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The Fefferman-Stein decomposition for the Constantin-Lax-Majda equation: regularity criteria for inviscid fluid dynamics revisited

In honour of Professor Peter Constantin's 60th birthday

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Abstract

The celebrated Beale-Kato-Majda (BKM) criterion for the 3D Euler equations has been updated by Kozono and Taniuchi (2000) by replacing the supremum with the Bounded Mean Oscillation (BMO) norm. We consider this generalized criterion in an attempt to understand it more intuitively by giving an alternative explanation. For simplicity, we first treat the Constantin-Lax-Majda equation $\dfrac{\partial \omega}{\partial t} = H(\omega)\omega$ for the vorticity ω in one-dimension and identify a mechanism underlying the update of such an estimate. In particular, we obtain an example of a set of dynamical equations for its Fefferman-Stein (FS) decomposition $\omega = \omega_0 + H[\omega_1]$. In its simplest form, it reads $\dfrac{\partial \omega_0}{\partial t} = \omega_0 H[\omega_0] - \omega_1 H[\omega_1]$ and $\dfrac{\partial \omega_1}{\partial t} = \omega_0 H[\omega_1] + \omega_1 H[\omega_0]$. The equation for the second component ω_1, responsible for a possible logarithmic blow-up, is linear and homogeneous; hence it remains zero if it is so initially until a stronger blow-up takes place. This rules out a logarithmic blow-up on its own and underlies the generalized BKM criterion. Numerical results are also presented to illustrate how each component of the FS decomposition evolves in time. Higher dimensional cases are also discussed. Without knowing fully explicit FS decompositions for the 3D Euler equations, we show that the second component of the FS decomposition will not appear if it is zero initially, thereby precluding a logarithmic blow-up.

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I. INTRODUCTION

It is not known whether smooth solutions of the 3D Euler equations persist globally in time or not (see, e.g. [8] for the problem and its relevance to turbulence). We do know a number of conditions under which they remain smooth over a finite time interval $[0, T]$. One of the fundamental result is the so-called Beale-Kato-Majda (BKM) criterion given in terms of a supremum of the vorticity [1]. Later, it has been improved by Kozono and Taniuchi replacing the L^∞ norm with the weaker BMO semi-norm [20, 21]. The proof is based on highly non-trivial applications of *inequalities* obtained in harmonic analysis and is rather lengthy. In particular, the absence of a logarithmic blow-up is an immediate consequence of this update.

The purpose of this paper is to make the update a bit more transparent by giving its alternative interpretation. More specifically, we consider a simpler 1D model vorticity equation of Constantin-Lax-Majda (CLM) [6] as a test problem. We write down the equations for the Fefferman-Stein (FS) decomposition [12] explicitly. On this basis we interpret the same result using *equalities*. The same method is applied to higher-dimensional equations such as the 3D Euler equations.

In Section II, we review the BKM criterion and its extension. In Section III, we briefly review the CLM model equation. In Section IV, we study the FS decomposition for the CLM model. In Section V, we present numerical results on the FS decomposition for the CLM model. In Section VI, we discuss the 3D Euler equations and SQG equations. Section VII is a summary and an outlook.

II. THE BKM ANALYSIS AND ITS EXTENSION

The Beale-Kato-Majda criterion states the condition of the existence of classical solutions in terms of higher Sobolev norms [1]. In $\mathbb{R}^3$, a non-dimensional time integral of the form

$$\int_0^T \|\boldsymbol{\omega}\|_{L^\infty} dt < \infty \tag{1}$$

is a condition for the existence of smooth solutions on $[0, T]$. A work by Kozono and Taniuchi [20] has updated this to

$$\int_0^T \|\boldsymbol{\omega}\|_{BMO} dt < \infty, \tag{2}$$

using the slightly weaker BMO norm, see also [21]. Basic techniques of proofs used therein may be found in [3, 4, 18, 19]

The BKM criterion is necessary and sufficient for the existence of classical solutions to the 3D Euler equations. Then what is the benefit of the update ? Let us consider a question whether the vorticity can blow up logarithmically in space variables. If we have only the BKM criterion, we cannot answer this question because the L^∞ norm of a logarithmic function is unbounded; such a blow-up may or may not occur. However, with the criterion generalized by BMO, we can safely rule out such a possibility because $\log|\boldsymbol{x}| \in$ BMO, which contradicts with (2).

III. THE CONSTANTIN-LAX-MAJDA MODEL

For simplicity and illustration, we consider the 1D model of vorticity equation, known as the Constantin-Lax-Majda model. The model reads

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \tag{3}$$

with a smooth initial condition

$$\omega(x,0) = \omega_0(x). \tag{4}$$

Here

$$H(\omega) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\omega(y)}{x-y} \mathrm{d}y \tag{5}$$

denotes the Hilbert transform and $⨍$ a principal-value integral. Its useful properties are

$$H[H[f]] = -f \quad \text{(reciprocity)}, \tag{6}$$

and

$$H[fg] = fH[g] + H[f]g + H[H[f]H[g]] \quad \text{(convolution property)} \tag{7}$$

for any functions f and g.

Blow-up of solutions to the model was proved by an ingenious non-constructive argument in an unpublished work[5]. Note that the same argument was used in the work [7].

