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GRAPHS THAT DO NOT CONTAIN A CYCLE WITH A NODE THAT HAS AT LEAST TWO NEIGHBORS ON IT*

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Abstract. We recall several known results about minimally 2-connected graphs and show that they all follow from a decomposition theorem. Starting from an analogy with critically 2-connected graphs, we give structural characterizations of the classes of graphs that do not contain as a subgraph and as an induced subgraph, a cycle with a node that has at least two neighbors on the cycle. From these characterizations we get polynomial time recognition algorithms for these classes and polynomial time algorithms for vertex-coloring and edge-coloring.

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1. Introduction. In this paper all graphs are finite, simple, and undirected. A *propeller* (C, x) is a graph that consists of a chordless cycle C, called the *rim*, and a node x, called the *center*, that has at least two neighbors on C. The aim of this work is to investigate the structure of graphs defined by excluding propellers as subgraphs and as induced subgraphs.

In section 2 we motivate the study of these two classes of graphs by revisiting several theorems concerning classes of graphs defined by constraints on connectivity, such as minimally and critically 2-connected graphs.

Our second motivation for the study of propeller-free graphs is our interest in wheel-free graphs. A *wheel* is a propeller whose rim has length at least 4 and whose center has at least 3 neighbors on the rim. We say that a graph G *contains* a graph F if F is isomorphic to a subgraph of G and G contains F as an induced subgraph if F is isomorphic to an induced subgraph of G. We say that G is F*-free* if G does not contain F as an induced subgraph, and for a family of graphs $\mathcal{F}$, G is $\mathcal{F}$-free if it is F-free for every $F \in \mathcal{F}$. Clearly, propeller-free graphs form a subclass of wheel-free graphs, because every wheel is a propeller.

Many interesting classes of graphs can be characterized as being $\mathcal{F}$-free for some family $\mathcal{F}$. The most famous such example is the class of perfect graphs. A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\chi(H)$ denotes

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the *chromatic number* of H, i.e., the minimum number of colors needed to color the nodes of H so that no two adjacent nodes receive the same color, and $\omega(H)$ denotes the size of a largest clique in H, where a *clique* is a graph in which every pair of nodes are adjacent. A *hole* in a graph is an induced cycle of length at least 4. The famous strong perfect graph theorem [7] states that a graph is perfect if and only if it does not contain an odd hole or the complement of an odd hole. (Such graphs are known as *Berge graphs.*) This proof is obtained through a decomposition theorem for Berge graphs, and in this study wheels and another set of configurations known as 3-*path configurations* ($3PC$s) play a key role. The $3PC$s are structures induced by three paths $P_1 = x_1 \dots y_1$, $P_2 = x_2 \dots y_2$, and $P_3 = x_3 \dots y_3$ such that $\{x_1, x_2, x_3\} \cap \{y_1, y_2, y_3\} = \emptyset$, $X = \{x_1, x_2, x_3\}$ induces either a triangle or a single node, $Y = \{y_1, y_2, y_3\}$ induces either a triangle or a single node, and the nodes of $P_i \cup P_j$, $i \neq j$, induce a hole. More specifically, a $3PC(\cdot, \cdot)$ is a $3PC$ in which both X and Y consist of a single node; a $3PC(\Delta, \cdot)$ is a $3PC$ in which X induces a triangle and Y consists of a single node; and a $3PC(\Delta, \Delta)$ is a $3PC$ in which both X and Y induce triangles. It is easy to see that Berge graphs are both $3PC(\Delta, \cdot)$-free and odd-wheel-free (where an *odd-wheel* is a wheel that induces an odd number of triangles). The remaining wheels and 3PCs form structures around which the decompositions occur in the decomposition theorem for Berge graphs in [7].

Wheels and 3PCs are called *Truemper configurations*, and they play a role in other classes of graphs. A well-studied example is the class of even-hole-free graphs. Here again, the decomposition theorems for this class [9, 24] are obtained by studying Truemper configurations that may occur as induced subgraphs. In both classes (Berge graphs and even-hole-free graphs), analysing what happens when the graph contains a wheel is a difficult task. This suggests that wheel-free graphs should have interesting structural properties. This is also suggested by three subclasses of wheel-free graphs described below.

- Say that a graph is *unichord-free* if it does not contain a cycle with a unique chord as an induced subgraph. The class of unichord-free graphs is a subclass of wheel-free graphs (because every wheel contains a cycle with a unique chord as an induced subgraph), and unichord-free graphs have a complete structural description; see [26] and also the end of section 2.1 below.
- It is easy to see that the class of K_4-free graphs that do not contain a subdivision of wheel as an induced subgraph is the class of graphs that do not contain a wheel or a subdivision of K_4 as induced subgraphs. Here again, this subclass of wheel-free graphs has a complete structural description; see [14].
- The class of graphs that do not contain a wheel (as a subgraph) does not have a complete structural description so far. However, in [25] (see also [1]), several structural properties for this class are given. It is also proved there that every graph that does not contain a wheel is 4-colorable and that every K_4-free graph that does not contain a wheel is 3-colorable.

In section 3 we continue this list of well-understood subclasses of wheel-free graphs by proving decomposition theorems for graphs that do not contain propellers, both in the subgraph and the induced subgraph sense. Based on the decomposition theorems, in section 4 we construct polynomial time recognition algorithms for these two classes of graphs. Note that the complexity of detecting a wheel as an induced subgraph is an open question, while the complexity of detecting the other Truemper's configurations is settled. ($3PC(\Delta, \cdot)$ is polynomial [6] and is one of the steps in the polynomial time recognition algorithm for Berge graphs [6]; $3PC(\cdot, \cdot)$ is polynomial [8]; $3PC(\Delta, \Delta)$ is NP-complete [16]). In the same section, we prove that deciding whether a graph contains, as an induced subgraph, a propeller such that the center has at least 4 neighbors on the rim is an NP-complete problem. It is easy to show directly that propeller-free graphs have a node of degree at most 2, which implies that the class can be vertex-colored in polynomial time; see Theorem 2.11. In section 5, we prove that propeller-free graphs admit what we call *extreme decompositions*, that are decompositions such that one of the blocks of decomposition is in some simple basic class to be defined later. Using this property we show that 2-connected propeller-free graphs have an edge both of whose endnodes are of degree 2. This property is used to give polynomial time algorithms for edge-coloring propeller-free graphs. Observe that since a clique on four nodes is a propeller, finding the size of a largest clique in a propeller-free graph can clearly be done in polynomial time. On the other hand, finding a maximum stable set of a propeller-free graph is NP-hard (follows easily from [21]; see also [26]).

Terminology and notation. Let G be a graph. For $x \in V(G)$, $N(x)$ denotes the set of neighbors of x. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S and $G \setminus S = G[V(G) \setminus S]$. For $x \in V(G)$ we also use notation $G \setminus x$ to denote $G \setminus \{x\}$. For $e \in E(G)$, $G \setminus e$ denotes the graph obtained from G by deleting edge e.

A set $S \subseteq V(G)$ is a *node cutset* of G if $G \setminus S$ has more than one connected component. Note that if $S = \emptyset$, then G is disconnected. When $|S| = k$ we say that S is a *k-cutset*. If $\{x\}$ is a node cutset of G, then we say that x is a *cutnode* of G. A 2-cutset $\{a, b\}$ is a K_2-cutset if $ab \in E(G)$ and an *S_2-cutset* otherwise. If a graph G has a node cutset S, then $V(G) \setminus S$ can be partitioned into two nonempty sets C_1, C_2 such that no edge of G has an end in C_1 and an end in C_2. In this situation, we say that (S, C_1, C_2) is a *split* of S.

A *path* P is a sequence of distinct nodes $p_1p_2 \dots p_k$, $k \geq 1$, such that p_ip_{i+1} is an edge for all $1 \leq i < k$. Edges p_ip_{i+1} for $1 \leq i < k$ are called the edges of P. Nodes p_1 and p_k are the *endnodes* of P, and $p_2 \dots p_{k-1}$ is the *interior* of P. P is refered to as a *p_1p_k-path*. For two nodes p_i and p_j of P, where $j \geq i$, the path $p_i \dots p_j$ is called the *p_ip_j-subpath* of P and is denoted by $P_{p_ip_j}$. We write $P = p_1 \dots p_{i-1}P_{p_ip_j}p_{j+1} \dots p_k$ or $P = p_1 \dots p_iP_{p_ip_j}p_j \dots p_k$. A cycle C is a sequence of nodes $p_1p_2 \dots p_kp_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and p_1p_k is an edge. Edges p_ip_{i+1} for $1 \leq i < k$ and edge p_1p_k are called the edges of C. Let Q be a path or a cycle. The node set of Q is denoted by $V(Q)$. The *length* of Q is the number of its edges. An edge $e = uv$ is a *chord* of Q if $u, v \in V(Q)$ but uv is not an edge of Q. A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q.

In all complexity analysis of the algorithms, n stands for the number of nodes of the input graph and m for the number of edges.

2. Classes defined by constraints on connectivity. The *connectivity* of a graph G is the minimum size of a node set S such that $G \setminus S$ is disconnected or has only one node. A graph is *k-connected* if its connectivity is at least k. A graph is *minimally k-connected* if it is k-connected and if the removal of any edge yields a graph of connectivity $k - 1$. A graph is *critically k-connected* if it is k-connected and if the removal of any node yields a graph of connectivity $k - 1$. Minimally and critically k-connected graphs were the object of much research; see [4], for instance. Note that minimally (and critically) k-connected graphs are classes of graphs that are not closed under any classical containment relation for graphs such as the subgraph and induced subgraph containment relations. But as we shall see, there are several ways to enlarge a class to make it closed under taking subgraphs or induced subgraphs. Here we consider the classes of minimally and critically 2-connected graphs and related hereditary classes that have similar structural properties but are algorithmically more convenient to work with.

2.1. Minimally 2-connected graphs. In this section we revisit several old results on minimally 2-connected graphs and establish the relationship between this class and a class that contains it and is closed under taking subgraphs. Two xy-paths P and Q in a graph G are *internally disjoint* if they have no internal nodes in common, i.e., $V(P) \cap V(Q) = \{x, y\}$. We will use the following classical result.

THEOREM 2.1 (Menger; see [5]). *A graph G on at least two nodes is 2-connected if and only if any two nodes of G are connected by at least two internally disjoint paths.*

Let $\mathcal{C}'_0$ be the class of graphs such that the nodes of degree at least 3 induce an independent set. Let $\mathcal{C}'_1$ be the class of *chordless graphs*, that are graphs whose cycles are all chordless (in other words, the class of graphs that do not contain a cycle with a chord). Observe that classes $\mathcal{C}'_0$ and $\mathcal{C}'_1$ are both closed under taking subgraphs. (And in particular, they are closed under taking induced subgraphs.) It is easy to check that $\mathcal{C}'_0 \subsetneq \mathcal{C}'_1$.

