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Identification of linear and nonlinear continuous time models

from sampled data sets

by

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Abstract

The identification of continuous time linear and nonlinear models from sampled-data records is investigated and a new recursive orthogonal algorithm is derived. An integral equation formulation is shown to be appropriate and enhancement of the parameter estimates using data prefiltering is demonstrated for noise corrupted data sets. Simulated examples are included to illustrate the methods.

1. Introduction

The problem of discrete-time model identification (DMI) has been thoroughly studied, and numerous parameter estimation methods have been developed for discrete-time models based on sampled data records [1,2,3,4,5]. The problem of continuous-time model identification (CMI) using digital computers has however received little attention. In recent years, the relevance and importance of the CMI problem has been increasingly recognized in several areas of identification and self-tuning adaptive control [6]. Unbehauen and Rao [7] recently gave a comprehensive review of the developments in the identification of continuous-time systems. One of the existing estimation methods for continuous-time model identification is the indirect approach based on discrete-time model identification [7,8]. This approach has the advantage that it utilizes existing parameter estimation methods for discrete-time models, but it requires extensive computations and is inappropriate for on-line or real-time applications. In contrast, the direct parameter estimation methods which use orthogonal functions such as block-pulse functions (BPF), state variable filters (SVF) and numerical integration have received more and more attention [9,10]. These methods are based on integration rather than the noise accentuating derivatives. The disadvantage of this is the initial condition problem which arises due to integration. Since the initial states of a system are usually neither known or zero, they have to be identified together with the system parameters. It has been found that estimation of the initial states introduces some problems and makes it difficult to apply the parameter estimation methods to on-line or real-time identification [10]. Recent developments [11,12] have focused on expansions of the system transients using an orthogonal function expansion, which is then used in an integral equation governing the system dynamics. The orthogonality of the basis functions leads to extensive simplification of the integral equation, and accurate identification of deterministic linear MIMO systems, in the absence of external disturbances, have been reported [12]. A major disadvantage of orthogonal-function expansion approaches is the requirement that all the data acquired over the observation interval is required to find the coefficients in the expansion.

In the present study a new orthogonal function approach is introduced. Sampled-data sequences are first approximated by an orthogonal time series using the three-term recurrence formula followed by the inclusion of approximation procedures for the identification of continuous time systems. Selection of the orthogonal time base and the usefulness of the error reduction ratio test in function approximation is discussed. The effects of derivative and integral action on the approximate functions is also considered. Since the orthogonal approximation can be time consuming, a recursive approximation procedure is formulated for obtaining the orthogonal series parameters for a uniformly sampled signal. Integral equations for the evaluation of the system dynamics using the orthogonal time series are presented and the problem of unsolved initial conditions is removed. Once a set of equations describing the system dynamics has been set up, the unknown coefficients can be obtained using any of the well-known estimation algorithms such as least squares or extended least squares estimation algorithms. Application of filters on the input and output signals in order to improve the final estimates is also investigated. The proposed estimation algorithm is then extended to the identification of continuous time nonlinear systems. Finally, illustrative examples are included to show the effectiveness of the proposed algorithms.

2. Orthogonal functions

A real valued function $f(t)$ is said to be square-integrable in an interval $[t_0, t_N]$ if the Lebesgue integral is less than infinity.

$$\int_{t_0}^{t_N} f^2(\tau)\, d\tau < +\infty \tag{1}$$

Almost all process signals in physical reality satisfy the Lebesgue integral of eqn.(1). Any signal which is square integrable can be represented by a summation of orthogonal time functions

$$f(t) = \sum_{i=0}^{\infty} f_i w_i(t) \tag{2}$$

where $w_i(t)$, $i=0,1,2,\ldots$ are orthogonal over the interval $[t_0, t_N]$ and f_i, $i=0,1,2,\ldots$ are some constant coefficients.

2.1 Approximation of sampled-data records

Suppose that $(N+1)$ data records are collected from the function $f(t)$ at time instants $\{t_0, t_1, \ldots, t_N\}$ and the sampling rate is higher than the Nyquist sampling frequency. The objective of approximation is to fit a continuous time function to the sampled-data records such that

$$f(t_j) \approx \sum_{i=0}^{M} \beta_i t_j^i \ , \quad j=0,1,2,\ldots,N \tag{3}$$

where β_i, $i=0,1,2,...$ are some constant coefficients and M is the order of the approximation. In the orthogonal function approach to approximation, eqn.(3) is first transformed into an auxiliary model

$$f(t_j) = \sum_{i=0}^{M} f_i w_i(t_j) + \zeta(t_j) \quad , \quad j = 0,1,...,N \tag{4}$$

where $w_i(t_j)$, $i=0,1,2,...,M$ is the orthogonal time series involved in the approximation and $\zeta(t_j)$ are modelling errors. The $w_i(t_j)$, $i=0,1,2,...,M$ are constructed to be orthogonal over the data record such that

$$\sum_{j=0}^{N} w_i(t_j) w_{r+1}(t_j) = 0 \quad , \quad i = 0,1,...,r \tag{5}$$

by using the three-term recurrence method [13]

$$\begin{aligned} w_0(t_j) &= 1 \\ w_i(t_j) &= t_j^i - \sum_{r=0}^{i-1} \alpha_{ri} w_r(t_j) \quad , \quad i=0,1,2,...,M \\ \alpha_{ri} &= \frac{\sum_{j=0}^{N} t_j^i w_r(t_j)}{\sum_{j=0}^{N} w_r^2(t_j)} \end{aligned} \tag{6}$$

The orthogonal parameters can then be recovered with

$$f_i = \frac{\sum_{j=0}^{N} f(t_j) w_i(t_j)}{\sum_{j=0}^{N} w_i^2(t_j)} \quad , \quad i=0,1,2,...,M \tag{7}$$

Once the orthogonal parameters f_i have been estimated, the original system parameters β_i in the original continuous time function can be computed as

$$\begin{aligned} \beta_M &= f_M \\ \beta_i &= f_i - \sum_{m=i+1}^{M} \alpha_{rm} \beta_m \quad , \quad i=M-1,...,0 \end{aligned} \tag{8}$$

Notice that the proposed orthogonal functions are defined in any arbitrary interval $t \in [t_0, t_N]$ and the coefficients f_i given in eqn.(7) have the property of minimising the mean square error

$$\epsilon^2 = \frac{1}{N+1}\sum_{j=0}^{N}\left(f(t_j) - \sum_{i=0}^{M} f_i w_i(t_j)\right)^2 \tag{9}$$

A by-product from the orthogonal approximation is the error reduction ratio (ϵRR) defined as [13,14]

$$\epsilon RR_i = \frac{\sum_{j=0}^{N} f_i^2 w_i^2(t_j)}{\sum_{j=0}^{N} f^2(t_j)} \times 100 \quad , \quad i=0,1,2,...,M \tag{10}$$

This can serve as an indicator of how important a particular term is and how close the overall approximated function is to the original system function. If the sum of the error reduction ratios reaches a certain threshold, therefore the approximation procedure can be terminated indicating that the fitted function is an adequate representation of the original function. If the sum of the error reduction ratios is closed to 100, this provides an indication that the fitted model is very close to the true system function.

2.2 Differentiation and integration

Define $\hat{f}(t_j) = \sum_{i=0}^{M} f_i w_i(t_j)$ so that eqn.(4) becomes

$$f(t_j) = \hat{f}(t_j) + \zeta(t_j) \quad , \quad j=0,1,2,...,N \tag{11}$$

The first order derivative of eqn.(11) is given as

$$f^{(1)}(t_j) = \hat{f}^{(1)}(t_j) + \zeta^{(1)}(t_j) \tag{12}$$

whereas the first order integral of eqn.(11) in the interval $[t_0,t_N]$ is given as

$$\int_{t_0}^{t_N} f(\tau)d\tau = \int_{t_0}^{t_N} \hat{f}(\tau)d\tau + \int_{t_0}^{t_N} \zeta(\tau)d\tau \tag{13}$$

If the modelling error $\zeta(t_j)$ has zero mean and the integration interval is sufficiently large

$$\int_{t_0}^{t_N} \zeta(\tau)d\tau \to 0$$

and eqn.(13) will reduce to

$$\int_{t_0}^{t_N} f(\tau)d\tau \approx \int_{t_0}^{t_N} \hat{f}(\tau)d\tau \tag{14}$$

From eqns.(12) and (14), two important remarks can be made. First, the derivative action operating on the approximated function of eqn.(11) has no effect on reducing or improving the approximation. Rather the error term is accentuated in some cases because of the high bandwidth of the modelling error. Conversely integral action has the effect of reducing and in some cases eliminating the modelling error term as seen from eqn.(14). Hence the integral approach is adopted for the identification of continuous time systems.

Since the orthogonal series $w_i(t)$, $i=0,1,2,\ldots,M$ satisfy the three term recurrence formula of eqn.(6), this can also be used for the definite integral of $w_i(t)$ in the interval $[t_0,t_N]$ as

$$\Im_i(t_{0,N}) = \begin{cases} t_N - t_0 & , \; i=0 \\ \dfrac{t_N^{i+1} - t_0^{i+1}}{i+1} - \displaystyle\sum_{r=0}^{i-1} \alpha_{ri} \Im_r(t_{0,N}) & , \; i=1,2,\ldots,M \end{cases} \tag{15}$$

where

$$\Im_i(t_{0,N}) = \int_{t_0}^{t_N} w_i(\tau)d\tau$$

Define the j'th order integral of the function $w_i(t)$ as

$$\Im_i^j(t_{0,N}) = \int\int\ldots\int_{t_0}^{t_N} w_i(\tau_1)\, d\tau_1 d\tau_2 \ldots d\tau_j$$

and use the three-term recurrence formula again to give

$$\Im_i^j(t_{0,N}) = \begin{cases} \dfrac{t_N^j - t_0^j}{j!} - \dfrac{t_0^{j-1}}{(j-1)!} - \ldots & , \; i=0 \\ \dfrac{i!\,(t_N^{i+j} - t_0^{i+j})}{(i+j)!} - \dfrac{i!\,t_0^{i+j-1}}{(i+j-1)!} - \ldots - \displaystyle\sum_{r=0}^{i-1} \alpha_{ri} \Im_r^j(t_{0,N}) & , \; i=1,2,\ldots,N \end{cases} \tag{16}$$

2.3 Examples

Twenty one data records collected from the continuous time function

$$f(t) = \sin(0.3t) + \sin(0.8t) \quad , \quad t \in [0,10] \tag{17}$$

at a sampling period $\Delta T = 0.5$ were used for the analysis of the signal $f(t)$. Two eight'th order continuous time models denoted as A and B were fitted to the data records using the orthogonal function approach (eqns.(6), (7) and (8)) with the two-sided orthogonal time base $t_s \in [-5,5]$ and the one-sided orthogonal time base $t \in [0,10]$ respectively. Table 1 shows the results of the two approximations. The sum of the error reduction ratios captured by both models A and B reach 99.9999% indicating that good approximations of the original function $f(t)$ have been obtained. The two continuous time functions and their corresponding parameters can be recovered by applying eqn.(8) to the orthogonal parameters obtained in Table 1 to give

$$\begin{aligned} f_A(t_s) = {} & 0.24093 - 0.49971\, t_s + 0.19677\, t_s^2 + 0.05430\, t_s^3 - 0.01240\, t_s^4 \\ & -0.16024E{-2}\, t_s^5 + 0.25242E{-3}\, t_s^6 + 0.16428E{-6}\, t_s^7 - 0.20228E{-7}\, t_s^8 \end{aligned} \tag{18}$$

$$\begin{aligned} f_B(t) = {} & -0.44997E{-3} + 1.12543\, t - 0.07121\, t^2 - 0.01604\, t^3 - 0.03804\, t^4 \\ & +0.01361\, t^5 - 0.17383E{-2}\, t^6 + 0.97324E{-6}\, t^7 - 0.20224E{-7}\, t^8 \end{aligned} \tag{19}$$

where $f_A(t_s)$ and $f_B(t)$ are the fitted models A and B respectively and $t_s = t-5$ is the shifted time base. Substituting $t = t_s+5$ into eqn.(18) gives

$$\begin{aligned} f_A(t) = {} & -0.46138E{-3} + 1.12545\, t - 0.07126\, t^2 - 0.01600\, t^3 - 0.03805\, t^4 \\ & +0.01361\, t^5 - 0.17386E{-2}\, t^6 + 0.97342E{-6}\, t^7 - 0.20228E{-7}\, t^8 \end{aligned} \tag{20}$$

Figure 1 shows the predicted output of the fitted models A and B superimposed on the actual function $f(t)$. The figure clearly demonstrates that both models A and B captured the original function very well in the region $t \in [0,10]$. A more detailed analysis of models A and B based on both derivative and integral approximations was carried out and the results are summarised in Table 2. Inspection of Table 2 shows that the percentage error of the derivative and integral approximations obtained from both models are less than 0.5% indicating that both models approximate the original function very well. Notice that the percentage errors obtained in the integral approximations are smaller than the errors obtained for the derivative approximations. There is no significant difference between the two fitted models A and B. However, the two-sided orthogonal time base approximation is preferable in general because it is numerically more robust and contributions from high order terms is significantly less compared to the one-sided orthogonal time base approximation.