Later, a method of exact solutions was discovered by complex functions [6]. Introduce a complex-valued function

$$F(x) = \omega(x) + iH[\omega](x) \tag{8}$$

and analytically extend it into the upper-half plane $z = x + iy$ $(y > 0)$. It can be checked that the real and imaginary parts of a complex-valued equation

$$\frac{\partial F(z)}{\partial t} = -\frac{i}{2} F(z)^2 \tag{9}$$

reproduce the CLM equation together with its Hilbert transform. It should be noted that (9) is a *local* ordinary differential equation and exactly solvable. Blowup is then proved for a certain class of initial data. We note that the CLM equation, originally introduced as a model for the vorticity equations, has some physical relevance in plasma physics [2, 33]. See also [26, 28] for related works on the CLM equation.

IV. THE FEFFERMAN-STEIN DECOMPOSITION

A. Fundamentals

It is possible to work out a BKM criterion or its BMO-extension for the CLM model. On the basis of the CLM model we consider why has the update been possible or what is the underlying mechanism which enables such an improvement. The proofs of such an update depends on *inequalities* obtained in harmonic analysis and are rather long. The rationale of this paper is to give an alternative view based on *equalities*.

To this end we recall the Fefferman-Stein decomposition in 1D, which states

$$\forall f \in \text{ BMO} \Longleftrightarrow f = f_0 + H[f_1], \tag{10}$$

for some $f_0, f_1 \in L^\infty$, H is the Hilbert transform. This was obtained by the $\mathcal{H}^\infty$-BMO duality and Hahn-Banach theorem, hence the proof is not constructive. Thus, we have no idea on where f_0, f_1 come from, or how they behave locally. This decomposition is far from unique; assuming that (10) is obtained, we may write it equivalently $f = f_0 - w + H[f_1 + v]$ for any v, w such that $w = H[v]$ holds. See [29–32] for related real analytic methods.

Our strategy is as follows. First we write down the dynamical equations for each component of the FS decomposition for the vorticity ω in the CLM model:

$$\omega = \underbrace{\omega_0}_{\text{bounded}} + \underbrace{H[\omega_1],}_{\text{potentially unbounded}} \tag{11}$$

where $\omega_0, \omega_1 \in L^\infty$. Second, we interpret what we get.

B. The Fefferman-Stein decomposition for the CLM model

Soon after Fefferman-Stein's paper on $\mathcal{H}^\infty$-BMO duality [12], attempts have been made to construct BMO explicitly. This was first successful in one spatial dimension. It is based on the following results see e.g. [35, 36, 38]. For completeness, we present the formulas needed for such a construction, while they are not going to be used explicitly in what follows. Only the idea of use of a complex function g is essential.

The first part is valid in any n dimensions. Setting

$$f(x) = \iint_{\mathbb{R}^2_+} P_t(x-y)d\mu(y,t), \tag{12}$$

we have

$$\|f\|_{\text{BMO}} \le C\|\mu\|_c, \tag{13}$$

where

$$P_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2} \tag{14}$$

denotes the Poisson kernel and $\|\mu\|_c$ the Carleson norm associated with the Carleson measure μ. For any compact supported $f \in \text{BMO}$, there is a Carleson measure μ and $b \in L^\infty(\mathbb{R}^n)$ such that

$$f(x) = \iint_{\mathbb{R}^{n+1}_+} P_t(x-y)d\mu(y,t) + b(x), \tag{15}$$

$$\|\mu\|_c \le C\|f\|_{\text{BMO}}, \tag{16}$$

$$\|b\|_\infty \le C\|f\|_{\text{BMO}} \tag{17}$$

hold. These show a connection of the Carleson measure to BMO.

A formula relating the Carleson measure μ and a complex function g was obtained by Jones [16]. Jones' powerful formula is the following. For $x \in \mathbb{R}^1, \zeta \in \mathbb{R}^2_+$, if we set

$$K(\mu,x,\zeta) = \frac{1}{\pi}\frac{\text{Im}\zeta}{(x-\zeta)(x-\bar\eta)}\exp\left[\iint_{0<\text{Im}\eta\le\text{Im}\zeta}\left(\frac{-i}{\zeta-\bar\eta}+\frac{i}{x-\bar\zeta}\right)\frac{d|\mu|(\eta)}{\|\mu\|_C}\right], \tag{18}$$

$$g(x) = \iint_{\zeta\in\mathbb{R}^2_+} K(\mu,x,\zeta)d\mu(\zeta), \tag{19}$$

then we have

$$\|g\|_\infty \le C\|\mu\|_C \tag{20}$$

and

$$\iint_{\mathbb{R}^2_+} P_t(x-y)d\mu(y,t) = \text{Re}\, g(x) + H(\text{Im}\, g)(x), \text{ modulo const.} \tag{21}$$

By combining the above results we can construct BMO explicitly. (It should be noted that unlike the function F above, the function g is not harmonic in half-plane; its real and imaginary part are independent.) It says that in handling a BMO-valued function, we may handle a complex function g instead, through a suitable Carleson measure.

Thus we need to evolve

$$g = \omega_0 + i\omega_1 \tag{22}$$

somehow to represent the dynamics of the CLM equation. As a simplest choice, we try

$$\frac{\partial g}{\partial t} = gH[g], \tag{23}$$

which is nothing but evolving g itself by the CLM equation. In fact, we can prove that this is a correct choice.

By taking the real and imaginary parts of (23), we find it is equivalent to the following set of equations

$$\begin{cases} \dfrac{\partial \omega_0}{\partial t} = \omega_0 H[\omega_0] - \omega_1 H[\omega_1], \\ \dfrac{\partial \omega_1}{\partial t} = \omega_0 H[\omega_1] + \omega_1 H[\omega_0]. \end{cases} \tag{24}$$

This is the set of equations we are after. We prove that (24) is a FS decomposition of the CLM equation.