LEMMA 2.2. *A graph G is chordless if and only if for every subgraph H of G, either H has connectivity at most 1 or H is minimally 2-connected.*

Proof. A cycle with a chord has connectivity 2 and is not minimally 2-connected since removing the chord yields a 2-connected graph. This proves the "if" part of the theorem. To prove the "only if" part, consider a chordless graph G and suppose for a contradiction that some subgraph H of G is 2-connected and not minimally 2-connected. So by deleting some edge e, a 2-connected graph H' is obtained. By Theorem 2.1, the two endnodes of e are contained in a cycle C of H'. But then C together with e forms in H a cycle with a chord, a contradiction. ◻

Class $\mathcal{C}'_1$ was studied by Dirac [10] and Plummer [20] in the 1960s.

THEOREM 2.3 (see Dirac [10] and Plummer [20]). *A 2-connected graph is chordless if and only if it is minimally 2-connected.*

Proof. If G is a 2-connected chordless graph, then by Lemma 2.2, it is minimally 2-connected. Conversely, suppose that G is a minimally 2-connected graph and let uv be an edge of G. So, $G \setminus uv$ has connectivity 1 and therefore contains a cutnode x. Since G is 2-connected, it follows that $(G \setminus uv) \setminus x$ has two connected components, one containing u, the other containing v. This implies that every cycle of G that contains u and v must go through uv, so uv cannot be a chord of any cycle of G. This proof can be repeated for all edges of G. It follows that G is chordless. ◻

It seems that it was not observed until recently that the class $\mathcal{C}'_1$ of chordless graphs admits a simple decomposition theorem with $\mathcal{C}'_0$ serving as a basic class. An S_2-cutset $\{a, b\}$ is *proper* if it has a split $(\{a, b\}, C_1, C_2)$ such that neither $G[\{a, b\} \cup C_1]$ nor $G[\{a, b\} \cup C_2]$ is a chordless ab-path. When we say that $(\{a, b\}, D_1, D_2)$ is a split of a proper S_2-cutset, we mean that neither $G[\{a, b\} \cup D_1]$ nor $G[\{a, b\} \cup D_2]$ is a chordless ab-path. The following theorem is implicitly proved in [26] and explicitly stated and proved in [14]. We include here a proof that is much shorter and simpler than the previous ones.

THEOREM 2.4. *A graph in $\mathcal{C}'_1$ is either in $\mathcal{C}'_0$ or has a 0-cutset, a 1-cutset, or a proper S_2-cutset.*

Proof. Let G be in $\mathcal{C}'_1 \setminus \mathcal{C}'_0$ and suppose that G has no 0-cutset and no 1-cutset. So in G there is an edge $e = uv$ such that u and v both have degree at least 3 and by Lemma 2.2, $G \setminus e$ is not 2-connected so it has a 0-cutset (so it is disconnected) or a 1-cutset.

If $G \setminus e$ is disconnected, then u (and v) would be a cutnode of G. So $G \setminus e$ has a cutnode $w \notin \{u, v\}$. Since w is not a cutnode of G, the graph $(G \setminus e) \setminus w$ has exactly two connected components C_u and C_v, containing u and v, respectively, and $V(G) = C_u \cup C_v \cup \{w\}$. Let $u' \notin \{v, w\}$ be a neighbor of u. (u' exists since u has degree at least 3.) So, $u' \in C_u$. In G, u is not a cutnode, so there is a path P_u from u' to w whose interior is in $C_u \setminus \{u\}$. Together with a path P_v from v to w with interior in C_v, P_u, uu', and e form a cycle, so $uw \notin E(G)$ for otherwise uw would be a chord of this cycle. Because of the degrees of u and v, $(\{u, w\}, C_u \setminus \{u\}, C_v)$ is a split of a proper S_2-cutset of G. □

Theorem 2.3 shows that the class of minimally 2-connected graphs is a subclass of some hereditary class that has a precise decomposition theorem, namely, Theorem 2.4. There is a more standard way to embed a class $\mathcal{C}$ into an hereditary class $\mathcal{C}'$: taking the closure of $\mathcal{C}$, that is, the class $\mathcal{C}'$ of all subgraphs (or induced subgraphs according to the containment relation under consideration) of graphs from $\mathcal{C}$. But as far as we can see, applying this method to minimally 2-connected graphs yields a class more difficult to handle than chordless graphs, as suggested by what follows. The classes $\mathcal{C}'_0$ and $\mathcal{C}'_1$ are both closed under taking subgraphs (and in particular under taking induced subgraphs), so the class of subgraphs of minimally 2-connected graphs is contained in $\mathcal{C}'_1$. On the other hand, a chordless graph or even a graph from $\mathcal{C}'_0$ may fail to be a subgraph of some minimally 2-connected graph. For instance consider the path on a, b, c, d and add the edge bd. The obtained graph is chordless in $\mathcal{C}'_0$, and no minimally 2-connected graph may contain it as a subgraph. Hence $\mathcal{C}'_1$ is a proper superclass of the class of subgraphs of minimally 2-connected graphs.

In the rest of this subsection, we show how Theorem 2.4 can be used to prove several known theorems. The first example is about edge and total coloring. (We do not reproduce the proof, which is a bit long.) Note that for the proof of the following theorem, the only approach we are aware of is to use Theorem 2.4.

THEOREM 2.5 (Machado, de Figueiredo, and Trotignon [15]). *Let G be a chordless graph of maximum degree at least* 3. *Then G is $\Delta(G)$-edge colorable and* $(\Delta(G)+1)$*-total-colorable.*

Dirac [10] and Plummer [20] independently showed that minimally 2-connected graphs have at least two nodes of degree at most 2 and chromatic number at most 3. We now show how Theorem 2.4 can be used to give simple proofs of these results for chordless graphs in general. In the rest of this subsection, when $(\{a, b\}, X, Y)$ is a split of a proper S_2-cutset of a graph G, we denote by G_X the graph obtained from $G[X \cup \{a, b\}]$ by adding a node y that is adjacent to both a and b. (G_Y is defined similarly from $G[Y \cup \{a, b\}]$ by adding a node x that is adjacent to both a and b.)

THEOREM 2.6. *Every chordless graph on at least two nodes has at least two nodes of degree at most* 2.

Proof. We prove the result by induction on the number of nodes. If $G \in \mathcal{C}'_0$, then clearly the statement holds. Let $G \in \mathcal{C}'_1 \setminus \mathcal{C}'_0$, and assume the statement holds for graphs with fewer than $|V(G)|$ nodes. Suppose G has a 0-cutset or 1-cutset S, and let $C_1, \dots, C_k$ be the connected components of $G \setminus S$. For $i = 1, \dots, k$, by induction applied to $G_i = G[V(C_i) \cup S]$, C_i contains a node of degree at most 2 in G_i. Note that such a node is of degree at most 2 in G as well, and hence G has at least two nodes of degree at most 2. So we may assume that G is 2-connected, and hence by Theorem 2.4, G has a proper S_2-cutset with split $(\{a, b\}, X, Y)$. We now show that both X and Y contain a node of degree at most 2.

Let $(\{a', b'\}, X', Y')$ be a split of a proper S_2-cutset of G such that $X' \subseteq X$, and out of all such splits assume that $|X'|$ is the smallest possible. We now show that both a' and b' have at least two neighbors in X'. Since G is 2-connected both a' and b' have a neighbor in every connected component of $G \setminus \{a', b'\}$. In particular $G[Y' \cup \{a', b'\}]$ contains an $a'b'$-path Q and a' has a neighbor a_1 in X'. Suppose $N(a') \cap X' = \{a_1\}$. If a_1b' is not an edge, then (since $G[X' \cup \{a', b'\}]$ is not a chordless path), $(\{a_1, b'\}, X' \setminus \{a_1\}, Y' \cup \{a'\})$ is a split of a proper S_2-cutset of G, contradicting our choice of $(\{a', b, \}, X', Y')$. So a_1b' is an edge. Then since $G[X' \cup \{a', b'\}]$ is not a chordless path, $X' \setminus \{a_1\}$ contains a node c. Since a_1 cannot be a cutnode of G, there is a $b'c$-path in $G \setminus a_1$ whose interior nodes are in X'. Since b' cannot be a cutnode of G, there is an a_1c-path in $G \setminus b'$ whose interior nodes are in X'. Therefore $G[X' \cup b'] \setminus a_1b']$ contains an a_1b'-path P. But then $V(P) \cup V(Q)$ induces a cycle with a chord, a contradiction. Therefore, a' has at least two neighbors in X' and by symmetry so does b'.

Note that $|V(G_{X'})| < |V(G)|$, and clearly since G is chordless so is $G_{X'}$. So by induction, there is a node $t \in V(G_{X'}) \setminus \{y'\}$ that is of degree at most 2 in $G_{X'}$. Since both a' and b' have at least two neighbors in X', it follows that $t \in X'$, and hence t is of degree at most 2 in G as well. So X contains a node of degree at most 2, and by symmetry so does Y, and the result holds. ☐

In the proof above, the key idea to make the induction work is to consider a split minimizing one of the sides. This can be avoided by using a stronger induction hypothesis: in every cycle of a 2-connected chordless graph that is not a cycle, there exist four nodes a, b, c, d that appear in this order and such that a, c have degree 2 and b, d have degree at least 3.

Note that for proving the theorem below, it is essential that the class we work on is closed under taking induced subgraphs. This is why proofs of 3-colorability in [10] and [20] are more complicated. (They consider only minimally 2-connected graphs that are not closed under taking subgraphs.)

COROLLARY 2.7. *If G is a chordless graph, then $\chi(G) \leq 3$.*

Proof. Let G be a chordless graph and by Theorem 2.6 let x be a node of G of degree at most 2. Inductively color $G \setminus x$ with at most 3 colors. This coloring can be extended to a 3-coloring of G since x has at most two neighbors in G. ☐

We now show how Theorem 2.4 may be used to prove the main result in [20], that is, Theorem 2.9 below. We need the next lemma whose simple proof is omitted.

LEMMA 2.8 (see [15]). *Let G be a 2-connected chordless graph not in $\mathcal{C}_0'$. Let (X, Y, a, b) be a split of a S_2-cutset of G such that $|X|$ is minimum among all possible such splits. Then G_X is in $\mathcal{C}_0'$. Moreover, a and b both have degree at least 3 in G and in G_X.*

THEOREM 2.9 (Plummer [20]). *Let G be a 2-connected graph. Then G is minimally 2-connected if and only if either*

(i) *G is a cycle; or*

(ii) *if S denotes the set of nodes of degree 2 in G, then there are at least two components in $G \setminus S$, each component of $G \setminus S$ is a tree, and if C is any cycle in G and T is any component of $G \setminus S$, then $(V(C) \cap V(T), E(C) \cap E(T))$ is empty or connected.*

Proof. Suppose first that G is minimally 2-connected (or equivalently chordless). If G is in $\mathcal{C}_0'$, then $G \setminus S$ contains only isolated nodes. Hence, either $G \setminus S$ is empty, in which case all nodes of G are of degree 2, meaning that G is a cycle, or $G \setminus S$ is not empty, in which case G contains at least two nodes of degree at least 3 and the second outcome holds.

