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Optimal Control and Stabilization for Nonlinear Systems

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Abstract

The optimal control and stabilization of nonlinear systems of the form $\dot{x} = A(x)x + B(x)u$ is considered and explicit expressions for the control are developed.

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1 Introduction

In this paper we shall consider the optimal control and stabilization of a non-linear system of the form

$$\dot{x} = A(x)x + B(x)u$$

If $(A(x), B(x))$ is a stabilizable pair for each x, we shall show in section 3 that a stabilizing control is given by solving the standard Riccati equation at each point x provided $\frac{\partial A(x)}{\partial x}, \frac{\partial B(x)}{\partial x}$ satisfy some bounded growth conditions. By using the infinite-time cost

$$J_\infty = \int_0^\infty (x^T Q x + u^T R u) dt$$

we shall show in section 4 that the local control in section 3 is actually the global control if A and B are analytic functions and $(Q, A(x))$ is observable, so that we may again control pointwise.

As a byproduct of these results we shall also obtain an explicit solution of the algebraic Riccati equation when A is symmetric, and $Q = qI$ for some q. This is useful in stabilizing systems since the choice of Q is somewhat arbirary and we can (almost always) diagonalize $A(x)$ into a symmetric form.

For a discussion of the linear- and bilinear-quadratic cases, see [1],[2],[3],[4] and [5] .

2 Notation

Notation in this paper will be mainly standard. However, one point to note is that if $P(x)$ is a matrix-valued function, we denote its derivative with respect to x by $P_x(x)$. This is interpreted as a vector of matrices, so that the notation $x^T P_x(x)$ is intended to mean $\sum_{i=1}^{n} x_i P_{x_i}(x)$ which is the sum of the n matrices $x_i P_{x_i}(x), 1 \leq i \leq n$. Moreover, $||.||$ will denote the standard Euclidean norm except when we define $||P_x(x)||$ which will be defined by regarding P_x, as stated above, as a vector of matrices.

At one point in the paper we shall need the concept of partition of unity. A *partition of unity* on R^n is a set of C^∞ functions $\{\alpha_U(x)\}_{U \in \mathcal{U}}$ where $\mathcal{U}$ is a locally finite open cover of R^n (i.e. $\cup_{U \in \mathcal{U}} U = R^n$, and there exists a neighbourhood of each point which meets only finitely many of the U's in $\mathcal{U}$) such that

$$(i) \qquad \alpha_U(x) \geq 0\ , \quad \text{for each } x \in R^n$$

$$(ii) \qquad Supp\ \alpha_U \in U \quad (\text{so that } \alpha_U(x) = 0 \quad \text{if} \quad x \notin U\ )$$

$$(iii) \qquad \sum_{U \ni x} \alpha_U(x) = 1 \quad \text{for each } x \in R^n$$

(See [6]).

3 Stabilizing Controls for Nonlinear Systems

In this section we shall consider the problem of designing a stabilizing feedback control for the system

$$\dot{x} = A(x)x + B(x)u \tag{3.1}$$

where $A(.) : R^n \longrightarrow R^{n^2}$ and $B(.) : R^n \longrightarrow R^{nm}$ are continuously differentiable matrix-valued functions. We shall do this by applying the standard linear-quadratic regulator control

$$u(t) = -R^{-1}B^T(x)P(x)x \tag{3.2}$$

at each point x, where

$$-\frac{\partial P}{\partial t}(t,x) = Q + P(x)A(x) + A^T(x)P(x) - P(x)B(x)R^{-1}B^T(x)P(x) \tag{3.3}$$

and

$$P(t_f, x) = F \tag{3.4}$$

Here, Q, R and F are symmetric and positive definite and t_f is some fixed time. The basic idea is that we are determining the control at any point $\overline{x} \in R^n$ by solving the linear-quadratic problem

$$\begin{aligned} \dot{x} &= A(\overline{x})x + B(\overline{x})u \\ J &= x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^TQx + u^TRu)dt \end{aligned}$$

with $\overline{x}$ fixed.

Lemma 1 Suppose that

$$Q \geq \frac{\partial P}{\partial x}(A(x)x - B(x)R^{-1}B^T(x)P(x)x) + \epsilon I \tag{3.5}$$

where $\epsilon > 0$ is independent of x. Then

$$||x||^2 \leq C||P^{-\frac{1}{2}}(t,x)||^2 exp\left(-\epsilon \int_0^t ||P^{\frac{1}{2}}(t,x)||^{-2}dt\right)$$

for some $C > 0$.

Proof Consider the function $V \triangleq x^T(t)P(t,x)x(t) = ||P^{\frac{1}{2}}(t,x)x(t)||^2$. We have

$$\begin{aligned}
\frac{dV}{dt} &= \dot{x}^T P x + x^T \frac{\partial P}{\partial t} + x^t P \dot{x} + x^T \left(\frac{\partial P}{\partial x}\dot{x}\right) x \\
&= (x^T A^T(x) - x^T P B(x) R^{-1} B^T(x)) P x \\
&\quad - x^T(Q + PA(x) + A^T(x)P - PB(x)R^{-1}B^T(x)P)x \\
&\quad + x^T(PA(x)x - PB(x)R^{-1}B^T(x)Px) \\
&\quad + x^T \left(\frac{\partial P}{\partial x}(A(x)x - B(x)R^{-1}B^T(x)Px\right) x \\
&= -x^T(Q + PB(x)R^{-1}B^T(x)P)x \\
&\quad + x^T \left(\frac{\partial P}{\partial x}(A(x)x - B(x)R^{-1}B^T(x)Px\right) x \\
&\leq -x^T \left(Q - \frac{\partial P}{\partial x}(A(x)x - B(x)R^{-1}B^T(x)Px)\right) x \\
&\leq -\epsilon||x||^2
\end{aligned}$$