Proof

Applying the Hilbert transform on $(24)_2$, we have

$$\begin{aligned} \frac{\partial H[\omega_1]}{\partial t} &= H\left[\omega_0 H[\omega_1] + \omega_1 H[\omega_0]\right] \\ &= H[\omega_0]H[\omega_1] - \omega_0\omega_1 \end{aligned}$$

by the convolution property. Taking the time derivative of the FS decomposition $\omega = \omega_0 + H[\omega_1]$, we have by $(24)_1$

$$\begin{aligned} \frac{\partial \omega}{\partial t} &= \frac{\partial \omega_0}{\partial t} + \frac{\partial H[\omega_1]}{\partial t} \\ &= \omega_0 H[\omega_0] - \omega_1 H[\omega_1] + H[\omega_0]H[\omega_1] - \omega_0\omega_1 \\ &= (\omega_0 + H[\omega_1])\,(H[\omega_0] - \omega_1) \\ &= (\omega_0 + H[\omega_1])\,H\left[\omega_0 + H[\omega_1]\right] \\ &= \omega H[\omega], \end{aligned}$$

where the penultimate line follows by the reciprocity property. Thus (24) reproduces the CLM equation correctly. $\square$

Now that we have found an expression for the FS decomposition of the CLM equation, we can generate infinitely many others. Consider the time derivative of $\omega = \omega_0 + H[\omega_1]$ we have

$$\begin{aligned}\frac{\partial \omega}{\partial t} &= \frac{\partial \omega_0}{\partial t} + H\left[\frac{\partial \omega_1}{\partial t}\right] \\ &= \frac{\partial \omega_0}{\partial t} - w + H\left[\frac{\partial \omega_1}{\partial t} + v\right],\end{aligned}$$

for any w, v such that $w = H[v]$. Here, we have made use of non-uniqueness of the FS decomposition for the time derivative $\dfrac{\partial \omega}{\partial t}$. This should be distinguished from similar non-uniqueness for ω itself used in numerical computations below.

We may consider the following more general decomposition

$$\begin{cases} \dfrac{\partial \omega_0}{\partial t} = \omega_0 H[\omega_0] - \omega_1 H[\omega_1] + w(\omega_0, \omega_1), \\ \dfrac{\partial \omega_1}{\partial t} = \omega_0 H[\omega_1] + \omega_1 H[\omega_0] + H[w(\omega_0, \omega_1)]. \end{cases} \tag{25}$$

Needless to mention, it is applicable even if ω_0, ω_1 are smooth, although in practice it is not efficient to solve the CLM equation numerically using two variables rather than one.

We make some observations in the simplest case $w = 0$ (24).

1. The first equation reduces to the original CLM equation if $\omega_1 \equiv 0$, as it should.

2. More importantly, the second one is *linear* in ω_1 and homogeneous. Hence we conclude that if $\omega_1 = 0$ at $t = 0$, it will continue to be so unless ω_0 goes singular. In other words, the second component can never drive a formation of singularity on its own. Thus, a logarithmic blow-up cannot take place without a breakdown of the first component.

3. The equation (24) becomes invalid when ω_0 becomes unbounded, or when $H[\omega_1]$ develops a logarithmic singularity (we know that the latter does not happen by the exact solution).

It is known [13, 34] that if $\omega \in BMO$, then $H[\omega] \in BMO$. But generally $\omega H[\omega] \notin BMO$ and notably

$$\operatorname{sgn} x \cdot \log|x| \notin BMO. \tag{26}$$

Hence, the right-hand-side of (24) is no longer in BMO once ω takes a genuine BMO value.

Now we are in a position to answer our original question: "Why has the update of the BKM criterion by the BMO norm been made possible ?". As far as the CLM model is concerned, it is because the equation for ω_1 has a simple structure of a *linear homogeneous* equation.

It is straightforward to obtain a FS decomposition for the 1D inviscid Burgers equation (omitted). We will see below that the same mechanism actually works in physically more relevant higher-dimensional cases.

V. NUMERICAL RESULTS

It is of interest to illustrate how each component of the FS decomposition (24) behaves for the CLM model. We consider the CLM equation on a periodic interval $[0, 2\pi]$ and a simple initial condition

$$\omega(x, t = 0) = \cos x. \tag{27}$$

Note that under periodic boundary conditions the Hilbert transform takes the form

$$H(\omega) = \frac{1}{2\pi} \fint_0^\pi \omega(y) \cot\left(\frac{x-y}{2}\right) \mathrm{d}y.$$

The exact solution for this initial condition was already given in the original paper [6].

Numerical computations are performed by 2/3-dealiased Fourier spectral method with time-stepping by the fourth-order Runge-Kutta scheme. The number of grid points is $N = 4096$ and a time step is $\Delta t = 1 \times 10^{-3}$.

By making use of the non-uniqueness of the FS decomposition on the initial data $\omega = \omega_0 + H[\omega_1]$, we can consider more general initial conditions of the form

$$\omega_0(x, 0) = \alpha \cos x - (1 - \alpha) H[\sin x], \tag{28}$$

where α is a parameter; $0 \leq \alpha \leq 1$. (Recall that $H[\cos x] = \sin x,\ H[\sin x] = -\cos x$.) We make the following choices for the numerical computations of (24).

