, ds.$$

Dividends are paid in terms of a perishable consumption good, with price normalized to one.

There are I investors with initial wealth $V_\nu^i(0) \geq 0$, $i = 1, \ldots, I$, such that $\bar{V}_\nu(0) = \sum_{i=1}^I V_\nu^i(0) > 0$. Each investor is represented by a (possibly random) investment strategy $\lambda^i(t_n^\nu) = (\lambda_1^i(t_n^\nu), \ldots, \lambda_K^i(t_n^\nu))$, $n \geq 0$. It is assumed that $\sum_{k=1}^K \lambda_k^i(t_n^\nu) = 1$ and $\lambda_k^i(t_n^\nu) > 0$ for all $k = 1, \ldots, K$. The component $\lambda_k^i(t_n^\nu)$ describes the investor's budget share invested in asset k at the point in time t_n^ν.

Every investor consumes a constant fraction, $c\nu$, of his wealth in every period with the remainder being invested in assets. The constant $c > 0$ (with $c\nu < 1$) is the same for all investors and represents the intensity of consumption. The market value of investor i's investment in asset k held between time t_n^ν and t_{n+1}^ν is given by $(1 - c\nu)\lambda_k^i(t_n^\nu)V_\nu(t_n^\nu)$.

The discrete-time dynamics is defined by an equation describing the evolution of the vector of investors' wealth between two consecutive points in time. The wealth of investor i, $i = 1, \ldots, I$, evolves as

$$V_\nu^i(t_{n+1}^\nu) = \sum_{k=1}^K \theta_{\nu,k}^i(t_n^\nu)\big[S_{\nu,k}(t_{n+1}^\nu) + D_{\nu,k}(t_{n+1}^\nu)\big] \tag{1}$$

with portfolio

$$\theta_{\nu,k}^i(t_n^\nu) = \frac{(1 - c\nu)\lambda_k^i(t_n^\nu)V_\nu^i(t_n^\nu)}{S_{\nu,k}(t_n^\nu)}, \qquad k = 1, \ldots, K. \tag{2}$$

The quantity $\theta_{\nu,k}^i(t_n^\nu)$ represents the number of shares of asset k owned by investor i at the beginning of the period $[t_n^\nu, t_{n+1}^\nu)$. The market for asset

k clears if the total number of shares owned by investors is equal to 1: $\sum_{i=1}^{I}\theta_{\nu,k}^{i}(t_n^\nu)=1$. The asset price $S_{\nu,k}(t_n^\nu)$ is then given by

$$S_{\nu,k}(t_n^\nu)=(1-c\nu)\langle\lambda_k(t_n^\nu),V_\nu(t_n^\nu)\rangle. \tag{3}$$

Here $\langle x,y\rangle=\sum_{i=1}^{I}x^iy^i$ denotes the scalar product.

Inserting (2) and (3) into (1) yields

$$V_\nu^i(t_{n+1}^\nu)=\sum_{k=1}^{K}\frac{\lambda_k^i(t_n^\nu)V_\nu^i(t_n^\nu)}{\langle\lambda_k(t_n^\nu),V_\nu(t_n^\nu)\rangle}\Big[(1-c\nu)\langle\lambda_k(t_{n+1}^\nu),V_\nu(t_{n+1}^\nu)\rangle+D_{\nu,k}(t_{n+1}^\nu)\Big], \tag{4}$$

with $i=1,...,I$ or, equivalently, in vector notation

$$V_\nu(t_{n+1}^\nu)=\Theta(\Lambda(t_n^\nu),V_\nu(t_n^\nu))\left[(1-c\nu)\Lambda(t_{n+1}^\nu)V_\nu(t_{n+1}^\nu)+D_\nu(t_{n+1}^\nu)\right], \tag{5}$$

where, for each $t=t_n^\nu$, the matrix $\Lambda(t)\in\mathbb{R}^{K\times I}$ is given by $\Lambda_{ki}(t)=\lambda_k^i(t)$ and

$$\Theta_{ik}(\Lambda,V)=\frac{\Lambda_{ki}V^i}{(\Lambda V)_k}. \tag{6}$$

Equation (5) is an implicit description of the wealth dynamics of heterogenous investors trading in a stock market. As we show in this paper, the dynamics can be represented in explicit form by solving the intertemporal problem in $V_\nu(t_{n+1}^\nu)$ which stems from the short-term equilibrium of the asset market, see Section 5. The assets traded between the investors are 'long-lived,' they provide a random payoff stream over time and, at any period in time, have a market price. Trade between the investors takes place as an exchange of assets and consumption good (dividend).

The behavior of investors is described by investment strategies which are 'primitives' (i.e., exogenous) while asset prices and portfolios are endogenous. This marks a departure from the conventional general equilibrium paradigm in which market dynamics is derived from the maximization of utility from consumption. Here the dynamics is driven by the interaction of investment strategies. These strategies are taken as fundamental characteristics of the investors, while the optimality of individual behavior and the coordination of beliefs are not reflected in formal terms but are rather left to the interpretation of the revealed behavior. The components of our model are observable and can be estimated empirically, which makes the theory closer to practical applications, see the discussion in [12, Chapter 1] and [13].

The evolutionary view to this model, which places the emphasis on the processes of selection and mutation of investment strategies, is explained in detail in [10, 11]. An interpretation of the approach from the perspective of demand theory, which revives the Marshall concept of temporary equilibrium, is provided in [12]. Indeed the models considered in these papers are obtained by setting, in (5), the length of the time period to $\nu=1$.

Remark. The derivation of the corresponding model with an arbitrary, strictly increasing sequence of time points (t_n) is straightforward. The dynamics is given by (5) when replacing t_n^ν by t_n, t_{n+1}^ν by t_{n+1}, $c\nu$ by $c(t_{n+1}-t_n)$ and $D_\nu(t_{n+1}^\nu)$ by $\int_{t_n}^{t_{n+1}} \delta(s)ds$.

Any solution to (5) possesses the following property on the aggregate wealth of investors. Summation of (4) over $i = 1, ..., I$ yields

$$\bar{V}_\nu(t_n^\nu) = \sum_{i=1}^{I} V_\nu^i(t_n^\nu) = \frac{1}{c\nu} \sum_{k=1}^{K} D_{\nu,k}(t_n^\nu) \tag{7}$$

with $n \geq 1$. This relation says that the market for the consumption good clears, which is simply Walras' law. The definition of prices ensures clearing of all K asset markets. Since investors exhaust their budgets, Walras' law implies market clearing for the consumption good.

Define the set of strictly positive budget shares

$$\mathcal{L} = \Big\{ \Lambda \in (0,1]^{K \times I} : \sum_{k=1}^{K} \Lambda_{ki} = 1 \text{ for all } i \Big\}.$$

Assumption. To simplify the presentation, we will assume from now on that the sum of the dividend intensities is constant and, without further loss of generality, set this value equal to one, i.e. $\bar{\delta}(t) = \sum_{k=1}^{K} \delta_k(t) = 1$.

The case of a time-dependent process $\bar{\delta}(t)$ is discussed in detail in Section 7.

Under the above assumption, (7) implies for all $n \geq 1$

$$\bar{V}_\nu(t_n^\nu) = \frac{1}{c}. \tag{8}$$

Define the set

$$\mathcal{D} = \Big\{ V \in [0, \infty)^I : \sum_{i=1}^{I} V^i = \frac{1}{c} \Big\}.$$

Theorem 1. *Fix any $\nu > 0$ with $0 < c\nu < 1$. Suppose $\Lambda(t_n^\nu) \in \mathcal{L}$ for all $n \geq 0$.*

(i) For every $V_\nu(t_n^\nu) \in [0, \infty)^I$ with $\bar{V}_\nu(t_n^\nu) > 0$, there exists a unique $V_\nu(t_{n+1}^\nu)$ that solves (5). This solution satisfies $V_\nu(t_{n+1}^\nu) \in \mathcal{D}$.

(ii) For every initial value $V_\nu(0) \in [0, \infty)^I$ with $\bar{V}_\nu(0) > 0$ and a realization of the dividend process $\delta(t)$, the discrete-time dynamics (5) generates a sample path $V_\nu(t_n^\nu) \in \mathcal{D}$, $n = 1, 2, ...$

The proof of Theorem 1, as well as that of any other result, is provided in the appendix.

This result ensures that the wealth dynamics (5) is well-defined for every length ν of the time period. The solution $V_\nu(t_n^\nu)$ has the property (8) at each moment t_n^ν when dividends are paid and investors rebalance their portfolios in a self-financing way. This does not apply at the time when investors enter the market. Therefore, initial values only have to satisfy $V_\nu(0) \in [0,\infty)^I$ with $\bar{V}_\nu(0) > 0$ while $V_\nu(t_n^\nu) \in \mathcal{D}$, $n \geq 1$.

The next step is the derivation of the limit of the discrete-time model's sample paths as $\nu \to 0$. Our main finding is that this limit is the solution of a continuous-time model which is a random differential equation. The most surprising property of this limit model, which is derived in the next two sections, is that it possesses a natural interpretation as the wealth dynamics of self-financing strategies with market interaction. The limit model corresponds to the workhorse model of mathematical finance (e.g. Björk [4] or Shreve [24]) but extends the usual wealth dynamics by having endogenous asset prices and heterogenous investors.

3. Heuristic derivation of the limit model

It is instructive to present first a short-cut leading to the limit of the discrete-time model (5) as $\nu \to 0$. While the continuous-time model obtained is correct, this approach is heuristic. The proper mathematical proof of the approximation property of sample paths has to proceed differently, see Section 6. The derivation presented here might also provide a valuable shortcut for classroom presentations.

We assume that investors describe their benchmark budget shares $\lambda^i(t)$, $i = 1, \ldots, I$, at every time moment $t \geq 0$, but portfolios are rebalanced only at the discrete points in time t_n^ν, $n = 1, 2, \ldots.$ The shorter the length of the time period ν, the higher the frequency of rebalancing. A smaller ν enables investors to better track their benchmark strategy. In the limit the benchmark is perfectly matched.

Using that $V_\nu(t) = \Theta(\Lambda(t), V_\nu(t))\,\Lambda(t)\,V_\nu(t)$ for all $t = t_n^\nu$, one can rewrite (5) as

$$\begin{aligned} V_\nu(t_{n+1}^\nu) - V_\nu(t_n^\nu) \;&= \\ &\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\Big[(1-c\nu)\big[\Lambda(t_{n+1}^\nu)V_\nu(t_{n+1}^\nu) - \Lambda(t_n^\nu)V_\nu(t_n^\nu)\big] + D_\nu(t_{n+1}^\nu)\Big] \\ &-c\nu\, V_\nu(t_n^\nu). \end{aligned} \tag{9}$$

This representation allows to express the change in each investor's wealth from time t_n^ν to t_{n+1}^ν as the sum of (from left to right) changes in the asset prices, income from asset payoffs and consumption expenditure.

