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Burgers equation with a passive scalar: dissipation anomaly and Colombeau calculus

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Abstract

A connection between dissipation anomaly in fluid dynamics and Colombeau's theory of products of distributions is exemplified by considering Burgers equation with a passive scalar. Besides the well-known viscosity-independent dissipation of energy in the steadily propagating shock wave solution, the lesser known case of passive scalar subject to the shock wave is studied. An exact dependence of the dissipation rate ϵ_θ of the passive scalar on the Prandtl number P_r is given by a simple analysis: we show in particular $\epsilon_\theta \propto 1/\sqrt{P_r}$ for large P_r. The passive scalar profile is shown to have a form of a sum of $\tanh^n x$ with suitably scaled x, thereby implying the necessity to distinguish H from H^n when P_r is large, where H is the Heaviside function. An incorrect result of $\epsilon_\theta \propto 1/P_r$ would otherwise be obtained. This is a typical example where Colombeau calculus for products of weak solutions is required for a correct interpretation. A Cole-Hopf-like transform is also given for the case of unit Prandtl number.

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I. INTRODUCTION

One of the most important properties of fully-developed turbulence is that its total kinetic energy is dissipated in a nontrivial fashion even in the limit of vanishing viscosity, see e.g. [12]. This empirical observation is called "dissipation anomaly" and is believed to form the basis for turbulence theory. In the case of the 3D Navier-Stokes equations this is just a conjecture and no mathematical proof is available to support it.

Here we consider a much simpler model of fluid equation to study a similar phenomenon. More precisely we consider the Burgers equation [5, 6] together with a passive scalar:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

$$\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} = \kappa\frac{\partial^2 \theta}{\partial x^2}, \tag{2}$$

where u denotes the velocity field, θ the passive scalar field, ν the kinematic viscosity and κ the diffusivity. As boundary conditions, we consider constant values of velocity and scalar at infinity (see below). We note that Burgers equation with a passive scalar has been considered in [17] in connection with non-Gaussian statistics.

In Section II, we study steadily propagating waves in u and θ and study whether the dissipation rate ϵ_θ of θ is independent of ν or of κ. In Section III we derive an expression ϵ_θ in terms of the Prandtl number $P_r = \nu/\kappa$ and investigate its dependency on P_r. In Section IV, we determine the profile for θ and note that Colombeau calculus for the product of distributions [7–9] is required to interpret the result. We also discuss a generalization of the so-called Cole-Hopf transform [10, 11, 13] to the case of the passive scalar in Section V. Section VI is a Summary.

II. STEADY-STATE SOLUTIONS IN A MOVING FRAME

We consider (1) under the boundary conditions $u(x = \pm\infty) = \mp u_1$. If we seek a solution steady in a frame moving with a constant speed U we find using a change of variables $X = x - Ut,\ T = t$ [1];

$$u = U - u_1 \tanh\frac{u_1}{2\nu}(x - Ut + c), \tag{3}$$

where c is a constant of integration. The dissipation rate of total kinetic energy

$$\epsilon = \nu\int_{-\infty}^{\infty}\left(\frac{\partial u}{\partial x}\right)^2 dx,$$

is given by

$$\epsilon = \frac{u_1^3}{2} \int_{-\infty}^{\infty} \frac{d\xi}{\cosh^4 \xi},$$

where $\xi = \frac{u_1}{2\nu}(x - Ut + c)$. Because this (convergent) integral no longer involves viscosity, we see that ϵ is independent of ν without evaluating the definite integral (actually $= 4/3$).

Now we consider θ. From (2), the steady-state should satisfy

$$(u - U)\frac{d\theta}{dX} = \kappa \frac{d^2\theta}{dX^2}. \tag{4}$$

Using (3), it follows from (4) that

$$\frac{d\theta}{dX} = c' \left[\cosh \frac{u_1}{2\nu}(X + c)\right]^{-2\frac{\nu}{\kappa}}. \tag{5}$$