for some $\epsilon > 0$, by assumption. Hence,

$$\begin{aligned}
\frac{d}{dt}||P^{\frac{1}{2}}(t,x)x(t)||^2 &< -\epsilon||P^{-\frac{1}{2}}(t,x)P^{\frac{1}{2}}(t,x)x(t)||^2 \\
&\leq -\epsilon||P^{\frac{1}{2}}(t,x)||^{-2}||P^{\frac{1}{2}}(t,x)x(t)||^2
\end{aligned}$$

since

$$-||Lx|| \leq -||L^{-1}||^{-1}||x|| \tag{3.6}$$

for any invertible matrix L, and so

$$||P^{\frac{1}{2}}(t,x)x(t)||^2 \leq Cexp\left(-\epsilon\int_0^t ||P^{\frac{1}{2}}(t,x)||^{-2}dt\right)$$

for some constant C. Using (3.6) again, we have

$$||x||^2 \leq C||P^{-\frac{1}{2}}(t,x)||^2 exp\left(-\epsilon\int_0^t ||P^{\frac{1}{2}}(t,x)||^{-2}dt\right)$$

and the result follows. □.

Corollary 1 If $(A(x), B(x))$ is stabilizable for each x, then

$$||x(t)||^2 \leq C||P^{-\frac{1}{2}}(x)||^2 exp(-\epsilon t||P^{\frac{1}{2}}(x)||^{-2})$$

where $P(x)$ satisfies the algebraic Riccati equation

$$Q + P(x)A(x) + A^T(x)P(x) - P(x)B(x)R^{-1}B^T(x)P(x) = 0 \tag{3.7}$$

□

The next result gives a bound on $P(x)$ in the infinite-time case.

Theorem 1 Let $P(x)$ satisfy (3.7) for each x. Then

$$P^{-1}(x) = Q^{-\frac{1}{2}}([A(x)Q^{-1}A^T(x) + B(x)R^{-1}B^T(x)]^{\frac{1}{2}} - A(x)Q^{-\frac{1}{2}}). \tag{3.8}$$

is a positive definite solution of (3.7).

Proof Let $W(x) = P^{-1}(x)$. From (3.7) we have

$$W(x)QW(x) + A(x)W(x) + W(x)A^T(x) - B(x)R^{-1}B^T(x) = 0 \tag{3.9}$$

Hence

$$(W(x)Q^{\frac{1}{2}} + A(x)Q^{-\frac{1}{2}})(W(x)Q^{\frac{1}{2}} + A(x)Q^{-\frac{1}{2}})^T$$
$$-A(x)Q^{-1}A^T(x) - B(x)R^{-1}B^T(x) = 0 \qquad (3.10)$$

Put

$$K(x) = A(x)Q^{-1}A^T(x) + B(x)R^{-1}B^T(x).$$

Then K is positive definite and so we can write

$$K(x) = L(x)L^T(x)$$

where $L(x) = (A(x)Q^{-1}A^T(x) + B(x)R^{-1}B^T(x))^{\frac{1}{2}}$. Now (3.10) is of the form

$$Z(x)Z^T(x) - L(x)L^T(x) = 0 \qquad (3.11)$$

where

$$Z(x) = W(x)Q^{\frac{1}{2}} + A(x)Q^{-\frac{1}{2}}$$

A solution of (3.11) is clearly given by

$$Z(x) = L(x).$$

Hence

$$\begin{aligned} W(x) &= P^{-1}(x) \\ &= Q^{-\frac{1}{2}}\left([A(x)Q^{-1}A^T(x) + B(x)R^{-1}B^T(x)]^{\frac{1}{2}} - A(x)Q^{-\frac{1}{2}}\right) \qquad (3.12) \end{aligned}$$

is a positive definite solution of (3.7). □.

Remark We have, of course, effectively assumed that P is invertible in the proof of theorem 1. This in fact follows easily from (3.7) if $Q > 0$, since

$$\langle x, Qx\rangle + \langle x, PAx\rangle + \langle x, A^T Px\rangle - \langle x, PBR^{-1}B^T Px\rangle = 0$$

for all x and so

$$\begin{aligned} \|Q^{\frac{1}{2}}x\|^2 &= 2\langle Px, Ax\rangle - \langle R^{-\frac{1}{2}}B^T Px, R^{-\frac{1}{2}}B^T Px\rangle \\ &\leq 2\|Px\|\,\|Ax\| + \|R^{-\frac{1}{2}}B^T\|^2\|Px\|^2 \end{aligned}$$

If $\lambda = \|Px\|$, then this inequality is of the form

$$a\lambda^2 + b\lambda - c \geq 0$$

where $a, c > 0, b \geq 0$. It follows from the elementary theory of quadratic functions that $\|Px\| \neq 0$ for $x \neq 0$.

Corollary 2 If $A(x)$ is symmetric and $Q = qI$ for some scalar q, then (3.8) is the unique positive symmetric solution of (3.7). □

In the following discussion we shall therefore assume that $A(x)$ is diagonalizable for each x; since this is the generic case it is not particularly restrictive. Let $V(\overline{x})$ be a diagonalizing matrix at a fixed point $\overline{x}$. Then, if

$$y = V(\overline{x})x$$

we have

$$\begin{aligned} \dot{y} = V(\overline{x})\dot{x} &= V(\overline{x})A(\overline{x})V^{-1}(\overline{x})y + V(\overline{x})B(\overline{x})u \\ &= \Lambda(\overline{x})y + \overline{B}(\overline{x})u \end{aligned} \tag{3.13}$$

where $\overline{B}(\overline{x}) = V(\overline{x})B(\overline{x})$. By (3.12), if we solve the problem (3.13) with the cost $J = \int_0^\infty (y^T Qy + u^T Ru)dt$ and $Q = qI$, we have

$$P^{-1}(\overline{x}) = Q^{-\frac{1}{2}}([V(\overline{x})A(\overline{x})V^{-1}(\overline{x})Q^{-1}(V^T(\overline{x}))^{-1}A^T(\overline{x})V^T(\overline{x})$$

$$+V(\overline{x})B(\overline{x})R^{-1}B^T(\overline{x})V^T(\overline{x})]^{\frac{1}{2}} - V(\overline{x})A(\overline{x})V^{-1}(\overline{x})Q^{\frac{1}{2}}) \qquad (3.14)$$