So, by Theorem 2.4, we may assume that G admits a proper S_2-cutset $\{a, b\}$. By Lemma 2.8, we consider G_X and G_Y so that G_X is in $\mathcal{C}_0'$ and a, b have degree 3 in G_X. Note that from the definition of a proper S_2-cutset, none of G_X, G_Y is a cycle.

Inductively, let S_Y be the set of nodes of degree 2 in G_Y and let $T_1, \dots, T_k$ be the components of $G_Y \setminus S_Y$. If a or b has degree 2 in G_Y, then it has neighbors in at most one of the T_i's (and in fact has a unique neighbor in it). So, if for some $i \in \{1, \dots, k\}$, T_i does not contain a (resp., b and a, b) and if T_i is linked by some edge to a (resp., b and a and b), then we define the tree T_i' to be the tree obtained by adding the pendent node a (resp., b and both a and b) to T_i. For all $j = 1, \dots, k$ such that T_j' is not defined above, we put $T_j' = T_j$. Now, if we remove the nodes of degree 2 of G, T_1', …, T_k' are connected components. (Here we use the fact that since G_X is in $\mathcal{C}_0'$, all neighbors of a or b in X have degree 2.) The other components are the nodes of degree at least 3 from X. They are all trees because G_X is in $\mathcal{C}_0'$.

It remains to be proved that if C is any cycle in G and T is any component of $G \setminus S$, then $(V(C) \cap V(T), E(C) \cap E(T))$ is empty or connected. Let C be a cycle of G. There are three cases. Either $V(C) \subseteq X \cup \{a, b\}$, $V(C) \subseteq Y \cup \{a, b\}$, or C is formed of a path P_X from a to b with interior in X and a path P_Y from a to b with interior in Y. In the first case, the trees intersected by C are all formed of one node, so (ii) holds. In the second case, C is also a cycle of G_Y. Let T be a tree of $G \setminus S$ such that $V(T) \subset Y \cup \{a, b\}$. (All the other trees of $G \setminus S$ are on 1 node.) Note that $a \in V(C) \cap V(T)$ implies that a has degree at least 3 in G_Y and so T is also a tree of $G_Y \setminus S_Y$. Hence, $(V(C) \cap V(T), E(C) \cap E(T))$ is connected by the induction hypothesis applied to G_Y. In the third case, we consider the cycle C_Y formed by P_Y and the marker node of G_Y. We suppose that T has more than one node (otherwise the proof is easy), so $V(T) \subseteq Y \cup \{a, b\}$. Note that if T goes through a, then it must go through some neighbor of a in Y. This means that if a has degree 2 in G_Y and $a \in V(C) \cap V(T)$, then the neighbor a' of a in G_Y has degree at least 3 and is therefore in a tree of $G_Y \setminus S_Y$, so $a' \in V(C) \cap V(T)$. The same remark holds for b. Hence, $(V(C) \cap V(T), E(C) \cap E(T))$ is connected by the induction hypothesis applied to G_Y.

Suppose conversely that one of (i) or (ii) is satisfied by some 2-connected graph G. (Here we reproduce the proof given by Plummer.) If G is a cycle, then it is obviously minimally 2-connected. Otherwise, let $e = uv$ be an edge of G if it is enough to prove that $G \setminus e$ is not 2-connected. If u or v has degree 2 in G this holds obviously. Otherwise, u and v are in the same component T of $G \setminus S$. If $G \setminus e$ is 2-connected, then some cycle C of $G \setminus e$ goes through u and v and $(V(C) \cap V(T), E(C) \cap E(T))$ is not connected or empty because it contains u and v but not $e = uv$ (and removing any edge from a tree disconnects it), a contradiction to (ii). $\square$

Note that we do not use the existence of nodes of degree 2 to prove the theorem above. Hence, a new proof of their existence can be given: if G is 2-connected, then by Theorem 2.9, the nodes of degree 2 of G form a cutset of G. Hence, there must be at least two of them; otherwise, the existence of two nodes of degree at most 2 follows easily by induction.

We close this subsection by observing that there is another well-studied hereditary class that properly contains the class $\mathcal{C}_1'$, namely, the class of graphs that do not contain a cycle with a unique chord as an induced subgraph. In [26], a precise structural description of this class is given and used to obtain efficient recognition and coloring algorithms. Interestingly, it was proved by McKee [17] that these graphs can be defined by constraints on connectivity: the graphs with no cycles with a unique chord are exactly the graphs such that all minimal separators are independent sets (where a *separator* in a graph G is a set S of nodes such that $G \setminus S$ has more connected components than G).

2.2. Critically 2-connected graphs. In this subsection we consider the class of critically 2-connected graphs that were studied by Nebeský [18] and investigate whether there exists an analogous sequence of theorems as in the previous subsection, starting with critically 2-connected graphs instead of minimally 2-connected graphs. An analogue of Lemma 2.2 exists with "critically" instead of "minimally" and "propeller" instead of "cycle with a chord."

LEMMA 2.10. *A graph G does not contain a propeller if and only if for every subgraph H of G, either H has connectivity at most 1 or it is critically 2-connected.*

Proof. A propeller has connectivity 2 and is not critically 2-connected since removing the center yields a 2-connected graph. This proves the "if" part of the theorem. To prove the "only if" part, consider a graph G that contains no propeller, and suppose for a contradiction that some subgraph H of G does not satisfy the requirement on connectivity that is to be proved. Hence H is 2-connected and not critically 2-connected. So by deleting a node v, a 2-connected graph H' is obtained. Note that $|V(H)| \geq 4$. Since v has at least two neighbors u and w in H' (because of the connectivity of H), by Theorem 2.1 H' contains a cycle C through u and w and (C, u) is a propeller of H, a contradiction. ∎

An analogue of Theorem 2.3 seems hopeless. A critically 2-connected graph can contain anything as a subgraph: the class of the subgraphs of critically 2-connected graphs is the class of all graphs. To see this, consider a graph G on $\{v_1, \dots, v_n\}$. If G is not connected, then add a node v_{n+1} adjacent to all nodes. For every node v_i, add a node a_i adjacent to v_i and a node b_i adjacent to a_i. Add a node c adjacent to all b_i's. It is easy to see that the obtained graph is critically 2-connected and contains G as a subgraph. So there cannot be a version of Theorem 2.3 with "critically" instead of "minimally": a critically 2-connected graph may contain a propeller, since it may contain anything. Also, any property of graphs closed under taking subgraphs, such as being k-colorable, is false for critically 2-connected graphs, unless it holds for all graphs. However, there is a sequence of theorems, proved here, that mimics the sequence obtained by thinking of minimally 2-connected graphs. Note that containing a cycle with a chord as a subgraph is equivalent to containing a cycle with a chord as an induced subgraph, while containing a propeller as a subgraph is not equivalent to containing a propeller as an induced subgraph. So, there are two ways to find an analogue of chordless graphs, and in this paper we consider both.

Let $\mathcal{C}_0$ be the class of graphs with no node having at least two neighbors of degree at least three. Let $\mathcal{C}_1$ be the class of graphs that do not contain a propeller. Let $\mathcal{C}_2$ be the class of graphs that do not contain a propeller as an induced subgraph. It is is easy to check that $\mathcal{C}_0 \subsetneq \mathcal{C}_1 \subsetneq \mathcal{C}_2$.

Before studying decomposition theorems for $\mathcal{C}_1$ and $\mathcal{C}_2$, let us see that an analogue of Theorem 2.6 can be proved directly for propeller-free graphs (and implies that they are 3-colorable). Nebeský [18] proved that every critically 2-connected graph contains a node of degree 2, but critically 2-connected graphs are not 3-colorable in general, since they may contain any subgraph of arbitrarily large chromatic number. Note that studying longest paths to obtain nodes of small degree in graphs where "propeller-like" structures are excluded can give much stronger results; see [27].

THEOREM 2.11. *If $G \in \mathcal{C}_2$, then G has a node of degree at most 2 and G is 3-colorable.*

Proof. Suppose that for every $v \in V(G)$, $d(v) \geq 3$. Let P be a longest chordless path in G and x and y the endnodes of P. As $d(x) \geq 3$, x has at least two neighbors u and v not in P and u (resp., v) has a neighbor in $P \setminus x$, since otherwise $V(P) \cup \{u\}$ (resp., $V(P) \cup \{v\}$) would induce a longer path in G. We choose u_1 and v_1, neighbors of, respectively, u and v in $P \setminus x$, that are closest to x on P. Without loss of generality (W.l.o.g.) let us assume that x, u_1, v_1 appear in this order on P. Then $(vP_{xv_1}v, u)$ is an induced propeller of G, a contradiction. This proves that G has a node of degree at most 2. It follows by an easy induction that every graph from $\mathcal{C}_2$ is 3-colorable. ∎

3. Decomposition theorems. In this section we present decomposition theorems for graphs that do not contain propellers and graphs that do not contain propellers as induced subgraphs.

A K_2-cutset S of a graph G is *proper* if $G \setminus S$ contains no node adjacent to all nodes of S.

LEMMA 3.1. *If G is a 2-connected graph from $\mathcal{C}_2$, then every K_2-cutset of G is proper.*

Proof. Let $(\{a, b\}, A, B)$ be a split of a K_2-cutset that is not proper. W.l.o.g. A contains a node x that is adjacent to both a and b. Since G is 2-connected, both a and b have a neighbor in the same connected component of $G[B]$, and hence $G[\{a, b\} \cup B]$ contains a chordless cycle C passing through edge ab. But then (C, x) is a propeller that is contained in G as an induced subgraph, a contradiction. ∎

THEOREM 3.2. *A graph in $\mathcal{C}_1$ is either in $\mathcal{C}_0$ or it has a 0-cutset, a 1-cutset, a proper K_2-cutset, or a proper S_2-cutset.*

Proof. Let G be a 2-connected graph in $\mathcal{C}_1 \setminus \mathcal{C}_0$. So G contains a node w that has two neighbors u and v that are both of degree at least 3. Suppose $uv \in E(G)$ and let $u' \notin \{v, w\}$ be a neighbor of u. Since u cannot be a cutnode, there is a path P from u' to $\{v, w\}$ in $G \setminus u$ and hence $G[V(P) \cup \{u, v, w\}]$ contains a propeller, a contradiction. So $uv \notin E(G)$.