Hence we find that

$$\theta = c_1 \int_0^{\xi} \frac{d\eta}{\cosh^{2P_r} \eta} + c_2, \tag{6}$$

where $c_1 = 2\nu c'/u_1$ and c_2 are constants of integration. Under the boundary condition $\theta(x = \pm\infty) - U = \mp\theta_1$ we may fix the constants as $c_1 = -\frac{\theta_1}{I_\alpha(\infty)}$ and $c_2 = U$. Here we have introduced for convenience

$$I_\alpha(\xi) \equiv \int_0^{\xi} \frac{d\eta}{\cosh^{2\alpha} \eta}.$$

III. DISSIPATION RATE OF A PASSIVE SCALAR

By (5), the dissipation rate of passive scalar variance ϵ_θ is evaluated as follows:

$$\begin{aligned}
\epsilon_\theta &= \kappa \int_{-\infty}^{\infty} \left(\frac{\partial\theta}{\partial x}\right)^2 dx \\
&= \kappa \tilde{c'}^2 \int_{-\infty}^{\infty} \left[\cosh \frac{u_1}{2\nu}(X + c)\right]^{-4P_r} dX \\
&= \kappa \frac{(u_1\theta_1)^2}{4\nu^2 I_{P_r}(\infty)^2} \frac{2\nu}{u_1} \int_{-\infty}^{\infty} \cosh^{-4P_r}(\xi) d\xi,
\end{aligned}$$

thus we find

$$\epsilon_\theta = u_1\theta_1^2 \frac{1}{P_r} \frac{I_{2P_r}(\infty)}{I_{P_r}(\infty)^2}. \tag{7}$$

Because the integral $I_{P_r}(\infty)$ depends on ν and κ through P_r, we must evaluate it in full.

For integer-numbered P_r, say $= n$ we may explicitly carry out the integration in $I_{P_r}(\xi)$. The first two ($n = 1, 2$) are

$$\int_0^{\xi} \frac{d\eta}{\cosh^2 \eta} = \tanh \xi, \quad \int_0^{\xi} \frac{d\eta}{\cosh^4 \eta} = \tanh \xi - \frac{1}{3} \tanh^3 \xi.$$

More generally, noting that:

$$\frac{1}{\cosh^{2n}\xi} = \frac{1}{\cosh^2\xi}(1-\tanh^2\xi)^{n-1} = \frac{1}{\cosh^2\xi}\sum_{r=0}^{n-1}(-1)^r \begin{pmatrix} n-1 \\ r \end{pmatrix} \tanh^{2r}\xi \tag{8}$$

we find

$$I_n(\xi) = \int_0^\xi \frac{d\eta}{\cosh^{2n}\eta} = \sum_{r=0}^{n-1}(-1)^r \begin{pmatrix} n-1 \\ r \end{pmatrix} \frac{\tanh^{2r+1}\xi}{2r+1}$$

and

$$I_n(\infty) = \sum_{r=0}^{n-1}(-1)^r \begin{pmatrix} n-1 \\ r \end{pmatrix} \frac{1}{2r+1}.$$

Actually we have [14]

$$I_n(\infty) = \frac{2^{2(n-1)}\{(n-1)!\}^2}{(2n-1)!}.$$

For $P_r = n$ we obtain an exact expression

$$\epsilon_\theta = u_1\theta_1^2 \frac{\{(2n)!\}^4}{(4n)!(n!)^4}.$$

(For more general real-valued $P_r = \alpha$, we have $I_\alpha(\infty) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha)}{\Gamma(\frac{1}{2}+\alpha)}$ and thus $\epsilon_\theta = u_1\theta_1^2\frac{2}{\sqrt{\pi}}\frac{\Gamma(2\alpha)\Gamma(\frac{1}{2}+\alpha)^2}{\alpha\Gamma(\alpha)^2\Gamma(\frac{1}{2}+2\alpha)}$, where $\Gamma(\alpha)$ is the gamma function.)

By Stirling's formula $n! \simeq \sqrt{2\pi n} n^n e^{-n}$ for $n \gg 1$, we deduce that

$$I_{n+1}(\infty) \simeq \frac{1}{2}\sqrt{\frac{\pi}{n}}.$$

Therefore the dissipation rate of θ in the limit of large P_r is

$$\epsilon_\theta \simeq u_1\theta_1^2\sqrt{\frac{2}{\pi P_r}}, \quad \text{as} \quad P_r \to \infty, \tag{9}$$

which decays as $P_r^{-\frac{1}{2}}$ with P_r. Even in this simple 1D model, the problem of dissipation anomaly is subtle, in that ϵ_θ does depend on P_r in a nontrivial fashion.