Suppose $\Lambda(t)$ is differentiable in t for every sample-path. Dividing both sides of (9) by $\nu > 0$ and letting $\nu \to 0$ gives a differential equation

$$dV(t) = \Theta(\Lambda(t), V(t)) \Big[d\left(\Lambda(t)V(t)\right) + \delta(t)dt \Big] - cV(t)dt. \qquad (10)$$

The dynamics described by (10), which is an implicit differential equation, is in continuous time. Analogously to the discrete-time case, marginal changes in the investors' wealth stem from price changes, asset payoffs and consumption.

Unfortunately the derivation is as incomplete as it is brief. One cannot conclude that the sample paths generated by the discrete-time system (9) converge, as $\nu \to 0$, to the solution of the differential equation (10) describing the continuous-time system. This would require the estimation of the distance between both solutions over finite time horizons which cannot be derived from the convergence of the difference quotients to the differentials.

In this paper we follow a route akin to the convergence of numerical schemes to prove that the sample paths of (10) actually are the limit of the sample paths of the discrete-time model (5) as the length of the time periods, ν, tends to zero. This approach relies on deriving a bound on the difference between the paths of continuous and discrete-time models and showing that this bound tends to zero on any compact time interval as $\nu \to 0$. The proof that the limit of the discrete-time model is actually described by the solution to (10) is provided in Section 6.

The interpretation of the continuous-time system (10) is put on a solid foundation in the next section. This will be done by presenting an alternative derivation of the limit model. The derivation is based on the continuous-time wealth dynamics of self-financing strategies with consumption and the market interaction of investors.

4. Continuous-time evolutionary model

This section shows that the evolutionary stock market model with continuous time (10) generalizes the wealth dynamics of an investor with a self-financing strategy as used in mathematical finance. The main innovation is the incorporation of the market interaction of traders by introducing endogenous prices in this standard framework. We will proceed in two steps. First it is proved that the dynamics (10) is well-defined. Then we show that this evolutionary model satisfies the standard definition of the wealth dynamics of a self-financing strategy in mathematical finance. The stock price process is defined through market clearing.

When referring to the continuous-time dynamics we will use the term differential equation throughout this paper; but it is important to point out that these equations are actually integral equations (i.e. solution is always meant in the sense of Carathédory). All functions are assumed to be measurable in the sense defined at the beginning of Section 2.

The dynamics (10) can be written as

$$dV^i(t) = \sum_{k=1}^{K} \frac{\lambda_k^i(t) V^i(t)}{\langle \lambda_k(t), V(t) \rangle} \big[d\langle \lambda_k(t), V(t) \rangle + \delta_k(t) dt \big] - cV^i(t) dt \tag{11}$$

for $i = 1, ..., I$.

Define the set

$$\mathcal{L}'_M = \Big\{ L' \in \mathbb{R}^{K \times I} : \sum_{k=1}^{K} L'_{ki} = 0 \text{ for all } i \text{ and } \|L'\| \leq M \Big\}$$

with $\|\cdot\|$ denoting the Euclidean norm.

Theorem 2. *Suppose $\Lambda(t) \in \mathcal{L}$ for all $t \geq 0$ and there is a constant $M \geq 0$ such that $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ for all $t \geq 0$. Then the system* (10) *has a unique solution $V(t) \in \mathcal{D}$ for every initial value $V(0) \in \mathcal{D}$. The solution is continuous and global, i.e. defined for all $t \geq 0$.*

The assumption $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ means that $\Lambda(t)$ is differentiable with uniformly bounded derivative. This rules out arbitrarily fast changes in the investors' strategies. It is essentially a technical condition needed to interpret the dynamics (10) as a random differential equation.

Suppose investment strategies $\lambda^i(t) = (\lambda_1^i(t), ..., \lambda_K^i(t))$, $i = 1, ..., I$, are given and each asset is in unit supply. Define the price of asset k at time t by

$$S_k(t) = \lambda_k^1(t)\, V^1(t) + ... + \lambda_k^I(t)\, V^I(t) = \langle \lambda_k(t), V(t) \rangle. \tag{12}$$

Under the assumptions of Theorem 2 the asset price is well-defined with $S_k(t) > 0$. Investor i's portfolio is defined as

$$\theta_k^i(t) = \frac{\lambda_k^i(t) V^i(t)}{S_k(t)} \tag{13}$$

for $k = 1, ..., K$. The market for each asset k clears because $\theta_k^1(t) + ... + \theta_k^I(t) = 1$, i.e. demand is equal to supply.

Finally denote the consumption process of investor i by

$$dC^i(t) = cV^i(t) dt. \tag{14}$$

Inserting (12)–(14) into (11), one obtains the dynamics

$$dV^i(t) = \sum_{k=1}^{K} \theta_k^i(t) \big[dS_k(t) + \delta_k(t) dt \big] - dC^i(t). \tag{15}$$

This equation is the standard mathematical finance textbook definition of the wealth dynamics of a trader employing a self-financing strategy with consumption in a market with K assets, see e.g. Björk [4] or Shreve [24].

Summarizing the discussion, our continuous-time model generalizes the workhorse of mathematical finance (15). In contrast to the standard model, the evolutionary stock market model derives endogenous asset prices from the market interaction of heterogenous investors.

5. Representation as random dynamical system

The evolutionary stock market model, both in discrete and in continuous time, is defined by an implicit equation. The wealth of investors, resp. the differential of the wealth, appears on both sides of the equation. In this section an explicit formulation of the dynamics is derived. The availability of a description of the dynamics as a random dynamical system (Arnold [2]) is indispensable for efficient numerical simulations and analytical studies of the dynamics.

The explicit representation of the discrete-time system is presented in Section 5.1. This result requires invertibility of the 'capital gains' matrix. It turns out that a reduction of the dimension of the dynamics is possible by using the market clearing condition for the consumption good. While this dimension-reduction is optional for the discrete-time model, it is necessary to derive an explicit representation of the continuous-time dynamics, Section 5.2. The explicit formulation of both models as random dynamical systems is used in the proof of existence and uniqueness of solution as well as in the main result on the convergence of sample paths as $\nu \to 0$.

5.1. Discrete-time system

Rewrite (5) as

$$\Big[\mathrm{Id} - (1-c\nu)\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\Lambda(t_{n+1}^\nu)\Big] V_\nu(t_{n+1}^\nu) = \Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) D_\nu(t_{n+1}^\nu).$$

An explicit representation of the dynamics can be obtained only if the matrix on the far left of this equation is invertible. This follows from Lemma E.1(i). Therefore the dynamics has the explicit form:

$$V_\nu(t_{n+1}^\nu) = \Big[\mathrm{Id} - (1-c\nu)\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))\Lambda(t_{n+1}^\nu)\Big]^{-1} \Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu)) D_\nu(t_{n+1}^\nu). \tag{16}$$

This representation is used to prove existence and uniqueness of solution which is asserted in Theorem 1 in Section 2.

The dimension of the system (16) can be reduced by one. Equation (8) implies that for $n \geq 1$

$$V_\nu^I(t_n^\nu) = \frac{1}{c} - \sum_{i=1}^{I-1} V_\nu^i(t_n^\nu). \tag{17}$$

Relation (17) might not be satisfied at the initial time, but can be assumed without loss of generality. The vector of initial wealth $V_\nu(0)$ can be scaled by any positive constant without changing the sample paths $V_\nu(t_n^\nu)$, $n = 1, 2, \dots$. This invariance property follows from equation (16) and the fact that the investors' portfolios $\Theta(\Lambda, V)$ are unchanged if the vector V is multiplied by a positive constant, i.e.

$$\Theta(\Lambda, \alpha V) = \Theta(\Lambda, V), \qquad \text{for all } \alpha > 0.$$

From now on we assume that $V_\nu(0)$ satisfies (17).

Define

$$\hat{\mathcal{D}} = \Big\{ \hat{V} \in [0, \infty)^{I-1} : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{1}{c} \Big\}.$$

There is a one-to-one correspondence between elements in $\mathcal{D}$ and $\hat{\mathcal{D}}$ using (17). Given $V \in \mathcal{D}$, define $\hat{V} = (V^1, ..., V^{I-1})$, which is in $\hat{\mathcal{D}}$. Conversely, for $\hat{V} \in \hat{\mathcal{D}}$ define $V = (\hat{V}^1, ..., \hat{V}^{I-1}, (1/c) - \sum_{i=1}^{I-1} \hat{V}^i) \in \mathcal{D}$.

Consider the discrete-time dynamics (5). The representation of the corresponding system in terms of $\hat{V}$ is given by

$$\begin{aligned} \hat{V}_\nu(t_{n+1}^\nu) = \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big) \Big[&(1 - c\nu) \Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu) \Big) \hat{V}_\nu(t_{n+1}^\nu) \\ &+ \frac{1 - c\nu}{c} \lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu) \Big] \end{aligned} \tag{18}$$

with function $\hat{\Theta} : \mathcal{L} \times \hat{\mathcal{D}} \to \mathbb{R}^{(I-1) \times K}$ defined as

$$\hat{\Theta}_{ik}(\Lambda, \hat{V}) = \frac{\Lambda_{ki} \hat{V}^i}{\sum_{j=1}^{I-1} (\Lambda_{kj} - \Lambda_{kI}) \hat{V}^j + \Lambda_{kI}/c} \tag{19}$$

with $i = 1, ..., I-1$ and $k = 1, ..., K$. The matrix $\hat{\Lambda}(t) \in \mathbb{R}^{K \times (I-1)}$ is obtained from $\Lambda(t)$ by omitting the last column:

$$\hat{\Lambda}_{ki}(t) = \lambda_k^i(t) \tag{20}$$

with $k = 1, ..., K$ and $i = 1, ..., I-1$. $\Lambda^I(t) \in \mathbb{R}^{K \times (I-1)}$ is a matrix with $I-1$ identical columns, each being equal to the vector $\lambda^I(t)$:

$$\Lambda_{ki}^I(t) = \lambda_k^I(t) \tag{21}$$

with $k = 1, ..., K$ and $i = 1, ..., I-1$.

As above one obtains the semi-explicit form

$$\begin{aligned} &\Big[\mathrm{Id} - (1 - c\nu) \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big) \Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu) \Big) \Big] \hat{V}_\nu(t_{n+1}^\nu) \\ &\qquad = \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big) \left[\frac{1 - c\nu}{c} \lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu) \right]. \end{aligned} \tag{22}$$

Invertibility of the matrix on the far left of the equation is ensured by Theorem E.1 under the assumption of fully diversified investment strategies, i.e. $\Lambda(t_n^\nu) \in \mathcal{L}$. One arrives at the explicit form

$$\begin{aligned}\hat{V}_\nu(t_{n+1}^\nu) = \Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big)\left(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\right)\Big]^{-1} \\ \hat{\Theta}\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu)\big)\left[\frac{1-c\nu}{c}\lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu)\right].\end{aligned} \tag{23}$$

This representation is used in the proof of the convergence result for the wealth dynamics as $\nu \to 0$.

An alternative proof of existence and uniqueness of solution for the discrete-time model can be given using the reduced version (23). The argument is as follows. The mapping of $\hat{V}_\nu(t_n^\nu)$ into $\hat{V}_\nu(t_{n+1}^\nu)$ defined by (23) is uniquely determined. The direct proof that $\hat{V}_\nu(t_{n+1}^\nu) \in \hat{\mathcal{D}}$ is more cumbersome in this case than in the one considered in Theorem 1.

