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DG algebras with exterior homology

W. G. Dwyer, J. P. C. Greenlees and S. B. Iyengar

ABSTRACT

We study differential graded algebras (DGAs) whose homology is an exterior algebra over a commutative ring R on a generator of degree n , and also certain types of differential modules over these DGAs. We obtain a complete classification with $R = \mathbb{Z}$ or $R = \mathbb{F}_p$ and $n \geq -1$. The examples are unexpectedly interesting.

1. Introduction

We mainly study differential graded algebras (DGAs) whose homology is an exterior algebra over either \mathbb{F}_p or \mathbb{Z} on a class of degree -1 , as well as (left) modules over these DGAs whose homology is \mathbb{Z}/p in degree 0. In the case of an exterior algebra over \mathbb{F}_p , we find many DGAs, each having one such module; in the case of an exterior algebra over \mathbb{Z} , one DGA having many such modules. In both cases, the enumeration of possibilities involves complete discrete valuation rings with residue field \mathbb{F}_p . There are remarks about other types of DGAs in Section 6.

In more detail, a DGA is a chain complex A of abelian groups together with a multiplication map $A \otimes A \rightarrow A$ that is both unital and associative. A morphism $f: A \rightarrow B$ of DGAs is a map of chain complexes which respects multiplication and unit; f is said to be an *equivalence* if it induces isomorphisms $H_i A \cong H_i B$, $i \in \mathbb{Z}$. The homology $H_* A$ is a graded ring, and A is said to be of *type* β_S , for S a commutative ring, if $H_* A$ is an exterior algebra over S on a class of degree -1 . The notation β_S is meant to suggest that A captures a type of Bockstein operation. In topology, the ‘Bockstein’ is an operation β on mod p homology, of degree -1 and square 0, which arises from a generator of $\text{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/p, \mathbb{Z}/p)$.

A *module* over A is a chain complex X together with an action map $A \otimes X \rightarrow X$ with the usual unital and associativity properties. Morphisms and equivalences between modules are defined in the evident way. The module X is of *type* $(M, 0)$ for an abelian group M if there are isomorphisms of abelian groups

$$H_i X \cong \begin{cases} M, & i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We are particularly interested in DGAs of type β_S for $S = \mathbb{F}_p$ or $S = \mathbb{Z}$, and modules over these DGAs of type $(\mathbb{Z}/p, 0)$.

THEOREM 1.1. *There is a natural bijection between*

- (i) *equivalence classes of DGAs of type $\beta_{\mathbb{F}_p}$ and*

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(ii) *isomorphism classes of complete discrete valuation rings with residue field \mathbb{F}_p .*

Up to equivalence, each such DGA has a unique module of type $(\mathbb{Z}/p, 0)$.

REMARK 1.2. A complete discrete valuation ring \mathcal{O} is a principal ideal domain with a unique nonzero prime ideal \mathfrak{m} , such that \mathcal{O} is complete with respect to the topology determined by powers of \mathfrak{m} . If $\mathcal{O}/\mathfrak{m} \cong \mathbb{F}_p$, then \mathcal{O} is isomorphic either to $\mathbb{F}_p[[t]]$ or to a totally ramified extension of finite degree of the ring \mathbb{Z}_p^\wedge of p -adic integers; see [8, II.4-5]. The surprise in Theorem 1.1 is the sheer profusion of DGAs. In Remark 3.3, there is an explicit description of how to pass from a ring \mathcal{O} to a DGA.

The tables are turned when it comes to DGAs of type $\beta_{\mathbb{Z}}$.

THEOREM 1.3. *Up to equivalence, there is only a single DGA A of type $\beta_{\mathbb{Z}}$. There is a natural bijection between*

- (i) *equivalence classes of modules over A of type $(\mathbb{Z}/p, 0)$ and*
- (ii) *isomorphism classes of pairs (\mathcal{O}, π) , where \mathcal{O} is a complete discrete valuation ring with residue field \mathbb{F}_p , and π is a uniformizer for \mathcal{O} .*

REMARK 1.4. A uniformizer for \mathcal{O} is a generator of the maximal ideal \mathfrak{m} . The surprise in Theorem 1.3 is the profusion of modules, since there are many pairs (\mathcal{O}, π) as above. For instance, if $\pi \in \mathbb{Z}_p^\wedge$ is divisible by p but not by p^2 , then $(\mathbb{Z}_p^\wedge, \pi)$ is such a pair. Since \mathbb{Z}_p^\wedge has nontrivial automorphisms, these pairs are distinct for different choices of π .