Consider the same function $f(t)$ with the collected data corrupted by a zero mean discrete white noise sequence $e(t_k)$, k=0,1,2,... of variance 0.01.

$$\begin{aligned} f(t) &= \sin(0.3t) + \sin(0.8t) \\ f^*(t_k) &= f(t_k) + e(t_k) \end{aligned} \tag{21}$$

Again twenty one records were collected from $f^*(t_k)$ at a sampling period of 0.5 seconds from t=0 to 10s. Table 3 shows the results of the orthogonal approximation using the two-sided orthogonal time base $t_s \in [-5,5]$. A fifth order model fitted to the data records captured 99.35% of the total output power and this is a good indication that the fitted model is a good approximation of the original function. The shifted continuous time function $f_A(t_s)$ can be obtained by applying eqn.(8) to the evaluated orthogonal parameters to give

$$\begin{aligned} f_A(t_s) &= 0.32411-0.51496\ t_s+0.13148\ t_s^2 \\ &+0.05196\ t_s^3-0.00484\ t_s^4-0.00109\ t_s^5 \end{aligned} \tag{22}$$

Substituting $t=t_s+5$ into eqn.(22) gives

$$\begin{aligned} f_A(t) &= 0.08704+1.06655\ t-0.00582\ t^2 \\ &-0.12482\ t^3+0.02251\ t^4-0.00109\ t^5 \end{aligned} \tag{23}$$

The predicted output of the fitted model eqn.(23) superimposed on the actual function $f(t)$ and the sampled data records $f^*(t_k)$ are shown in Figure 2. The performance of the fitted model for the derivative and integral approximations is shown in Table 4. As expected, the derivative estimates are poor while the integral estimates still retain an acceptable accuracy.

The above simulations clearly demonstrate the effectiveness of the orthogonal approximation procedures and the power of integral action in noise and error reduction. Therefore, it is natural to select the orthogonal approximation procedures and the integral approach in the identification of continuous time system models.

3. Recursive approximation

One of the main disadvantages of the orthogonal function approach is the requirement to operate on all the data over the observation interval. When additional data becomes available, the whole operation has to repeated. Direct operation over all data records using eqns.(6), (7) and (8) is a time consuming task and cannot be easily implemented on-line if the number of data records involved is large and/or the order of orthogonal series is high. An improvement in the speed of operation is definitely required for on-line application.

For a fixed number of data records $(N+1)$ and a uniform sampling interval ΔT, the orthogonal series $w_i(t_j)$, $i=0,1,2,...,M$, summation of the square of the orthogonal data records $\sum_{j=0}^{N} w_i^2(t_j)$, and the coefficients α_{ri} of eqn.(6) are fixed if the time origin t_0 for the orthogonalisation remains unchanged. Also the single and multiple integrals of the orthogonal series $\Im_i^j(t_{0,N})$ in eqns.(15) and (16) which depend only on the coefficients α_{ri} and the integral time interval $[t_0,t_N]$ are also fixed for a fixed sampling interval ΔT and fixed length of data sequence $(N+1)$.

Consider the case when an additional data record enters the approximation process. The proposed procedure is to discard the first data point in the original sequence and pack the new data record at the end such that the total number of data records remains unchanged at $(N+1)$. The new data sequence $f(t_j)$, $j=1,2,...,N+1$ can then be approximated with the same orthogonal time base and series. To obtain the orthogonal coefficients f_i, $i=0,1,2,...,M$ at time t_{N+1}, the major obstacle to the speed of operation is the evaluation of the terms

$$\sum_{j=1}^{N+1} f(t_j)w_i(t_{j-1}) \quad , \quad i=0,1,2,...,M \tag{24}$$

which depend on the number of data involved in the operation.

Define

$$\begin{aligned} P_{0,N}^i &= \sum_{j=0}^{N} f(t_j)w_i(t_j) \quad , \quad P_{1,N+1}^i = \sum_{j=1}^{N+1} f(t_j)w_i(t_{j-1}) \\ S_{0,N}^i &= \sum_{j=0}^{N} f(t_j)t_j^i \quad , \quad S_{1,N+1}^i = \sum_{j=1}^{N+1} f(t_j)t_{j-1}^i \end{aligned} \tag{25}$$

Substituting the three term recurrence relationship of eqn.(6) into eqn.(25) gives

$$\begin{aligned} P_{0,N}^i &= \begin{cases} S_{0,N}^0 & , \ i=0 \\ S_{0,N}^i - \sum_{r=0}^{i-1} \alpha_{ri}P_{0,N}^r & , \ i=1,2,...,M \end{cases} \\ P_{1,N+1}^i &= \begin{cases} S_{1,N+1}^0 & , \ i=0 \\ S_{1,N+1}^i - \sum_{r=0}^{i-1} \alpha_{ri}P_{1,N+1}^r & , \ i=1,2,...,M \end{cases} \end{aligned} \tag{26}$$

The relationship between the term $S_{0,N}^{i}$ and the term $S_{1,N+1}^{i}$ is given by

$$S_{1,N+1}^{i} = \bar{S}_{0,N}^{i} - f(t_0)t_{-1}^{i} + f(t_{N+1})t_N^{i} \quad , \quad i=0,1,2,\ldots,M \tag{27}$$

where

$$\begin{aligned} \bar{S}_{0,N}^{i} &= \sum_{j=0}^{N} f(t_j)t_{j-1}^{i} \\ &= \sum_{j=0}^{N} f(t_j)(t_j-\Delta T)^{i} \end{aligned} \tag{28}$$

and ΔT is the fixed sampling period. From eqns.(27) and (28), the evaluation of the first few $P_{1,N+1}^{i}$ terms from the previous estimation requires a few subtractions, additions, multiplications and divisions. The number of data records involved in the approximation process no longer affects the speed of operation. If the order M of the orthogonal series is small, this significantly reduces the computational burden for orthogonal series approximation. Expansion of eqn.(28) for the first three terms gives

$$\begin{aligned} \bar{S}_{0,N}^{0} &= S_{0,N}^{0} \\ \bar{S}_{0,N}^{1} &= S_{0,N}^{1}-\Delta T S_{0,N}^{0} \\ \bar{S}_{0,N}^{2} &= S_{0,N}^{2}-2\Delta T S_{0,N}^{1}+\Delta T^{2} S_{0,N}^{0} \end{aligned}$$

For low order orthogonal series, the expansion of eqn.(28) results in just a few terms such that the computational time is greatly reduced. For example, if a fourth order orthogonal series is used to approximate a sequence of 100 data records, the direct operation of eqn.(11) requires 500 multiplications and 495 additions whereas the proposed recursive approximation algorithm requires only 30 multiplications and 30 additions. That is over fifteen times improvement in speed and this clearly demonstrates the effectiveness of the proposed algorithm. The recursive approximation algorithm is compared with the direct calculation in Table 5 where ratios of the speed between the recursive approximation algorithm and the direct calculation of eqn.(24) are shown. If the number of data N involved in the approximation is high and the order of the orthogonal series M is low, there is a vast improvement in the speed of operation using the recursive approximation algorithm. As the order of the orthogonal series M increases and the number of data N is reduced, the improvement in speed is less obvious. On some occasions the improvement in speed can become negative because the expansion of the terms in eqn.(28) can become very high and the number of operations on those terms may surpass the actual effort applied directly on eqn.(24).

Another way to improve the speed of operation is to realise a parallel implementation of the orthogonal approximation algorithm. That is individual orthogonal coefficients f_i can be evaluated independently. Hence the estimation algorithm can be implemented on parallel processing hardware. The number of parallel processing elements depends on the number of data involved in the operation and on the order of the orthogonal series. This normally requires massive hardware if a reasonable approximation is required.

4. Linear systems

Consider a single input single output (SISO) continuous system that can be described by the linear differential equation

$$A(D)y(t) = B(D)u(t) \tag{29}$$

where $u(t)$ is the input signal and $y(t)$ the output response; $D = \frac{d}{dt}$ is the differential operator, A and B are polynomials in D.

$$\begin{aligned} A(D) &= a_0 D^n + a_1 D^{n-1} + \ldots + 1 \\ B(D) &= b_1 D^{n-1} + \ldots + b_n \\ D^n &= \frac{d^n}{dt^n} \\ n &= \textit{order of system} \end{aligned} \tag{30}$$

and the system eqn.(29) is subject to arbitrary initial conditions

$$\begin{aligned} u_0 &= \left(u(t_0), \ldots, u^{(n-2)}(t_0)\right)^T \\ y_0 &= \left(y(t_0), \ldots, y^{(n-1)}(t_0)\right)^T \end{aligned} \tag{31}$$

One way to set up the equations for the identification of the system parameters is to replace the differential operator s in eqn.(29) with an approximation. In section 2, it has been found that derivative approximations are noise accentuating while the integral approximations are noise resistant. Therefore eqn.(29) is transformed to an integral equation before system identification is carried out.

Define

$$Y^{(j)}(t_{0,N}) = \int_{t_0}^{t_N}\int\cdots\int y(\tau_1)d\tau_1 d\tau_2 \ldots d\tau_j \quad ; \quad U^{(j)}(t_{0,N}) = \int_{t_0}^{t_N}\int\cdots\int u(\tau_1)d\tau_1 d\tau_2 \ldots d\tau_j$$

$$Y^{(1)}(t_{0,N}) = \int_{t_0}^{t_N} y(\tau)d\tau \quad ; \quad U^{(1)}(t_{0,N}) = \int_{t_0}^{t_N} u(\tau)d\tau$$

For a linear SISO system described by eqn.(29), consider evaluation at an arbitrary time τ_1 and integrate from t_0 to t_N with respect to τ_1:

$$a_0\left[y^{(n-1)}(t_N)-y^{(n-1)}(t_0)\right] + a_1\left[y^{(n-2)}(t_N)-y^{(n-2)}(t_0)\right] + \ldots + a_{n-1}\left[y(t_N)-y(t_0)\right] + Y^{(1)}(t_{0,N})$$
$$= b_1\left[u^{(n-2)}(t_N)-u^{(n-2)}(t_0)\right] + \ldots + b_{n-1}\left[u(t_N)-u(t_0)\right] + b_n U^{(1)}(t_{0,N})$$

The latter equation holds at all points in time, and hence consideration of the evaluation at τ_2 followed by integration from t_0 to t_N with respect to τ_2 gives

$$a_0\left[y^{(n-2)}(t_N)-y^{(n-2)}(t_0)-(t_N-t_0)y^{(n-1)}(t_0)\right] + a_1\left[y^{(n-3)}(t_N)-y^{(n-3)}(t_0)-(t_N-t_0)y^{(n-2)}(t_0)\right] + \ldots$$
$$+ a_{n-1}\left[Y^{(1)}(t_{0,N})-(t_N-t_0)y(t_0)\right] + Y^{(2)}(t_{0,N})$$
$$= b_1\left[u^{(n-3)}(t_N)-u^{(n-3)}(t_0)-(t_N-t_0)u^{(n-2)}(t_0)\right] + \ldots + b_{n-1}\left[U^{(1)}(t_{0,N})-(t_N-t_0)u(t_0)\right] + b_n U^{(2)}(t_{0,N})$$