Note that (3.14) is the solution of the problem:

$$\dot{x} = A(\overline{x})x + B(\overline{x})u$$

subject to

$$J = \int_0^\infty (x^T V^T(\overline{x})QV(\overline{x})x + u^T Ru)\, dt$$

In order to estimate $\frac{\partial P}{\partial x}$ note from (3.7) that

$$\begin{aligned} \frac{\partial A^T}{\partial x_i}P + A^T\frac{\partial P}{\partial x_i} + \frac{\partial P}{\partial x_i}A \quad &+ \quad P\frac{\partial A}{\partial x_i} - \frac{\partial P}{\partial x_i}BR^{-1}B^TP - P\frac{\partial B}{\partial x_i}R^{-1}B^TP \\ &- \quad PBR^{-1}\frac{\partial B^T}{\partial x_i}P - PBR^{-1}B^T\frac{\partial P}{\partial x_i} = 0 \end{aligned}$$

for $1 \leq i \leq n$, and so

$$\begin{aligned} (A - BR^{-1}B^TP)^T\frac{\partial P}{\partial x_i} + \frac{\partial P}{\partial x_i}(A - BR^{-1}B^TP) \quad &= \quad \left(\frac{\partial B}{\partial x_i}R^{-1}B^TP - \frac{\partial A}{\partial x_i}\right)^T P \\ &+ \quad P\left(\frac{\partial B}{\partial x_i}R^{-1}B^TP - \frac{\partial A}{\partial x_i}\right) \end{aligned}$$

i.e.

$$\Gamma^T\frac{\partial P}{\partial x_i} + \frac{\partial P}{\partial x_i}\Gamma = \Pi_i^T P + P\Pi_i \qquad (3.15)$$

where

$$\begin{aligned} \Gamma \quad &= \quad A - BR^{-1}B^TP \\ \Pi_i \quad &= \quad \frac{\partial B}{\partial x_i}R^{-1}B^TP - \frac{\partial A}{\partial x_i} \end{aligned} \qquad (3.16)$$

However, (3.15) is a Lyapunov equation with stable Γ and so

$$\frac{\partial P}{\partial x_i} = \int_0^\infty e^{\Gamma t}(\Pi_i^T P + P\Pi_i)e^{\Gamma^T t}dt. \tag{3.17}$$

Define

$$\left\|\frac{\partial P}{\partial x}\right\|^2 = \sum_{i=1}^{n}\left\|\frac{\partial P}{\partial x_i}\right\|^2. \tag{3.18}$$

Since Γ is stable, we can write

$$||e^{\Gamma t}|| = Me^{-\omega t} \tag{3.19}$$

for some positive numbers M, ω (depending on x). We therefore have

Lemma 2

$$\left\|\frac{\partial P}{\partial x}\right\| \leq 2||P||^2\frac{M^2}{\omega^2}\left\{\sum_{i=1}^{n}\left(\left\|\frac{\partial B}{\partial x_i}\right\| ||R^{-1}||\ ||B||\ ||P|| + \left\|\frac{\partial A}{\partial x_i}\right\|\right)^2\right\}^{\frac{1}{2}}$$

Proof From (3.14) and (3.16) we have

$$\begin{aligned}\left\|\frac{\partial P}{\partial x_i}\right\| &\leq 4||\Pi_i||^2||P||^2\frac{M^4}{\omega^4} \\ &\leq 4||P||^2\frac{M^4}{\omega^4}\left(\left\|\frac{\partial B}{\partial x_i}\right\| ||R^{-1}||||B||||P|| + \left\|\frac{\partial A}{\partial x_i}\right\|^2\right)\end{aligned}$$

□.

Theorem 2 If $Q = qI$ and

$$\begin{aligned} q-\epsilon &\geq 2||P||^2\frac{M^2}{\omega^2}\left\{\sum_{i=1}^{n}\left(\left\|\frac{\partial B}{\partial x_i}\right\| ||R^{-1}||||B||||P|| + \left\|\frac{\partial A}{\partial x_i}\right\|^2\right)\right\}^{\frac{1}{2}} \times \\ & \quad ||A(x)x - B(x)R^{-1}B^T(x)P(x)x|| \end{aligned} \tag{3.20}$$

for each x and some $0 < \epsilon < q$, then the local control $-R^{-1}B^T(x)P(x)x$, where $P(x)$ is given by (3.7), is stabilizing.

Proof By lemmas 1 and 2 it suffices to show that if a matrix L satisfies $||L|| \leq 1$, then $L \leq I$. However, this follows from

$$\langle x, Lx \rangle \leq ||x||^2 ||L|| \leq ||x||^2 = \langle x, Ix \rangle.$$

for each x. □.

Remark If $q > 0$ if fixed then (3.20) places restrictions on the sizes of $\frac{\partial A}{\partial x_i}, \frac{\partial B}{\partial x_i}$, $1 \leq i \leq n$. Of course, if these are zero then we merely obtain the classical linear solution.

4 The Optimal Solution

In the previous section we have obtained a stabilizing feedback control for a nonlinear system with bounded growth conditions by applying an approximate form of the infinite-time problem. In this section we shall examine the exact problem via Bellman's dynamic programming method. We shall show that the local solution is actually an exact solution under some mild conditions.

Consider the finite-time problem

$$min\ J = x^T(t_f)Fx(t_f) + \int_0^{t_f} (x^T Qx + u^T Ru)dt \tag{4.1}$$

subject to the nonlinear dynamics

$$\dot{x} = A(x)x + B(x)u.$$

As in the linear case we obtain the Bellman equation

$$V_t + x^T Qx + V_x^T A(x)x - \frac{1}{4} V_x^T B(x) R^{-1} B^T(x) V_x = 0$$

$$V(x,t_f) = x^T F x.$$

If we assume that $A(x)$ and $B(x)$ are analytic functions then we may write

$$V(x,t) = x^T P(t,x) x$$

for some analytic function P (since it is easily shown that no terms of order 0 or 1 occur in V in this case).