On the other hand, it can be checked that

$$\lim_{P_r\to 0}\frac{1}{P_r}\frac{I_{2P_r}(\infty)}{I_{P_r}(\infty)^2} = 1$$

so

$$\epsilon_\theta \to u_1\theta_1^2, \quad \text{as} \quad P_r \to 0.$$

In the cases $P_r \ll 1$ or $P_r = O(1)$, ϵ_θ remain finite, that is, there is anomaly in the dissipation of the passive scalar in the limit

IV. CONNECTION TO COLOMBEAU CALCULUS

In the case of $\nu \to 0$, care should be taken in the interpretation. Indeed, in the expression

$$\theta(\xi) = -\frac{\theta_1}{I_n(\infty)} \sum_{r=0}^{n-1} (-1)^r \begin{pmatrix} n-1 \\ r \end{pmatrix} \frac{\tanh^{2r+1}\xi}{2r+1} + U, \tag{10}$$

formally $\tanh\xi \to H(\xi)$ as $\nu \to 0$, where H is the Heaviside function, but this does not necessarily mean that $\tanh^n \xi \to H(\xi)$ for $n(\neq 1)$.

Colombeau theory has been developed to account for multiplication of distributions to some extent [7–9], by generalizing Schwartz theory of distributions. For details, see the references cited therein. Later its connection to non-standard analysis has been pointed out [2]. We note that this theory has been applied to the Burgers equation, e.g. [8, 15, 16] but not to the problem with a passive scalar.

A notable feature of Colombeau theory is that it can tell apart H from H^n $(n \neq 1)$. In this sense the problem in question is a typical example to which Colombeau theory applies. If we naively identify $\tanh^{2r+1}\xi$ with $\tanh\xi$ in the limit of vanishing viscosity $\nu \to 0$, we would get $I_n(\xi) \approx I_n(\infty)\tanh\xi$, or

$$\theta(\xi) \approx -\theta_1 \tanh\xi + U.$$

It follows that

$$\epsilon_\theta \approx \frac{1}{P_r}\theta_1 u_1^2 \int_0^\infty \frac{d\xi}{\cosh^4\xi} = \theta_1 u_1^2 \frac{2}{3P_r}, \tag{11}$$

or $\epsilon_\theta \propto 1/P_r$ rather than the correct asymptotic dependence $\epsilon_\theta \propto 1/\sqrt{P_r}$. Therefore the above naive identification leads to a completely wrong dependence on P_r.

In Fig. 1 we plot the dependency of ϵ_θ on P_r as given by (7). (For numerical purposes it is convenient to write $I_\alpha(\infty) = \int_0^1 (1-\tau^2)^{\alpha-1}\, d\tau$.) It shows how quickly ϵ_θ asymptotes to (9) and that how poor a job the naive (11) does.

The above results on dissipation anomaly suggests that Colombeau calculus is required for a correct description of the present problem. In order to check this view we see how jump conditions [8] come out of Colombeau calculus. Below the symbol $\sim$ denotes *association* which is a weaker relationship than equality (=).

Case 1.

We start from

$$u_t + uu_x \sim 0,\ \theta_t + u\theta_x \sim 0,$$

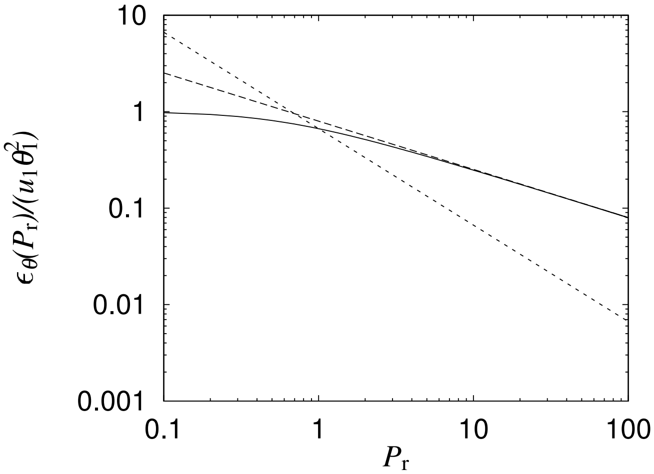

FIG. 1: Non-dimensionalized dissipation rate of the passive scalar $\frac{\epsilon_\theta(P_r)}{u_1\theta_1^2}$ as a function of P_r (solid line) and the large-P_r asymptotics $\sqrt{\frac{2}{\pi P_r}}$ (dashed line). The dotted line shows the *incorrect* behavior $\frac{2}{3P_r}$ obtained by discarding the subtle differences among $\tanh^n \xi$.