The object A of Theorem 1.3 can be taken to be the DGA \mathcal{F} that contains copies of \mathbb{Z} in degrees 0 and -1 , is trivial elsewhere, and has zero differential. In Remark 5.5, there is an explicit description of how to pass from a pair (\mathcal{O}, π) to a module over \mathcal{F} .

REMARK 1.5. If A is a connective DGA, that is, $H_i A = 0$ for $i < 0$, and X is a module over A such that $H_i X = 0$ for $i \neq 0$, then X is determined up to equivalence by the isomorphism class of $H_0 X$ as an (ordinary) module over $H_0 A$; see, for instance, [5, 3.9]. This is in strong contrast to what happens in the nonconnective setting of Theorem 1.3.

Generalizations. We have some remarks in Section 6 about DGAs with other types of exterior algebra homology.

The arguments below can be interpreted in the setting of stable homotopy theory, and they lead to a classification of associative ring spectra of type $\beta_{\mathbb{Z}}$ or $\beta_{\mathbb{F}_p}$ (appropriately interpreted) and of module spectra over these ring spectra of type $(\mathbb{Z}/p, 0)$. No new examples come up; all of these ring spectra and module spectra are obtained in a standard way [9] from DGAs.

1.1. DG-objects, equivalences, and formality

As suggested above, a map between differential graded (DG) objects of any kind is said to be an *equivalence* if it induces an isomorphism on homology groups. Two objects are *equivalent* if they are related by a zig-zag $\leftarrow \rightarrow \leftarrow \cdots \rightarrow$ of equivalences. A DG-object X is *formal* if it is equivalent to a DG-object Y of the same kind that has zero differentials. Of course, in this case the graded constituent Y_i of Y must be isomorphic to $H_i X$. For instance (cf. Remark 1.5), if A is a connective DGA and X is a module over A such that $H_i X$ vanishes except for a single value of i , then X is formal as an A -module.

If A is a DGA such that $H_i A = 0$ for $i \neq 0$, then A is formal as a DGA. To see this, let $A' \subset A$ be the subcomplex given by

$$A'_i = \begin{cases} A_i, & i > 0, \\ \ker(\partial: A_0 \rightarrow A_{-1}), & i = 0, \\ 0, & i < 0. \end{cases}$$

Then A' is a DGA, and there is a zig-zag of DGA-equivalences

$$A \xleftarrow{\sim} A' \xrightarrow{\sim} H_0 A,$$

where the object on the right is a ring, treated as a DGA concentrated in degree 0.

If A is a DGA such that $H_* A$ is a polynomial algebra $\mathbb{Z}[x]$ on a class x of (arbitrary) degree n , then A is also formal: choose a cycle $\chi \in A_n$ representing x and construct an equivalence $(\mathbb{Z}[x], 0) \rightarrow (A, \partial)$ by $x \mapsto \chi$. On the other hand, if S is a commutative ring other than \mathbb{Z} and $H_* A \cong S[x]$, then it is not necessarily the case that A is formal, even for $S = \mathbb{F}_p$ (see Section 6 for examples); the above argument applies only if A itself is an algebra over S , or at least equivalent to an algebra over S .

1.2. Relationship to Moore–Koszul duality

Suppose that k is some chosen field. Say that a DGA A over k is *admissible* if $H_i A$ is finite-dimensional over k for all i and $H_0 A \cong k$. Applying duality over k to the bar constructions in [6] produces bijections between equivalence classes of the admissible DGAs indicated below.

$$\{A \mid H_i A = 0, i > 0 \text{ \& } i = -1\} \begin{matrix} \xrightarrow{A \mapsto \text{End}_A(k)} \\ \xleftarrow{B \mapsto \text{End}_B(k)} \end{matrix} \{B \mid H_i B = 0, i < 0\}.$$

In topology, for instance, this gives the relationship between the cochain algebra A of a simply connected space X and the chain algebra B of the loop space ΩX .

Our technique is to push the boundaries of the above Moore–Koszul duality construction. In proving Theorem 1.1 (Sections 2 and 3), we start with a DGA A of type $\beta_{\mathbb{F}_p}$, more or less construct by hand an action of A on (something equivalent to) \mathbb{Z}/p , and show that the DGA $B = \text{End}_A(\mathbb{Z}/p)$ has its homology concentrated in degree 0, and so is essentially an ordinary ring \mathcal{O} (Subsection 1.1). It turns out that \mathcal{O} is a complete discrete valuation ring, and that it determines A up to equivalence via the formula

$$A \xrightarrow{\sim} \text{End}_{\mathcal{O}}(\mathbb{Z}/p).$$

Note that it is *not* necessarily the case that A is a DGA over \mathbb{F}_p . In Section 4, the same technique succeeds in classifying DGAs of type $\beta_{\mathbb{Z}}$. The proof in Section 5 of the second part of Theorem 1.3 relies on ideas from [4] which fit modules into a type of Moore–Koszul duality setting.