Case 1: a natural choice ($\alpha = 1$)

$$\omega_0(x, 0) = \cos x, \omega_1(x, 0) = 0 \tag{29}$$

Case 2: an awkward choice ($\alpha = 1/2$)

$$\omega_0(x,0) = \frac{1}{2}\cos x, \omega_1(x,0) = -\frac{1}{2}\sin x \tag{30}$$

Case 3: an extreme choice ($\alpha = 0$)

$$\omega_0(x,0) = 0, \omega_1(x,0) = -\sin x \tag{31}$$

We start with the usual choice 1). We show the time evolution of the vorticity ω and the strain rate $H[\omega]$ in Fig.1. The solution is known to break down at $t = 2$. At that time, ω behaves like $\propto \frac{1}{x-\pi/2}$ and $H[\omega]$ like the Dirac delta function at $x = \pi/2$.

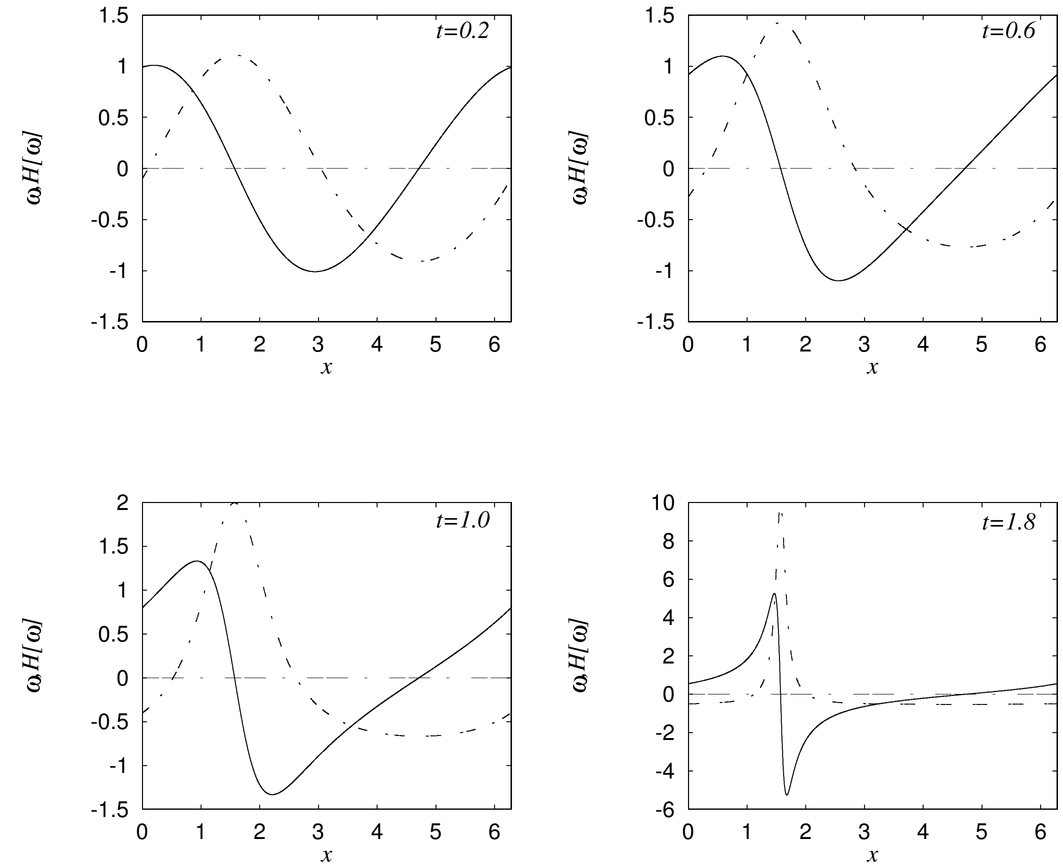

FIG. 1: The vorticity ω(solid) and the strain rate $H[\omega]$ (dashed) for Case 1.

For the choice 2) we plot in Fig.2 the time evolution of the enstrophy (that is , squared L^2-norm of the vorticity) for each component defined by

$$Q_0(t) = \frac{1}{2}\left\langle \omega_0(x,t)^2 \right\rangle \text{ and } Q_1(t) = \frac{1}{2}\left\langle \omega_1(x,t)^2 \right\rangle,$$

where the brackets denote a spatial average. We observe a clear equi-partition $Q_0(t) = Q_1(t)$ for all time, whose sum is of course equal to the total enstrophy

$$Q(t) = \frac{1}{2}\left\langle \omega(x,t)^2 \right\rangle .$$

We show the time evolution of ω_0 and $H[\omega_0]$ in Fig.3. Actually, they coincide completely throughout the evolution.

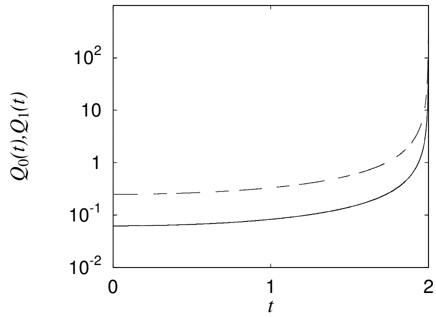

FIG. 2: The evolution of the enstrophy $Q_0(t)$(solid) and $Q_1(t)$(dashed) and their sum (dotted) for Case 2. Note that $Q_0(t)$ and $Q_1(t)$ are indistinguishable as they coincide.