If no node of $G \setminus w$ is a cutnode separating u from v, then by Theorem 2.1, there is a cycle of $G \setminus w$ going through u and v so that in G, w is the center of a propeller, a contradiction. Hence there is such a cutnode w'. So, in $G \setminus \{w, w'\}$, there are distinct components C_u and C_v containing u and v, respectively, and possibly other components whose union is denoted by C. But then $(\{w, w'\}, C \cup C_u, C_v)$ is a split of either an S_2-cutset of G (when $ww' \notin E(G)$), which is proper because of the degrees of u and v, or a split of a K_2-cutset (when $ww' \in E(G)$), which is proper by Lemma 3.1. ∎

A 3-cutset $\{u, v, w\}$ of a graph G is an *I-cutset* if the following hold:

- $G[\{u, v, w\}]$ contains exactly one edge.
- There is a partition $(\{u, v, w\}, K', K'')$ of $V(G)$ such that
  - (i) no edge of G has an endnode in K' and an endnode in K'';
  - (ii) for some connected component C' of $G[K']$, u, v, and w all have a neighbor in C'; and
  - (iii) for some connected component C'' of $G[K'']$, u, v, and w all have a neighbor in C''.

In these circumstances, we say that $(\{u, v, w\}, K', K'')$ is a *split* of the I-cutset $\{u, v, w\}$.

THEOREM 3.3. *If a graph G is in $\mathcal{C}_2$, then either $G \in \mathcal{C}_1$ or G has an I-cutset.*

Proof. Let G be a graph in $\mathcal{C}_2 \setminus \mathcal{C}_1$, and let (C, x) be a propeller of G whose rim has the fewest number of chords. Note that C must have at least one chord.

CLAIM 1. *Let $y'y''$ be a chord of C and P_1 and P_2 the two $y'y''$-subpaths of C. If a node $u \in V(G) \setminus V(C)$ has more than one neighbor on C, then it has exactly two neighbors on C, one in the interior of P_1 and the other in the interior of P_2.*

Proof of Claim 1. Let $u \in V(G)\setminus V(C)$ and suppose that u has at least two neighbors on C. If u has at least two neighbors on P_i for some $i \in \{1, 2\}$, then $G[V(P_i)\cup\{u\}]$ contains a propeller that contradicts our choice of (C, x). This completes the proof of Claim 1.

By Claim 1, x has exactly two neighbors x' and x'' on C.

CLAIM 2. *If $u \in V(G) \setminus (V(C) \cup \{x\})$, then u has at most one neighbor on C.*

Proof of Claim 2. Assume not. Then by Claim 1, u has exactly two neighbors u' and u'' on C. Let P_1 and P_2 be the two $u'u''$-subpaths of C. Note that since C has a chord, by Claim 1 that chord has one endnode in the interior of P_1 and the other in the interior of P_2. In particular, neither P_1 nor P_2 is an edge. If $\{x', x''\} \subset V(P_i)$ for some $i \in \{1, 2\}$, then the graph induced by $V(P_i) \cup \{u, x\}$ contains a propeller with center x that contradicts our choice of (C, x). So w.l.o.g. x' is contained in the interior of P_1 and x'' in the interior of P_2. Let $y'y''$ be a chord of C. Then by Claim 1 we may assume that nodes u', x', y', u'', x'', y'' are all distinct and appear in this order when traversing C clockwise. If $u'y''$ is an edge, then the graph induced by $V(P_1) \cup \{u, y''\}$ contains a propeller with center y'' that contradicts our choice of (C, x). So $u'y''$ is not an edge, and by symmetry neither is $u''y'$. Let P'_1 (resp., P'_2) be the $u'y'$-subpath (resp., $u''y''$-subpath) of C that contains x' (resp., x''). Then the graph induced by $V(P'_1) \cup V(P'_2) \cup \{u, x\}$ contains a propeller with center x that contradicts our choice of (C, x). This completes the proof of Claim 2.

Let $y'y''$ be a chord of C. By Claim 1, nodes x', y', x'', y'' are all distinct and w.l.o.g. appear in this order when traversing C clockwise. Let P' (resp., P'') be the $y'y''$-subpath of C that contains x' (resp., x'').

CLAIM 3. *C cannot have a chord $z'z''$ such that $z' \in V(P') \setminus \{y', y''\}$ and $z'' \in V(P'') \setminus \{y', y''\}$.*

Proof of Claim 3. Assume it does. W.l.o.g. z' is on the $x'y'$-subpath of P'. Then, by Claim 1, z'' is on the $x''y''$-subpath of P''. Let C' be the cycle obtained by following P' from z' to y'', going along edge $y''y'$, following P'' from y' to z'', and going along edge $z''z'$. Since C' cannot have fewer chords than C (by the choice of (C, x)), it follows that both $z'y'$ and $z''y''$ are edges. But then $G[V(P') \cup \{z''\}]$ contains a propeller with center y' that contradicts our choice of (C, x). This completes the proof of Claim 3.

Assume that S is not a cutset of G that separates x' from x''. Then there exists a shortest path $P = p_1p_2 \dots p_k$ in $G\setminus S$ such that p_1 has a neighbor $u \in P'\setminus\{y', y''\}$ and p_k has a neighbor $v \in P'' \setminus \{y', y''\}$. Finding a contradiction will complete the proof, since conditions (ii) and (iii) in the definition of an I-cutset are satisfied because of C and x. By Claim 3, P has length at least 2. By Claim 2 and the definition of P, P is a chordless path, $N(p_1) \cap V(C) = \{u\}$, $N(p_k) \cap V(C) = \{v\}$, and the only nodes of (C, x) that may have a neighbor in the interior of P are x, y', and y''.

Let $P_{uy''}$ (resp., $P_{y''v}$) be the uy''-subpath (resp., $y''v$-subpath) of C that does not contain y'. Let $P_{uy'}$ (resp., $P_{y'v}$) be the uy'-subpath (resp., $y'v$-subpath) of C that does not contain y''.

CLAIM 4. *y' and y'' have no neighbors in P.*

Proof of Claim 4. First suppose that both y' and y'' have a neighbor in P.

Let p_i (resp., p_j) be the node of P with smallest index adjacent to y' (resp., y''). W.l.o.g. $i \le j$. Let Q be a chordless path from u to y'' in $G[V(P_{uy''})]$. Then $V(Q) \cup \{p_1, p_2, \ldots, p_j, y'\}$ induces in G a propeller with center y', a contradiction.

So we may assume w.l.o.g that y'' does not have a neighbor in P. Suppose y' does. Let Q be the uv-subpath of C that contains y''. Let Q' be a chordless uv-path in $G[V(Q)]$. By Claim 3, Q' contains y''. But then $G[V(Q') \cup V(P) \cup \{y'\}]$ is a propeller with center y', a contradiction. This completes the proof of Claim 4.

By symmetry it suffices to consider the following two cases.

Case 1. $x' \in V(P_{uy''})$ and $x'' \in V(P_{y''v})$. Let C' be the cycle that consists of $P_{uy''}$, $P_{y''v}$, and P. Then by Claim 4 (C', x) is a propeller that contradicts our choice of (C, x).

Case 2. $x' \in V(P_{uy'})$ and $x'' \in V(P_{y''v})$. Suppose $y'v$ is an edge. Let C' be the cycle that consists of $P_{uy''}$, $P_{y''v}$, and P. Then by Claim 4 (C', y') is a propeller that contradicts our choice of (C, x). So $y'v$ is not an edge, and by symmetry neither is uy''. Now let C' be the cycle that consists of $P_{uy'}$, $y'y''$, $P_{y''v}$, and P. Then by Claim 4, (C', x) is a propeller that contradicts our choice of (C, x). □

4. Recognition algorithms. Deciding whether a graph contains a propeller can be done directly as follows: for every 3-node path xyz, check whether there are two internally disjoint xz-paths in $G \setminus y$. Since checking whether there are two internally disjoint xz-paths can be done in $\mathcal{O}(n)$ time [19] (see also [23]), this leads to an $\mathcal{O}(n^4)$ recognition algorithm for class $\mathcal{C}_1$.

Recognizing whether a graph contains a propeller as an induced subgraph is a more difficult problem, and we are not aware of any direct method for doing that. Observe that the above method would not work since checking whether there is a chordless cycle through two specified nodes of an input graph is NP-complete [2, 4]. In section 4.1, we give an NP-completeness result showing that the detection of a "propeller-like" induced subgraph may be hard. In section 4.2, an $\mathcal{O}(nm)$ decomposition based recognition algorithm for $\mathcal{C}_1$ (using Theorem 3.2) is given. In section 4.3, an $\mathcal{O}(n^2m^2)$ decomposition based recognition algorithm for $\mathcal{C}_2$ (using Theorem 3.3) is given.

4.1. Detecting 4-propellers. A 4-*propeller* is a propeller whose center has at least four neighbors on the rim.

THEOREM 4.1. *The problem whose instance is a graph G and whose question is "does G contain a 4-propeller as an induced subgraph?" is NP-complete.*

Proof. Let H be a graph of maximum degree 3 with 2 nonadjacent nodes x and y of degree 2. Detecting an induced cycle through x and y in H is an NP-complete problem (see Theorem 2.7 in [13]). We now show how to reduce this problem to the detection of a 4-propeller. Let x' and x'' (resp., y' and y'') be the neighbors of x (resp., of y). Subdivide the edges xx', xx'', yy', and yy''. Call a, b, c, d the four nodes created by these subdivisions. Add a node v adjacent to a, b, c, and d. Call G this new graph. Note that since H has maximum degree 3, v is the only node of degree at least 4 in G, so every 4-propeller of G must be centered at v. Hence, G contains a 4-propeller if and only if H contains an induced cycle through x and y. □

Note that detecting (as an induced subgraph) a propeller whose center has exactly two neighbors on the rim is mentioned in [13, section 3.3] as an open problem (the first of the seven open problems).

4.2. Recognition algorithm for $\mathcal{C}_1$. We first define blocks of decomposition w.r.t. different cutsets.

If G has a 0-cutset, i.e., it is disconnected, then its *blocks of decomposition* are the connected components of G. If G has a 1-cutset $\{u\}$ and $C_1, \dots, C_k$ are the connected components of $G \setminus u$, then the blocks of decomposition w.r.t. this cutset are graphs $G_i = G[C_i \cup \{u\}]$ for $i = 1, \dots, k$.

Let (S, A, B) be a split of a proper K_2-cutset of G. The blocks of decomposition of G with respect to this split are graphs $G' = G[S \cup A]$ and $G'' = G[S \cup B]$.

Let $(\{u, v\}, K', K'')$ be a split of a proper S_2-cutset of G. The blocks of decomposition of G with respect to this split are graphs G' and G'' defined as follows. Block G' is the graph obtained from $G[V(K') \cup \{u, v\}]$ by adding new nodes u' and v' and edges uu', $u'v'$, and $v'v$. Block G'' is the graph obtained from $G[V(K'') \cup \{u, v\}]$ by adding new nodes u'' and v'' and edges uu'', $u''v''$, and $v''v$. Nodes u', v', u'', v'' are called the *marker nodes* of their block of decomposition.