Continuing this process to the point at which eqn.(29) has been integrated n+1 times and collecting terms gives

$$Y^{(n+1)}(t_{0,N}) = -\sum_{j=0}^{n-1} a_j Q_y^{(j+1)}(t_{0,N}) + \sum_{j=1}^{n} b_j Q_u^{(j+1)}(t_{0,N}) \tag{32}$$

where

$$Q_y^{(j+1)}(t_{0,N}) = Y^{(j+1)}(t_{0,N}) - \sum_{i=1}^{n-j}\left(\frac{(t_N^{n-i+1}-t_0^{n-i+1})}{(n-i+1)!} - \frac{t_0^{n-i}}{(n-i)!} - \ldots\right) y^{(n-j-i)}(t_0)$$

$$Q_u^{(j+1)}(t_{0,N}) = U^{(j+1)}(t_{0,N}) - \sum_{i=1}^{n-j}\left(\frac{(t_N^{n-i+1}-t_0^{n-i+1})}{(n-i+1)!} - \frac{t_0^{n-i}}{(n-i)!} - \ldots\right) u^{(n-j-i)}(t_0)$$

$$n > i,j$$

The system parameters a_j and b_j can be evaluated if $Q_y^{(k)}(t_{0,N})$ and $Q_u^{(k)}(t_{0,N})$ are available. In normal practice, all system states and initial conditions are seldom known. Estimates are therefore required to identify the system dynamics. If the system output and input are replaced with the corresponding orthogonal estimates then

$$y(t) \approx \sum_{i=0}^{M} f_{y_i} w_i(t) = \sum_{i=0}^{M} \beta_{y_i} t^{i}$$
$$u(t) \approx \sum_{i=0}^{M} f_{u_i} w_i(t) = \sum_{i=0}^{M} \beta_{u_i} t^{i} \qquad (33)$$

where f_{y_i} and f_{u_i} are the orthogonal parameter estimates and β_{y_i} and β_{u_i} are the actual parameter estimates for the output and input signals respectively. The first and higher order derivative approximations for the system input and output can be written as

$$\hat{y}^{(1)}(t) = \sum_{i=1}^{M} i\beta_{y_i} t^{i-1}$$
$$\hat{u}^{(1)}(t) = \sum_{i=1}^{M} i\beta_{u_i} t^{i-1} \qquad (34)$$

and the j'th order derivative approximations are given by

$$\hat{y}^{(j)}(t) = \sum_{i=j}^{M} \frac{i!}{(i-j)!} \beta_{y_i} t^{i-j}$$
$$\hat{u}^{(j)}(t) = \sum_{i=j}^{M} \frac{i!}{(i-j)!} \beta_{u_i} t^{i-j} \qquad (35)$$

The input and output integral can also be replaced with the orthogonal estimates as

$$\hat{Y}^{(j)}(t_{0,N}) = \sum_{i=0}^{M} f_{y_i} \Im_i^{j}(t_{0,N})$$
$$\hat{U}^{(j)}(t_{0,N}) = \sum_{i=0}^{M} f_{u_i} \Im_i^{j}(t_{0,N}) \qquad (36)$$

Replacing the unknown states and initial conditions in eqn.(32) with the corresponding approximations of eqns.(34), (35) and (36) gives

$$\sum_{i=0}^{M} f_{y_i} \Im_i^{n+1}(t_{0,N}) = -\sum_{j=0}^{n-1} a_j \hat{Q}_y^{(j+1)}(t_{0,N}) + \sum_{j=1}^{n} b_j \hat{Q}_u^{(j+1)}(t_{0,N}) + \zeta(t_{0,N}) \qquad (37)$$

where

$$\hat{Q}_y^{(j+1)}(t_{0,N}) = \sum_{i=0}^{M} f_{y_i}\Im_i^{j+1}(t_{0,N})$$

$$- \sum_{m=1}^{n-j}\left(\frac{(t_N^{n-m+1}-t_0^{n-m+1})}{(n-m+1)!} - \frac{t_0^{n-m}}{(n-m)!} - \dots\right)\sum_{i=n-j-m}^{M}\frac{i!}{(i-n+j+m)!}\,\beta_{y_i}t^{i-n+j+m} \qquad (38)$$

$$\hat{Q}_u^{(j+1)}(t_{0,N}) = \sum_{i=0}^{M} \hat{f}_{u_i}\Im_i^{j+1}(t_{0,N})$$

$$- \sum_{m=1}^{n-j}\left(\frac{(t_N^{n-m+1}-t_0^{n-m+1})}{(n-m+1)!} - \frac{t_0^{n-m}}{(n-m)!} - \dots\right)\sum_{i=n-j-m}^{M}\frac{i!}{(i-n+j+m)!}\,\beta_{u_i}t^{i-n+j+m}$$

and $\zeta(t_{0,N})$ is the modelling error. If $t_0 = 0$, eqn.(38) is reduced to

$$\hat{Q}_y^{(j+1)}(t_{0,N}) = \sum_{i=0}^{M} f_{y_i}\Im_i^{j+1}(t_{0,N}) - \sum_{m=1}^{n-j}\frac{t_N^{n-m+1}}{(n-m+1)!}(n-j-m)!\,\beta_{y_{n-j-m}}$$

$$\hat{Q}_u^{(j+1)}(t_{0,N}) = \sum_{i=0}^{M} f_{u_i}\Im_i^{j+1}(t_{0,N}) - \sum_{m=1}^{n-j}\frac{t_N^{n-m+1}}{(n-m+1)!}(n-j-m)!\,\beta_{u_{n-j-m}} \qquad (39)$$

For an additional R data records, a set of simultaneous equations can then be set up as

$$\sum_{i=0}^{M} f_{y_i}\Im_i^{n+1}(t_{0,N}) = -\sum_{j=0}^{n-1} a_j\hat{Q}_y^{(j+1)}(t_{0,N}) + \sum_{j=1}^{n} b_j\hat{Q}_u^{(j+1)}(t_{0,N}) + \zeta(t_{0,N})$$

$$\sum_{i=0}^{M} f_{y_i}\Im_i^{n+1}(t_{1,N+1}) = -\sum_{j=0}^{n-1} a_j\hat{Q}_y^{(j+1)}(t_{1,N+1}) + \sum_{j=1}^{n} b_j\hat{Q}_u^{j+1}(t_{1,N+1}) + \zeta(t_{1,N+1}) \qquad (40)$$

$$\vdots$$

$$\sum_{i=0}^{M} f_{y_i}\Im_i^{n+1}(t_{R,N+R}) = -\sum_{j=0}^{n-1} a_j\hat{Q}_y^{(j+1)}(t_{R,N+R}) + \sum_{j=1}^{n} b_j\hat{Q}_u^{(j+1)}(t_{R,N+R}) + \zeta(t_{R,N+R})$$

If the modelling error terms can be suitably approximated by a moving average (MA) noise model unbiased parameter estimates for a_j and b_j can be obtained using the extended least squares estimation algorithm

$$\theta = (\Phi^T\Phi)^{-1}\Phi^T\Gamma \qquad (41)$$

where

$$\Phi = \begin{bmatrix} \hat{Q}_y^{(1)}(t_{0,N}) & \cdots & \hat{Q}_y^{(n)}(t_{0,N}) & \hat{Q}_u^{(2)}(t_{0,N}) & \cdots & \hat{Q}_u^{(n+1)}(t_{0,N}) & \xi(t_{-1,N-1}) & \cdots & \xi(t_{-n,N-n}) \\ \hat{Q}_y^{(1)}(t_{1,N+1}) & \cdots & \hat{Q}_y^{(n)}(t_{1,N+1}) & \hat{Q}_u^{(2)}(t_{1,N+1}) & \cdots & \hat{Q}_u^{(n+1)}(t_{1,N+1}) & \xi(t_{0,N}) & \cdots & \xi(t_{-n+1,N-n+1}) \\ \vdots & & & & & & & & \\ \hat{Q}_y^{(1)}(t_{R,N+R}) & \cdots & \hat{Q}_y^{(n)}(t_{R,N+R}) & \hat{Q}_u^{(2)}(t_{R,N+R}) & \cdots & \hat{Q}_u^{(n+1)}(t_{R,N+R}) & \xi(t_{R-1,N+R-1}) & \cdots & \xi(t_{R-n,N+R-n}) \end{bmatrix}$$

$$\Gamma = \left[\sum_{i=0}^{M} f_{y_i}\mathfrak{S}_i^{n+1}(t_{0,N}) \;\; \sum_{i=0}^{M} f_{y_i}\mathfrak{S}_i^{n+1}(t_{1,N+1}) \;\; \cdots \;\; \sum_{i=0}^{M} f_{y_i}\mathfrak{S}_i^{n+1}(t_{R,N+R})\right]^T$$

$$\theta = \left[a_0 \; a_1 \cdots a_{n-1} \; b_1 \cdots b_n \; d_1 \cdots d_n\right]^T$$

and $\xi(t_{k,N+k})$ is the estimated MA noise sequence given by

$$\zeta(t_{k,N+k}) = \xi(t_{k,N+k}) + d_1\xi(t_{k-1,N+k-1}) + \ldots + d_n\xi(t_{k-n,N+k-n}) \tag{42}$$

The continuous time system parameters can therefore be obtained from discrete time data records using eqns.(39), (40) and (41) and the initial condition problem associated with the integral equation approach has been solved.

Similarly, for a stochastic linear system described by

$$A(D)z(t) = B(D)u(t) + C(D)e(t) \tag{43}$$

where $z(t)$ is the measured output response, $u(t)$ is the input signal, $e(t)$ is a zero mean white noise, $D = \frac{d}{dt}$ is the differential operator, A, B and C are polynomials in D:

$$A(D) = a_0D^n + a_1D^{n-1} + \ldots + 1$$
$$B(D) = b_1D^{n-1} + \ldots + b_n$$
$$C(D) = c_0D^n + c_1D^{n-1} + \ldots + 1$$

Integrating eqn.(43) n+1 times and collecting terms gives

$$Z^{(n+1)}(t_{0,N}) = -\sum_{j=0}^{n-1} a_jQ_z^{(j+1)}(t_{0,N}) + \sum_{j=1}^{n} b_jQ_u^{(j+1)}(t_{0,N}) + \sum_{j=0}^{n} c_jQ_e^{(j+1)}(t_{0,N}) \tag{44}$$

Setting $t_0=0$ and replacing the unknown states in eqn.(44) with the corresponding estimates gives

$$\sum_{i=0}^{M} f_{z_i}\mathfrak{S}_i^{n+1}(t_{0,N}) = -\sum_{j=0}^{n-1} a_j\hat{Q}_z^{(j+1)}(t_{0,N}) + \sum_{j=1}^{n} b_j\hat{Q}_u^{(j+1)}(t_{0,N}) + \zeta(t_{0,N}) \tag{45}$$

where the stochastic terms $c_j Q_e^{(j+1)}(t_{0,N})$ and the approximation errors are lumped into a modelling error term $\zeta(t_{0,N})$,

$$\hat{Q}_z^{(j+1)}(t_{0,N}) = \sum_{i=0}^{M} f_{z_i} \mathfrak{F}^{j+1}(t_{0,N}) - \sum_{m=1}^{n-m} \frac{t_N^{n-m+1}}{(n-m+1)!} (n-j-m)! \ \beta_{z_{n-j-m}}$$

and

$$z(t) = \sum_{i=0}^{M} f_{z_i} w_i(t) = \sum_{i=0}^{M} \beta_{z_i} t^i$$

The parameter estimates for a_j and b_j can again be obtained using the extended least squares estimation algorithm of eqns.(41) and (42) if an additional R data records are available and the modelling error terms can be approximated by a MA noise model. The parameter estimates can be refined further if the input and output signals are pre-filtered using a low pass filter to give

$$A(D)z^F(t) = B(D)u^F(t) + C(D)F(D)e(t) \qquad (46)$$

where $F(D)$ is a filter function and

$$z^F(t) = F(D)z(t)$$
$$u^F(t) = F(D)u(t)$$

Here, the filtering operation only affects the noise model while the system transfer function remains unchanged. Therefore, if the filter function is correctly chosen, this will considerably reduce the unwanted high frequency noise in the original input and output signals and the accuracy of the approximations on the filtered signals will be improved. Orthogonal approximation and system identification can then be carried out on the filtered signals $u^F(t)$ and $z^F(t)$ instead of on the raw signals $u(t)$ and $z(t)$.

5. Nonlinear systems

The preceding integral equation approach can also be extended to cover the identification of certain nonlinear systems with polynomial type of nonlinearities. Consider a SISO system governed by an ordinary differential equation of the form

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \ldots + y(t) + \sum_{l=2}^{L} g_l \left[y^{(n)}(t),\ldots,y(t),u^{(n-1)}(t),\ldots,u(t),t\right] \qquad (47)$$
$$= b_1 u^{(n-1)}(t) + \ldots + b_n u(t)$$

where the nonlinearity can be expanded as a polynomial type function $g_l[y^{(n)}(t),...,y(t),u^{(n-1)}(t),...,u(t),t]$, l is the order of the nonlinearity and L is the maximum order of nonlinearity.