Finally we consider the case 3). We plot the evolution of enstrophy in Fig.4. We observe that $Q_0(t)$, starting from 0, catches up with $Q_1(t)$ towards the end of the computation and they collapse in the final stage, again showing an equi-partition in L^2.

The time evolution of ω_0 and $H[\omega_1]$ is shown Fig.5, which is of some interest. The vorticity ω_0 starts to grow generating oscillations with a half period of the initial ω_1. We observe a perfect cancellation of the strain and the vorticity in the vicinity of $x = \frac{3}{2}\pi$.

We summarize the features of the numerical solutions as follows:

1. We can confirm that if $\omega_1(0) = 0$, then $\omega_1(t) = 0$ for any $t > 0$ up to a blowup, consistent with the mathematical fact. (not shown in Fig.1) This means that the dynamics of ω_1 does not suffer from serious instability.

2. If $\omega_1(0) \neq 0$ both $\omega_0(t)$ and $\omega_1(t)$ blow up at the same time, showing an equipartition in the L^2 norm, both in Cases 2 and 3. In particular, a complete equipartition in Case 2 is striking.

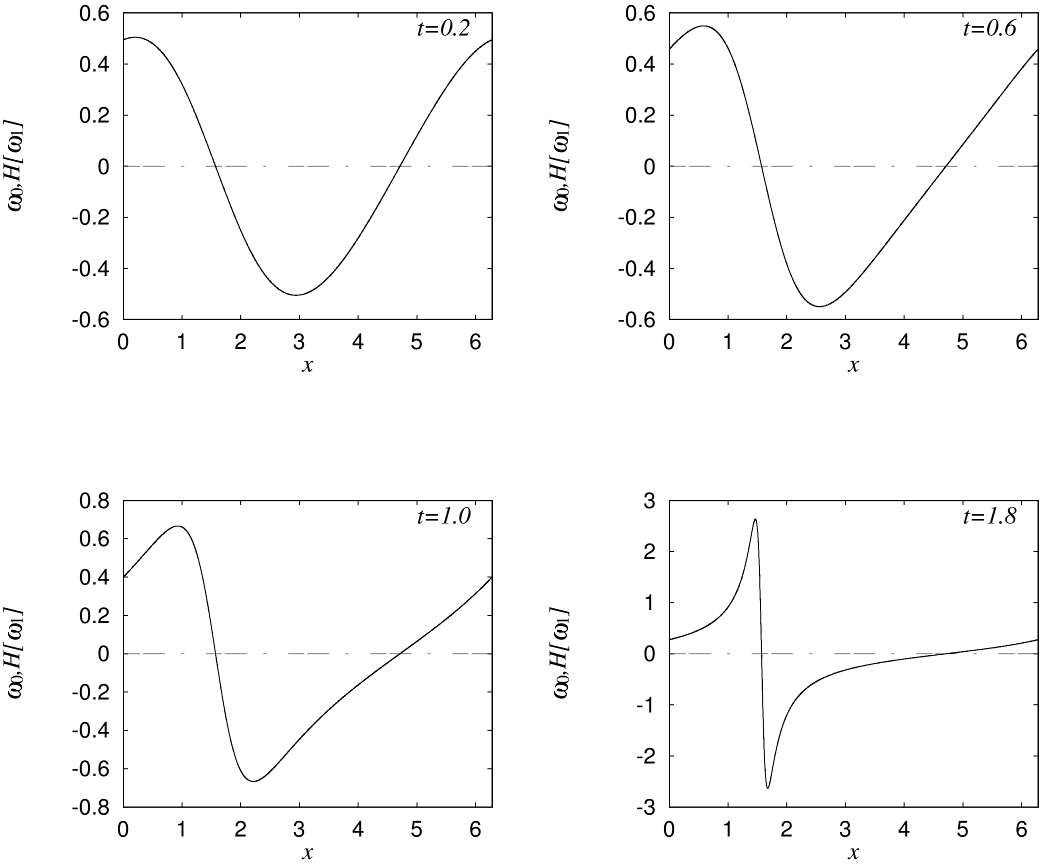

FIG. 3: The vorticity ω_0(solid) and $H[\omega_1]$ (dashed) for Case 2. They are indistinguishable because they coincide. Note that the amplitudes are scaled down by a factor of 2 in comparison with Fig.1. Note that $\omega = \omega_0 + H[\omega_1]$.

3. In particular, even if $\omega_0(0) = 0$, $\omega_0(t) \neq 0$ at later times. Because unlike the second component ω_1, the equation for the first component is nonlinear and inhomogeneous, it can emerge from $\omega_1(0) = 0$.

VI. HIGHER DIMENSIONAL CASES

A. Fundamentals

We have identified a mechanism underlying the update of the BKM criterion by the BMO norm for the CLM model. As mentioned above, the BKM criterion for the 3D Euler equations has been generalized by the BMO norm in [20, 21]. In two dimensions, there is a

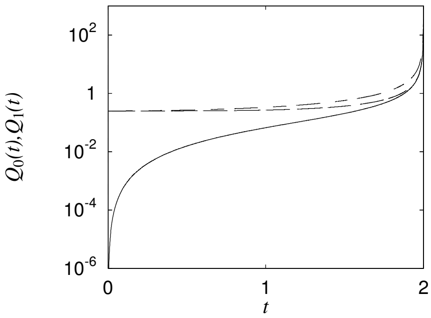

FIG. 4: The evolution of the enstrophy $Q_0(t)$(solid) and $Q_1(t)$(dashed) and their sum (dotted) for Case 3.

similar result on the SQG equation [17]. We expect that a similar mechanism underlies these generalized estimates. One way to confirm this view is to seek the dynamical equations for the FS decomposition of the 3D Euler equations or the SQG equation.