Then we obtain the partial differential equation

$$\begin{aligned} -P_t &= Q + (P + \frac{1}{2}x^T P_x)^T A(x) + A^T(x)(P + \frac{1}{2}x^T P_x) \\ &\quad -(P + \frac{1}{2}x^T P_x)^T B(x) R^{-1} B^T(x)(P + \frac{1}{2}x^T P_x) \end{aligned} \tag{4.2}$$

We now wish to prove, as in the linear case, that the solution of (4.2) converges as $t_f \longrightarrow \infty$ to the positive definite solution of the matrix 'Riccati'equation

$$\begin{aligned} &Q + (P + \frac{1}{2}x^T P_x)^T A(x) + A^T(x)(P + \frac{1}{2}x^T P_x) \\ &-(P + \frac{1}{2}x^T P_x)^T B(x) R^{-1} B^T(x)(P + \frac{1}{2}x^T P_x) = 0 \end{aligned} \tag{4.3}$$

Let J_∞ denote the infinite-time cost

$$J_\infty = \int_0^\infty (x^T Q x + u^T R u) dt \tag{4.4}$$

The proof follows in a similar way to the linear case- see [5] . The main difficulty is to show that there exists a control such that (4.4) is finite. We prove this in

Lemma 4.1 Suppose that $(A(x), B(x))$ is a stabilizable pair for each x. Then the set of control and trajectory pairs $(u(t), x(t))$ for which the infinite-time cost (4.4) is finite is nonempty for each x_0.

Proof We first show that a stabilizing matrix function $F(x)$ may be chosen to be continuous. For any fixed point x^* the pair $(A(x^*), B(x^*))$ is stabilizable and so there exists a matrix $\overline{F}(x^*)$ such that $\overline{T}(x^*) \triangleq A(x^*) + B(x^*)\overline{F}(x^*)$ is stable, i.e.

$$||e^{\overline{T}(x^*)t}|| \leq \overline{M}(x^*)e^{-\overline{\lambda}(x^*)t}$$

for some $\overline{M}(x^*) \geq 0$ and $\overline{\lambda}(x^*) > 0$. By continuity of $A(x)$ and $B(x)$, there exists a neighbourhood $\overline{U}_{x^*}$ of x^* such that $\overline{T}(x) \triangleq A(x) + B(x)\overline{F}(x^*)$ is stable and

$$||e^{\overline{T}(x)t}|| \leq \overline{\overline{M}}(x^*)e^{-\overline{\overline{\lambda}}(x^*)t}$$

for some other constants $\overline{\overline{M}}$ and $\overline{\overline{\lambda}}$. It follows that $\overline{F}$ can be chosen to be locally constant. Let

$$\overline{\mathcal{U}} = \{\overline{U}_{x^*}\}$$

be a covering of R^n by sets of the form $\overline{U}_{x^*}$ and let $\mathcal{U} = \{U_{x^*}\}$ be a locally finite cover of R^n obtained by shrinking the sets in $\overline{\mathcal{U}}$. (Since R^n is paracompact this is always possible.) Now let $\{\alpha_U(x)\}_{U \in \mathcal{U}}$ be a partition of unity subordinate to the cover $\mathcal{U}$. Let $\overline{F}_U$ denote the constant stabilizing matrix defined in the neighbourhood U and put

$$F(x) = \sum_{U \ni x} \alpha_U(x)\overline{F}_U \quad .$$

The sum is finite and so exists; moreover F is continuous. Now

$$A(x) + B(x)F(x) \quad = \quad \sum_{U \ni x} \alpha_U(x)A(x) + \sum_{U \ni x} \alpha_U(x)B(x)\overline{F}_U$$

$$= \sum_{U \ni x} \alpha_U(x)(A(x) + B(x)\overline{F}_U)$$

and the right hand side is stable.

Next let x_0 be any vector and let $\mathcal{B}$ denote the closed ball with centre 0 and radius $||x_0||$. Choose a continuous function $F(x)$ as above so that

$$||e^{T(x)t}|| \leq M(x)e^{-\lambda(x)t}$$

for each $x \in \mathcal{B}$ where

$$T(x) = A(x) + B(x)F(x) \ .$$

Since A, B and F are continuous, so is $\lambda(x)$ which therefore has a minimum value λ' on the compact set $\mathcal{B}$. Similarly, $M(x)$ has a maximum value M' on $\mathcal{B}$. Let $\epsilon > 0$ be given such that $M'\epsilon < \lambda'$. By continuity of A, B and F we see that T is uniformly continuous on $\mathcal{B}$ and so

$$||T(x) - T(y)|| < \epsilon$$

for x and y in any sufficiently small ball in $\mathcal{B}$. Choose a finite number of such balls $\mathcal{B}_1, \cdots, \mathcal{B}_L$ covering $\mathcal{B}$ and fix points $x(i)$ in $\mathcal{B}_i$ for $1 \leq i \leq L$. Then, in $\mathcal{B}_i$ we have

$$\dot{x}(t) = T(x(i))x(t) + (T(x(t)) - T(x(i)))x(t)$$

and so

$$||x(t)|| \leq M'e^{-\lambda' t}||x(0)|| + \int_0^t M'e^{-\lambda'(t-s)}\epsilon||x(s)||ds \ .$$

Hence, by Gronwall's inequality, we have

$$||x(t)|| \leq e^{(-\lambda' + \epsilon M')t} ||x(0)|| \ .$$

Since this is true in any ball $\mathcal{B}_i$ it holds throughout $\mathcal{B}$ and so $x(t) \in L^2(0, \infty)$ and the result follows. □.

Remark It follows from the above proof that if $\overline{A}(x)$ is a continuous matrix-valued function which is asymptotically stable for each x, then the equation

$$\dot{x} = \overline{A}(x)x \quad , \quad x(0) = x_0$$

is asymptotically stable for all x_0.