For a first order dynamical system with a second order polynomial type nonlinearity for example, the nonlinear function becomes

$$g_2[y^{(1)}(t),y(t),u(t),t]$$
$$= \gamma_1 y^{(1)}(t)y^{(1)}(t)+\gamma_2 y^{(1)}(t)y(t)+\gamma_3 y^{(1)}(t)u(t)+\gamma_4 y(t)y(t)+\gamma_5 y(t)u(t)+\gamma_6 u(t)u(t)$$

where γ_i are constant coefficients.

Integrating eqn.(47) n+1 times gives

$$Y^{(n+1)}(t_{0,N})+\sum_{j=0}^{n-1} a_j Q_y^{(j+1)}(t_{0,N})+\sum_{l=2}^{L} G_l^{(n+1)}[y^{(n)}(t),...,y(t),u^{(n-1)}(t),...,u(t),t]=\sum_{j=1}^{n} b_j Q_u^{(j+1)}(t_{0,N}) \quad (48)$$

where

$$G_l^{(n+1)}[y^{(n)}(t),...,y(t),u^{(n-1)}(t),u(t),t] = \int\int_{t_0}^{t_N}...\int g_l[y^{(n)}(\tau_1),...,y(\tau_1),u^{(n-1)}(\tau_1),u(\tau_1),\tau_1]d\tau_1...d\tau_{n+1}$$

When samples of the input and output are obtained, estimates of $u(t)$ and $y(t)$ can be obtained using the proposed orthogonal approximation algorithm. A set of equations similar to those of eqn.(40) can therefore be set up. Since the model is linear-in-the-parameters, the extended least squares algorithm can then be applied to estimate the unknown parameters.

For example, consider a dynamical system which is governed by the nonlinear differential equation

$$a_0\frac{dy(t)}{dt} + y(t) + \gamma_1 y^2(t) = b_1 u(t) \quad (49)$$

Integrating twice and rearranging gives the integral equation

$$Y^{(2)}(t_{0,N}) = - a_0 Q_y^{(1)}(t_{0,N}) - \gamma_1 \int\int_{t_0}^{t_N} y^2(\tau_1)d\tau_1 d\tau_2 + b_1 Q_u^{(2)}(t_{0,N}) \quad (50)$$

Assuming the system input and output can be approximated by the orthogonal estimates

$$u(t) \approx \sum_{i=0}^{M} f_{u_i} w_i(t) = \sum_{i=0}^{M} \beta_{u_i} t^i$$
$$y(t) \approx \sum_{i=0}^{M} f_{y_i} w_i(t) = \sum_{i=0}^{M} \beta_{y_i} t^i$$

and $t_0=0$, then the unknown variables in eqn.(50) can be replaced by their corresponding orthogonal approximations as

$$\hat{Y}^{(2)}(t_{0,N}) = \sum_{i=0}^{M} f_{y_i} \Im_i^2(t_{0,N})$$

$$\hat{Q}_y^{(1)}(t_{0,N}) = \sum_{i=0}^{M} f_{y_i} \Im_i(t_{0,N}) - \beta_{y_1} t_N$$

$$\hat{Q}_u^{(2)}(t_{0,N}) = \sum_{i=0}^{M} f_{u_i} \Im_i^2(t_{0,N})$$

$$\iint_{t_0}^{t_N} \hat{y}^2(\tau_1) d\tau_1 d\tau_2 = \iint_{t_0}^{t_N} \left(\sum_{i=0}^{M} \beta_{y_i} \tau_1^i \right)^2 d\tau_1 d\tau_2$$

$$= \iint_{t_0}^{t_N} \left(\beta_{y_0}^2 + (2\beta_{y_0}\beta_{y_1})\tau_1 + (2\beta_{y_0}\beta_{y_2} + \beta_{y_1}^2)\tau_1^2 + \ldots \right) d\tau_1 d\tau_2$$

Additional approximations can be obtained when more data records are available and a set of simultaneous equations similar to those of eqn.(40) can be set up. Since the system is linear-in-the-parameters, the unknown coefficients can be estimated using the extended least squares estimation algorithm.

For stochastic nonlinear systems, procedures similar to those described in the previous section can be applied. The filtering operation can no longer be applied because of the nonlinear terms.

6. Continuous time model estimation algorithm

The continuous time model estimation algorithm can be summarised as follows:

1) Define the fixed sampling time ΔT, the orthogonal time base $[t_0, t_N]$ and the number of data points $N+1$ involved in the orthogonal approximation.
2) Define the structure of the system such as linear or nonlinear, order of dynamics n, order of nonlinearity L and all possible candidate terms.
3) Evaluate α_{ri}, $w_i(t_j)$, $\sum_{j=0}^{N} w_i^2(t_j)$ and $\Im_i^j(t_{0,N})$ according to eqns.(6) and (16).
4) Filter the input and output signals if necessary.
5) Calculate f_{y_i}, f_{u_i}, β_{y_i} and β_{u_i} according to eqns.(7) and (8).

6) Construct $\hat{Y}^{(l)}(t_{k,N+k})$, $\hat{U}^{(l)}(t_{k,N+k})$, $\hat{Q}_y^{(l)}(t_{k,N+k})$, $\hat{Q}_u^{(l)}(t_{k,N+k})$, and $\hat{Q}_z^{(l)}(t_{k,N+k})$ according to eqns.(36) and (39).

7) Construct the noise sequence as

$$\xi(t_{k,N+k}) = \zeta(t_{k,N+k}) - d_1\xi(t_{k-1,N+k-1}) - \dots - d_n\xi(t_{k-n,N+k-n})$$

8) Apply the extended least squares estimation algorithm of eqn.(41) for the identification of the system parameters.

7. Illustrative examples

The operation and the effectiveness of the proposed algorithm are best illustrated by examples. These include the identification of continuous time linear and nonlinear systems and the identification of a continuous time stochastic linear system.

7.1 Linear systems

A linear system S_1 given by

$$0.25y^{(2)}(t) + 0.7y^{(1)}(t) + y(t) = 1.25u(t) \tag{51}$$

was excited by an input random noise of bandwidth 1.5Hz. 290 data records were collected at a sampling period of 0.04s for the analysis. Figure 3 shows the collected input and output data records. Segments of 41 data records with the orthogonal time base $t_s \in [-0.8, 0.8]$ were chosen to perform the orthogonal approximation and a threshold value on the sum of error reduction ratios was set to 99.999%. A total of 250 values for $\hat{Y}^{(3)}(.)$, $\hat{Q}_y^{(1)}(.)$, $\hat{Q}_y^{(2)}(.)$, $\hat{Q}_u^{(1)}(.)$ and $\hat{Q}_u^{(2)}(.)$ were created for the identification of the system dynamics, these are illustrated in Figure 4. When the extended least squares estimation algorithm of eqn.(41) was applied to the 5 sets of data shown in Fig. 4 using a second order dynamical model, the model

$$0.250204y^{(2)}(t) + 0.690539y^{(1)}(t) + y(t) = -0.005534u^{(1)}(t) + 1.24873u(t) + \zeta(t) \tag{51}$$
$$\zeta(t) = \textit{modelling error}$$

was obtained. A comparison with eqn.(51) shows that this is very close to the original system S_1.

The identification of a stochastic linear system will be illustrated using system S_2 defined by

$$0.25y^{(2)}(t) + 0.7y^{(1)}(t) + y(t) = 1.25u(t) \tag{52}$$
$$z(t) = y(t) + e(t)$$

where $e(t)$ is a zero mean white noise of variance 0.0001. Again 290 data records were collected at a sampling time of 0.04s and the collected data records are shown in Fig. 5. The initial threshold value for approximation was set to 99% and the same orthogonal approximation procedures were applied. Figure 6 shows the five sets of estimated states. When the extended least squares algorithm was applied to the five estimated states and a second order dynamical model was specified, the model

$$0.224726z^{(2)}(t) + 0.640633z^{(1)}(t) + z(t) = -0.022604u^{(1)}(t) + 1.14972u(t) + \zeta(t) \quad (53)$$

was obtained. Comparing eqn.(53) with the original system equation (52) shows that a fair estimate has been obtained. However, this estimate can be refined further if the input and output signals are pre-filtered before the approximation and identification process. The filter was chosen to be the transfer function of the initial fitted model

$$F(s) = \frac{-0.022604s + 1.14972}{0.224726s^2 + 0.640633s + 1} \quad (54)$$

such that the cut-off frequency would not be too high or too low. Also signals falling outside the passband of the system will be rejected while signals falling within the passband of the system will be retained for the analysis. Filtering reduced the high frequency noise dramatically as shown in Fig. 7. The threshold value was set to 99.99% for the approximation of the filtered input and output signals and the five estimated filtered states are shown in Fig. 8. The estimated second order model based on the five estimated filtered states was given by

$$0.251894z^{(2)}(t) + 0.690737z^{(1)}(t) + z(t) = -0.0177982u^{(1)}(t) + 1.24818u(t) + \zeta(t) \quad (55)$$

Comparing eqn.(55) with eqn.(52) shows that the estimated model has improved considerably.

Figure 9 shows 290 data records collected from system S_3 obtained by increasing the variance of the added noise to 0.0025 (roughly equal to increasing the noise by 14dB). The initial threshold value for approximation was set to 90% because of the high noise level. Figure 10 shows the five sets of initial estimated states. The fitted second order model to these five states was

$$0.076461z^{(2)}(t) + 0.369982z^{(1)}(t) + z(t) = -0.072566u^{(1)}(t) + 0.712858u(t) + \zeta(t) \quad (56)$$

a rather poor estimate. The fitted model can be improved by filtering. Table 6 shows the fitted model parameters after each filtering, approximation and identification stages. Figure 11 shows the filtered input and output signals after three filtering stages and the corresponding estimated states are shown in Fig. 12. The final estimated model was given as

$$0.233684z^{(2)}(t) + 0.685152z^{(1)}(t) + z(t) = -0.065332u^{(1)}(t) + 1.24663u(t) + \zeta(t) \quad (57)$$

Comparing eqn.(57) with eqns.(56) and (52), there is a significant improvement in the estimates when the filtering operation was implemented.

The above three examples clearly demonstrate the effectiveness of the proposed algorithm in the identification of continuous time systems and the improvements that can be obtained with the application of filtering operations.

7.2 Nonlinear system

A nonlinear system S_4 described by the nonlinear differential equation

$$0.2y^{(1)}(t) + y(t) + 0.16y(t)y(t) = u(t) \tag{58}$$

was excited by an input random noise of bandwidth 1.5Hz. Figure 13 shows the 290 collected data records sampled at a period of 0.04s. A total of 41 data records with the orthogonal time base $t_s \in [-0.8, 0.8]$ were chosen to perform the orthogonal approximation and a threshold value of 99.999% was specified for the approximation. The five sets of 250 estimated states are shown in Fig. 14. When a first order model with a nonlinear term $y^2(t)$ was specified for the identification, the model

$$0.20003y^{(1)}(t) + y(t) + 0.160075y(t)y(t) = 1.00007u(t) + \zeta(t) \tag{59}$$

was obtained. Clearly the fitted model of eqn.(59) matches with the original system eqn. (58). This result illustrates that the proposed algorithm can be applied to the identification of both linear and nonlinear systems.

8. Conclusions

The orthogonal approximation of a continuous time function using the three term recurrence formula has been studied and shown to produce good approximations. Both derivative and integral approximations have been investigated and as expected integral approximations have been shown to be more resistant to noise. A new recursive approximation algorithm has been devised and shown to improve the speed of operation when the number of data records is large and the order of the orthogonal series is low. Incorporating the orthogonal approximations into the integral formulation for linear dynamical systems resolves the problem of initial conditions. The application of the extended least squares algorithm then provides a direct way of identifying the parameters of continuous time dynamical systems based on sampled data records. The use of filters on the raw input and output signals further improves the estimates because errors in approximating the derivatives are reduced. Extension of the algorithms to deterministic nonlinear systems produced good results but the application to stochastic nonlinear systems requires further investigation because the filtering operations will modify the system equations and cannot be applied.