5.2. Continuous-time system

While the reduction of dimension is optional for the discrete-time system, it turns out to be a necessary step in deriving an explicit differential equation in the continuous-time case. To verify this claim, rewrite the continuous-time model (10) in semi-explicit form:

$$[\mathrm{Id} - \Theta(\Lambda(t), V(t))\Lambda(t)]\, dV(t) = \Theta(\Lambda(t), V(t))\, \delta(t)dt - cV(t)dt. \tag{24}$$

The matrix on the far left is not invertible because all column sums are equal to zero. The system suffers from an 'over-specification' of solution through the existence of at least one linearly dependent row in the capital gains matrix. Therefore one cannot simply proceed as in the discrete-time case. We will show that reducing the dimension by one leads to an invertible matrix in the new system and therefore gives an explicit system.

The relation (17) holds for any solution of the continuous-time model: simply take the sum of (11) over $i = 1, ..., I$ to verify that $V(t) \in \mathcal{D}$. In addition one has $dV^I(t) = -\sum_{i=1}^{I-1} dV^i(t)$.

Using these two properties, (24) can be written (analogously to the discrete-time case) in semi-explicit form:

$$\begin{aligned}&\Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \Lambda^I(t))\Big]\, d\hat{V}(t) \\ &\quad = \hat{\Theta}(\Lambda(t), \hat{V}(t))\Big[\big(d[\hat{\Lambda}(t) - \Lambda^I(t)]\big)\hat{V}(t) + \frac{1}{c}d\lambda^I(t) + \delta(t)dt\Big] - c\hat{V}(t)dt.\end{aligned} \tag{25}$$

As in Section 5.1, Theorem E.1 allows to take the inverse of the matrix on the far left of this equation. Therefore the dynamics (25) can be expressed in explicit form:

$$d\hat{V}(t) = \Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \hat{\Lambda}^I(t))\Big]^{-1}$$
$$\Big[\hat{\Theta}(\Lambda(t), \hat{V}(t))\Big(\big(d[\hat{\Lambda}(t) - \hat{\Lambda}^I(t)]\big)\hat{V}(t) + \frac{1}{c}d\lambda^I(t) + \delta(t)dt\Big) - c\hat{V}(t)dt\Big]. \tag{26}$$

Analytical results on the long-term dynamics of the random dynamical system generated by (26) are obtained in Palczewski and Schenk-Hoppé [20].

6. Convergence of sample paths

This section presents the main result on the convergence of sample paths generated by the discrete-time model to that of the continuous-time model as the length ν of the time period tends to zero. We also give a numerical illustration of the approximation method.

6.1. *The result*

Fix an initial value $V(0) = V_\nu(0) \in \mathcal{D}$ for both models. Then Theorem 1 ensures, for any fixed $\nu > 0$, existence and uniqueness of a sample path $V_\nu(t_n^\nu)$, $n = 0, 1, 2, ...,$ while Theorem 2 gives the corresponding result for the sample path $V(t)$, $t \geq 0$. Both sample paths take values in the set $\mathcal{D}$.

Define the distance between the two sample paths at time t_n^ν by

$$\alpha_n^\nu = \|V(t_n^\nu) - V_\nu(t_n^\nu)\| \tag{27}$$

with Euclidean norm $\|\cdot\|$. Our aim is to derive an upper bound on α_n^ν (independent of the dividend intensity process $\delta(t)$) that uniformly converges to zero on compact time intervals of the form $[0, T]$ as $\nu \to 0$.

The convergence result requires a slightly stronger assumption (discussed in detail after the theorem) on the investment strategies than is needed for the existence and uniqueness of solution. Define for every $\varepsilon > 0$ the set of fully ε-diversified strategies

$$\mathcal{L}_\varepsilon = \Big\{\Lambda \in [\varepsilon, 1]^{K\times I} : \sum_{k=1}^K \Lambda_{ki} = 1 \text{ for all } i\Big\}.$$

One has the following result on the convergence of sample paths of the discrete-time model as the length of the time period tends to zero.

Theorem 3. *Suppose there exist $\varepsilon, M > 0$ such that $\Lambda(t) \in \mathcal{L}_\varepsilon$ and $\frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ for all $t \geq 0$. Let $V_\nu(0) = V(0) \in \mathcal{D}$. Then, for every $T > 0$, there exists a constant $C_1 > 0$, depending on T, ε, M, but independent of $V(0)$, $\nu > 0$ and $(\delta(t))_{t\in[0,T]}$, such that $\alpha_n^\nu \leq C_1\nu$ for $n = 0, 1, ..., \lfloor T/\nu \rfloor$.*

This result implies that the continuous-time model (10) is *approximated* by the model (5). For small time steps the sample path generated by the discrete-time system is close to the continuous-time sample path. This approximation property shows that the random differential equation (10) is the correct limit model and that the heuristic derivation provides the right answer.

The discrete-time model does not correspond to a standard numerical approximation scheme (such as Euler or Runge-Kutta). Its main advantage over standard methods is that asset markets clear at any point in time along all sample paths. No correction procedure (such as projection) is required to ensure that the numerical solution stays in the set $\mathcal{D}$.

The condition of fully ε-diversified investment strategies, which bounds the budget shares strictly away from zero, is needed to ensure a minimal degree of 'niceness' in the behavior of sample paths. Under this condition, prices are bounded away from zero and, thus, dividend-yields are bounded away from infinity. This restricts the local movement of the vector of investors' wealth from being arbitrarily fast. Conditions with a similar spirit are standard in approximation results for numerical schemes of deterministic and stochastic differential equations. It is noteworthy that the convergence in Theorem 3 is uniform in $\delta(t)$. Viewed as a numerical approximation, the upper bound on the difference of the sample paths holds true for any potential realization of the dividend intensity.

Let us give a short intuition for the proof of the convergence result. It is based on an estimate of the distance between sample paths of the discrete- and continuous-time system. The dynamics are compared at the times t_n^ν, $n = 1, 2, \ldots$. The explicit representations for the system with reduced dimension is employed, see Section 5.

Lemma B.2 states that the change of the wealth vector between two points in time t_n^ν and t_{n+1}^ν can be expressed as follows: in the discrete-time case

$$\hat{V}_\nu(t_{n+1}^\nu) - \hat{V}_\nu(t_n^\nu) = \int_0^\nu F\big(\delta(t_n^\nu + h), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda(t_{n+1}^\nu), \Lambda'(t_n^\nu + h)\big)\, dh, \tag{28}$$

and in the continuous-time case

$$\hat{V}(t_{n+1}^\nu) - \hat{V}(t_n^\nu) = \int_0^\nu F\big(\delta(t_n^\nu + h), 0, \hat{V}(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda'(t_n^\nu + h)\big)\, dh, \tag{29}$$

where $\Lambda'(t) = \frac{\partial}{\partial t}\Lambda(t)$ denotes the matrix of marginal changes of the components of the investment strategies.

An upper bound on the absolute distance between both systems' sample paths can be defined in terms of the difference of the function F over subsets of the domain. This difference can be bounded from above. Finally Gronwall's lemma is applied to derive the upper bound whose existence is asserted in Theorem 3.

6.2. Example

We consider an example to illustrate the continuous-time dynamics and the convergence property of the discrete-time model. Suppose there are three investors and three dividend-bearing long-lived assets (shares). The dividend intensities $\delta(t)$ are driven by a continuous-time Markov process $X(t)$ with two states $\{1, 2\}$. The Markov process has an intensity of transition from state 1 to 2 equal to 2.2 and an intensity of the opposite transition equal to 1.05. The dividend intensity process is determined by $X(t)$ in the following way:

$$\delta(t) = \begin{cases} (0.7, 0.3, 0), & X(t) = 1, \\ (0.3, 0.2, 0.5), & X(t) = 2. \end{cases}$$

Investors follow constant proportions investment strategies:

$$\Lambda(t) = \begin{pmatrix} 0.5 & 0.3 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.3 & 0.7 \end{pmatrix}$$

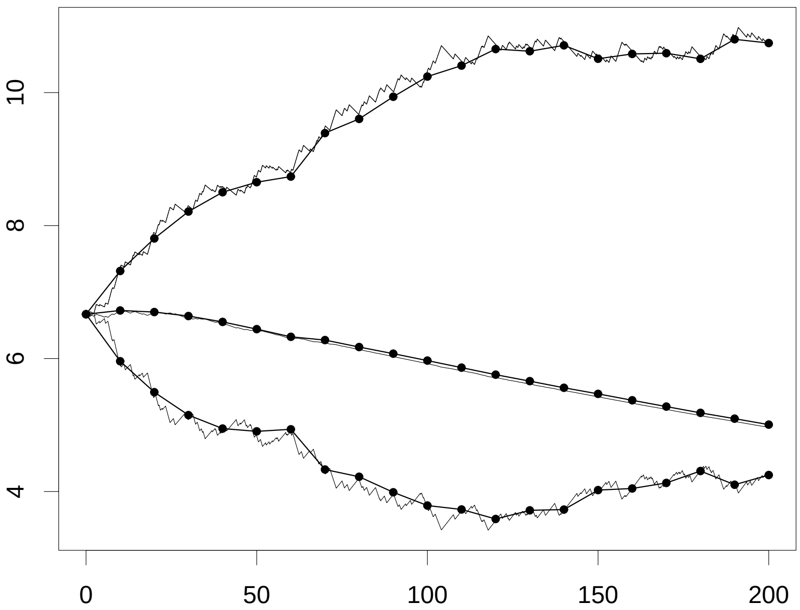
The strategy of investor i is described by column i. The consumption rate is set to $c = 0.05$, and the initial wealth is evenly distributed among the investors.

The time series of the wealth dynamics and stock prices for one realization of the dividend process are presented in Figure 1. The solid lines represent the solution to the continuous-time system (computed using a time step of 0.1) and the bold lines are obtained for the discrete-time model with time step length $\nu = 10$. This approximation consists of 20 points over the time horizon depicted in the figure.