In n dimensions, we have formally

$$BMO(\mathbb{R}^n) = L^\infty(\mathbb{R}^n) + \sum_{j=1}^{n} R_j[L^\infty(\mathbb{R}^n)] \tag{32}$$

as the dual of

$$\|f\|_{\mathcal{H}^\infty(\mathbb{R}^\backslash)} = \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|R_j[f]\|_{L^1(\mathbb{R}^n)}, \tag{33}$$

where $\mathcal{H}^\infty$ denotes Hardy space, R_j the j-th component of Riesz transform. The FS decomposition states that for $f \in \mathrm{BMO}(\mathbb{R}^{\mathrm{n}})$, we can find $f_0, f_1, \ldots, f_n$ such that

$$f = f_0 + \sum_{j=1}^{n} R_j[f_j] \tag{34}$$

and its norm is given by

$$\|f\|_{BMO} \approx \|f_0\|_\infty + \sum_{j=1}^{n} \|f_j\|_\infty. \tag{35}$$

A constructive proof of the FS decomposition was given by [37] in any spatial dimensions, but it entirely rests on the real analytic methods. For this reason, that result cannot be applied directly to obtain FS decompositions e.g. for the 3D Euler equations. It seems difficult to determine the FS decompositions for higher-dimensional equations explicitly.

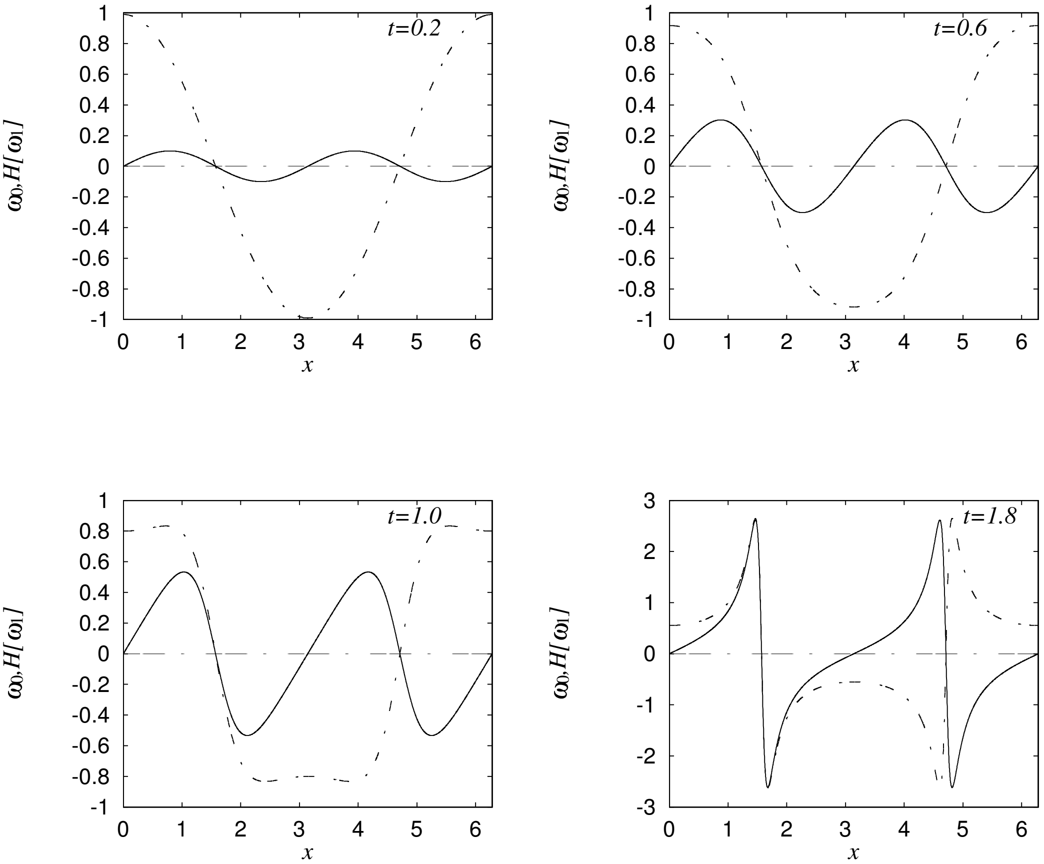

FIG. 5: The vorticity ω_0(solid) and $H[\omega_1]$ (dashed) for Case 3. Note that $\omega = \omega_0 + H[\omega_1]$.

B. The 3D Euler equations

Before considering the 3D problem, we reconsider the FS decomposition for the CLM model by setting

$$\omega = \omega_0 + \omega_*, \quad \text{with} \quad \omega_* \equiv H[\omega_1] \tag{36}$$

to write

$$\begin{cases} \dfrac{\partial \omega_0}{\partial t} = \omega_0 H[\omega_0] + \omega_* H[\omega_*] + w(\omega_0, \omega_*), \\ \\ \dfrac{\partial \omega_*}{\partial t} = H[\omega_0]\omega_* + \omega_0 H[\omega_*] - w(\omega_0, \omega_*). \end{cases} \tag{37}$$

The function $w(\omega_0, \omega_*)$, quadratic in ω_0, ω_*, must satisfy the condition that

$$w(\omega_0, \omega_*) = 0, \quad \text{if} \quad \omega_* = 0,$$

because $(37)_1$ should reduce to the original CLM equation when ω_* is missing. In other words, the quadratic term in ω_0 should be absent from w. Except for that, $w(\omega_0, \omega_*)$ is arbitrary. We know in the case of the CLM equation that the simple choice $w(\omega_0, \omega_*) \equiv 0$ does give one decomposition by direct computations.