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	model A $t_s \in [-5,5]$	ϵRR	model B $t \in [0,10]$	ϵRR
f_0	0.79202	68.5113	0.79202	68.5113
f_1	-.91950E-2	0.0846	-.91950E-2	0.0846
f_2	0.01593	1.8509	0.01593	1.8509
f_3	0.01879	17.8803	0.01879	17.8803
f_4	-.52792E-2	9.5183	-.52792E-2	9.5183
f_5	-.89468E-3	1.7949	-.89468E-3	1.7949
f_6	0.15283E-3	0.3337	0.15283E-3	0.3337
f_7	0.16428E-6	0.0237	0.16428E-6	0.0237
f_8	-.20228E-7	0.0021	-.20228E-7	0.0021
	Total	99.9999	Total	99.9999

Table 1 Estimated orthogonal parameters and corresponding error reduction ratios for the trigonometric function eqn.(17)

	f(t)	$f_A(t)$	% error	$f_B(t)$	% error
$f(5)$	0.2407	0.2409	0.100	0.2410	0.125
$f^{(1)}(5)$	-.5017	-.4997	0.395	-.4997	0.405
$f^{(2)}(5)$	0.3946	0.3935	0.264	0.3936	0.256
$\int_0^{10} f(t)dt$	8.6052	8.0654	0.002	8.0674	0.027
$\int_0^{10}\int f(t)dt^2$	42.7195	42.7253	0.014	42.7288	0.022

Table 2 Percentage errors for derivative and integral approximations for the trigonometric function eqn.(17)

	fitted model	eRR
f_0	0.80011	68.4830
f_1	-.01156	0.1311
f_2	0.01835	2.4050
f_3	0.01899	17.8780
f_4	-.00483	7.8239
f_5	-.00109	2.6288
	Total	99.3498

Table 3 Estimated orthogonal parameters and corresponding error reduction ratios for the stochastic trigonometric function eqn.(21)

	f(t)	$f_A(t)$	% error
$f(5)$	0.2407	0.3241	34.659
$f^{(1)}(5)$	-.5017	-.5150	2.645
$f^{(2)}(5)$	0.3946	0.2630	33.354
$\int_0^{10} f(t)dt$	8.6052	8.1528	1.086
$\int_0^{10}\int f(t)dt^2$	42.7195	43.1457	0.998

Table 4 Percentage errors for derivative and integral approximations for the stochastic trigonometric function eqn.(21)

M \ N	20	50	100	500
4	0.30	0.12	0.06	0.012
9	0.55	0.22	0.11	0.022
14	0.80	0.32	0.16	0.032
19	1.05	0.42	0.21	0.042

Table 5 Ratios of the speed of the recursive approximation algorithm to direct evaluation (N = data length, M = order of orthogonal series)

	Stages of filtering			
	0	1	2	3
a_0	0.076461	0.185461	0.220175	0.233684
a_1	0.369982	0.642019	0.684993	0.685152
b_1	-.072566	-.093439	-.080437	-.065332
b_2	0.712858	1.146010	1.246130	1.246630

Table 6 Traces of the estimated parameters for system S_3

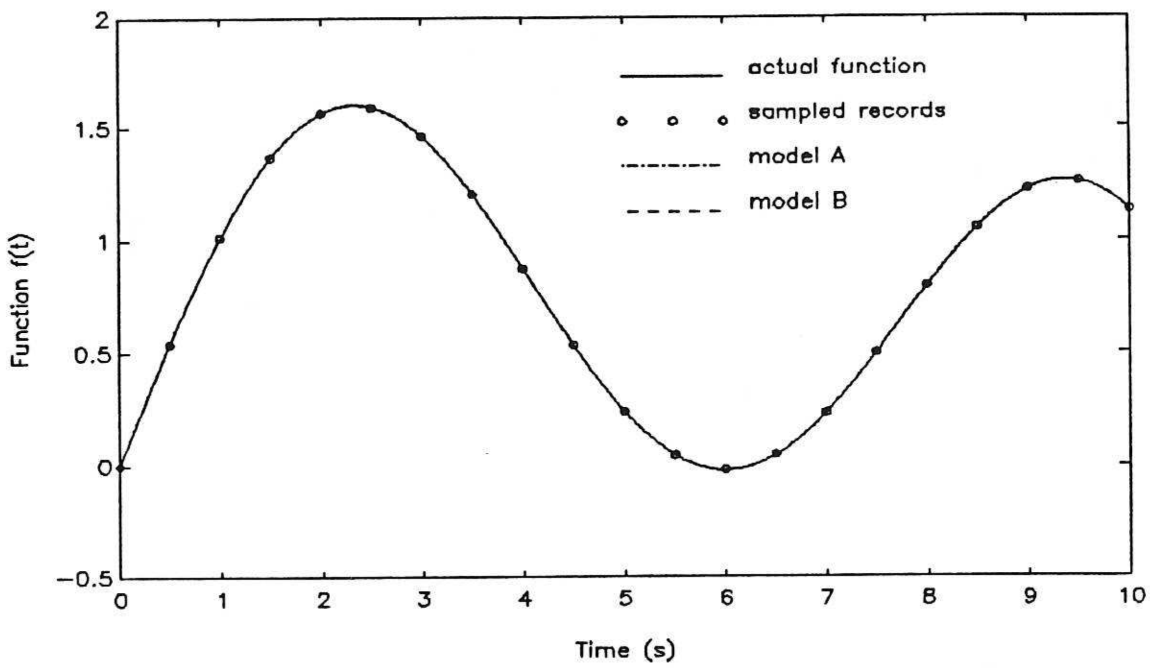

Figure 1 Orthogonal approximations superimposed on the actual function for the trigonometric function eqn.(17)

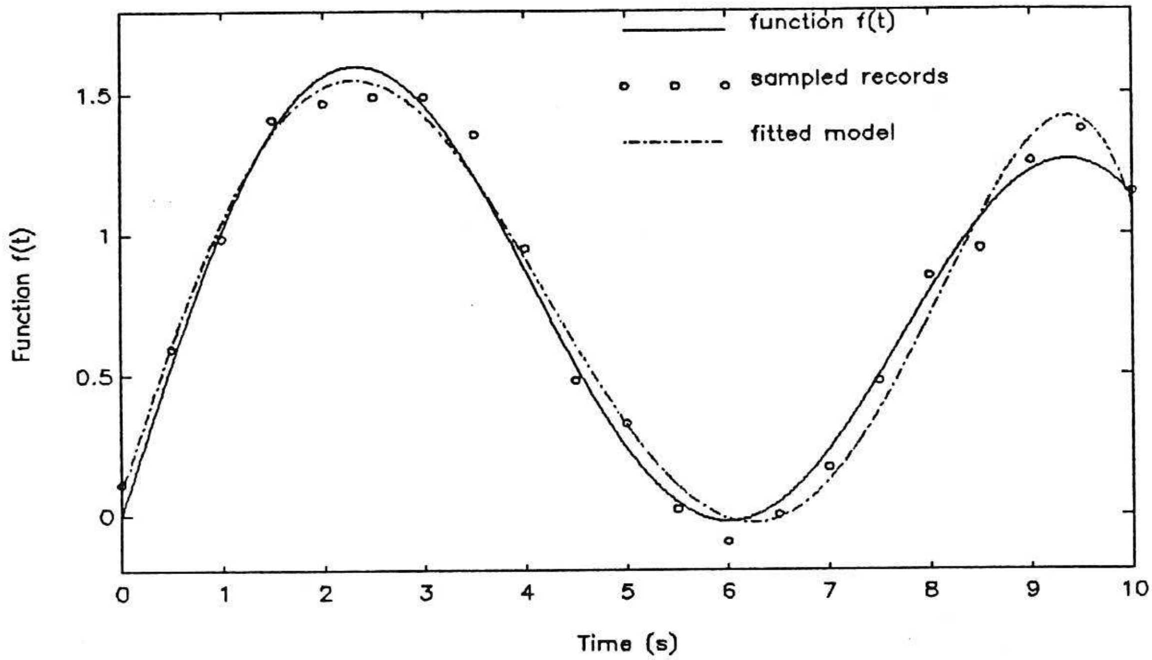

Figure 2 Orthogonal approximation superimposed on the actual function for the stochastic trigonometric function eqn.(21)