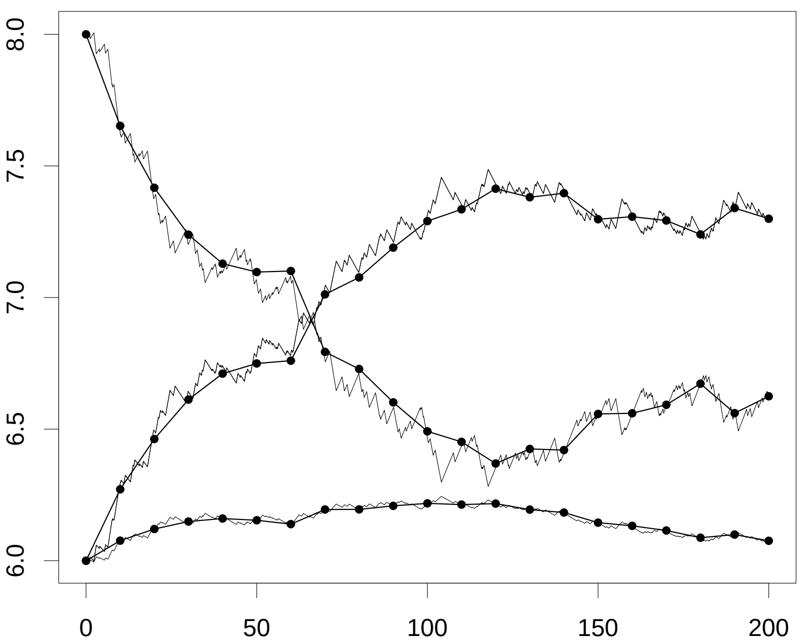
Figure 1(a) collects the time series of all three investors' wealth $V^i(t)$ resp. $V^i_\nu(t^\nu_n)$ with $i = 1, 2, 3$: investor 1 (top), investor 2 (middle) and investor 3 (bottom). Figure 1(b) shows the corresponding asset prices $S_k(t)$ resp. $S_{\nu,k}(t^\nu_n)$ with $k = 1, 2, 3$. Over the time horizon $[0, 200]$, the price of asset 1 increases from 6.0 to about 7.3, the price of asset 2 is nearly constant and the price of asset 3 falls from 8.0 to about 6.5.

The results reported in Figure 1 suggest that the discrete-time approximation of the continuous-time dynamics works well. This statement can be quantified by measuring the approximation error for different lengths of the time step.

The approximation error measured by the norm of the difference between the exact (continuous-time) solution and the approximate (discrete-time) solution of the wealth dynamics is reported in Figure 2. Figure 2(a) depicts the error for different lengths of the time step. Each simulation run uses the same trajectory of the dividend process as in Figure 1. Over the short time horizon $[0, 200]$ the error α^ν_n seems to grow slowly and linearly. Figure 2(b) looks at a time horizon ten times longer than in the previous case. The

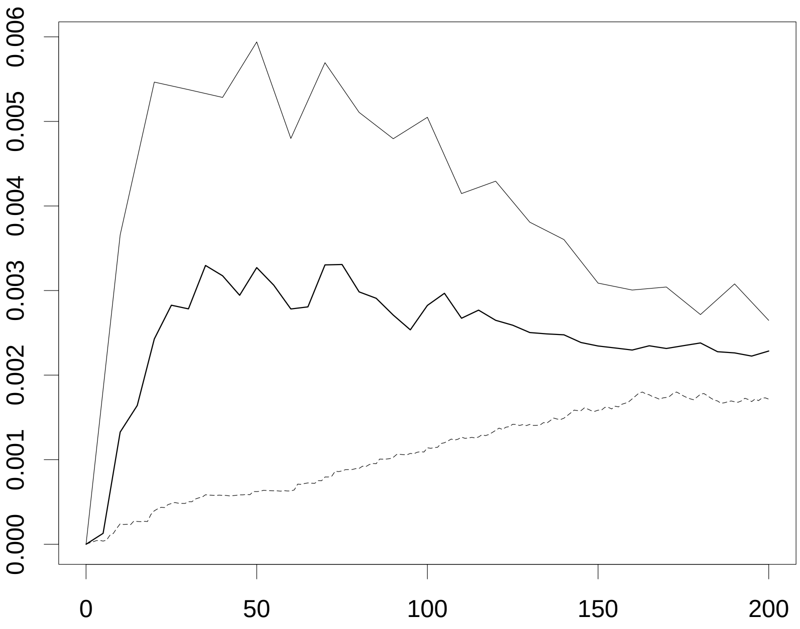


(a) Investors' wealth: exact solution (solid line) and approximation with the step size $\nu = 10$ (bold line).

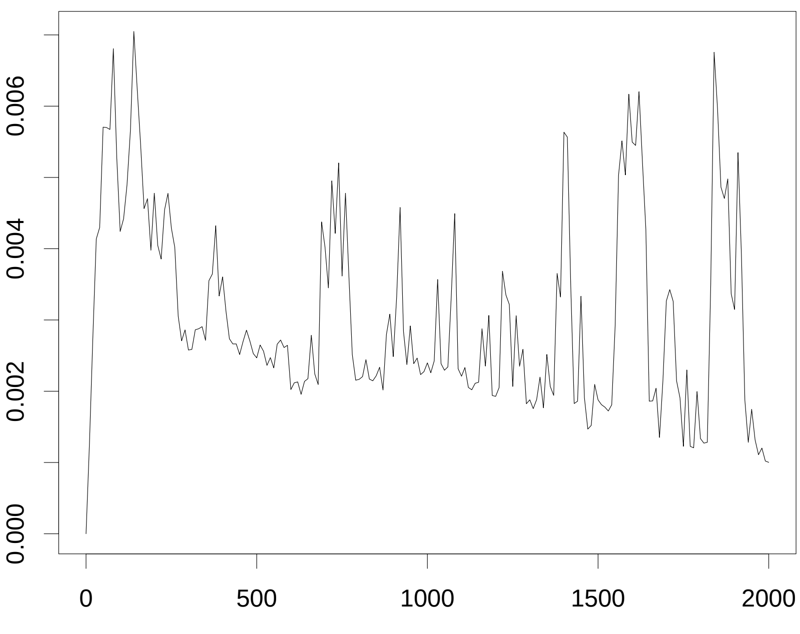


(b) Stock prices: exact solution (solid line) and approximation with the step size $\nu = 10$ (bold line).

Figure 1: Time series of investors' wealth and stock prices over the horizon $[0, 200]$.



(a) Approximation error for $\nu = 1$ (dashed line), $\nu = 5$ (bold line) and $\nu = 10$ (solid line) over the time horizon $[0, 200]$.



(b) Approximation error for $\nu = 10$ over the time horizon $[0, 2000]$.

Figure 2: Approximation error $\alpha_n^\nu = \|V(t_n^\nu) - V_\nu(t_n^\nu)\|$ for runs of different lengths.

error appears to be bounded rather than to grow with time. This observation confirms the common wisdom that the numerical approximation of continuous-time dynamical systems is often much better than theory predicts. Despite the fact that Theorem 3 only ensures convergence of sample paths on compact time intervals, the simulation results indicate that the discrete-time dynamics approximates the solution to the continuous-time system well even over very long time horizons. Indeed this feature turns out to be consistent across runs.

The reason for the high precision of the numerical approximation over long time horizons could be the existence of a random attractor (consisting only of one point) of the continuous-time dynamical system (10). Some related results on asymptotic behavior of the continuous-time model can be found in Palczewski and Schenk-Hoppé [20].

7. Time-dependent aggregate dividend intensity

This section discusses the validity of the above results under a weaker condition on the sum of the dividend intensities $\bar{\delta}(t) = \sum_{k=1}^{K} \delta_k(t)$. Assume that every sample path of $\bar{\delta}(t)$ is strictly positive and differentiable. Denote its derivative by $\bar{\delta}'(t)$. Suppose also that every sample path of the investment strategy $\Lambda(t)$ is differentiable—with its derivative denoted by $\Lambda'(t)$.

7.1. Discrete-time system

The discrete-time dynamics (5) as well as its representation in explicit form (16) are both valid when the aggregate dividend intensity is time-dependent. Summation of (4) over $i = 1, ..., I$ gives $\bar{V}_\nu(t_n^\nu) = \bar{D}_\nu(t_n^\nu)/(c\nu)$ for $n \geq 1$ (see (7)), where

$$\bar{D}_\nu(t_n^\nu) = \sum_{k=1}^{K} D_{\nu,k}(t_n^\nu) = \int_{t_{n-1}^\nu}^{t_n^\nu} \bar{\delta}(t)dt. \tag{30}$$

Define the set

$$\mathcal{D}_\nu(t_n^\nu) = \Big\{ V \in [0,\infty)^I : \sum_{i=1}^{I} V^i = \frac{\bar{D}_\nu(t_n^\nu)}{c\nu} \Big\}$$

for each $n = 1, 2, ...$ Note that $\mathcal{D}_\nu(t_n^\nu) = \mathcal{D}$ for $n \geq 1$ if $\bar{\delta}(t) = 1$ for $t \in [0, \infty)$.

One has the following assertion which generalizes Theorem 1.

Lemma 1. *Fix any $\nu > 0$ with $0 < c\nu < 1$. Suppose $\Lambda(t_n^\nu) \in \mathcal{L}$ for all $n \geq 0$. Assume that every sample path of $\bar{\delta}(t)$ is strictly positive.*

(i) For every $V_\nu(t_n^\nu) \in [0,\infty)^I$ with $\bar{V}_\nu(t_n^\nu) > 0$, there exists a unique $V_\nu(t_{n+1}^\nu)$ that solves (5). This solution satisfies $V_\nu(t_{n+1}^\nu) \in \mathcal{D}_\nu(t_{n+1}^\nu)$.

(ii) For every initial value $V_\nu(0) \in [0,\infty)^I$ with $\bar{V}_\nu(0) > 0$ and a realization of the dividend process $\delta(t)$, the discrete-time dynamics (5) *generates a sample path $V_\nu(t_n^\nu) \in \mathcal{D}_\nu(t_n^\nu)$, $n = 1, 2, \ldots$.*

The reduction of the dimension is similar to the procedure outlined in Section 5.1. Here we employ the relation

$$V_\nu^I(t_n^\nu) = \bar{D}_\nu(t_n^\nu)/(c\nu) - \sum_{i=1}^{I-1} V_\nu^i(t_n^\nu) \tag{31}$$

for all $n \geq 1$ and the fact that the sample path is invariant to the scaling of the initial wealth vector $V_\nu(0)$.

We will assume without loss of generality that $V_\nu(0) \in \mathcal{D}_\nu(0)$, where $\mathcal{D}_\nu(0)$ is given by (30) with $\bar{D}_\nu(0) = \nu\bar{\delta}(0)$ (which is strictly positive by our assumption on the dividend intensity). Using (31) one has a one-to-one correspondence between $V_\nu \in \mathcal{D}_\nu(t_n^\nu)$ and $\hat{V}_\nu = (V_\nu^1, \ldots V_\nu^{I-1}) \in \hat{\mathcal{D}}_\nu(t_n^\nu)$, where

$$\hat{\mathcal{D}}_\nu(t_n^\nu) = \Big\{\hat{V} \in [0,\infty)^{I-1} : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{\bar{D}_\nu(t_n^\nu)}{c\nu}\Big\}.$$

The system (5) can be written in terms of $\hat{V}$ as

$$\begin{aligned}\hat{V}_\nu(t_{n+1}^\nu) = \hat{\Theta}_\nu\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu), \bar{D}_\nu(t_n^\nu)\big)\Big[&(1-c\nu)\big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\big)\hat{V}_\nu(t_{n+1}^\nu) \\ &+ \frac{1-c\nu}{c\nu}\lambda^I(t_{n+1}^\nu)\bar{D}_\nu(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu)\Big],\end{aligned} \tag{32}$$

where the function $\hat{\Theta}_\nu : \mathcal{H}_\nu \to \mathbb{R}^{(I-1)\times K}$, with

$$\mathcal{H}_\nu = \Big\{(\Lambda, \hat{V}, \bar{D}) \in \mathcal{L} \times [0,\infty)^{I-1} \times (0,\infty) : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{\bar{D}}{c\nu}\Big\},$$

is defined as

$$\hat{\Theta}_{\nu,ik}(\Lambda, \hat{V}, \bar{D}) = \frac{\Lambda_{ki}\hat{V}^i}{\sum_{j=1}^{I-1}(\Lambda_{kj} - \Lambda_{kI})\hat{V}^j + \Lambda_{kI}\bar{D}/(c\nu)}.$$

By virtue of Theorem E.1, the explicit representation of (32) is given by

$$\begin{aligned}\hat{V}_\nu(t_{n+1}^\nu) = \Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}_\nu\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu), \bar{D}_\nu(t_n^\nu)\big)\Big(\hat{\Lambda}(t_{n+1}^\nu) - \hat{\Lambda}^I(t_{n+1}^\nu)\Big)\Big]^{-1} \\ \hat{\Theta}_\nu\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu), \bar{D}_\nu(t_n^\nu)\big)\Big[\frac{1-c\nu}{c}\lambda^I(t_{n+1}^\nu) + D_\nu(t_{n+1}^\nu)\Big].\end{aligned} \tag{33}$$

This equation generalizes (23) to the case of a time-dependent aggregate dividend intensity process.