Returning to equation (4.2) we see by the Kowalewski theorem that a global solution $P(t, x)$ exists which is differentiable for all x. Let $P_{[t_f]}(t, x)$ denote the solution of (4.2) on the interval $[0, t_f]$, with $P_{[t_f]}(t_f, x) = F$. Then, as in the linear case, by lemma 4.1, we see that

$$P_{[t_f]}(0, x) \leq q(x)I \quad , \quad 0 \leq t_f < \infty$$

for some continuously differentiable function $q(x)$. Again, as in the linear case, the matrix-valued function $\tau \longrightarrow P_{[\tau]}(0, x)$ is increasing, i.e.

$$P_{[\tau_2]}(0, x) \geq P_{[\tau_1]}(0, x) \text{for } \tau_2 \geq \tau_1 \ .$$

Then $P_{[\tau]}(0, x)$ converges uniformly on compact sets to a differentiable matrix-valued function $P(x)$ and $P(x)$ satisfies equation (4.3).

The control associated with $P(x)$ which minimizes (4.4) is given by

$$u(t) = -R^{-1}B^T(x)(P(x) + \frac{1}{2}x^T P_x)x \tag{4.5}$$

With this control our system becomes

$$\begin{aligned} \dot{x} &= [A(x) - R^{-1}B^T(x)(P(x) + \frac{1}{2}x^T P_x)]x \\ &\triangleq \overline{A}(x)x \end{aligned} \tag{4.6}$$

where

$$\overline{A}(x) = A(x) - R^{-1}B^T(x)(P(x) + \frac{1}{2}x^T P_x)$$

Theorem 4.1 If, for each x, $(A(x), B(x))$ is stabilizable and $(Q, A(x))$ is observable, then with the control (4.5), the system

$$\dot{x} = A(x)x + B(x)u$$

is asymptotically stable.

Proof By the above remark it suffices to show that $\overline{A}(x)$ is a stable matrix for each x. To show this consider the infinite- time problem

$$min \int_0^\infty (x^T Q x + u^T R u)dt$$

subject to

$$\dot{x} = A(\overline{x})x + B(\overline{x})u$$

for any given (fixed) $\overline{x}$. By the classical infinite-time problem, the control

$$\overline{u}(t) = -R^{-1}B^T(\overline{x})\Pi(\overline{x})x$$

is stabilizing, where Π satisfies the Riccati equation

$$Q + \Pi^T(\overline{x})A(\overline{x}) + A^T(\overline{x})\Pi(\overline{x}) - \Pi^T(\overline{x})B(\overline{x})R^{-1}B^T(\overline{x})\Pi(\overline{x}) = 0 \tag{4.7}$$

Comparing (4.3) and (4.7) the result follows since for each $\overline{x}$, P and Π are related by

$$\Pi(\overline{x}) = P(\overline{x}) + \frac{1}{2}\overline{x}^T P_x(\overline{x})$$

We have now therefore shown that for the nonlinear control problem

$$min\ J_\infty = \int_0^\infty (x^T Q x + u^T R u) dt$$

subject to the dynamics

$$\dot{x} = A(x)x + B(x)u\ ,$$

the optimal control is given by

$$u(t) = -R^{-1}B^T(x)(P(x) + \frac{1}{2}x^T P_x(x))x$$

where P satisfies the implicit partial differential equation (4.3). However, since we only need the function $P(x) + \frac{1}{2}x^T P_x(x)$ we can solve the ordinary Riccati equation

$$Q + \Pi^T(x)A(x) + A^T(x)\Pi(x) - \Pi^T(x)B(x)R^{-1}B^T(x)\Pi(x) = 0 \tag{4.8}$$

for $\Pi = P + \frac{1}{2}x^T P_x$.

As a simple example consider the stabilization of a system

$$\dot{x} = A(x)x + B(x)u \tag{4.9}$$

in which $A(x_1), A(x_2)$ commute for all $x_1, x_2 \in R^n$. Then we can diagonalize $A(x)$ by a diagonalizing matrix independent of x. Hence me may suppose that in

(4.9), $A(x)$ is diagonal and therefore symmetric. If $(A(x), B(x))$ is stabilizable and $(I, A(x))$ is observable, then by (3.12) and (4.8) a stabilizing control is given by

$$u(t) = -R^{-1}B^T(x)\Pi(x)x$$

where

$$\Pi(x) = \left(\left[A(x)A^T(x) + B(x)B^T(x)\right]^{\frac{1}{2}} - A(x)\right)^{-1}$$

5 Example

As an example, we shall consider a nonlinear model of the F-8 fighter aircraft. The equations representing the dynamics of the aircraft are ([7]):

$$\begin{aligned}
\dot{x}_1 &= -0.877x_1 + x_3 - 0.088x_1x_3 + 0.47x_1^2 \\
&\quad -0.019x_2^2 - x_1^2x_3 + 3.846x_1^3 - 0.215u \\
&\quad +0.28x_1^2u + 0.47x_1u^2 + 0.63u^3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -4.208x_1 - 0.396x_3 - 0.47x_1^2 - 3.564x_1^3 \\
&\quad - 20.967u + 6.265x_1^2u + 46x_1u^2 + 61.4u^3
\end{aligned}$$

where x_1 is the angle of attack in radians, x_2 is the pitch angle, x_3 is the pitch rate and u is the control input.

In an attempt to solve the control problem for the above system, Garrard and Jordan [7], presented an approach for computing a linear control law based on the linearized version of the nonlinear model and using linear quadratic regulator theory. Second and third-order nonlinear controllers were also derived. In their approach, which is computationally difficult, the three controllers presented only work for a small angle of attack. Even the third-order controller cannot recover from stall if the angle of attack is greater than about 0.4655 rads. (26.7 degrees).