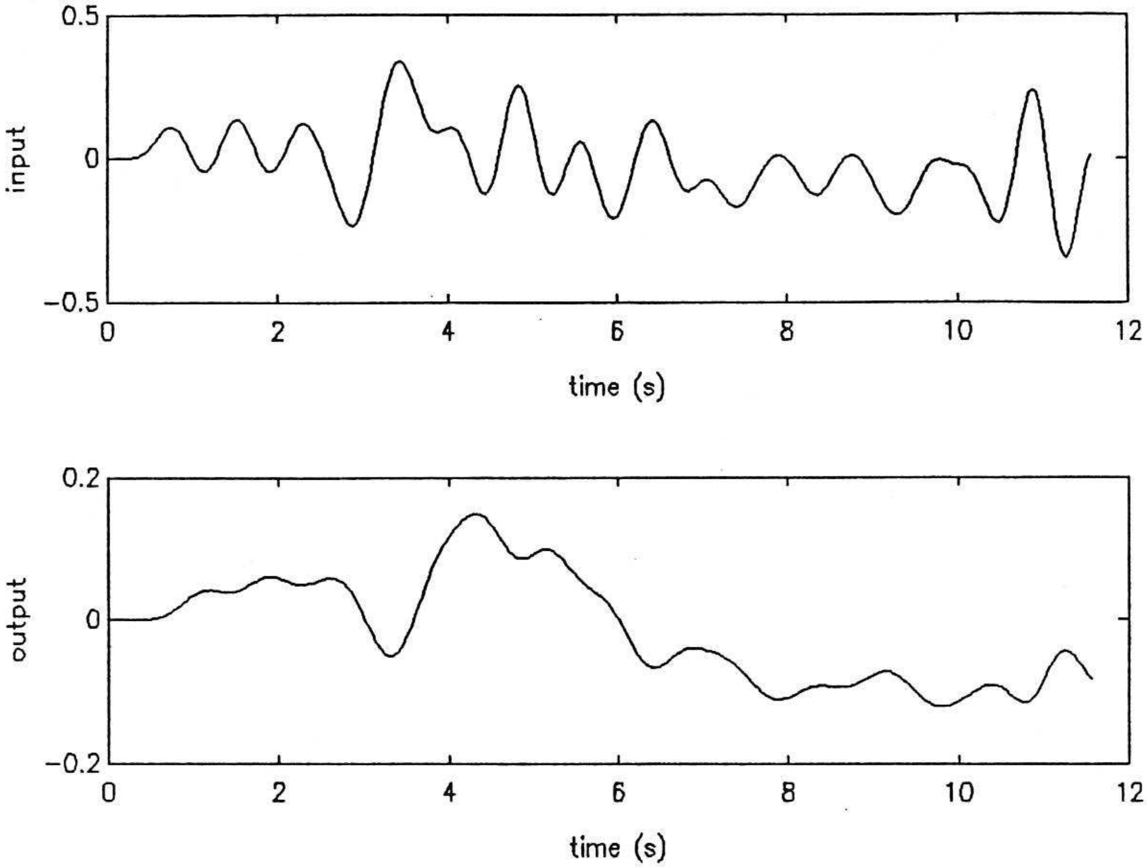

Figure 3 Input and output for system S_1

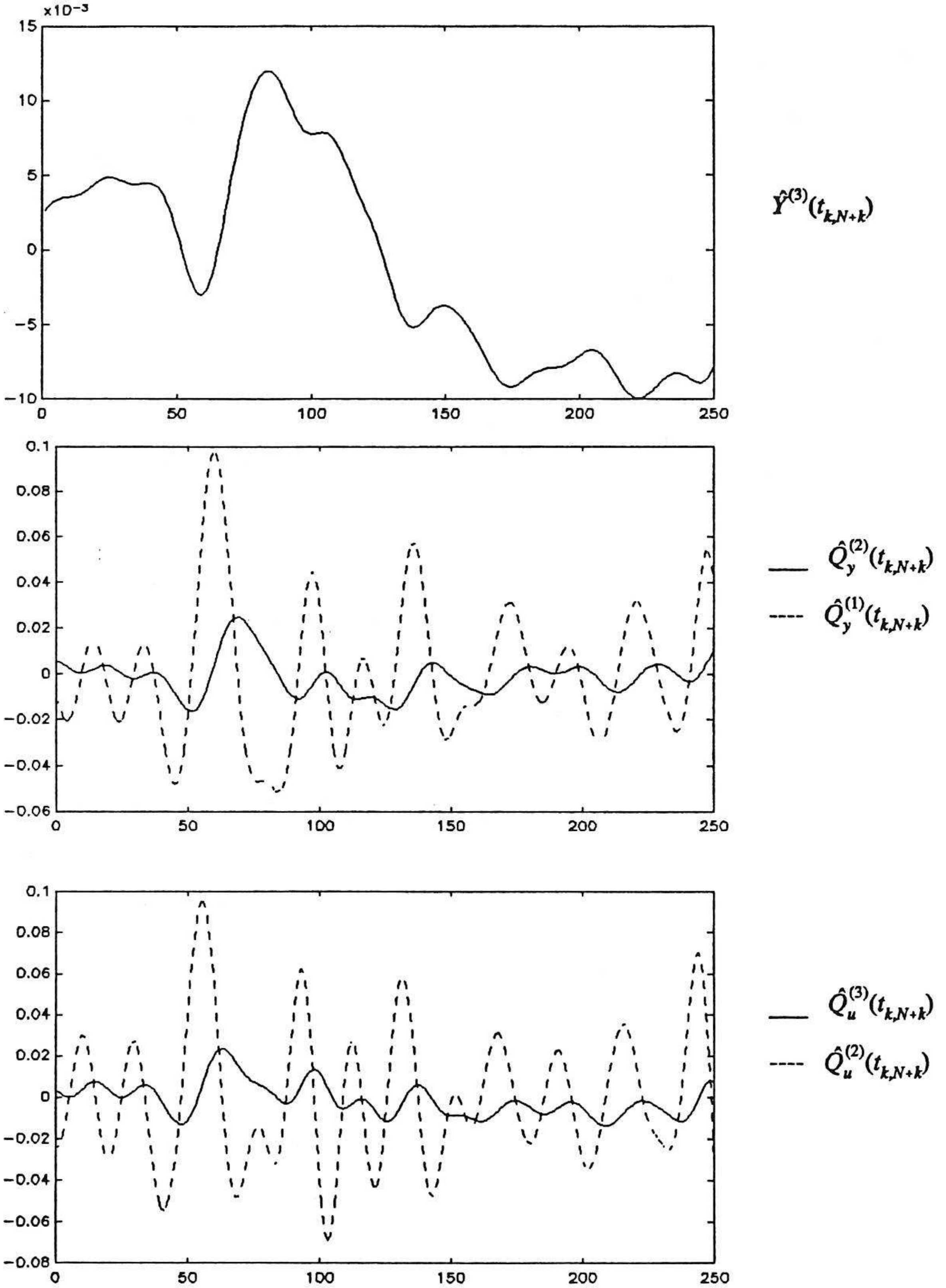

Figure 4 Estimated system states for system S_1

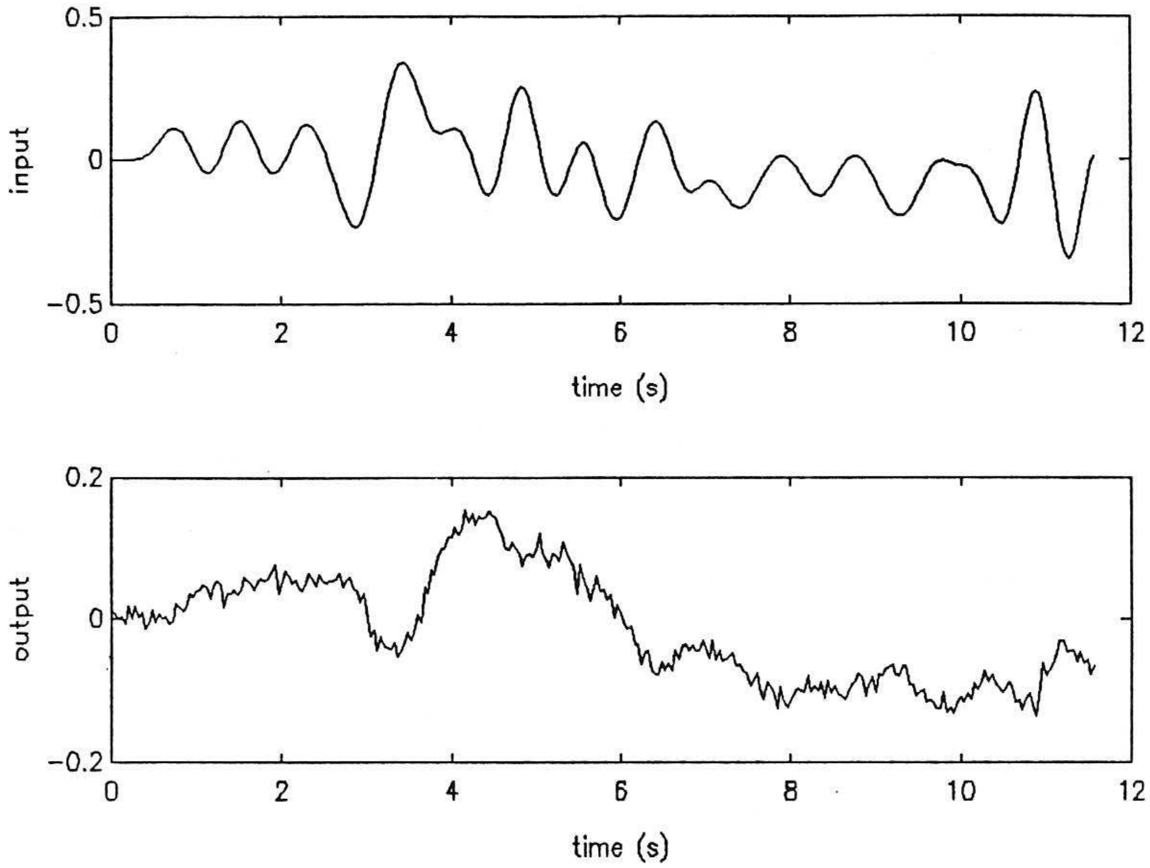

Figure 5 Input and output for system S_2

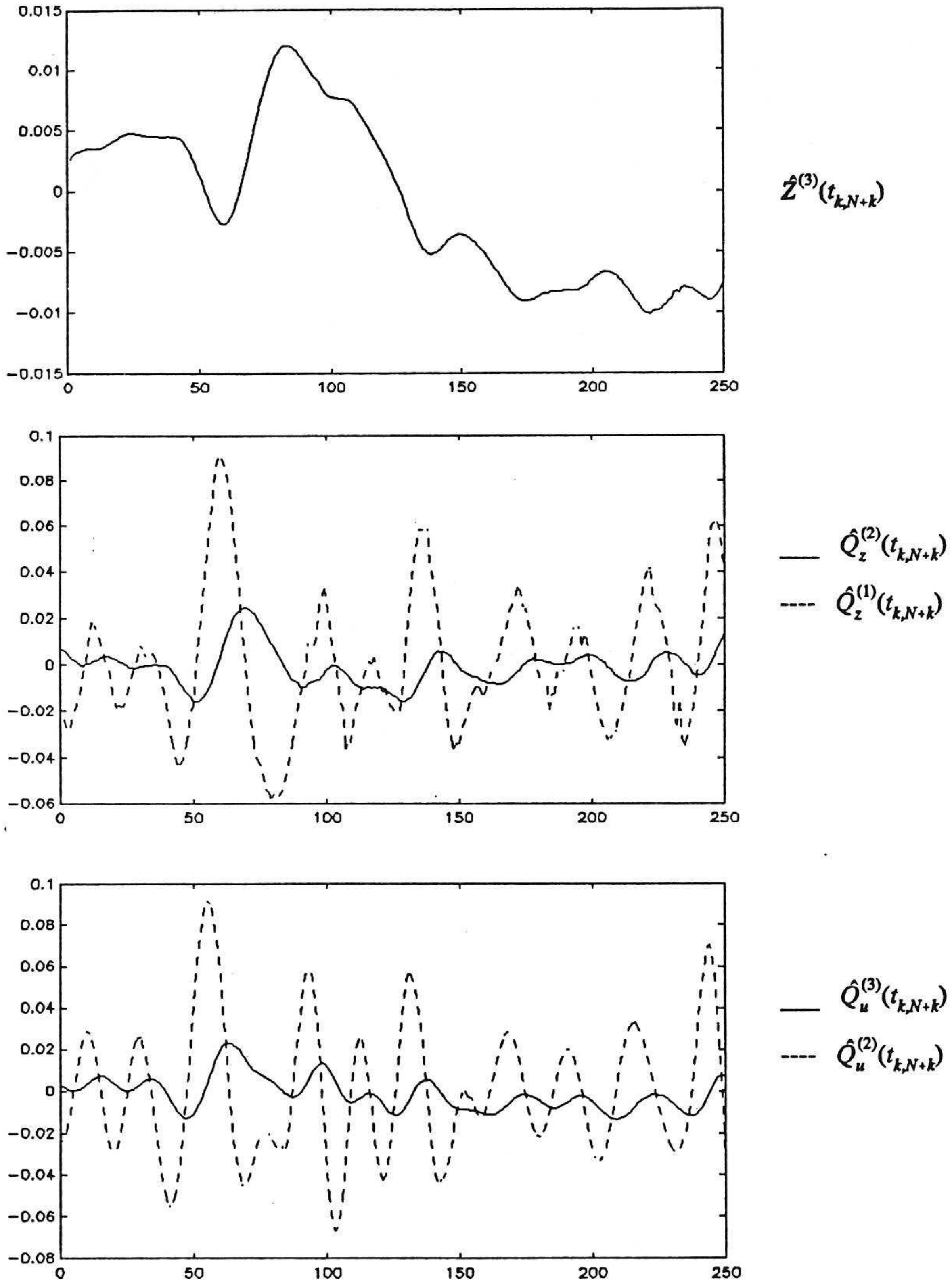

Figure 6 Initial estimated system states for system S_2