7.2. Continuous-time system

We now show how to extend the results in Section 5.2. First note that the dynamics (10) and (11) are both correct for a time-dependent aggregate dividend intensity.

The aggregate wealth at time t in the continuous-time model is given by $\bar{\delta}(t)/c$ (sum (11) over $i = 1, ..., I$), i.e. $V(t) \in \mathcal{D}(\bar{\delta}(t))$, where

$$\mathcal{D}(\bar{\delta}) = \Big\{ V \in [0, \infty)^I : \sum_{i=1}^{I} V^i = \bar{\delta}/c \Big\}.$$

The following lemma generalizes Theorem 2.

Lemma 2. *Suppose $\Lambda(t) \in \mathcal{L}$ for all $t \geq 0$ and there exists an $M > 0$ such that $\Lambda'(t) \in \mathcal{L}'_M$ and $|\bar{\delta}'(t)| \leq M$ for all $t \geq 0$. Then the system* (10) *has a unique solution $V(t) \in \mathcal{D}(\bar{\delta}(t))$, $t \geq 0$, for every initial value $V(0) \in \mathcal{D}(\bar{\delta}(0))$. The solution is continuous and global.*

Lemma 2 implies that for all $t \geq 0$

$$V^I(t) = \bar{\delta}(t)/c - \sum_{i=1}^{I-1} V^i(t).$$

Using this relation, the dimension of the system (10) can be reduced by one. It is straightforward to show that

$$d\hat{V}(t) = \hat{\Theta}\big(\Lambda(t), \hat{V}(t), \bar{\delta}(t)\big) \bigg[\big(\hat{\Lambda}(t) - \Lambda^I(t)\big)\, d\hat{V}(t) + \big(d[\hat{\Lambda}(t) - \Lambda^I(t)]\big)\, \hat{V}(t) + \frac{1}{c} d\big[\bar{\delta}(t)\lambda^I(t)\big] + \delta(t)dt \bigg] - c\hat{V}(t)dt, \tag{34}$$

where the function $\hat{\Theta} : \mathcal{H} \to \mathbb{R}^{(I-1)\times K}$, with

$$\mathcal{H} = \Big\{ (\Lambda, \hat{V}, \bar{\delta}) \in \mathcal{L} \times [0, \infty)^{I-1} \times (0, \infty) : \sum_{i=1}^{I-1} \hat{V}^i \leq \frac{\bar{\delta}}{c} \Big\},$$

is given by

$$\hat{\Theta}_{ik}(\Lambda, \hat{V}, \bar{\delta}) = \frac{\Lambda_{ki}\hat{V}^i}{\sum_{j=1}^{I-1} (\Lambda_{kj} - \Lambda_{kI})\hat{V}^j + \Lambda_{kI}\bar{\delta}/c}.$$

Equation (34) can be written in explicit form (see Theorem E.1):

$$d\hat{V}(t) = \Big[\mathrm{Id} - \hat{\Theta}(\Lambda(t), \hat{V}(t))(\hat{\Lambda}(t) - \hat{\Lambda}^I(t)) \Big]^{-1} \times \bigg[\hat{\Theta}(\Lambda(t), \hat{V}(t)) \Big(\big(d[\hat{\Lambda}(t) - \hat{\Lambda}^I(t)]\big)\hat{V}(t) + \frac{1}{c} d\big[\bar{\delta}(t)\lambda^I(t)\big] + \delta(t)dt \Big) - c\hat{V}(t)dt \bigg]. \tag{35}$$

7.3. Convergence result

The dynamics of the discrete-time model (33) and the continuous-time model (35) (both with time-dependent aggregate dividend intensity $\bar{\delta}(t)$) are close in the sense that the distance between both paths at times t_n^ν,

$$\alpha_n^\nu = \|V(t_n^\nu) - V_\nu(t_n^\nu)\|,$$

converges to zero as $\nu \to 0$.

The convergence result, Theorem 3, is generalized by the following theorem.

Theorem 4. *Suppose $\bar{\delta}(t)$ is twice differentiable and there exist $\varepsilon, M > 0$ such that $\Lambda'(t) \in \mathcal{L}'_M$, $|\bar{\delta}'(t)| \leq M$, $|\bar{\delta}''(t)| \leq M$, $\bar{\delta}(t) \geq \varepsilon$ and $\Lambda(t) \in \mathcal{L}_\varepsilon$. Let $V_\nu(0) = V(0) \in \mathcal{D}(\bar{\delta}(0))$. For every $T > 0$ there exists a constant $C_1 > 0$ depending on T, ε, M and $\bar{\delta}(0)$ (but independent of $V(0)$, $\nu > 0$ and $(\delta(t))_{t\in[0,T]}$), such that $\alpha_n^\nu \leq C_1 \nu$ for $n = 0, 1, ..., \lfloor T/\nu \rfloor$.*

Note that one has $\mathcal{D}_\nu(0) = \mathcal{D}(\bar{\delta}(0))$ for $\nu > 0$. Therefore, the set of initial conditions for the discrete- and continuous-time models are identical.

8. Conclusion

This paper marks the birth of continuous-time evolutionary finance. It opens an avenue for the study of the wealth dynamics of interacting investment strategies (and the endogenous asset price dynamics it entails) in continuous time. We derive the continuous-time limit of a generalization of the discrete-time evolutionary stock market model by Evstigneev, Hens and Schenk-Hoppé [11, 12]. This limit model, which is obtained by letting the length of the time period tend to zero, has an explicit representation as a random dynamical system and possesses a meaningful interpretation from an economics and finance point of view. The continuous-time model extends the standard framework of mathematical finance by introducing (endogenous) stock prices which are driven by the market interaction of investors. An efficient numerical simulation of the limit model is possible because our approximation results provide an explicit scheme which converges uniformly on finite time intervals. Our numerical method has the advantage of delivering market clearing in every time step without the need of projection techniques. Future research will focus on analytical and numerical studies of the dynamics of the continuous-time model. First results are obtained in Palczewski and Schenk-Hoppé [20].

Appendix A

Proof of Theorem 1. Lemma E.1(i) implies that the discrete-time dynamics (5) is equivalent to the explicit representation (16). The assumption

$\Lambda(t_n^\nu) \in \mathcal{L}$ and $\bar{V}_\nu(t_n^\nu) > 0$ imply that $\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))$ is well-defined. Therefore, (16) (and thus (5)) has a unique solution.

Lemma E.1(i) also states that the inverse matrix in (16) maps the non-negative orthant $[0,\infty)^I$ into itself. Since the vector $\Theta(\Lambda(t_n^\nu), V_\nu(t_n^\nu))D_\nu(t_{n+1}^\nu)$ has only non-negative coordinates, one finds that $V_\nu(t_{n+1}^\nu)$ is non-negative. Equation (8) further gives that the coordinates of $V_\nu(t_{n+1}^\nu)$ sum up to $1/c$. This implies $V_\nu(t_{n+1}^\nu) \in \mathcal{D}$. □

Proof of Theorem 2. The proof of existence and uniqueness of the solution to the model in continuous time employs standard results from the theory of random differential equations. Lemma B.2(ii) states that the explicit system (26) is equivalent to

$$d\hat{V}(t) = F(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda(t), \Lambda'(t))dt. \tag{36}$$

Lemma B.1 implies local Lipschitz continuity of F in the argument $\hat{V}$ and local integrability in all arguments (due to their boundedness). Arnold [2, Theorem 2.2.1] ensures existence and uniqueness of the solution. The solution is global because all components are non-negative and their sum is finite for all t. In fact, the set $\hat{\mathcal{D}}$ is forward invariant. □

Appendix B

Define the set of dividend intensities

$$\mathcal{S} = \Big\{\delta \in [0,\infty)^K : \sum_{k=1}^{K} \delta_k = 1\Big\}.$$

Define the function $F : \mathcal{S} \times [0,1] \times \hat{\mathcal{D}} \times \mathcal{L} \times \mathcal{L} \times \mathbb{R}^{K\times I} \to \mathbb{R}^{I-1}$ as

$$\begin{aligned} F(\delta, \alpha, \hat{V}, \Lambda_1, \Lambda_2, \Lambda') = {}& \Big[\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda_1, \hat{V})(\hat{\Lambda}_2 - \hat{\Lambda}_2^I)\Big]^{-1} \times \\ & \times \Big[\hat{\Theta}(\Lambda_1, \hat{V})\Big(\delta + (1-\alpha)(\hat{\Lambda}' - \hat{\Lambda}'^I)\hat{V} + \frac{1-\alpha}{c}\Lambda'_{\cdot,I}\Big) - c\hat{V}\Big]. \end{aligned} \tag{37}$$

The vector $\Lambda'_{\cdot,I}$ is the last column of Λ'. $\hat{\Lambda} \in \mathbb{R}^{K\times(I-1)}$ denotes the matrix obtained from Λ by omitting the last column, and the matrix $\hat{\Lambda}^I \in \mathbb{R}^{K\times(I-1)}$ has $I-1$ identical columns, each being equal to the last column of Λ.