Another approach to the problem has been carried out by Desrochers and Al-Jaar [8]. In their method they design a controller for a reduced-order or simplified nonlinear model of the nonlinear plant. Then they use this controller for the original plant. This controller is also shown to work only for a small angle of attack similar to the previous one.

To use the approach presented in this paper we write the dynamical equations in the form

$$\dot{x} = A(x)x + B(x)u$$

where

$$A(x) = \begin{bmatrix} -0.877 + 0.47x_1 - x_1x_3 + 3.846x_1^2 & -0.019x_2 & 1 - 0.088x_1 \\ 0 & 0 & 0 \\ -4.208 - 0.47x_1 - 3.564x_1^2 & 0 & -0.396 \end{bmatrix}$$

$$B(x) = \begin{bmatrix} -0.215 + 0.28x_1^2 \\ 0 \\ -20.967 + 6.265x_1^2 \end{bmatrix}$$

Terms involving nonlinearities in U are eliminated as the approach used here cannot account for nonlinear control terms. However, these terms have only a small effect on the dynamics as shown in [7].

Figs.1-6 show the response curves for the various techniques mentioned above. It is clearly seen that the method developed here can cope with much larger angles of attack than the techniques cited in [7] and [8].

6 Conclusions

In this paper we have studied the optimal control and stabilization of nonlinear systems of the form

$$\dot{x} = A(x)x + B(x)u$$

We have shown that if $(A(x), B(x))$ is stabilizable for all x and the matrix-valued functions $A(x)$ and $B(x)$ satisfy certain boundedness conditions on their derivatives, then a local infinite-time control will stabilize the system, and we have obtained an explicit formula for the control by solving the Riccati equation when $A(x)$ is symmetric (which is achieved by locally diagonalizing $A(x)$). In the second part of the paper we have shown that the control developed in the first part is also an optimal control if $(Q, A(x))$ is observable for all x and $A(x)$ and $B(x)$ are analytic. These results should be useful in many types of nonlinear control; for example, flight control systems of the kind discussed above.

References

[1] S.P.Banks, 'Control Systems Engineering ', Prentice-Hall,1988.

[2] S.P.Banks, 'Mathematical Theories of Nonlinear Systems ', Prentice-Hall,1988.

[3] S.P.Banks and M.K.Yew, 'On a Class of Suboptimal Controls for Infinite-Dimensional Bilinear Systems, Sys. Control Letts.,**5**,327-333,1985.

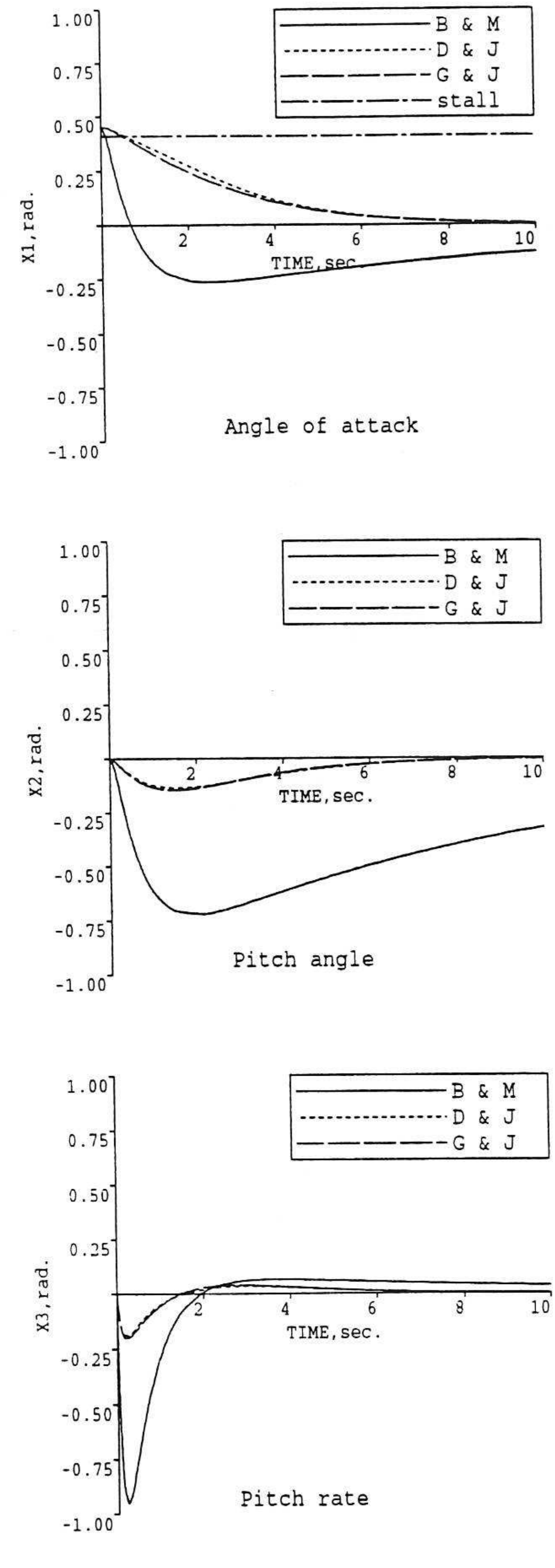
[4] S.P.Banks, 'Optimal Control of Distributed Bilinear Systems ', Sys. Control Letts.,**19**,629-636,1988.

[5] D.L.Russell, 'Mathematics of Finite-Dimensional Control Systems ', Dekker.