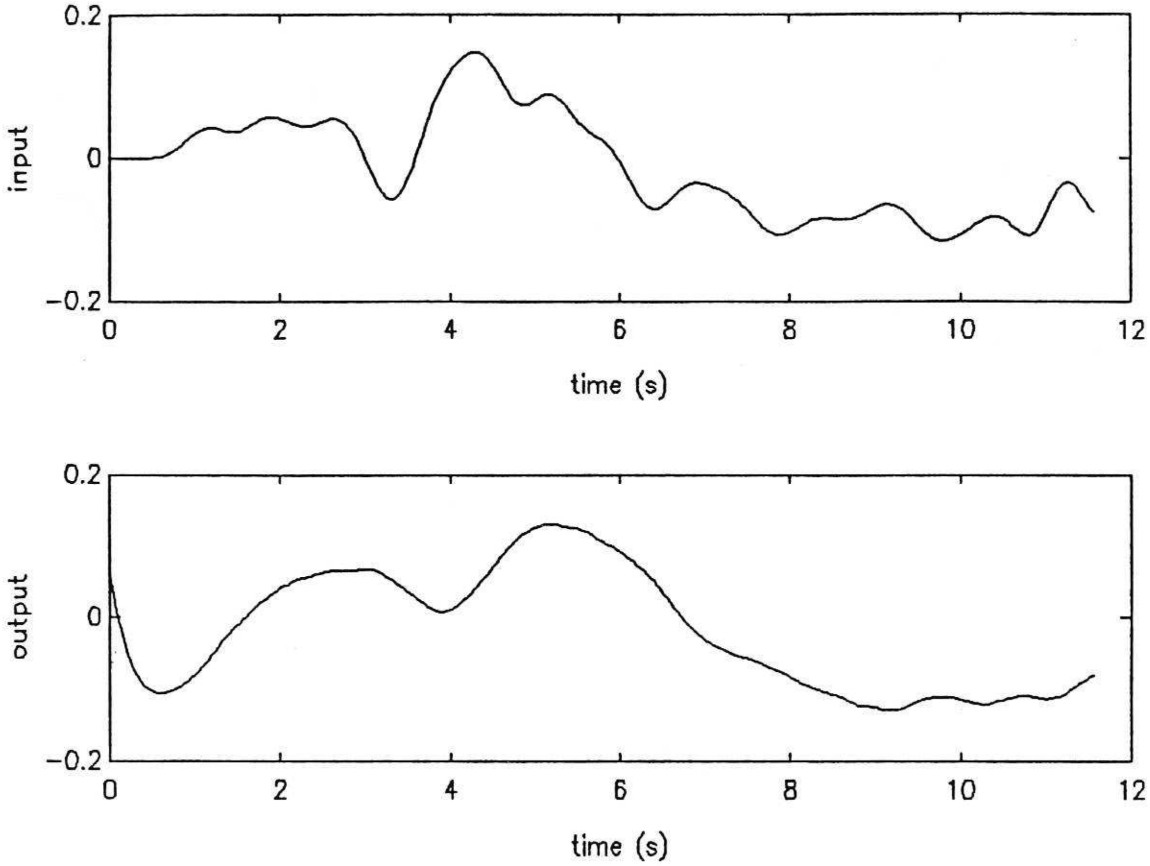

Figure 7 Filtered input and output for system S_2

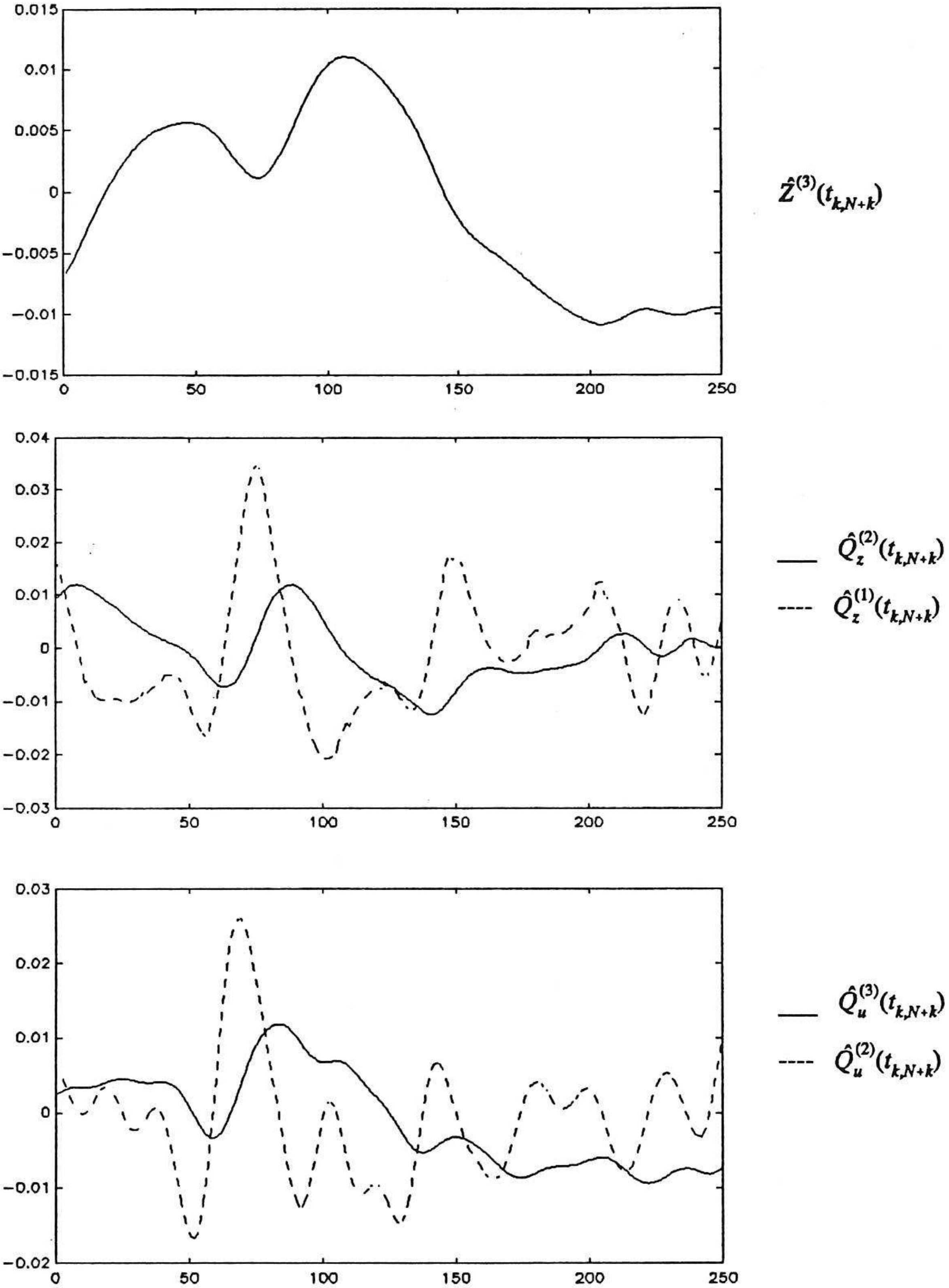

Figure 8 Estimated filtered system states for system S_2

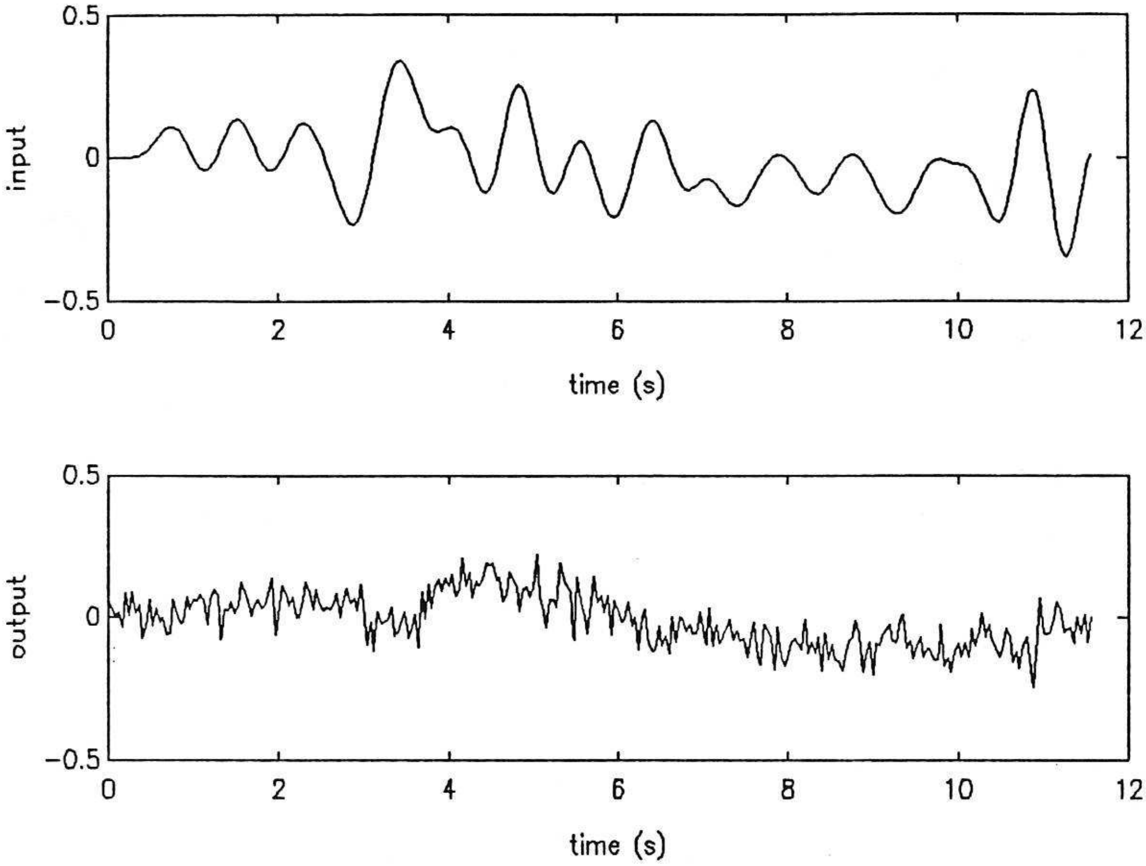

Figure 9 Input and output for system S_3

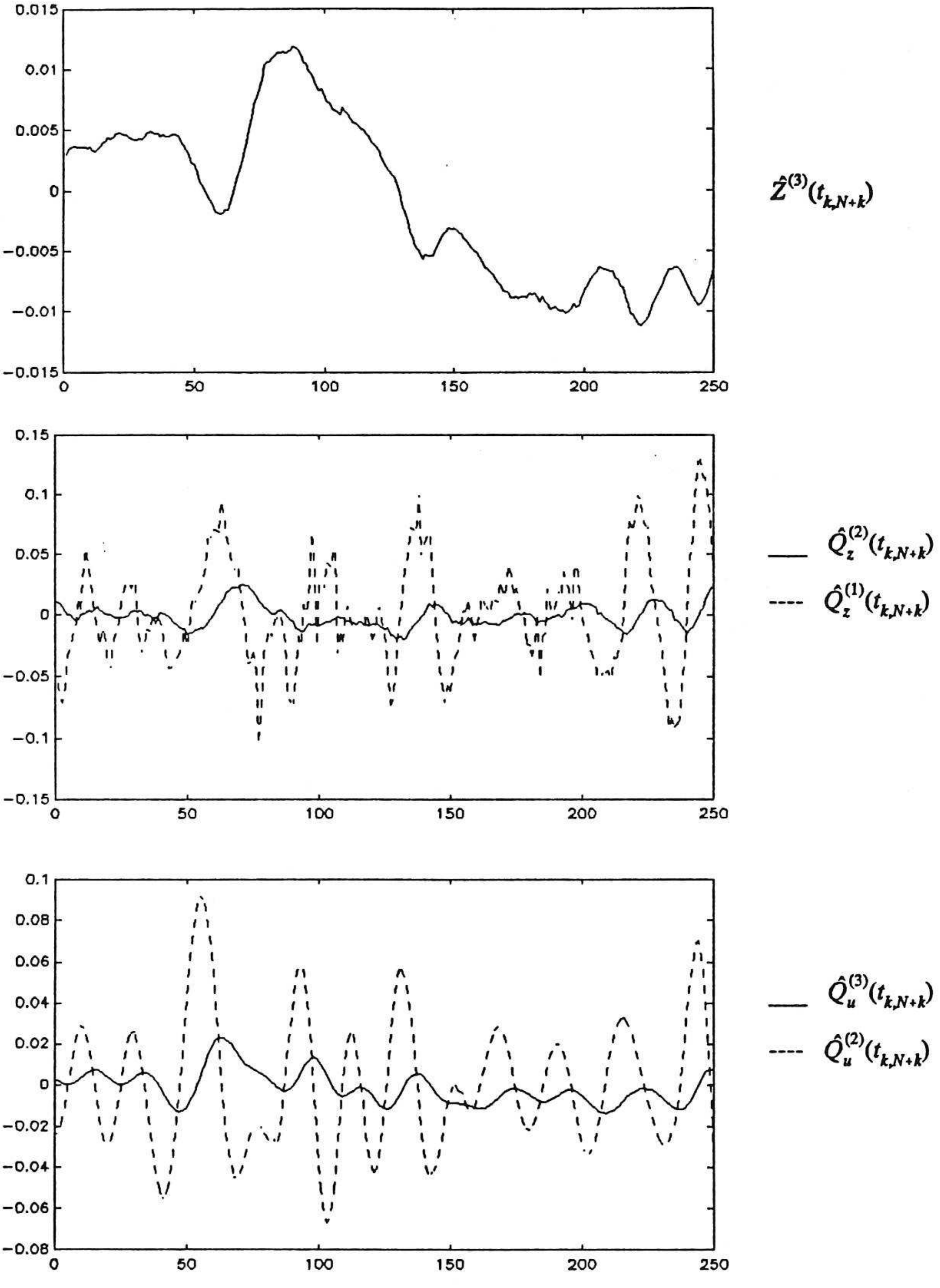

Figure 10 Estimated system states for system S_3