Lemma B.1. *The function F is continuously differentiable on its domain.*

Proof of Lemma B.1. Theorem E.1 implies that the matrix $\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda_1, \hat{V})(\hat{\Lambda}_2 - \hat{\Lambda}_2^I)$ is invertible. Therefore the function F is well-defined. Direct computation shows that F is continuously differentiable. □

Lemma B.2. *Suppose $\Lambda(t) \in \mathcal{L}$ and its derivative $\Lambda'(t)$ exists for all $t \geq 0$ along every sample path.*

(i) The solution to (23) *fulfills*

$$\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = \int_0^\nu F\big(\delta(t^\nu_n + h), c\nu, \hat{V}_\nu(t^\nu_n), \Lambda(t^\nu_n), \Lambda(t^\nu_{n+1}), \Lambda'(t^\nu_n + h)\big)dh.$$

(ii) The solution to (26) *satisfies*

$$d\hat{V}(t) = F(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda(t), \Lambda'(t))dt.$$

Proof of Lemma B.2. (i): The investor's wealth $\hat{V}_\nu(t^\nu_n)$, less his consumption, is fully invested in the available assets, i.e.

$$\hat{V}_\nu(t^\nu_n) = \frac{1}{1 - c\nu}\hat{\Theta}\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n)\big)S_\nu(t^\nu_n),$$

see (2). Using the market clearing condition (3), this implies

$$\hat{V}_\nu(t^\nu_n) = \hat{\Theta}\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n)\big)\left[\Big(\hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) + \frac{1}{c}\lambda^I(t^\nu_n)\right].$$

Inserting into (18) yields

$$\begin{aligned}
\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) &= -c\nu\hat{V}_\nu(t^\nu_n) \\
&+ \hat{\Theta}(\cdot)\Big[(1 - c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\Big)\big(\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n)\big) \\
&\quad + (1 - c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\
&\quad + \frac{1 - c\nu}{c}\big(\lambda^I(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\big) + D_\nu(t^\nu_{n+1})\Big],
\end{aligned}$$

where $\hat{\Theta}(\cdot)$ is used as a shorthand notation for $\hat{\Theta}\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n)\big)$. Lemma E.1(ii) implies that this equation can be written in explicit form:

$$\begin{aligned}
\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) &= \Big[\mathrm{Id} - (1 - c\nu)\hat{\Theta}(\cdot)\big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\big)\Big]^{-1} \times \\
&\times \Big[(1 - c\nu)\hat{\Theta}(\cdot)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\
&\quad + \frac{1 - c\nu}{c}\hat{\Theta}(\cdot)\big(\lambda^I(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\big) + \hat{\Theta}(\cdot)D_\nu(t^\nu_{n+1}) - c\nu\hat{V}_\nu(t^\nu_n)\Big].
\end{aligned}$$

This proves (i).

(ii): Differentiability of $\Lambda(t)$ for *every* t implies that integration with respect to $d\Lambda(t)$ can be substituted by the integration with respect to $\Lambda'(t)dt$, Rudin [22, p. 325]. This also applies to $\hat{\Lambda}(t)$ and $\hat{\Lambda}^I(t)$. Therefore, the dynamics (26) is equivalent to the formulation in (ii). □

Appendix C

Proof of Theorem 3. Denote the distance between the two sample paths of the models with reduced dimension derived in Section 5 by

$$\hat{\alpha}_n^\nu = \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm.

One has $(\alpha_n^\nu)^2 \leq I(\hat{\alpha}_n^\nu)^2$ because

$$\begin{aligned}\|V(t_n^\nu) - V_\nu(t_n^\nu)\|^2 &= \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|^2 + \Big(\sum_{i=1}^{I-1}(\hat{V}^i(t_n^\nu) - \hat{V}_\nu^i(t_n^\nu))\Big)^2 \\ &\leq I\|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|^2,\end{aligned}$$

which implies that it suffices to obtain the convergence result for the 'hat'-system.

According to the assumptions of the theorem there are $\varepsilon, M > 0$ such that $\Lambda(t) \in \mathcal{L}_\varepsilon$ for all $t \geq 0$ and its derivative satisfies $\Lambda'(t) = \frac{\partial}{\partial t}\Lambda(t) \in \mathcal{L}'_M$ for all $t \geq 0$. The function F (which is defined in (37)) is continuously differentiable by Lemma B.1 and the set $\mathcal{S} \times [0,1] \times \hat{\mathcal{D}} \times \mathcal{L}_\varepsilon \times \mathcal{L}_\varepsilon \times \mathcal{L}'_M$ is compact. This implies existence of a constant C_2 such that for any $\delta \in \mathcal{S}$, $\alpha, \alpha_* \in [0,1]$, $\hat{V}, \hat{V}_* \in \hat{\mathcal{D}}$, $\Lambda_1, \Lambda_{1,*}, \Lambda_2, \Lambda_{2,*} \in \mathcal{L}_\varepsilon$ and $\Lambda' \in \mathcal{L}'_M$

$$\|F(\delta, \alpha, \hat{V}, \Lambda_1, \Lambda_2, \Lambda')\| \leq C_2, \tag{38}$$

$$\begin{aligned}&\big\|F(\delta, \alpha, \hat{V}, \Lambda_1, \Lambda_2, \Lambda') - F(\delta, \alpha_*, \hat{V}_*, \Lambda_{1,*}, \Lambda_{2,*}, \Lambda')\big\| \\ &\qquad \leq C_2\big(\|\hat{V} - \hat{V}_*\| + |\alpha - \alpha_*| + \|\Lambda_1 - \Lambda_{1,*}\| + \|\Lambda_2 - \Lambda_{2,*}\|\big).\end{aligned} \tag{39}$$

This result plays an important role in the derivation of estimates in the remainder of this proof.

Subtracting equation (28) for the discrete-time system $\hat{V}_\nu(t_{n+1}^\nu)$ from equation (29) for the continuous-time system $\hat{V}(t_{n+1}^\nu)$ and taking norms on both sides, yields

$$\begin{aligned}\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \Big\| \int_0^\nu \Big[&F\big(\delta(t_n^\nu + h), 0, \hat{V}(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda(t_n^\nu + h), \Lambda'(t_n^\nu + h)\big) \\ &- F\big(\delta(t_n^\nu + h), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda(t_{n+1}^\nu), \Lambda'(t_n^\nu + h)\big)\Big] dh \Big\|.\end{aligned}$$

This inequality remains valid if the norm is pulled inside the integral (due to Jensen's inequality). The estimate (39) implies that

$$\begin{aligned}\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \int_0^\nu C_2\Big(&\|\hat{V}(t_n^\nu + h) - \hat{V}_\nu(t_n^\nu)\| + \|\Lambda(t_n^\nu + h) - \Lambda(t_n^\nu)\| \\ &+ \|\Lambda(t_n^\nu + h) - \Lambda(t_{n+1}^\nu)\| + c\nu\Big) dh.\end{aligned}$$

Boundedness of the derivative $\Lambda'(t)$ gives

$$\|\Lambda(t_n^\nu + h) - \Lambda(t_n^\nu)\| \leq Mh,$$
$$\|\Lambda(t_n^\nu + h) - \Lambda(t_{n+1}^\nu)\| \leq M(\nu - h).$$

The triangle inequality and the relation (38) yield

$$\|\hat{V}(t_n^\nu + h) - \hat{V}_\nu(t_n^\nu)\| \leq \|\hat{V}(t_n^\nu + h) - \hat{V}(t_n^\nu)\| + \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\| \leq C_2 h + \hat{\alpha}_n^\nu.$$

Inserting these estimates provides the following upper bound on the approximation error at time t_{n+1}^ν:

$$\begin{aligned}
\hat{\alpha}_{n+1}^\nu &\leq \hat{\alpha}_n^\nu + \int_0^\nu C_2 \Big(C_2 h + \hat{\alpha}_n^\nu + Mh + M(\nu - h) + c\nu \Big) dh \\
&\leq \hat{\alpha}_n^\nu + \frac{1}{2} {C_2}^2 \nu^2 + \nu C_2 \hat{\alpha}_n^\nu + C_2 M \nu^2 + \frac{1}{2} C_2 c \nu^2 \\
&= (1 + \nu C_2) \hat{\alpha}_n^\nu + \frac{C_2}{2} \nu^2 (C_2 + 2M + c).
\end{aligned}$$

At the initial time, $\hat{V}(0) = \hat{V}_\nu(0) \in \hat{\mathcal{D}}$. This implies $\hat{\alpha}_0^\nu = 0$. Gronwall's lemma gives

$$\hat{\alpha}_n^\nu \leq \frac{(1 + \nu C_2)^n}{\nu C_2} \frac{C_2}{2} \nu^2 (C_2 + 2M + c) \leq e^{n\nu C_2} \frac{1}{2} (C_2 + 2M + c)\, \nu,$$

where the second estimate uses the inequality $(1 + a)^n \leq e^{na}$ for $a \geq 0$. If $n\nu \leq T$, the expression $e^{n\nu C_2}$ is bounded by e^{TC_2}. Thus, the constant C_1 in the assertion of the theorem is given by

$$C_1 = \frac{\sqrt{I}}{2} e^{TC_2} (C_2 + 2M + c).$$

This completes the proof. □

Appendix D

Proof of Lemma 1. The proof is analogous to that of Theorem 1. □

Proof of Lemma 2. The proof is analogous to that of Theorem 2. □

Proof of Theorem 4. We derive integral representations of the wealth increments of the continuous- and discrete-time dynamics. These formulations generalize those obtained in Lemma B.2.

Define $G : \mathcal{A}_\varepsilon \to \mathbb{R}^{I-1}$ by

$$\begin{aligned}
G(\delta, \alpha, \hat{V}, \Lambda_1, \Lambda_2, \Lambda', \bar{d}, \bar{d}', w) &= \Big[\mathrm{Id} - (1 - \alpha) \hat{\Theta}(\Lambda_1, \hat{V}, w)(\hat{\Lambda}_2 - \hat{\Lambda}_2^I) \Big]^{-1} \times \\
&\times \Big[\hat{\Theta}(\Lambda_1, \hat{V}, w) \Big(\delta + (1 - \alpha)(\hat{\Lambda}' - \hat{\Lambda}'^I) \hat{V} + \frac{1 - \alpha}{c} \big(\bar{d} \hat{\Lambda}'_{\cdot,I} + \bar{d}' \hat{\Lambda}_{\cdot,I} \big) \Big) - c\hat{V} \Big],
\end{aligned} \tag{40}$$

where

$$\hat{\Theta}_{ik}(\Lambda, \hat{V}, w) = \frac{\Lambda_{ki}\hat{V}^i}{\sum_{j=1}^{I-1}(\Lambda_{kj} - \Lambda_{kI})\hat{V}^j + \Lambda_{kI} w/c}$$

and

$$\begin{aligned}\mathcal{A}_\varepsilon = \Big\{(\delta, \alpha, \hat{V}, \Lambda_1, \Lambda_2, \Lambda', \bar{d}, \bar{d}', w) \in [0,\infty)^K \times [0,1] \times [0,\infty)^{I-1} \times \mathcal{L}_\varepsilon \times \mathcal{L}_\varepsilon \\ \times \mathcal{L}'_M \times [\varepsilon,\infty) \times [-M,M] \times [\varepsilon,\infty) : \\ \sum_{i=1}^{I-1} \hat{V}^i \le w/c \ \text{ and } \ \bar{d}, w, \sum_{k=1}^{K} \delta_k \le \bar{\delta}(0) + TM\Big\}.\end{aligned}$$

Using the function G, the continuous-time model (35) can be written as

$$d\hat{V}(t) = G\big(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda(t), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{\delta}(t)\big)dt.$$

Therefore,

$$\hat{V}(t^\nu_{n+1}) - \hat{V}(t^\nu_n) = \int_{t^\nu_n}^{t^\nu_{n+1}} G\big(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda(t), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{\delta}(t)\big)dt. \tag{41}$$