LEMMA 4.2. *For* 0*-cutsets,* 1*-cutsets, and proper* K_2*-cutsets the following holds:* G *is in* $\mathcal{C}_1$ *(resp.,* $\mathcal{C}_2$*) if and only if all the blocks of decomposition are in* $\mathcal{C}_1$ *(resp.,* $\mathcal{C}_2$*).*

Proof. Since a propeller is 2-connected, the theorem obviously holds for 0-cutsets and 1-cutsets. Suppose that $(\{u, v\}, K', K'')$ is a split of a proper K_2-cutset of G, and let G' and G'' be the blocks of decomposition w.r.t. this split. Since G' and G'' are induced subgraphs of G, it follows that if $G \in \mathcal{C}_1$ (resp., $\mathcal{C}_2$), then $G', G'' \in \mathcal{C}_1$ (resp., $\mathcal{C}_2$). If G' and G'' are in $\mathcal{C}_2$, then clearly (since $\{u, v\}$ is proper) G is in $\mathcal{C}_2$. Finally assume that G' and G'' are in $\mathcal{C}_1$ but that a propeller (C, x) is a subgraph of G. Since $\{u, v\}$ is proper, it follows that C contains a node of K' and a node of K''. Hence, C contains both u and v, and so w.l.o.g. x is in K'. But then the uv-subpath of C whose interior nodes are in K', together with edge uv and node x, induces a propeller that is a subgraph of G', a contradiction. ∎

LEMMA 4.3. *Let* G *be a* 2*-connected graph that does not have a proper* K_2*-cutset. Let* $(\{u, v\}, K', K'')$ *be a split of a proper* S_2*-cutset of* G*, and let* G' *and* G'' *be the blocks of decomposition w.r.t. this split. Then* $G \in \mathcal{C}_1$ *if and only if* $G' \in \mathcal{C}_1$ *and* $G'' \in \mathcal{C}_1$*.*

Proof. Assume that $G \in \mathcal{C}_1$ and w.l.o.g. that a propeller (C, x) is a subgraph of G'. Since (C, x) is not a subgraph of G, (C, x) must contain at least one of the marker nodes u' or v'. By the definition of u' and v' it is clear that $u', v' \in V(C)$. Since G has no 1-cutset both u and v have a neighbor in every connected component of $G \setminus \{u, v\}$, and hence $G[V(K'') \cup \{u, v\}]$ contains a path P from u to v. If in C we replace path $uu'v'v$ with P, we get a propeller (C', x), which is a subgraph of G, a contradiction.

To prove the converse assume that $G' \in \mathcal{C}_1$ and $G'' \in \mathcal{C}_1$, but that a propeller (C, x) is a subgraph of G. Since (C, x) cannot be a subgraph of G' or G'', (C, x) must contain nodes from both K' and K''. Then clearly $x \notin \{u, v\}$, so w.l.o.g. we may assume that $x \in K'$. If there is a node from $V(C)$ in K', then there are uv-paths P_1 and P_2 in $G[V(K') \cup \{u, v\}]$ and $G[V(K'') \cup \{u, v\}]$, respectively, such that $V(P_1) \cup V(P_2) = V(C)$. Replacing in C path P_2 with $uu'v'v$, we get a propeller (C', x) which is a subgraph of G', a contradiction. Therefore, C is contained in $G[V(K'') \cup \{u, v\}]$. Since $x \in V(K')$, it has no neighbor in K''. Since (C, x) is a propeller, it follows that C contains both u and v and x is adjacent to both u and v. By definition of the proper S_2-cutsets, $G[V(K') \cup \{u, v\}]$ is not a path, and hence K' must contain a node y distinct from x. If there is a node w adjacent to both x and v, then w is in K' and hence $\{u, v, u', v', x, w\}$ induces a propeller in G' with center w. Therefore, no node of G is adjacent to both x and v, and by symmetry, no node of G is adjacent to both x and u. Since $\{x, v\}$ cannot be a proper K_2-cutset in G, there is a path from u to y in $G[(V(K') \cup \{u\}) \setminus \{x\}]$. Similarly, since $\{x, u\}$ cannot be a proper

K_2-cutset in G, there is a path from y to v in $G[(V(K') \cup \{v\}) \setminus \{x\}]$. Therefore, there is a path P from u to v in $G[(V(K') \cup \{u, v\}) \setminus \{x\}]$. But then $V(P) \cup \{u', v', x\}$ induces a graph in G' that contains a propeller with center x, a contradiction: □

THEOREM 4.4. *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *G is correctly identified as not belonging to $\mathcal{C}_2$ or a list $\mathcal{L}$ of induced subgraphs of G such that*

(i) *$G \in \mathcal{C}_1$ if and only if for every $L \in \mathcal{L}$, $L \in \mathcal{C}_1$;*

(ii) *$G \in \mathcal{C}_2$ if and only if for every $L \in \mathcal{L}$, $L \in \mathcal{C}_2$;*

(iii) *for every $L \in \mathcal{L}$, L is 2-connected and does not have a K_2-cutset;*

(iv) *$\sum_{L \in \mathcal{L}} |V(L)| \leq 6n$ and $\sum_{L \in \mathcal{L}} |E(L)| \leq 2n + m$.*

Running time: $\mathcal{O}(nm)$

Proof. Consider the following algorithm.

Step 1: Let $\mathcal{L} = \mathcal{F} = \emptyset$.

Step 2: Find maximal 2-connected components of G (i.e., decompose G using 0-cutsets and 1-cutsets) and add them to $\mathcal{F}$.

Step 3: If $\mathcal{F} = \emptyset$, then return $\mathcal{L}$ and stop. Otherwise, remove a graph F from $\mathcal{F}$.

Step 4: Decompose F using proper K_2-cutsets as follows.

Step 4.1: Let $\mathcal{F}' = \{F\}$ and $\mathcal{L}' = \emptyset$.

Step 4.2: If $\mathcal{F}' = \emptyset$, then merge $\mathcal{L}'$ with $\mathcal{L}$ and go to Step 3. Otherwise, remove a graph H from $\mathcal{F}'$.

Step 4.3: Check whether H has a K_2-cutset. If it does not, then add H to $\mathcal{L}'$ and go to Step 4.2. Otherwise, let S be a K_2-cutset of H. Check whether S is proper. If it is not, then output "$G \notin \mathcal{C}_2$" and stop. Otherwise, construct blocks of decomposition w.r.t. S, add them to $\mathcal{F}'$, and go to Step 4.2.

We first prove the correctness of this algorithm. Suppose the algorithm terminates in Step 4.3 because it has identified a K_2-cutset of H that is not proper. By Step 2, the graph F that is placed in $\mathcal{F}'$ in Step 4.1 is 2-connected, and since blocks of decomposition of a 2-connected graph w.r.t. a K_2-cutset are also 2-connected, all graphs that are ever placed in list $\mathcal{F}'$ are 2-connected, and in particular H is 2-connected. Therefore, by Lemma 3.1, H is correctly identified as not belonging to $\mathcal{C}_2$.

We may now assume that the algorithm terminates in Step 3 by returning the list $\mathcal{L}$. By Lemma 4.2, (i) and (ii) hold. By the construction of the algorithm, clearly (iii) holds.

Let $\mathcal{F}^*$ be the list $\mathcal{F}$ at the end of Step 2. Since every node of G is in at most two graphs of $\mathcal{F}^*$ and every edge of G is in exactly one graph of $\mathcal{F}^*$, the following holds:

$$(1) \quad \sum_{F \in \mathcal{F}^*} |V(F)| \leq 2n \text{ and } \sum_{F \in \mathcal{F}^*} |E(F)| = m.$$

Let F be a graph placed in the list $\mathcal{F}'$ in Step 4.1, and let $\mathcal{L}_F$ be the list $\mathcal{L}'$ at the time it is merged with $\mathcal{L}$ in Step 4.2. We now show that

$$(2) \quad |\mathcal{L}_F| \leq |V(F)|, \sum_{L \in \mathcal{L}_F} |V(L)| \leq 3|V(F)|, \text{ and } \sum_{L \in \mathcal{L}_F} |E(L)| \leq |V(F)| + |E(F)|.$$

For any graph T define $\phi(T) = |V(T)| - 2$ and $\psi(T) = |E(T)| - 1$. Suppose (S, A, B) is a split of a K_2-cutset of H and let $H_A = H[S \cup A]$ and $H_B = H[S \cup B]$ be the blocks of decomposition. Clearly $\phi(H) = |A| + |B| = \phi(H_A) + \phi(H_B)$ and $\psi(H) = \psi(H_A) + \psi(H_B)$. Since a block is of size at least 3, it follows that $\phi(H), \phi(H_A)$,

$\phi(H_B), \psi(H), \psi(H_A), \psi(H_B)$ are all at least 1. Therefore $\phi(F) = \sum_{L\in\mathcal{L}_F} \phi(L) \geq |\mathcal{L}_F|$ and hence $|\mathcal{L}_F| \leq |V(F)|$. Furthermore, $|V(F)| - 2 = \sum_{L\in\mathcal{L}_F}(|V(L)| - 2)$, and so $\sum_{L\in\mathcal{L}_F} |V(L)| = |V(F)| - 2 + 2|\mathcal{L}_F| \leq 3|V(F)|$. By a similar argument, but using ψ, $|E(F)| - 1 = \sum_{L\in\mathcal{L}_F}(|E(L)| - 1)$, and so $\sum_{L\in\mathcal{L}_F} |E(L)| = |E(F)| - 1 + |\mathcal{L}_F| \leq |V(F)| + |E(F)|$. Therefore (2) holds.

Let $\mathcal{L}^*$ be the list $\mathcal{L}$ outputted by the algorithm. Then

$$\begin{aligned}\sum_{L\in\mathcal{L}^*} |V(L)| &= \sum_{F\in\mathcal{F}^*}\sum_{L\in\mathcal{L}_F} |V(L)| \\ &\leq \sum_{F\in\mathcal{F}^*} 3|V(F)| \quad \text{by (2)} \\ &\leq 6n \quad \text{by (1)}.\end{aligned}$$

Also

$$\begin{aligned}\sum_{L\in\mathcal{L}^*} |E(L)| &= \sum_{F\in\mathcal{F}^*}\sum_{L\in\mathcal{L}_F} |E(L)| \\ &\leq \sum_{F\in\mathcal{F}^*} (|V(F)| + |E(F)| \quad \text{by (2)} \\ &\leq 2n + m \quad \text{by (1)}\end{aligned}$$

and hence (iv) holds.