In view of this, for the 3D Euler equations in vorticity form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}, \tag{38}$$

where

$$\boldsymbol{u} = -\nabla \times \triangle^{-1}\boldsymbol{\omega}, \quad (\nabla \cdot \boldsymbol{u} = 0), \tag{39}$$

we consider

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \underbrace{\sum_{j=1}^{3} R_j[\boldsymbol{\omega}_j]}_{=\omega_*} \tag{40}$$

by re-grouping of the FS decomposition for $\boldsymbol{\omega}$. Here, care should be taken that the index j, $(j = 0, 1, 2, 3)$ in $\boldsymbol{\omega}_j$ denotes components of the FS decomposition, not those of the vorticity vector. See [22] for a related work.

We consider (40) at each instant of time and its time derivative

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \frac{\partial \boldsymbol{\omega}_0}{\partial t} + \sum_{j=1}^{3} R_j \left[\frac{\partial \boldsymbol{\omega}_j}{\partial t} \right].$$

Because of non-uniqueness of the FS decomposition, it can be written as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \left(\frac{\partial \boldsymbol{\omega}_0}{\partial t} - \boldsymbol{w} \right) + \sum_{j=1}^{3} R_j \left[\frac{\partial \boldsymbol{\omega}_j}{\partial t} + \boldsymbol{w}_j \right]$$

for any $\boldsymbol{w}, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3$ such that

$$\boldsymbol{w} = \sum_{j=1}^{3} R_j[\boldsymbol{w}_j].$$

In terms of $\boldsymbol{\omega}_*$, we have simply

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \left(\frac{\partial \boldsymbol{\omega}_0}{\partial t} - \boldsymbol{w} \right) + \left(\frac{\partial \boldsymbol{\omega}_*}{\partial t} + \boldsymbol{w} \right).$$

We can thus write for the FS decomposition of the 3D Euler equations

$$\begin{cases} \dfrac{\partial \boldsymbol{\omega}_0}{\partial t} + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{\omega}_0 + (\boldsymbol{u}_* \cdot \nabla)\boldsymbol{\omega}_* \\ = (\boldsymbol{\omega}_0 \cdot \nabla)\boldsymbol{u}_0 + (\boldsymbol{\omega}_* \cdot \nabla)\boldsymbol{u}_* + \boldsymbol{w}(\boldsymbol{u}_0, \boldsymbol{\omega}_0; \boldsymbol{u}_*, \boldsymbol{\omega}_*), \\ \\ \dfrac{\partial \boldsymbol{\omega}_*}{\partial t} + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{\omega}_* + (\boldsymbol{u}_* \cdot \nabla)\boldsymbol{\omega}_0 \\ = (\boldsymbol{\omega}_0 \cdot \nabla)\boldsymbol{u}_* + (\boldsymbol{\omega}_* \cdot \nabla)\boldsymbol{u}_0 - \boldsymbol{w}(\boldsymbol{u}_0, \boldsymbol{\omega}_0; \boldsymbol{u}_*, \boldsymbol{\omega}_*), \end{cases} \tag{41}$$

where $\boldsymbol{u}_0 = -\nabla \times \triangle^{-1}\boldsymbol{\omega}_0$, $\boldsymbol{u}_* = -\nabla \times \triangle^{-1}\boldsymbol{\omega}_*$ and $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{u}_* = 0$. The equation (41) says nothing but that $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \boldsymbol{\omega}_*$ solves the 3D Euler equations. Here $\boldsymbol{w}$ denotes an arbitrary quadratic function of $\boldsymbol{u}_0, \boldsymbol{\omega}_0, \boldsymbol{u}_*$ and $\boldsymbol{\omega}_*$. We do not know how to determine its complete functional form, but we do know that

$$\boldsymbol{w}(\boldsymbol{u}_0, \boldsymbol{\omega}_0; 0, 0) = 0,$$

because the equation $(41)_1$ must reduce to the original 3D Euler equations if $\boldsymbol{u}_* = \boldsymbol{\omega}_* = 0$. That is, the quadratic terms in $\boldsymbol{u}_0$ or $\boldsymbol{\omega}_0$ themselves should not appear in $\boldsymbol{w}$. It is crucial to note that this information alone suffices to deduce the mechanism for excluding the possibility of logarithmic blow-up. If $\boldsymbol{u}_* = \boldsymbol{\omega}_* = 0$ at $t = 0$, then $\boldsymbol{u}_* = \boldsymbol{\omega}_* = 0$ at $t > 0$, because of $(41)_2$.

The FS theorem guarantees that we have at least one $\boldsymbol{w}$ that satisfies (41), but we do not know how to determine it. In particular, we do not know if $\boldsymbol{w} \equiv 0$ gives a FS decomposition or not.