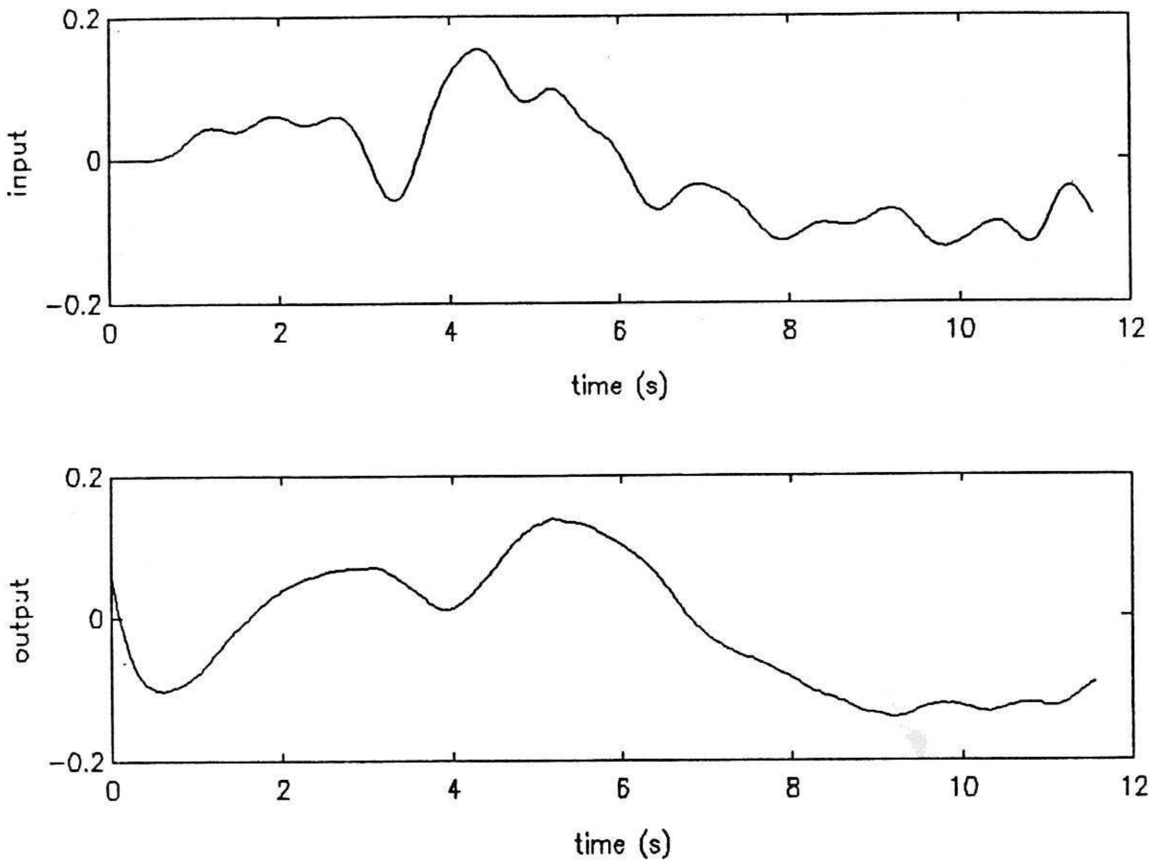

Figure 11 Filtered input and output for system S_3

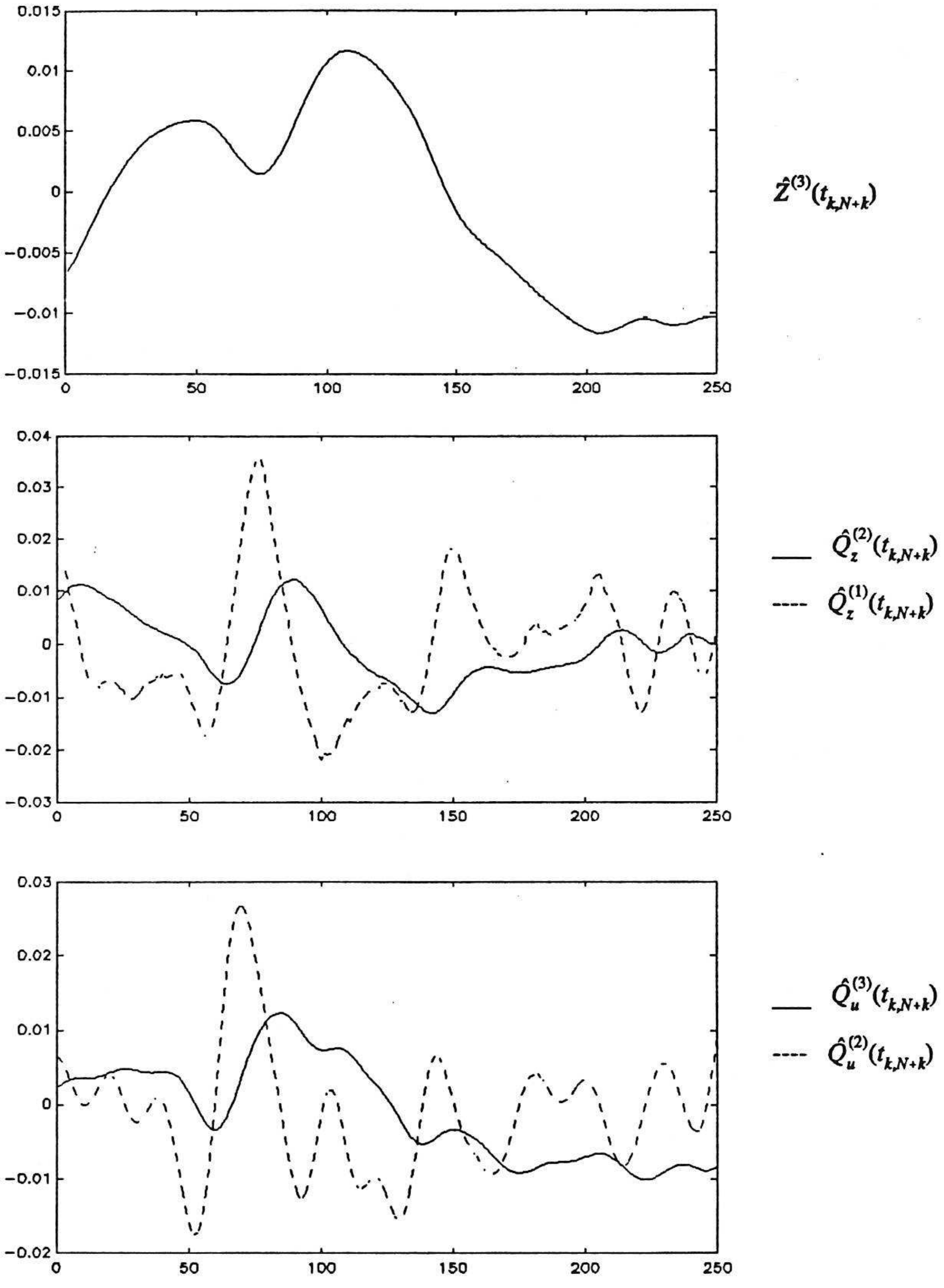

Figure 12 Estimated filtered system states for system S_3

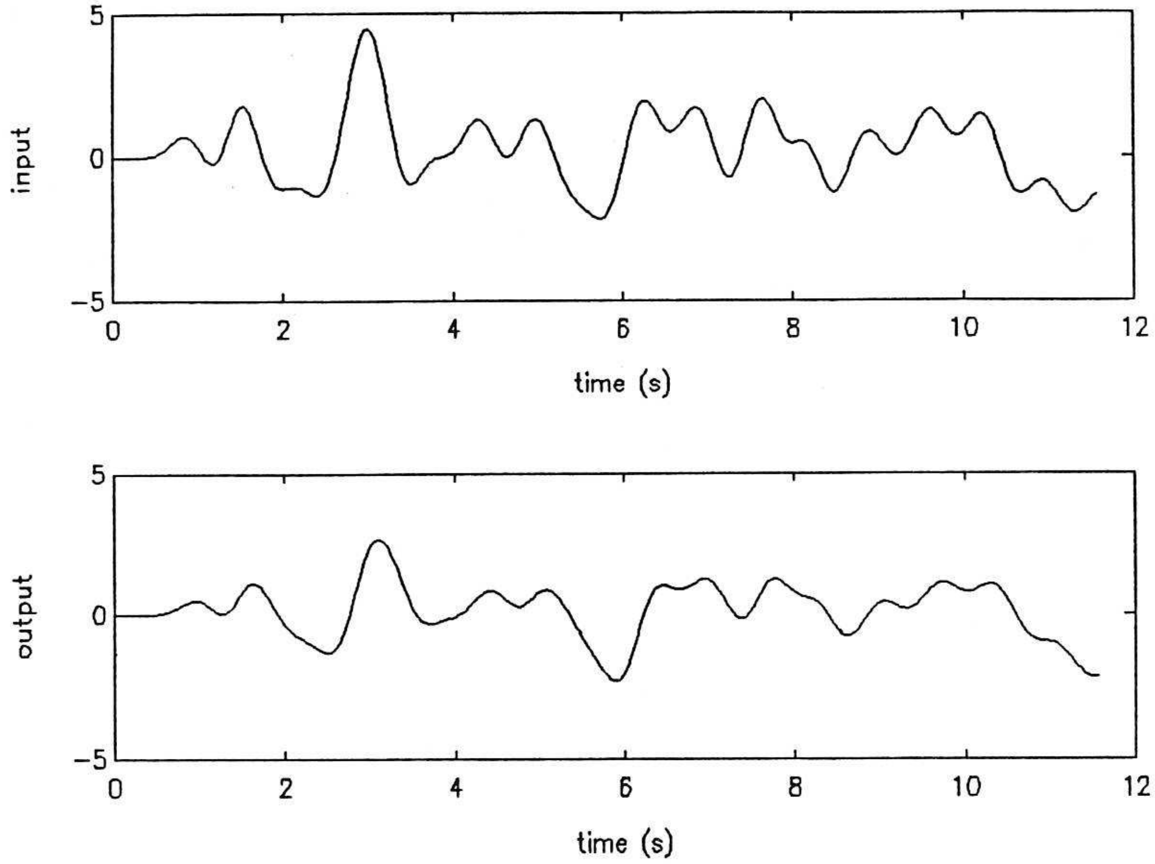

Figure 13 Input and output for system S_4

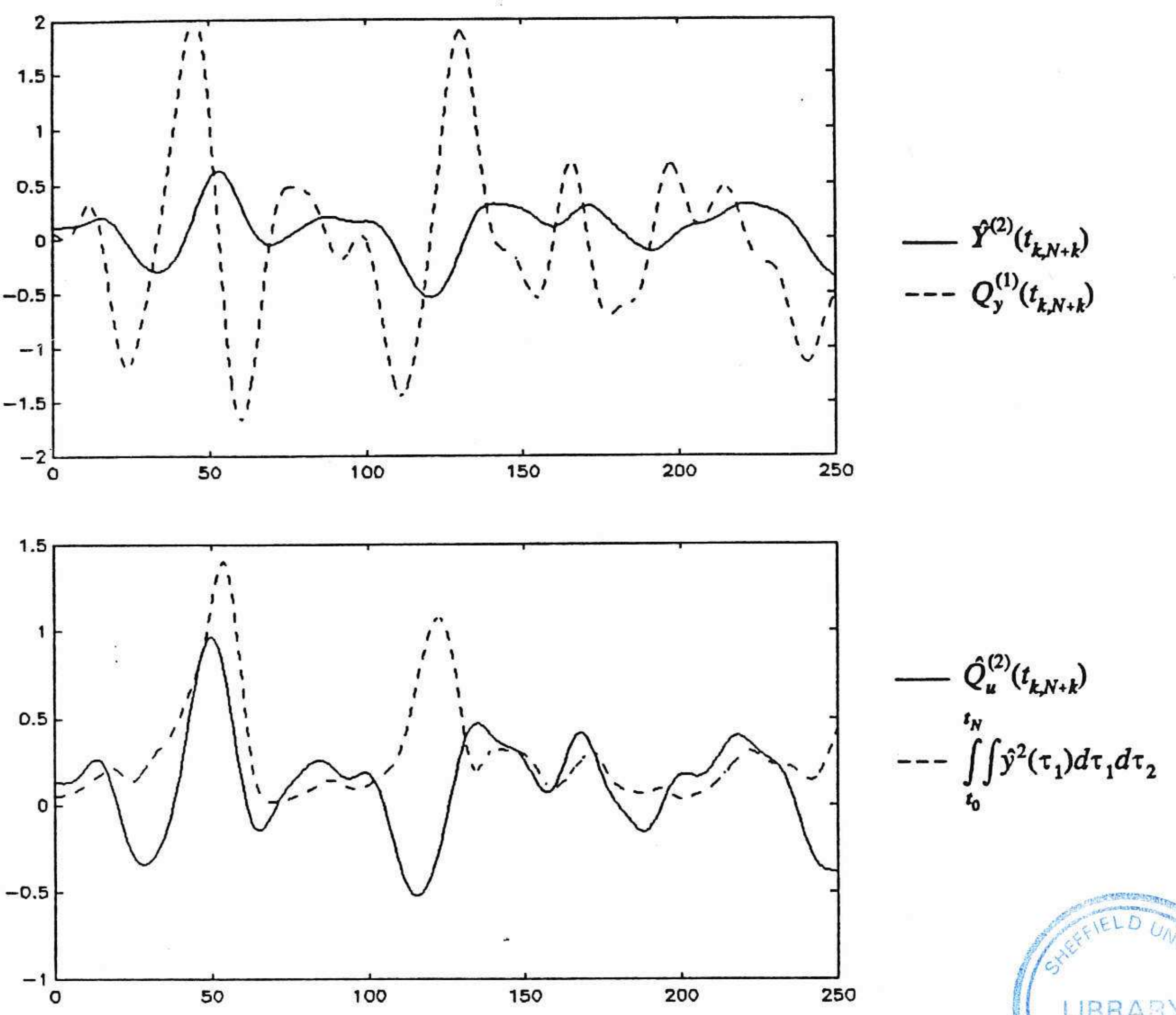

Figure 14 Estimated system states for system S_4