We now show that this algorithm can be implemented to run in $\mathcal{O}(nm)$ time. Step 2 can be implemented to run in $\mathcal{O}(n + m)$ time [11, 22]. Checking whether a graph H has a K_2-cutset can be done by the algorithm in [12] in $\mathcal{O}(|V(H)| + |E(H)|)$ time. This algorithm finds triconnected components of H in linear time, and in particular it finds all K_2-cutsets of H (and some S_2-cutsets). By (2) it follows that Step 4 can be implemented to run in $\mathcal{O}(|V(F)||E(F)|)$ time. Step 4 is applied to every graph $F \in \mathcal{F}^*$. Since

$$\begin{aligned}\sum_{F\in\mathcal{F}^*} |V(F)||E(F)| &\leq \sum_{F\in\mathcal{F}^*} |V(F)| \sum_{F\in\mathcal{F}^*} |E(F)| \\ &\leq 2nm \quad \text{by (1)},\end{aligned}$$

it follows that the total running time is $\mathcal{O}(nm)$. ∎

A cycle of length k is denoted by C_k.

LEMMA 4.5. *Let G be a 2-connected graph that does not have a K_2-cutset. If $G \in \mathcal{C}_2$ and it contains a C_k for some $k \in \{3, 4, 5\}$ as an induced subgraph, then $G = C_k$.*

Proof. Let $G \in \mathcal{C}_2$ and suppose that G contains a $C_k = x_1x_2 \ldots x_kx_1$ as an induced subgraph for some $k \in \{3, 4, 5\}$. Assume $G \neq C_k$ and that G has no 1-cutset or K_2-cutset. Let K be a connected component of $G \setminus C_k$.

If a node $x \in K$ is adjacent to more than one node of C_k, then $V(C_k) \cup \{x\}$ induces a propeller of G. So a node of K can have at most one neighbor in C_k. Since G has no 1-cutset nor K_2-cutset, $|N(K) \cap V(C_k)| \geq 2$, and if $|N(K) \cap V(C_k)| = 2$, then the two nodes of $N(K) \cap V(C_k)$ are nonadjacent.

Suppose $k = 3$, and let P be a minimal path of K such that its endnodes are adjacent to different nodes of C_k. Then $V(P) \cup V(C_k)$ induces a propeller. Therefore $k \in \{4, 5\}$, and hence $N(K) \cap V(C_k)$ contains nonadjacent nodes. Let P be a minimal path of K such that its endnodes are adjacent to nonadjacent nodes of C_k. We may assume w.l.o.g. that the endnodes of P are adjacent to x_1 and x_3. By the choice of P, we may assume w.l.o.g. that nodes of $V(C_k) \setminus \{x_1, x_2, x_3\}$ have no neighbors in P. But then $V(C_k) \cup V(P)$ induces a propeller. ∎

LEMMA 4.6. *Let G be a 2-connected graph with no K_2-cutset. Let $(\{u, v\}, A, B)$ be a split of a proper S_2-cutset of G, and G_A and G_B the corresponding blocks of decomposition. Then the following hold:*

(i) *G_A and G_B are 2-connected and have no K_2-cutset.*

(ii) *If $|A| \leq 2$ or $|B| \leq 2$, then $G \notin \mathcal{C}_2$.*

Proof. Since G is connected, then clearly by the construction of blocks, so are G_A and G_B. To prove (i) assume w.l.o.g. that S is a 1-cutset or a K_2-cutset of G_A. Since G is 2-connected, both u and v have a neighbor in every connected component of $G \setminus \{u, v\}$. So we may assume that $S \cap \{u', v'\} = \emptyset$ (where u' and v' are the marker nodes of G_A). Then w.l.o.g. we may assume that $v \notin S$. Let C and D be connected components of $G_A \setminus S$ such that $v \in C$. Then $u', v' \in C$, and if $u \notin S$, then $u \in C$. Therefore $D \subseteq A$, and hence S is a cutset of G, a contradiction. Therefore (i) holds.

To prove (ii) assume w.l.o.g. that $|A| \leq 2$ and $G \in \mathcal{C}_2$. Since G is 2-connected, there is a chordless uv-path P in $G[A \cup \{u, v\}]$. Since $\{u, v\}$ is a proper S_2-cutset, it follows that P has length 3, say $P = uxv$, and $|A| = 2$, say $A = \{x, y\}$. Since G is 2-connected, $|N(y) \cap \{u, v, x\}| \geq 2$, so G contains a cycle of length at most 4. But then by Lemma 4.5, G is a chordless cycle, a contradiction. ∎

THEOREM 4.7. *There is an algorithm with the following specifications:*

Input: *A 2-connected graph G that does not have a K_2-cutset.*

Output: *YES if $G \in \mathcal{C}_1$, and NO otherwise.*

Running time: $\mathcal{O}(nm)$

Proof. Consider the following algorithm:

Step 1: If G has fewer than seven nodes, then check directly whether $G \in \mathcal{C}_1$, return the answer, and stop.

Step 2: Let $\mathcal{L} = \{G\}$.

Step 3: If $\mathcal{L} = \emptyset$, then output YES and stop. Otherwise, remove a graph F from $\mathcal{L}$.

Step 4: Check whether $F \in \mathcal{C}_0$. If it is, then go to Step 3. Otherwise, let w be a node of F and u and v its neighbors that are of degree at least 3.

Step 5: If uv is an edge then output NO and stop. Otherwise, check whether $F \setminus w$ has a cutnode w' that separates u from v. If it does not, then output NO and stop. Otherwise, let C_u and C_v be the connected components of $F \setminus \{w, w'\}$ that contain u and v, respectively, and denote by C the union of the remaining components. If $|C_u \cup C| \leq 2$ or $|C_v| \leq 2$, then output NO and stop. Otherwise for the split $(\{w, w'\}, C \cup C_u, C_v)$ of the proper S_2-cutset $\{w, w'\}$ construct the corresponding blocks of decomposition, add them to $\mathcal{L}$, and go to Step 3.

We first prove the correctness of this algorithm. We may assume that the algorithm does not terminate in Step 1. By Step 1 and Lemma 4.6, all the graphs that are ever put on list $\mathcal{L}$ are 2-connected, have no K_2-cutset, and have at least 7 nodes. Note that by Lemma 4.3, at every stage of the algorithm when blocks of decomposition are added to $\mathcal{L}$ in Step 5 the following holds: G belongs to $\mathcal{C}_1$ if and only if all the graphs in $\mathcal{L}$ belong to $\mathcal{C}_1$. If the algorithm terminates in Step 5, then by Lemma 4.6 and the proof of Theorem 3.2 it does so correctly. If the algorithm terminates in Step 3, then by Lemma 4.3 it does so correctly.

We now show that this algorithm can be implemented to run in $\mathcal{O}(nm)$ time. Step 1 can clearly be implemented to run in constant time. For a graph F that is removed from list $\mathcal{L}$ in Step 3, clearly Steps 4 and 5 can be implemented to run in

$\mathcal{O}(|V(F)|+|E(F)|)$ time. If we show that the number of times these steps are applied is at most n, then it follows that the total running time is $\mathcal{O}(nm)$.

Let $\mathcal{F}$ be the set of graphs that are identified as belonging to $\mathcal{C}_0$ in Step 4. Then the number of times Steps 4 and 5 are applied is at most $|\mathcal{F}|$. We now show that $|\mathcal{F}| \leq n$. For any graph H define $\phi(H) = |V(H)| - 6$. Now let F be a graph that is decomposed by a proper S_2-cutset in Step 5. Denote by (S, A, B) the split used for the decomposition and by F_A and F_B the corresponding blocks of decomposition. Clearly $\phi(F) = |A| + |B| - 4 = \phi(F_A) + \phi(F_B)$. Since all graphs that are ever placed on list $\mathcal{L}$ have at least seven nodes, it follows that $\phi(F), \phi(F_A)$, and $\phi(F_B)$ are all at least 1. Hence $\phi(G) = \sum_{L\in\mathcal{F}} \phi(L) \geq |\mathcal{F}|$. Therefore $|\mathcal{F}| \leq n$. □

THEOREM 4.8. *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *YES if $G \in \mathcal{C}_1$, and NO otherwise.*

Running time: $\mathcal{O}(nm)$

Proof. First apply the algorithm from Theorem 4.4. If this algorithm returns $G \notin \mathcal{C}_2$, then return NO and stop. Otherwise, let $\mathcal{L}$ be the outputted list. Now apply the algorithm from Theorem 4.7 to every graph in $\mathcal{L}$. If any of the outputs is NO, then return NO and stop, and otherwise return YES and stop. Since $\sum_{L\in\mathcal{L}} |V(L)||E(L)| \leq \sum_{L\in\mathcal{L}} |V(L)| \sum_{L\in\mathcal{L}} |E(L)|$, it follows by Theorems 4.4 and 4.7 that the running time is $\mathcal{O}(nm)$. □

4.3. Recognition algorithm for $\mathcal{C}_2$. We say that an I-cutset $\{u, v, w\}$ is *proper* if no node of $G\setminus\{u, v, w\}$ has at least two neighbors in $\{u, v, w\}$. Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of a graph G, and assume uv is an edge. The blocks of decomposition of G w.r.t. this split are graphs G' and G'' defined as follows. Block G' is the graph obtained from $G[V(K') \cup \{u, v, w\}]$ by adding new nodes u'_1, u'_2, v'_1, and v'_2 (called the marker nodes of G') and edges uu'_1, $u'_1u'_2$, u'_2w, vv'_1, $v'_1v'_2$, and v'_2w. Block G'' is the graph obtained from $G[V(K'') \cup \{u, v, w\}]$ by adding new nodes u''_1, u''_2, v''_1, and v''_2 (called the marker nodes of G'') and edges uu''_1, $u''_1u''_2$, u''_2w, vv''_1, $v''_1v''_2$, and v''_2w.

We use the following notation in the proofs that follow. By the definition of an I-cutset, $G[K' \cup \{u, v, w\}] \setminus uv$ contains a chordless uv-path P'_{uv}, a chordless uw-path P'_{uw}, and a chordless vw-path P'_{vw}, whose interiors belong to the same connected component of $G[K']$. Define P''_{uv} , P''_{uw} , P''_{vw} for K'' in the obvious analogous manner.