1.3. Notation and terminology

We work in the context of [4, 5]. Rings are tacitly identified with DGAs concentrated in degree 0 (Subsection 1.1). An ordinary module M over a ring R is similarly tacitly identified with the chain complex X over R with $X_0 = M$ and $X_i = 0$ for $i \neq 0$. Hom is the derived homomorphism complex and \otimes is the derived tensor product. If X is a module over the DGA (or ring) A , then $\text{End}_A(X)$ denotes the DGA obtained by taking a projective model for X and forming the usual DGA of endomorphisms of this model. See [10, 2.7.4], but re-index to conform to our convention that differentials always reduce degree by 1. (Up to equivalence, the DGA $\text{End}_A(X)$ depends only on the equivalence types of A and of X ; this can be proved for instance with the bimodule argument of [7, 3.7.6].) If R is a ring and M is an ordinary

R -module, then there are isomorphisms

$$H_i \operatorname{End}_R(M) \cong \operatorname{Ext}_R^{-i}(M, M).$$

We write $\operatorname{Ext}_R^0(M, M)$ or $H_0 \operatorname{End}_R(M)$ for the ordinary endomorphism ring of M over R .

If X is a chain complex or graded abelian group, then we write $\Sigma^i X$ for its i -fold shift: $(\Sigma^i X)_j = X_{j-i}$, $\partial(\Sigma^i x) = (-1)^i \Sigma^i(\partial x)$. For the sake of clarity, we attempt as much as reasonably possible to make a notational distinction between the field \mathbb{F}_p and the abelian group $\mathbb{Z}/p = \mathbb{Z}/p\mathbb{Z}$.

2. A module of type $(\mathbb{Z}/p, 0)$

In this section, A is a DGA of type $\beta_{\mathbb{F}_p}$. We construct a module X over A of type $(\mathbb{Z}/p, 0)$ and show that for any module Y over A of type $(\mathbb{Z}/p, 0)$ there is an equivalence $e_Y : X \rightarrow Y$. However, the equivalence e_Y is not unique in any sense, even up to homotopy. See [5, 3.3, 3.9] for more general constructions of this type.

The construction is inductive. Suppose that Z is a module over A with

$$H_i Z \cong \begin{cases} \mathbb{Z}/p, & i = 0, -1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Choose a map $\kappa : \Sigma^{-1}A \rightarrow Z$ that induces an isomorphism

$$\mathbb{Z}/p \cong H_0 A \cong H_{-1} \Sigma^{-1}A \xrightarrow{H_{-1}\kappa} H_{-1}Z,$$

and let JZ be the mapping cone of κ . There is a natural map $Z \rightarrow JZ$ that is an isomorphism on H_0 and trivial on the other homology groups (in particular $H_{-1}Z \rightarrow H_{-1}JZ$ is zero), and so the homology exact sequence of the triangle

$$\Sigma^{-1}A \longrightarrow Z \longrightarrow JZ$$

shows that $H_* JZ$ again vanishes except for copies of \mathbb{Z}/p in degrees 0 and -1 . Starting with $Z = A$, iterate the process to obtain a sequence

$$A \longrightarrow JA \longrightarrow J^2 A \longrightarrow \dots$$

of chain complexes and chain maps, and let $X = \operatorname{hocolim}_k J^k A$. (In this case, the homotopy colimit can be taken to be a colimit, that is, the ascending union.) It is immediate that X has type $(\mathbb{Z}/p, 0)$.

Suppose that Y is an arbitrary module of type $(\mathbb{Z}/p, 0)$. There is certainly a map $A \rightarrow Y$ that induces an isomorphism on H_0 , so to construct an equivalence $X \rightarrow Y$ it is enough to show that if Z satisfies (2.1) and $f : Z \rightarrow Y$ induces an isomorphism on H_0 , then f extends to $f' : JZ \rightarrow Y$. (By induction, this will guarantee that the map $A \rightarrow Y$ extends to a map $X = \operatorname{hocolim}_k J^k A \rightarrow Y$.) The map f extends to f' if and only if the composite

$$\Sigma^{-1}A \xrightarrow{\kappa} Z \xrightarrow{f} Y$$

is null homotopic. But the group of homotopy classes of A -module maps $\Sigma^{-1}A \rightarrow Y$ vanishes, since it is isomorphic to $H_{-1}Y$.