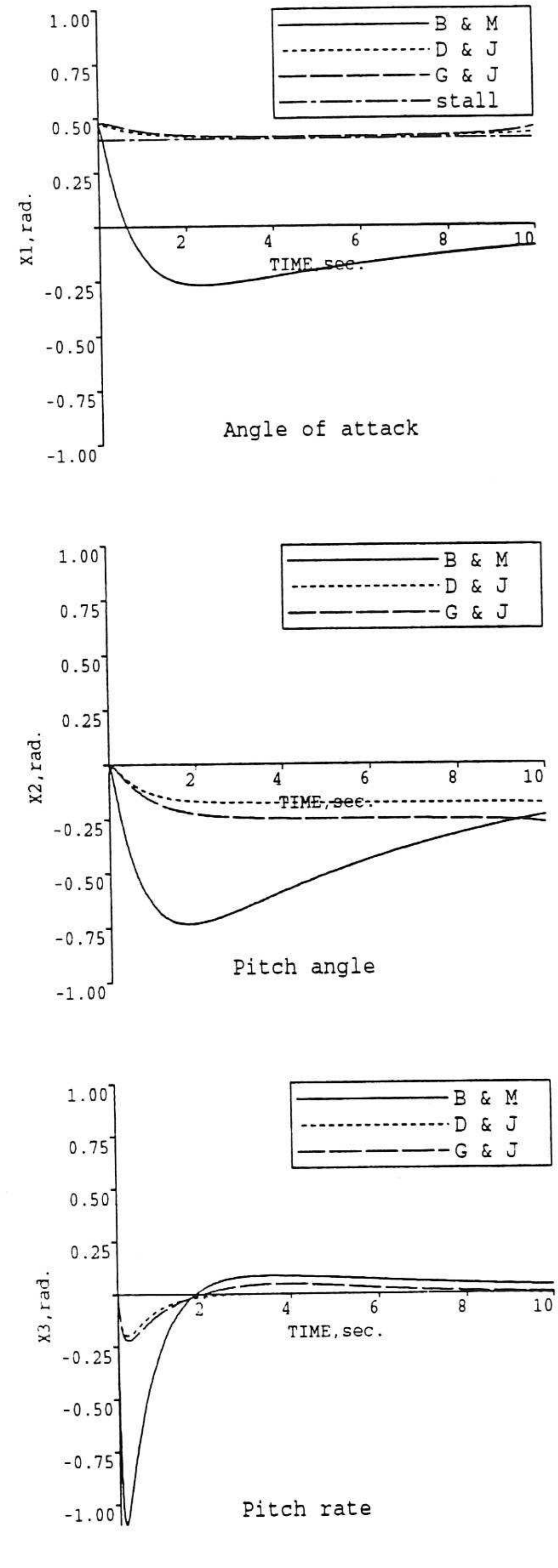
[6] C.Brickell and D.Clark, 'Differentiable Manifolds ', Van Nostrand, 1970.

[7] W.L.Garrard and J.M.Jordan, 'Design of Nonlinear Automatic Flight Control Systems ',Automatica, **13**,497-505,1977.

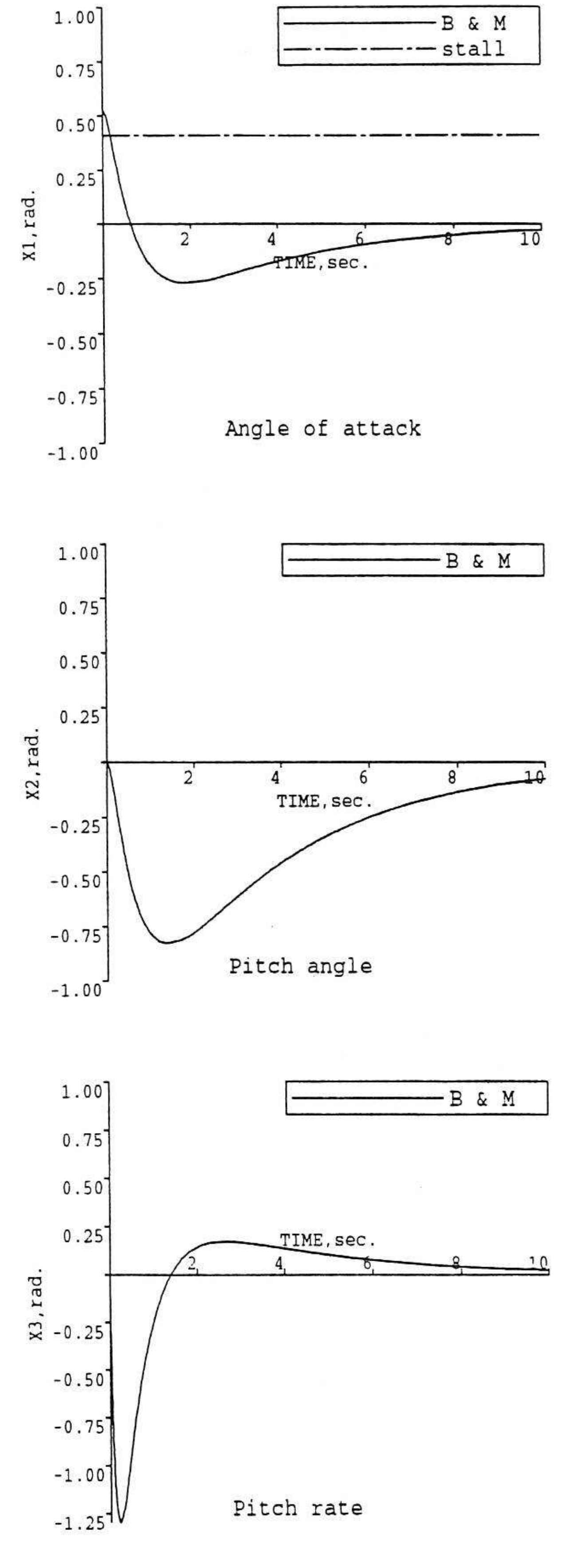
[8] A.A.Desrochers and R.Y.Al-Jaar, 'Nonlinear Model Simplification in Flight Control System Design ',J.Guidance, **7**, 684-689,1984.