LEMMA 4.9. *If G is a 2-connected graph in $\mathcal{C}_2$ that has no K_2-cutset, then every I-cutset of G is proper.*

Proof. Let $(\{u, v, w\}, K', K'')$ be a split of an I-cutset such that uv is an edge. Suppose that $x \in V(G) \setminus \{u, v, w\}$ has at least two neighbors in $\{u, v, w\}$. W.l.o.g. $x \in K'$, and by Lemma 4.5 w.l.o.g. $N(x) \cap \{u, v, w\} = \{u, w\}$. If $x \notin V(P'_{uw})$, then $G[V(P'_{uw}) \cup V(P''_{uw}) \cup \{x\}]$ is a propeller, and hence $G \notin \mathcal{C}_2$. So we may assume that $x \in V(P'_{uw})$, i.e., $P'_{uw} = uxw$. Let C' be the connected component of $G[K']$ that contains x, and let P be a shortest xv-path in $G[C' \cup \{v\}]$. If w does not have a neighbor in $P \setminus x$, then $G[V(P) \cup V(P''_{vw}) \cup \{u\}]$ is a propeller with center u, and hence $G \notin \mathcal{C}_2$. So we may assume that w has a neighbor in $P \setminus x$. If u does not have a neighbor in $P \setminus x$, then $G[V(P) \cup \{u, v, w\}]$ is a propeller, and hence $G \notin \mathcal{C}_2$. So, we may assume that u has a neighbor in $P \setminus x$. Then $G[(V(P) \setminus \{x, v\}) \cup \{u, w\}]$ contains a chordless uw-path Q, and hence $G[V(Q) \cup V(P''_{uw}) \cup \{x\}]$ is a propeller, implying that $G \notin \mathcal{C}_2$. □

LEMMA 4.10. *Let G be a 2-connected graph that does not have a K_2-cutset. Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of G, and let G' and G'' the* *corresponding blocks of decomposition. Then* $G \in \mathcal{C}_2$ *if and only if* $G' \in \mathcal{C}_2$ *and* $G'' \in \mathcal{C}_2$.

Proof. Let $G \in \mathcal{C}_2$ and assume w.l.o.g. that G' contains a propeller (C, x) as an induced subgraph. Clearly $x \in K' \cup \{u, v, w\}$. Since (C, x) cannot be contained in G, $V(C) \cap \{u'_1, u'_2, v'_1, v'_2\} \neq \emptyset$. W.l.o.g. we may assume that $uu'_1u'_2w$ is a subpath of C. If $V(C) \cap \{v'_1, v'_2\} \neq \emptyset$, then $V(C) = \{u, v, u'_1, u'_2, v'_1, v'_2, w\}$, and since x is adjacent to at least two nodes of C it follows that x is adjacent to at least two nodes of $\{u, v, w\}$, contradicting the assumption that $\{u, v, w\}$ is a proper I-cutset. Therefore $V(C) \cap \{v'_1, v'_2\} = \emptyset$. Let P' be the uw-subpath of C that does not contain u'_1. If $v \notin V(P')$, then $V(P') \cup V(P''_{uw}) \cup \{x\}$ induces a propeller in G with center x, a contradiction. Hence $v \in V(P')$. If x has at least two neighbors in $P' \setminus u$, then $V(P') \cup V(P''_{vw}) \cup \{x\}$ induces a propeller in G, a contradiction. So $N(x) \cap V(P') = \{u, a\}$, where a is a node of $P' \setminus \{u, v, w\}$. If v does not have a neighbor in $P''_{uw} \setminus u$, then $V(P''_{uw}) \cup V(P') \cup \{x\}$ induces a propeller in G with center x, a contradiction. Hence v has a neighbor in $P''_{uw} \setminus u$. But then the wa-subpath of P' together with $V(P''_{uw}) \cup \{x, v\}$ induces a propeller in G with center v, a contradiction.

To prove the converse assume that $G' \in \mathcal{C}_2$ and $G'' \in \mathcal{C}_2$ but that G contains as an induced subgraph a propeller (C, x). Let us first assume that C is contained in G' or G'', w.l.o.g $V(C) \subset V(G')$. If $x \in K' \cup \{u, v, w\}$, then (C, x) is in G', a contradiction. Otherwise $x \in K''$, and hence it has at least two neighbors in $\{u, v, w\}$, contradicting the assumption that $\{u, v, w\}$ is a proper I-cutset. So C must contain nodes from both K' and K'', and therefore it contains w and at least one node from the set $\{u, v\}$. W.l.o.g. we may assume that it contains u and that $x \in V(G')$. Let P be the uw-subpath of C contained in G'. First let us assume that $x \neq v$. But then the node set $V(P) \cup \{x, u'_1, u'_2\}$ induces a propeller in G', a contradiction. So $x = v$, and therefore v is adjacent to a node y of C different from u. We may assume w.l.o.g. that $y \in G'$. But then the node set $V(P) \cup \{x, u'_1, u'_2\}$ induces a propeller in G', a contradiction. ∎

LEMMA 4.11. *Let G be a 2-connected graph that does not have a K_2-cutset. Let $(\{u, v, w\}, K', K'')$ be a split of a proper I-cutset of G, and let G' and G'' the corresponding blocks of decomposition. Then the following hold:*

(i) *G' and G'' are 2-connected and have no K_2-cutset.*

(ii) *If $|K'| \leq 4$ or $|K''| \leq 4$, then $G \notin \mathcal{C}_2$.*

Proof. W.l.o.g. uv is an edge. Since G is connected, then clearly by the construction of blocks, so are G' and G''. To prove (i) assume w.l.o.g. that S is a 1-cutset or a K_2-cutset of G'. Since G is 2-connected and has no K_2-cutset, every connected component of $G \setminus \{u, v, w\}$ must contain a neighbor of w and a neighbor of u or v. So we may assume that S does not contain any of the marker nodes of G'. Then w.l.o.g. we may assume that $v \notin S$. Let C and D be connected components of $G' \setminus S$ such that $v \in C$. Then all the marker nodes and nodes of $\{u, w\} \setminus S$ are in C. Therefore $D \subseteq K'$, and hence S is a cutset of G, a contradiction. Therefore (i) holds.

To prove (ii) assume w.l.o.g. that $|K'| \leq 4$. Then P'_{uw} is of length at most 5. If v has a neighbor on $P'_{uw} \setminus u$, then since $\{u, v, w\}$ is a proper I-cutset and by Lemma 4.5, $G \notin \mathcal{C}_2$. So we may assume that v does not have a neighbor on $P'_{uw} \setminus u$. Since $\{u, v, w\}$ is a proper I-cutset, P'_{uw} and P'_{vw} are both of length at least 3. Suppose that the interior nodes of P'_{uw} and P'_{vw} are disjoint. Then $P'_{uw} = ux_1x_2w$ and $P'_{vw} = vy_1y_2w$, and hence since x_1, x_2, y_1, y_2 all belong to the same connected component of $G \setminus \{u, v, w\}$, there must be an edge between a node of $\{x_1, x_2\}$ and a node of $\{y_1, y_2\}$. But then by Lemma 4.5, $G \notin \mathcal{C}_2$. Finally we may assume w.l.o.g.

that $P'_{uw} = ux_1x_2x_3w$ and $P'_{vw} = uy_1x_3w$ (else by Lemma 4.5, $G \notin \mathcal{C}_2$). But then either $G[K' \cup \{u,v,w\} \cup V(P''_{vw})]$ is a propeller with center y_1 (if u does not have a neighbor on $P''_{vw} \setminus v$) or $G[\{u,v,w\} \cup V(P'_{vw}) \cup V(P''_{vw})]$ is a propeller with center u (if u does have a neighbor on $P''_{vw} \setminus v$). ∎

THEOREM 4.12. *There is an algorithm with the following specifications:*

Input: *A 2-connected graph G that does not have a K_2-cutset.*

Output: *YES if $G \in \mathcal{C}_2$, and NO otherwise.*

Running time: $\mathcal{O}(n^2m^2)$

Proof. Consider the following algorithm:

Step 1: If G has fewer than 12 nodes, then check directly whether $G \in \mathcal{C}_2$, return the answer, and stop.

Step 2: Let $\mathcal{L} = \{G\}$.

Step 3: If $\mathcal{L} = \emptyset$, then output YES and stop. Otherwise, remove a graph F from $\mathcal{L}$.

Step 4: Use the algorithm from Theorem 4.7 to check whether $F \in \mathcal{C}_1$. If it is then go to Step 3.

Step 5: For every edge uv and node w of F, check whether $\{u,v,w\}$ is a proper I-cutset of F. If such a cutset does not exist, return NO and stop. Otherwise let $(\{u,v,w\}, K', K'')$ be a split of a proper I-cutset of G. If $|K'| \leq 4$ or $|K''| \leq 4$, then return NO and stop. Otherwise construct blocks of decomposition, add them to $\mathcal{L}$, and go to Step 3.

We first prove the correctness of this algorithm. We may assume that the algorithm does not terminate in Step 1. By Step 1 and Lemma 4.11, all the graphs that are ever put on list $\mathcal{L}$ are 2-connected and have no K_2-cutset. So the correctness of the algorithm follows from Theorem 3.3, Lemma 4.9, Lemma 4.10, and Lemma 4.11.

We now show that this algorithm can be implemented to run in $\mathcal{O}(n^2m^2)$ time. Step 1 can clearly be implemented to run in constant time. For a graph F that is removed from list $\mathcal{L}$ in Step 3, Step 4 runs in $\mathcal{O}(|V(F)||E(F)|)$ time and Step 5 can be implemented to run in $\mathcal{O}(|V(F)||E(F)|^2)$ time (since there are $\mathcal{O}(|V(F)||E(F)|)$ sets $\{u,v,w\}$ that need to be checked, and for each one of them checking whether it is a proper I-cutset can clearly be done in $\mathcal{O}(|V(F)|+|E(F)|)$ time). If we show that the number of times these steps are applied is at most n, then it follows that the total running time is $\mathcal{O}(n^2m^2)$.

Let $\mathcal{F}$ be the set of graphs that are identified as belonging to $\mathcal{C}_1$ in Step 4. Then the number of times Steps 4 and 5 are applied is at most $|\mathcal{F}|$. We now show that $|\mathcal{F}| \leq n$. For any graph H define $\phi(H) = |V(H)| - 11$. Now let F be a graph that is decomposed by a proper I-cutset in Step 5. Denote by (S, K', K'') the split used for the decomposition and by F' and F'' the corresponding blocks of decomposition. Clearly $\phi(F) = |A| + |B| - 8 = \phi(F') + \phi(F'')$. Since by Step 1 and Lemma 4.11, all graphs that are ever placed on list $\mathcal{L}$ have at least 12 nodes, it follows that $\phi(F), \phi(F')$ and $\phi(F'')$ are all at least 1. Hence $\phi(G) = \sum_{L \in \mathcal{F}} \phi(L) \geq |\mathcal{F}|$. Therefore $|\mathcal{F}| \leq n$. ∎

THEOREM 4.13. *There is an algorithm with the following specifications:*

Input: *A graph G.*

Output: *YES if $G \in \mathcal{C}_2$, and NO otherwise.*

Running time: $\mathcal{O}(n^2m^2)$

Proof. First apply the algorithm from Theorem 4.4. If this algorithm returns $G \notin \mathcal{C}_2$, then return NO and stop. Otherwise, let $\mathcal{L}$ be the outputted list. Now apply the algorithm from Theorem 4.12 to every graph in $\mathcal{L}$. If any of the outputs is NO, then return NO and stop, and otherwise return YES and stop. Since

$\sum_{L\in\mathcal{L}} |V(L)|^2|E(L)|^2 \leq (\sum_{L\in\mathcal{L}} |V(L)|)^2(\sum_{L\in\mathcal{L}} |E(L)|)^2$, it follows by Theorems 4.4 and 4.12 that the running time is $\mathcal{O}(n^2m^2)$. ∎

5. Flat edges and edge-coloring. A *flat edge* of a graph G is an edge both of whose endnodes are of degree 2. In this section we show that every 2-connected propeller-free graph has a flat edge and use this property to edge-color it. To do this we first show the existence of an *extreme decomposition*, i.e., a decomposition in which one of the blocks is in $\mathcal{C}_0$.