We now consider the discrete-time system. The identity

$$\hat{V}_\nu(t^\nu_n) = \hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}_\nu(t^\nu_n)\big)\Big[\big(\hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_n)\big)\hat{V}_\nu(t^\nu_n) + \frac{1}{c\nu}\lambda^I(t^\nu_n)\bar{D}_\nu(t^\nu_n)\Big]$$

inserted into (32) gives

$$\begin{aligned}\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = &- c\nu\hat{V}_\nu(t^\nu_n) + \hat{\Theta}_\nu\big(\Lambda(t^\nu_n), \hat{V}_\nu(t^\nu_n), \bar{D}_\nu(t^\nu_n)\big) \times \\ &\times \Big[(1-c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\Big)\big(\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n)\big) \\ &\quad + (1-c\nu)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\ &\quad + \frac{1-c\nu}{c\nu}\big(\lambda^I(t^\nu_{n+1})\bar{D}_\nu(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\bar{D}_\nu(t^\nu_n)\big) + D_\nu(t^\nu_{n+1})\Big].\end{aligned}$$

This equation can be written in explicit form (see Theorem E.1):

$$\begin{aligned}\hat{V}_\nu(t^\nu_{n+1}) - \hat{V}_\nu(t^\nu_n) = &\Big[\mathrm{Id} - (1-c\nu)\hat{\Theta}_\nu(\cdot)\big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}^I(t^\nu_{n+1})\big)\Big]^{-1} \times \\ &\times \Big[(1-c\nu)\hat{\Theta}_\nu(\cdot)\Big(\hat{\Lambda}(t^\nu_{n+1}) - \hat{\Lambda}(t^\nu_n) - \hat{\Lambda}^I(t^\nu_{n+1}) + \hat{\Lambda}^I(t^\nu_n)\Big)\hat{V}_\nu(t^\nu_n) \\ &\quad + \frac{1-c\nu}{c\nu}\hat{\Theta}_\nu(\cdot)\big(\lambda^I(t^\nu_{n+1})\bar{D}_\nu(t^\nu_{n+1}) - \lambda^I(t^\nu_n)\bar{D}_\nu(t^\nu_n)\big) \\ &\quad + \hat{\Theta}_\nu(\cdot)D_\nu(t^\nu_{n+1}) - c\nu\hat{V}_\nu(t^\nu_n)\Big],\end{aligned}$$

where $\hat{\Theta}_\nu(\cdot)$ denotes $\hat{\Theta}_\nu\big(\Lambda(t_n^\nu), \hat{V}_\nu(t_n^\nu), \bar{D}_\nu(t_n^\nu)\big)$. Using the function G, this dynamics has the integral representation for $n \geq 1$:

$$\hat{V}_\nu(t_{n+1}^\nu) - \hat{V}_\nu(t_n^\nu) = \int_0^\nu G\Big(\delta(t_n^\nu + h), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda(t_{n+1}^\nu), \Lambda'(t_n^\nu + h), \frac{\bar{D}_\nu(t_n^\nu + h)}{\nu}, \frac{\bar{\delta}(t_n^\nu + h) - \bar{\delta}(t_n^\nu + h - \nu)}{\nu}, \frac{\bar{D}_\nu(t_n^\nu)}{\nu}\Big) dh, \tag{42}$$

where the definition (30) of $\bar{D}_\nu$ is extended to arbitrary values $t \geq \nu$ by setting

$$\bar{D}_\nu(t) = \int_{t-\nu}^t \bar{\delta}(s) ds.$$

The relation (42) does not apply for $n = 0$ because $\bar{\delta}(t)$ is defined for positive arguments only.

Subtracting (42) from (41), taking norms on both sides and applying Jensen's inequality, one obtains for $n \geq 1$

$$\begin{aligned}\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \int_{t_n^\nu}^{t_{n+1}^\nu} \Big\| & G\big(\delta(t), 0, \hat{V}(t), \Lambda(t), \Lambda(t), \Lambda'(t), \bar{\delta}(t), \bar{\delta}'(t), \bar{\delta}(t)\big) \\ & - G\Big(\delta(t), c\nu, \hat{V}_\nu(t_n^\nu), \Lambda(t_n^\nu), \Lambda(t_{n+1}^\nu), \Lambda'(t), \\ & \frac{\bar{D}_\nu(t)}{\nu}, \frac{\bar{\delta}(t) - \bar{\delta}(t-\nu)}{\nu}, \frac{\bar{D}_\nu(t_n^\nu)}{\nu}\Big)\Big\| dt,\end{aligned}$$

where $\hat{\alpha}_n^\nu = \|\hat{V}(t_n^\nu) - \hat{V}_\nu(t_n^\nu)\|$.

The same arguments as in Lemma B.1 can be applied to show that the function G is continuously differentiable on its domain $\mathcal{A}_\varepsilon$. Compactness of $\mathcal{A}_\varepsilon$ implies existence of a constant C_3 such that $\|G(a) - G(a')\| \leq C_3\|a - a'\|$ and $\|G(a)\| \leq C_3$ for any $a, a' \in \mathcal{A}_\varepsilon$. Therefore

$$\begin{aligned}\hat{\alpha}_{n+1}^\nu \leq \hat{\alpha}_n^\nu + \int_{t_n^\nu}^{t_{n+1}^\nu} C_3\Big(& c\nu + \|\hat{V}(t) - \hat{V}_\nu(t_n^\nu)\| + \|\Lambda(t) - \Lambda(t_n^\nu)\| \\ & + \|\Lambda(t) - \Lambda(t_{n+1}^\nu)\| + |\bar{\delta}(t) - \bar{D}_\nu(t)/\nu| \\ & + \Big|\bar{\delta}'(t) - \frac{\bar{\delta}(t) - \bar{\delta}(t-\nu)}{\nu}\Big| + |\bar{\delta}(t) - \bar{D}_\nu(t_n^\nu)/\nu|\Big) dt.\end{aligned}$$

Boundedness of the derivative $\Lambda'(t)$ yields $\|\Lambda(t) - \Lambda(t_n^\nu)\| \leq M(t - t_n^\nu)$ and $\|\Lambda(t) - \Lambda(t_{n+1}^\nu)\| \leq M(t_{n+1}^\nu - t)$ for all $t \in [t_n^\nu, t_{n+1}^\nu]$. The triangle inequality and boundeness of G on $\mathcal{A}_\varepsilon$ imply

$$\|\hat{V}(t) - \hat{V}_\nu(t_n^\nu)\| \leq \hat{\alpha}_n^\nu + C_3(t - t_n^\nu).$$

By Jensen's inequality

$$|\bar{\delta}(t) - \bar{D}_\nu(t_n^\nu)/\nu| = \Big|\bar{\delta}(t) - \frac{1}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} \bar{\delta}(s) ds\Big| \leq \frac{1}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} |\bar{\delta}(t) - \bar{\delta}(s)| ds.$$

Since the function $\bar\delta(t)$ has a bounded derivative $|\bar\delta'(t)| \le M$ for every $t \ge 0$, one finds $|\bar\delta(t) - \bar\delta(s)| \le M|t-s|$ and, for all $t \ge t_n^\nu$,

$$|\bar\delta(t) - \bar D_\nu(t_n^\nu)/\nu| \le \frac{M}{\nu}\int_{t_{n-1}^\nu}^{t_n^\nu} (t-s)ds = \frac{M}{2}\nu + M(t - t_n^\nu).$$

A similar argument gives

$$|\bar\delta(t) - \bar D_\nu(t)/\nu| \le M\nu/2.$$

Boundedness of the second derivative $|\bar\delta''(t)| \le M$ implies that for $t \ge \nu$

$$\Big|\bar\delta'(t) - \frac{\bar\delta(t) - \bar\delta(t-\nu)}{\nu}\Big| \le M\nu/2.$$

Inserting these estimates and changing variables by setting $h = t - t_n^\nu$, provides the following estimate for $\hat\alpha_{n+1}^\nu$:

$$\begin{aligned}
\hat\alpha_{n+1}^\nu &\le \hat\alpha_n^\nu + C_3\int_0^\nu \Big(c\nu + \hat\alpha_n^\nu + C_3 h + Mh + M(\nu - h) \\
&\qquad\qquad + \frac{M}{2}\nu + Mh + \frac{M}{2}\nu + \frac{M}{2}\nu\Big)dh \\
&= \hat\alpha_n^\nu + C_3\int_0^\nu \Big(\hat\alpha_n^\nu + (5M/2 + c)\nu + (C_3 + M)h\Big)dh \\
&= (1 + C_3\nu)\hat\alpha_n^\nu + C_3\nu^2(c + C_3/2 + 3M).
\end{aligned}$$

Recursive application of this inequality gives, for all $n \ge 1$, the upper bound

$$\begin{aligned}
\hat\alpha_n^\nu &\le \hat\alpha_1^\nu(1 + C_3\nu)^{n-1} + \frac{(1 + C_3\nu)^{n-1}}{C_3\nu} C_3\nu^2(c + C_3/2 + 3M) \\
&\le \hat\alpha_1^\nu e^{(n-1)C_3\nu} + e^{(n-1)C_3\nu}(c + C_3/2 + 3M)\,\nu,
\end{aligned} \tag{43}$$

where one has to use the inequality $(1 + a)^n \le e^{na}$ for $a \ge 0$ to obtain the last estimate.

The estimation of $\hat\alpha_1^\nu$ requires a different approach as equation (42) is not defined for $n = 0$ ($\bar\delta(t)$ is given only for $t \ge 0$). The increment of the discrete-time system between $t_0^\nu = 0$ and $t_1^\nu = \nu$ is given by

$$\begin{aligned}
\hat V_\nu(t_1^\nu) - \hat V_\nu(0) = \int_0^\nu G\Big(&\delta(t), c\nu, \hat V_\nu(0), \Lambda(0), \Lambda(t_1^\nu), \Lambda'(t), \\
&\frac{1}{\nu}\big[(\nu - t)\bar\delta(0) + \int_0^t \bar\delta(s)ds\big], \frac{\bar\delta(t) - \bar\delta(0)}{\nu}, \bar\delta(0)\Big)dt
\end{aligned} \tag{44}$$

because $\bar D_\nu(0) = \nu\bar\delta(0)$ and

$$\begin{aligned}
&\lambda^I(t_1^\nu)\bar D_\nu(t_1^\nu) - \lambda^I(0)\bar D_\nu(0) \\
&\quad = \int_0^\nu \Big\{\lambda^I(h)\big(\bar\delta(h) - \bar\delta(0)\big) + \Big(\frac{d}{dh}\lambda^I(h)\Big)\Big((\nu - h)\bar\delta(0) + \int_0^h \bar\delta(s)ds\Big)\Big\}dh.
\end{aligned}$$