C. The SQG equation

We also consider the SQG equations

$$\frac{\partial \theta}{\partial t} + (\boldsymbol{u} \cdot \nabla)\theta = 0, \quad \boldsymbol{u} = -\boldsymbol{R}^{\perp}[\theta], \tag{42}$$

where $\boldsymbol{u}_0 = -\nabla \times \triangle^{-1}\boldsymbol{\omega}_0, \boldsymbol{u}_* = -\nabla \times \triangle^{-1}\boldsymbol{\omega}_*$ and $\nabla \cdot \boldsymbol{u}_* = 0$. $\boldsymbol{R}^{\perp} = -\nabla^{\perp}\Lambda^{-1}$ denotes a skewed Riesz transform defined with $\nabla^{\perp} = (\partial_2, -\partial_1)$ and $\Lambda = (-\triangle)^{1/2}$. See, e.g. [9–11, 23–25] for this equation. We introduce the FS decompositions as

$$\theta = \theta_0 + \sum_{j=1}^{2} R_j[\theta_j] \tag{43}$$

together with

$$\boldsymbol{u} = -\boldsymbol{R}^{\perp}[\theta_0] - \sum_{j=1}^{2} \boldsymbol{R}^{\perp}[R_j[\theta_j]]. \tag{44}$$

We write these as $\theta = \theta_0 + \theta_*$, $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_*$, respectively by re-grouping of the terms. For the time derivative, we have

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta_0}{\partial t} + \sum_{j=1}^{2} R_j\left[\frac{\partial \theta_j}{\partial t}\right] = \left(\frac{\partial \theta_0}{\partial t} - \boldsymbol{w}\right) + \sum_{j=1}^{2} R_j\left[\frac{\partial \theta_j}{\partial t} + \boldsymbol{w}_j\right]$$

for any $\boldsymbol{w}, \boldsymbol{w}_1, \boldsymbol{w}_2$ such that

$$\boldsymbol{w} = \sum_{j=1}^{2} R_j[\boldsymbol{w}_j].$$

In terms of θ_*, we have trivially

$$\frac{\partial \theta}{\partial t} = \left(\frac{\partial \theta_0}{\partial t} - \boldsymbol{w}\right) + \left(\frac{\partial \theta_*}{\partial t} + \boldsymbol{w}\right).$$

We can write the FS decomposition for the SQG equation as

$$\begin{cases} \dfrac{\partial \theta_0}{\partial t} + (\boldsymbol{u}_0 \cdot \nabla)\theta_0 + (\boldsymbol{u}_* \cdot \nabla)\theta_* = \boldsymbol{w}(\theta_0, \boldsymbol{u}_0; \theta_*, \boldsymbol{u}_*), \\ \dfrac{\partial \theta_*}{\partial t} + (\boldsymbol{u}_0 \cdot \nabla)\theta_* + (\boldsymbol{u}_* \cdot \nabla)\theta_0 = -\boldsymbol{w}(\theta_0, \boldsymbol{u}_0; \theta_*, \boldsymbol{u}_*). \end{cases} \tag{45}$$

In principle, $\boldsymbol{w}$ depends on $\theta_0, \boldsymbol{u}_0, \theta_*$ and $\boldsymbol{u}_*$. Again, we have

$$\boldsymbol{w}(\theta_0, \boldsymbol{u}_0; 0, 0) = 0,$$

because $(45)_1$ reduces to the original SQG equation if $\boldsymbol{u}_* = \theta_* = 0$. Because the products of $\boldsymbol{u}_0$, θ_0 should be absent from $\boldsymbol{w}$, if $\boldsymbol{u}_* = \theta_* = 0$ initially, $\dfrac{\partial \theta_*}{\partial t} = 0$ persists, as long as the smooth solution persists.

The FS theorem guarantees that we have at least one $\boldsymbol{w}$ that satisfies (45), but we do not know how to determine it. In particular, we do not know if $\boldsymbol{w} \equiv 0$ gives a FS decomposition or not.

VII. SUMMARY AND OUTLOOK

In this paper two things have been done. i) We have obtained a complete form of the FS decomposition for the CLM equation and on this basis we have shown why the potentially singular component (the second component) in the FS decomposition does not appear before the first component blows up. ii) For the 3D Euler and SQG equations, we have considered similar FS decompositions and deduced the same conclusion by the requirement that the equation for the first component of the FS decomposition must reduce to the original equations in the absence of the second component. A complete determination the forms of the FS decompositions for the 3D Euler and 2D SQG equations is left for future study. In particular, we may ask whether or not $\boldsymbol{w} = 0$ is one choice of decomposition. The latter 2D case may be tractable because two-dimensional BMO is related with Riemann surface [27] and chances are that we can make use of complex function theory.

We recall that the class of BMO is slightly larger than L^∞; actually it is the smallest class of functions which includes L^∞ and which is closed with respect to the Hilbert transform (or Riesz transform in higher dimensions). In this sense, BMO is 'thin' on top of L^∞.

In this connection we recall the John-Nirenberg inequality [15]. It states that for any $f \in BMO$, its growth is at most logarithmic:

$$|\{x \in Q : |f - f_Q| > \lambda\}| \le c_1 |Q| \exp\left(-c_2 \frac{\lambda}{\|f\|_{BMO}}\right), \tag{46}$$

where

$$\|f\|_{BMO} \equiv \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q| dx \text{ with } f_Q \equiv \frac{1}{|Q|}\int_Q f dx \tag{47}$$

is the BMO norm. On the basis of (46), we have the so-called Garnett-Jones distance [14], which measures the distance of a BMO valued function to L^∞. It may be of interest to try the linearization of the dynamics of the second component on the basis of the 'thinness' of the genuine BMO in L^∞.

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