LEMMA 5.1. *Let G be a 2-connected graph in $\mathcal{C}_2 \setminus \mathcal{C}_0$. Then, there exists $S \subseteq V(G)$ such that* (i) *S is either a proper I-cutset or a proper S_2-cutset of G,* (ii) *there exists a split (S, K', K'') such that at least one of the blocks of decomposition, say, G', is in $\mathcal{C}_0$, and* (iii) *all nodes in S are of degree at least three in G'.*

Proof. By Lemma 3.1, Theorem 3.2, and Theorem 3.3, G has an I-cutset or a proper S_2-cutset. Note that by Lemma 4.9, any I-cutset is proper. Let (S, K', K'') be a split of an I-cutset or a proper S_2-cutset of G such that among all such splits, $|K'|$ is minimized. Let G' be the block of decomposition that contains K'. If S is a proper S_2-cutset we let $S = \{u, v\}$, and if S is an I-cutset we let $S = \{u, v, w\}$ and assume that uv is an edge.

CLAIM 1. *G' is 2-connected, has no K_2-cutset, and belongs to $\mathcal{C}_2$.*

Proof of Claim 1. G' is 2-connected and has no K_2-cutset by Lemma 4.6 and Lemma 4.11. If S is an I-cutset, then $G' \in \mathcal{C}_2$ by Lemma 4.10. So suppose that S is a proper S_2-cutset. Since G is 2-connected, $G[S \cup K'']$ contains a uv-path P. If G' contains a propeller as an induced subgraph, then so does $G[S \cup K' \cup V(P)]$. Therefore, $G \in \mathcal{C}_2$. This completes the proof of Claim 1.

CLAIM 2. *If S is a proper S_2-cutset, then both u and v have at least two neighbors in K'. In particular, all nodes of S have degree at least 3 in G'.*

Proof of Claim 2. Suppose not and let u_1 be the unique neighbor of u in K'. By Claim 1 and Lemma 4.5 (applied to G'), u_1v is not an edge. But then $(\{u_1, v\}, K' \setminus \{u_1\}, K'' \cup \{u\})$ is a split of a proper S_2-cutset of G, contradicting our choice of (S, K', K''). This completes the proof of Claim 2.

We now show that $G' \in \mathcal{C}_0$. Assume not. By Claim 1, Lemma 3.1, Theorem 3.2, and Theorem 3.3, G' has an I-cutset or a proper S_2-cutset with split (C, C_1, C_2). W.l.o.g. we may assume that (C, C_1, C_2) is chosen so that $|C_i|$ for some $i \in \{1, 2\}$ is minimized. Let M be the set of marker nodes of G'. By Claims 1 and 2 (applied to G' and C), all nodes of C have degree at least 3 in G', and hence $C \cap M = \emptyset$. We now consider the following two cases.

Case 1. S is a proper S_2-cutset of G. W.l.o.g. $M \subseteq C_2$. Note that C_1 is a proper subset of K'. But then $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset or a proper S_2-cutset of G, contradicting our choice of (S, K', K'').

Case 2. S is an I-cutset of G.

By the choice of (S, K', K''), $G[K']$ is connected. In particular, $|C \cap S| \leq 2$. If $|C \cap S| \leq 1$, then w.l.o.g. $(S \cup M) \setminus C \subseteq C_2$, and hence $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset or a proper S_2-cutset of G, contradicting our choice of (S, K', K''). So $|C \cap S| = 2$. Since each node of S has a neighbor in K', it follows that C is an I-cutset. Suppose that marker nodes u'_1, u'_2 are in C_1 and v'_1, v'_2 are in C_2. Then, w.l.o.g. $\{u, w\} \subseteq C$, so $(C, C_1 \setminus \{u'_1, u'_2\}, C_2 \cup \{u'_1, u'_2\})$ is also a split of an I-cutset of G'. So we may assume that w.l.o.g. $(S \cup M) \setminus C \subseteq C_2$, and hence $(C, C_1, (C_2 \setminus M) \cup K'')$ is a split of an I-cutset of G, contradicting our choice of (S, K', K''). ∎

A *flat pair* in a graph G is a pair of distinct flat edges e, f such that $e = uv$, $f = xy$, and $G[\{u, v, x, y\}]$ has exactly two edges: e and f.

LEMMA 5.2. *Let $G \in \mathcal{C}_0$ be a 2-connected graph. If $x \in V(G)$ is a node of degree at least 3, then there exists a flat pair e, f of G such that e and f both contain a node adjacent to x.*

Proof. Since G is 2-connected, all nodes have degree at least two. Since $G \in \mathcal{C}_0$, x has at least two neighbors u and v of degree 2. Since G is 2-connected, $uv \notin E(G)$ (otherwise x is a cutnode). Let u' (resp., v') be the neighbor of u (resp., v) that is distinct from x. Since $G \in \mathcal{C}_0$ and x has degree at least 3, both u' and v' are of degree 2. Since G is 2-connected, $u' \neq v'$ and $u'v' \notin E(G)$ (otherwise x is a cutnode). It follows that uu', vv' is a flat pair. □

THEOREM 5.3. *If $G \in \mathcal{C}_2$ is 2-connected, then either G is a chordless cycle or G has a flat pair.*

Proof. We prove the result by induction on $|V(G)|$. It is true when $|V(G)| \leq 3$.

Case 1. $G \in \mathcal{C}_0$. This follows directly from Lemma 5.2. This completes the proof in Case 1.

Case 2. G has a K_2-cutset.

Suppose $(\{a, b\}, C_1, C_2)$ is a split of a K_2-cutset of G, and let G_1 and G_2 be the corresponding blocks of decomposition. Note that by Lemma 3.1, $\{a, b\}$ is a proper K_2-cutset. For $i = 1, 2$, G_i is clearly 2-connected, by Lemma 4.2 $G_i \in \mathcal{C}_2$, and hence, by the induction hypothesis, G_i is either an induced cycle or it has a flat pair. Since $\{a, b\}$ is proper, G_i cannot be a triangle. Therefore, G_i has a flat edge entirely contained in C_i. Hence, G has a flat pair formed by a flat edge in C_1 and a flat edge in C_2. This completes the proof in Case 2.

From here on, we assume that G has no K_2-cutset. By Lemma 5.1, we may now assume that G has an I-cutset or a proper S_2-cutset S. Moreover, there exists a split (S, K', K'') such that the block of decomposition G' that contains K' belongs to $\mathcal{C}_0$, and all nodes of S have degree at least three in G'. This leads us to the following two cases.

Case 3. S is an I-cutset.

Suppose $S = \{u, v, w\}$ and $uv \in E(G)$. Note that w has degree at least 3 in G'. Since $G' \in \mathcal{C}_0$, u and v both have a neighbor in K', respectively, u' and v', of degree 2. Since G has no K_2-cutset, $u' \neq v'$ and $u'v' \notin E(G)$ (otherwise $\{u, v\}$ is a K_2-cutset). Since $G \in \mathcal{C}_0$ and u, v have degree at least 3, u' (resp., v') has one neighbors u'' (resp., v'') of degree 2. Since G has no K_2-cutset, $u'' \neq v''$ and $u''v'' \notin E(G)$. It follows that $u'u'', v'v''$ is a flat pair in G. This completes the proof in Case 3.

Case 4. S is an S_2-cutset.

Suppose $S = \{u, v\}$. By Lemma 4.6, G' is 2-connected. If u and v are the only nodes of degree at least 3 in G', then G is formed by at least three uv-paths. Since $G' \in \mathcal{C}_0$, all these paths have length at least 3. Therefore, they all have an internal flat edge, and G' has a flat pair entirely contained in K' that is therefore also a flat pair of G. Otherwise, there is a node $x \in K'$ of degree at least 3. By Lemma 5.2, G' has a flat pair entirely contained in K' that is therefore also a flat pair of G. This completes the proof in Case 4. □

An edge of a graph is *pending* if it contains at least one node of degree 1.

COROLLARY 5.4. *Every graph G in $\mathcal{C}_2$ with at least one edge contains an edge that is pending or flat.*

Proof. We consider the classical decomposition of G into blocks, in the sense of 2-connectivity (see [5]). So, G has a block B that is either a pending edge of G or a 2-connected graph containing at most one vertex x that has neighbors in $V(G) \setminus V(B)$. In the latter case, by Theorem 5.3, B is either a chordless cycle or it has a flat pair, and so at least one flat edge of B is nonincident to x and is therefore a flat edge of G. ∎

An edge-coloring of G is a function $\pi : E \to C$ such that no two adjacent edges receive the same color $c \in C$. If $C = \{1, 2, \ldots, k\}$, we say that π is a k-edge coloring. The chromatic index of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring.

Vizing's theorem states that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$, where $\Delta(G)$ is maximum degree of nodes in G. The edge-coloring problem or chromatic index problem is the problem of determining the chromatic index of a graph. The problem is NP-hard for several classes of graphs, and its complexity is unknown for several others. In this section we solve the edge-coloring problem for the class $\mathcal{C}_2$.

THEOREM 5.5. *If G is a graph in $\mathcal{C}_2$ such that $\Delta(G) \geq 3$, then $\chi'(G) = \Delta(G)$.*

Proof. Induction on $|E(G)|$. If $|E(G)| = 0$, the result clearly holds. By Corollary 5.4, G has an edge ab that is pending or flat. Note that $\mathcal{C}_2$ is not closed under removing edges in general, but it is closed under removing flat or pending edges. Set $G' = (V(G), E(G) \setminus \{ab\})$. If $\Delta(G') \geq 3$, then by the induction hypothesis, we can edge-color G' with $\Delta(G')$ colors. Otherwise, $\Delta(G') \leq 2$, so G' is 3-edge colorable. In either cases, G' is $\Delta(G)$-colorable. We can extend the edge-coloring of G' to an edge-coloring of G as follows: when ab is pending, by assigning a color to ab not used among the edges incident to ab, and when ab is flat, by assigning to ab a color not used for the two edges adjacent to ab. ∎

Note that when $\Delta(G) \leq 2$, G is a disjoint union of cycles and paths, so χ' is easy to compute. The proof above is easy to transform into a polynomial time algorithm that outputs the coloring whose existence is proved.

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