Subtracting (44) from (41) and proceeding as above, one obtains

$$
\begin{aligned}
\hat{\alpha}_1^\nu \le \int_0^\nu C_3\Big(c\nu &+ \|\hat{V}(t) - \hat{V}_\nu(0)\| + \|\Lambda(t) - \Lambda(0)\| \\
&+ \|\Lambda(t) - \Lambda(\nu)\| \\
&+ \Big|\frac{1}{\nu}\big[(\nu - t)\bar{\delta}(0) + \int_0^t \bar{\delta}(s)ds\big] - \bar{\delta}(t)\Big| \\
&+ \Big|\bar{\delta}'(t) - \frac{\bar{\delta}(t) - \bar{\delta}(0)}{\nu}\Big| + |\bar{\delta}(t) - \bar{\delta}(0)|\Big)\,dt.
\end{aligned}
\tag{45}
$$

We now estimate the terms on the right-hand side of (45). The first three terms that contain norms are bounded as above (using that $\alpha_0^\nu = 0$). The remaining three terms with absolute values can be bounded as follows:

$$
\Big|\frac{1}{\nu}\big[(\nu - t)\bar{\delta}(0) + \int_0^t \bar{\delta}(s)ds\big] - \bar{\delta}(t)\Big| \le Mt - \frac{Mt^2}{2\nu},
$$

$$
\Big|\bar{\delta}'(t) - \frac{\bar{\delta}(t) - \bar{\delta}(0)}{\nu}\Big| \le M + \frac{Mt^2}{2\nu}
$$

and

$$
|\bar{\delta}(t) - \bar{\delta}(0)| \le Mt.
$$

Inserting these upper bounds into (45), one obtains the estimate

$$
\hat{\alpha}_1^\nu \le C_3(c + C_3/2 + 2M)\nu^2 + C_3 M\nu.
$$

Therefore, using the inequality $C_3\nu \le e^{C_3\nu}$, estimate (43) gives for $n \ge 0$

$$
\hat{\alpha}_n^\nu \le e^{nC_3\nu}(c + C_3/2 + 2M)\nu + e^{(n-1)C_3\nu}(C_3M + c + C_3/2 + 3M)\,\nu.
$$

Thus the constant C_1 in the assertion of the theorem is given by

$$
C_1 = \sqrt{I}e^{TC_3}\big[2c + C_3(1 + M) + 5M\big],
$$

because $n\nu \le T$ for $n \le \lfloor T/\nu \rfloor$. □

Appendix E

This appendix collects auxiliary results on the invertibility of certain matrixes. They might be of independent interest.

Lemma E.1. *Suppose $A \in \mathbb{R}^{N\times K}$ and $B, C \in \mathbb{R}^{K\times N}$ are matrices with non-negative entries, $N \ge 1$, $K \ge 1$. Assume*

(a) all column sums of A are strictly less than 1:

$$
\sum_{i=1}^N A_{ij} < 1 \quad \textit{for all } j = 1, \ldots, K;
$$

(b) all column sums of B are equal to 1; and

(c) C has identical columns and the column sum is equal to 1.

Then

(i) the matrix $\mathrm{Id} - AB$ is invertible and its inverse maps the non-negative orthant into itself; and

(ii) the matrix $\mathrm{Id} - A(B - C)$ is invertible.

Proof of Lemma E.1. The matrix $D = \mathrm{Id} - AB$ has a column-dominant diagonal. Each diagonal entry strictly dominates the sum of absolute values of the remaining entries in the corresponding column:

$$D_{ii} > \sum_{j=1, j\neq i}^{N} |D_{ji}|, \qquad i = 1, \ldots, N. \tag{46}$$

Indeed, the (i, j) entry of the matrix D is given by

$$\mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik} B_{kj}.$$

All off-diagonal entries are non-positive and the entries on the diagonal are non-negative. The condition (46) is equivalent to

$$\sum_{i=1}^{N} \sum_{k=1}^{K} A_{ik} B_{kj} < 1, \qquad j = 1, \ldots, N.$$

The following computation proves this inequality:

$$\sum_{i=1}^{N} \sum_{k=1}^{K} A_{ik} B_{kj} = \sum_{k=1}^{K} \Big(\sum_{i=1}^{N} A_{ik} \Big) B_{kj} < \sum_{k=1}^{K} B_{kj} = 1,$$

where the strict inequality follows from assumption (a).

Property (46) implies that the matrix D is invertible and D^{-1} maps the non-negative orthant into itself (see Corollary, p. 22, and Theorem 23, p. 24, in Murata [19]).

Invertibility of $[\mathrm{Id} - A(B - C)]$ is equivalent to the invertibility of

$$[\mathrm{Id} - AB + AC]D^{-1} = \mathrm{Id} + ACD^{-1}.$$

It suffices to prove that $x = 0$ is the only solution to the linear equation

$$x = -ACD^{-1}x. \tag{47}$$

For any $y \in \mathbb{R}^N$, the particular form of the matrix C implies $ACy = b\bar{y}$, where

$$b = \Big[\sum_{k=1}^{K} A_{1k}C_{k1}, \dots, \sum_{k=1}^{K} A_{Nk}C_{k1}\Big]^T, \quad \text{and} \quad \bar{y} = \sum_{i=1}^{N} y_i.$$

The linear equation (47) therefore can only have solutions of the form $x = \beta b$ with $\beta \in \mathbb{R}$. All coordinates of b are non-negative because the matrices A and C have non-negative entries. Assume that $x = \beta b$ is the solution to (47), with $\beta \neq 0$ and $b^i > 0$ for at least one $i = 1, ..., N$. The condition $\beta \neq 0$ implies that $b = -b\,\overline{D^{-1}b}$, where $\overline{D^{-1}b} = \sum_{i=1}^{N}(D^{-1}b)_i$. This further yields $\overline{D^{-1}b} = -1$. Since the matrix D^{-1} maps the non-negative orthant into itself, all coordinates of $D^{-1}b$ are non-negative and $\overline{D^{-1}b} \geq 0$—a contradiction. This implies that the only solution to (47) is $x = 0$, which proves the invertibility of the matrix $[\mathrm{Id} - A(B - C)]$. □

Theorem E.1. *Assume that all investment strategies are fully diversified, i.e.* $\Lambda \in \mathcal{L}$. *Then the matrix*

$$\mathrm{Id} - (1-\alpha)\hat{\Theta}(\Lambda, \hat{V})\,(\hat{\Lambda} - \Lambda^I)$$

is invertible for every $\hat{V} \in \hat{\mathcal{D}}$ *and* $\alpha \in [0, 1]$. *(The matrices* $\hat{\Theta}$, $\hat{\Lambda}$ *and* Λ^I *are defined in* (19)–(21)*.)*

Proof of Theorem E.1. Let $A = (1-\alpha)\hat{\Theta}(\Lambda, \hat{V})$. This matrix has only non-negative entries. The sum of all entries in the k-th column is given by

$$\begin{aligned}\sum_{i=1}^{I-1} A_{ik} &= (1-\alpha)\frac{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}(\Lambda_{ki} - \Lambda_{kI})\hat{V}^i + \Lambda_{kI}/c} \\ &= (1-\alpha)\frac{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i}{\sum_{i=1}^{I-1}\Lambda_{ki}\hat{V}^i + \Lambda_{kI}\big(1/c - \sum_{i=1}^{I-1}\hat{V}^i\big)}.\end{aligned}$$

If $\alpha > 0$, the above sum is strictly less than 1. For $\alpha = 0$ and $\sum_{i=1}^{I-1}\hat{V}^i < 1/c$, the full diversification of the I-th investor's strategy (i.e. $\Lambda_{kI} > 0$ for all k) implies that the sum of entries in every column of A is strictly less than 1. In both cases, Lemma E.1(ii) asserts that the matrix $\mathrm{Id} - A(\hat{\Lambda} - \Lambda^I)$ is invertible.

The proof in the case of $\alpha = 0$ and $\sum_{i=1}^{I-1}\hat{V}^i = 1/c$ requires a different argument. Invertibility of the matrix $D = \mathrm{Id} - A(\hat{\Lambda} - \Lambda^I)$ is equivalent to D having full rank. By reordering rows, which does not change the rank, we can assume that the last row of the matrix A contains only strictly positive entries (i.e. by placing an investor with non-zero wealth in the last row which is possible because $\hat{V}^i > 0$ for at least one $i = 1, ..., I-1$). We now use that adding rows does not change the rank of a matrix. First, adding the sum of rows 1 to $I-2$ to row $I-1$ gives

$$\begin{pmatrix} D_{11} & \dots & D_{1,I-2} & D_{1,I-1} \\ D_{21} & \dots & D_{2,I-2} & D_{2,I-1} \\ \vdots & & \vdots & \vdots \\ D_{I-2,1} & \dots & D_{I-2,I-2} & D_{I-2,I-1} \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

Second, we subtract from each row j, $j = 1, ..., I-2$, the last row multiplied by $D_{j,I-1}$. This leads to a matrix that has zeros in the last entry in each row $j = 1, ..., I-2$. The resulting matrix is given by

$$\begin{pmatrix} D_{11} - D_{1,I-1} & \dots & D_{1,I-2} - D_{1,I-1} & 0 \\ D_{21} - D_{2,I-1} & \dots & D_{2,I-2} - D_{2,I-1} & 0 \\ \vdots & & \vdots & \vdots \\ D_{I-2,1} - D_{I-2,I-1} & \dots & D_{I-2,I-2} - D_{I-2,I-1} & 0 \\ 1 & \dots & 1 & 1 \end{pmatrix}$$

The rank of the above matrix is equal to 1 plus the rank of the matrix $\tilde{D}$ where

$$\tilde{D} = \begin{pmatrix} D_{11} - D_{1,I-1} & \dots & D_{1,I-2} - D_{1,I-1} \\ D_{21} - D_{2,I-1} & \dots & D_{2,I-2} - D_{2,I-1} \\ \vdots & & \vdots \\ D_{I-2,1} - D_{I-2,I-1} & \dots & D_{I-2,I-2} - D_{I-2,I-1} \end{pmatrix}$$

The (i,j) entry in $\tilde{D}$ is given by

$$\begin{aligned} \tilde{D}_{ij} &= \mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik}(\Lambda_{kj} - \Lambda_{kI}) + \sum_{k=1}^{K} A_{I-1,k}(\Lambda_{k,I-1} - \Lambda_{kI}) \\ &= \mathbf{1}_{i=j} - \sum_{k=1}^{K} A_{ik}(\Lambda_{kj} - \Lambda_{k,I-1}). \end{aligned}$$

In matrix notation,

$$\tilde{D} = \mathrm{Id} - \tilde{A}(\tilde{B} - \tilde{C}),$$

where $\tilde{A} \in \mathbb{R}^{(I-2)\times K}$ is given by the matrix A omitting the last row, $\tilde{B} \in \mathbb{R}^{K\times(I-2)}$ is the matrix $\hat{\Lambda}$ omitting the last column, and $\tilde{C} \in \mathbb{R}^{K\times(I-2)}$ has all columns equal to $(\Lambda_{1,I-1}, ..., \Lambda_{K,I-1})$. Each column sum of $\tilde{A}$ is strictly less than 1 because the last row of the matrix A contains only strictly positive entries and the sum of entries in each column of A equals one. Lemma E.1(ii) implies that the matrix $\tilde{D}$ is invertible. Therefore D has full rank, i.e. it is invertible. $\square$

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