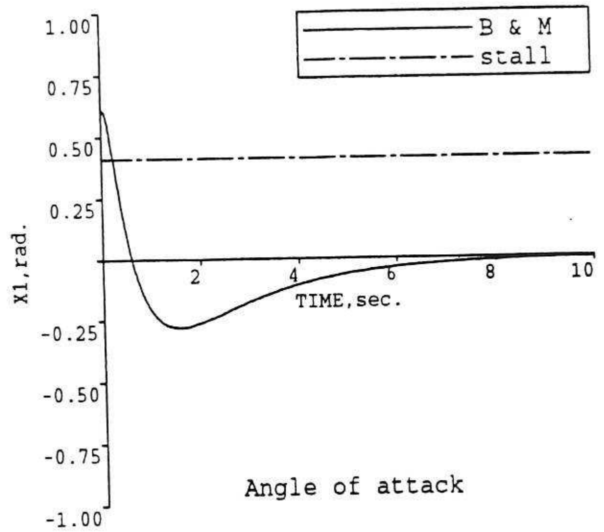
Fig(1), The response for x1(0) = 0.452.

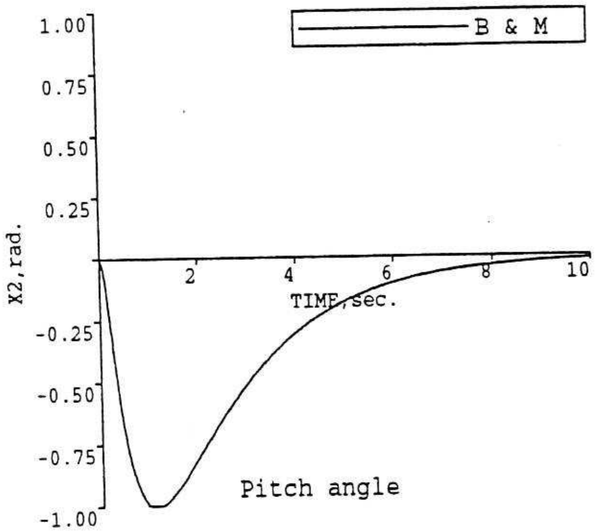


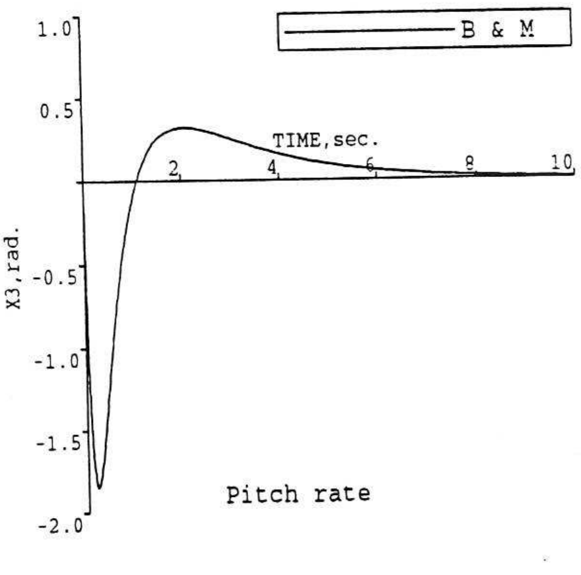
Fig(2), The response for x1(0) = 0.4655.



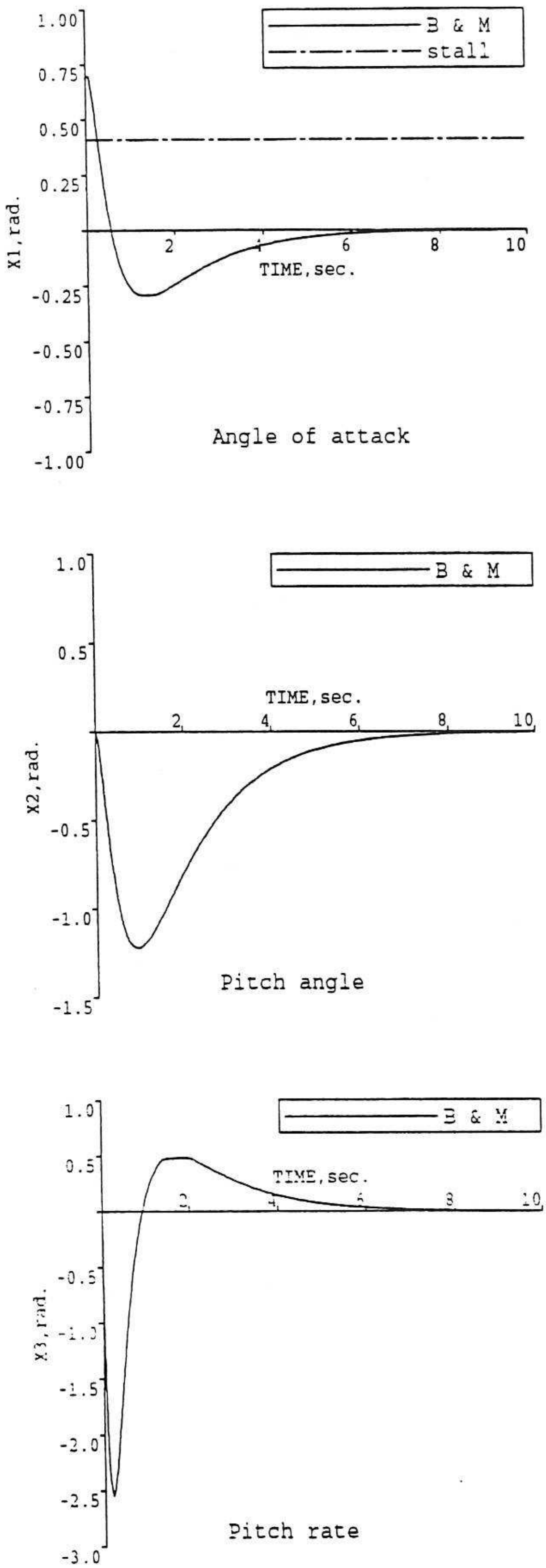
Fig(3), The response for x1(0) = 0.5236.







Fig(4), The response for x1(0) = 0.6108.



Fig(5), The response for x1(0) = 0.698.