$$u(x,t) = \Delta u H(x - Ut) + U + u_1,$$

$$\theta(x,t) = \Delta\theta K(x - Ut) + U + \theta_1,$$

where $\Delta u = u(\infty) - u(-\infty) = -2u_1$ and $\Delta\theta = \theta(\infty) - \theta(-\infty) = -2\theta_1$. Here H and K are Heaviside step functions. Recall that H and K may not agree with each other, as Colombeau's theory resolves degeneracy of the Heaviside function in classical calculus. From the first condition we have

$$-U\Delta u\, H' + (\Delta u)^2 HH' + (U + u_1)\Delta u H' \sim 0.$$

Since $HH' \sim \frac{1}{2}H'$, we obtain

$$\frac{1}{2}\Delta u + u_1 = 0.$$

We have from the second relation

$$-UK' + \Delta u HK' + (U + u_1)K' \sim 0.$$

Now $K' \sim \delta$ and $HK' \sim A\delta$ with some function $A = A(\Delta u, \Delta\theta, u_1, U)$, we have

$$A\Delta u + u_1 = 0.$$

From these we can fix $A = 1/2$, consistently.

Case 2.

On the other hand, if we start from imposing a more stringent condition on θ, that is,

$$u_t + uu_x \sim 0,\ \theta_t + u\theta_x = 0,$$

we have from the second equation

$$(\Delta u H + u_1)K' = 0$$

or

$$(1 - 2H)K' = 0,$$

which is absurd because $1 - 2H(x)$ is not proportional to x. This implies that the set of both equations in the non-dissipative case must be interpreted in Colombeau's sense.

V. GENERALIZATION OF THE COLE-HOPF TRANSFORM

In this section we consider a flow with finite total kinetic energy. For the Burgers equation (1) it is well known that the Cole-Hopf transform [10, 11, 13]:

$$u(x,t) = -2\nu \frac{\partial}{\partial x} \log \psi = -2\nu \frac{\psi_x}{\psi}$$

linearizes (1) to the diffusion equation

$$\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2}.$$

For some historical backgrounds on the Cole-Hopf transform, see e.g. [3, 4].

The equation (2) for a passive scalar is already linear, but it is of interest to seek a similar transform which expresses its solution in a closed form.

We assume

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2}, \tag{12}$$

and attempt to find a solution in the quotient form

$$\theta = \frac{\phi}{\psi}.$$

Then we find that

$$\phi_t = (\theta\psi)_t = 2\nu\psi_x\theta_x + \kappa\psi\theta_{xx} + \nu\theta\psi_{xx}.$$

Because $\phi_{xx} = \theta_{xx}\psi + 2\theta_x\psi_x + \theta\psi_{xx}$ and

$$\theta_{xx} = \frac{\phi_{xx}\psi - \phi\psi_{xx}}{\psi^2} - 2\frac{\phi_x\psi - \phi\psi_x}{\psi^3}\psi_x,$$

we obtain

$$\phi_t = \kappa\phi_{xx} + (\nu - \kappa)\left[2\frac{\psi_x\phi_x}{\psi} + \left(\frac{\psi_{xx}}{\psi} - 2\frac{\psi_x^2}{\psi^2}\right)\phi\right].$$

Therefore when $\nu = \kappa$ (i.e. $P_r = 1$) we may reduce the equation for the passive scalar to a heat diffusion equation (12). Note that u and θ, (or equivalently ψ and ϕ), can be chosen independently. In particular, for the special case $u = \theta$ we have $\phi = -2\nu\psi_x$ and recover the original Cole-Hopf transform.

In the general case $P_r \neq 1$, it is not known whether we may reduce (2) to a diffusion equation although it is known that (2) is regular for all time. To search for such a transformation left for future study. That might help in clarifying whether there is anomaly in passive scalar dissipation for the case of finite total energy and passive scalar variance.

VI. SUMMARY

In this paper we treat a steadily propagating solution of a passive scalar subject to Burgers equation. We have two results on this model.

First, there is anomaly in the dissipation of the passive scalar. In spite of its simplicity (after all, what we have solved is an ODE by a quadrature), it manifests a nontrivial behavior in its dissipation rate. Second, a lesson to be learned here is that if we do not distinguish $\tanh^n \xi$ for different n, we would obtain a wrong answer for the dissipation rate. This suggests that Colombeau calculus plays an important role even for this simple example.

It may be in order to recall that in the case of 2D Navier-Stokes equations, the dissipation rate η of enstrophy is estimated from above [18] as $\eta \propto (\log Re)^{-1/2}$, where Re is the Reynolds number. In the large-Re limit, η decays to zero, but does so very slowly (as a transcendental function). In contrast, the decay of ϵ_θ with Prandtl number is much more rapid.

Dissipation anomaly is a subtle problem; a special care is required even in this linear, 1D model problem, let alone possible dissipation anomaly in the 3D Navier-Stokes equations for which we have only experimental or numerical evidence.

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