3. Exterior algebras over \mathbb{F}_p

In this section, we prove Theorem 1.1. The proof depends on two lemmas.

Obtaining a ring from a DGA. Suppose that A is a DGA of type $\beta_{\mathbb{F}_p}$. According to Section 2, up to equivalence there is a unique module X over A of type $(\mathbb{Z}/p, 0)$. Let $\mathcal{E} = \operatorname{End}_A(X)$ be the derived endomorphism algebra of X .

LEMMA 3.1. *Let $A, X,$ and \mathcal{E} be as above.*

- (1) *The homology group $H_i\mathcal{E}$ vanishes for $i \neq 0,$ and the ring $H_0\mathcal{E}$ is a complete discrete valuation ring with residue field $\mathbb{F}_p.$*
- (2) *The natural map $A \rightarrow \text{End}_{\mathcal{E}}(X)$ is an equivalence.*

Obtaining a DGA from a ring. Suppose that \mathcal{O} is a complete discrete valuation ring with residue class field $\mathbb{F}_p.$ Let X denote \mathbb{Z}/p with the unique possible \mathcal{O} -module structure, and let $A = \text{End}_{\mathcal{O}}(X)$ be the derived endomorphism algebra of $X.$

LEMMA 3.2. *Let $\mathcal{O}, X,$ and A be as above.*

- (1) *The DGA A is of type $\beta_{\mathbb{F}_p}.$*
- (2) *The natural map $\mathcal{O} \rightarrow \text{End}_A(X)$ is an equivalence.*

Proof of Theorem 1.1. The existence and uniqueness of the module of type $(\mathbb{Z}/p, 0)$ is from Section 2. For the rest, Subsection 1.1 and Lemmas 3.1 and 3.2 provide inverse constructions matching up appropriate DGAs with appropriate rings. □

For minor efficiency reasons, we first prove Lemma 3.2 and then Lemma 3.1.

3.1. Proof of Lemma 3.2

For part (1), observe that as usual there are isomorphisms

$$H_i \text{End}_{\mathcal{O}}(X) \cong \text{Ext}_{\mathcal{O}}^{-i}(\mathbb{Z}/p, \mathbb{Z}/p).$$

But there is a short projective resolution of \mathbb{Z}/p over \mathcal{O}

$$0 \longrightarrow \mathcal{O} \xrightarrow{\pi} \mathcal{O} \longrightarrow \mathbb{Z}/p \longrightarrow 0. \tag{3.1}$$

By inspection, then, $\text{Ext}_{\mathcal{O}}^i(\mathbb{Z}/p, \mathbb{Z}/p)$ vanishes unless $i = 0$ or $i = 1,$ and in these two exceptional cases the group is isomorphic to $\mathbb{Z}/p.$

Since $\mathcal{O}/\mathfrak{m} \cong \mathbb{F}_p$ is a field and hence a regular ring, Lemma 3.2(2) is [5, 4.20].

REMARK 3.3. If \mathcal{O} is as in Lemma 3.2, an explicit model for $\text{End}_{\mathcal{O}}(\mathbb{Z}/p)$ can be derived from (3.1) as follows:

$$\begin{array}{ccc}
 1 & \langle L \rangle & \\
 & \downarrow \partial L = \pi D_1 + \pi D_2 & \\
 0 & \langle D_1, D_2 \rangle & \\
 & \downarrow \partial D_i = (-1)^i \pi U & \\
 -1 & \langle U \rangle &
 \end{array}
 \quad
 \begin{array}{l}
 D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
 L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
 \end{array}$$

This is a DGA which is nonzero only in degrees 1, 0, -1; the notation $\langle - \rangle$ denotes the free \mathcal{O} -module on the enclosed generators. From a multiplicative point of view, the DGA is as indicated a graded form of the ring of 2×2 matrices over $\mathcal{O}.$

Proof of Lemma 3.1(1). Let Λ denote the graded algebra H_*A and $x \in \Lambda_{-1}$ an additive generator. Write \mathbb{Z}/p for the (ordinary) graded Λ -module $\Lambda/\langle x \rangle \cong H_*X.$ For general reasons

(see Subsection 3.2) there is a left-half plane Eilenberg–Moore spectral sequence

$$E^2_{-i,j} = \text{Ext}^i_{\Lambda}(\Sigma^j \mathbb{Z}/p, \mathbb{Z}/p) \Rightarrow H_{j-i} \mathcal{E}.$$

(Note that Ext here is computed in the category of graded modules over a graded ring.) This is a conditionally convergent spectral sequence of bigraded algebras with differentials

$$d_r : E^r_{i,j} \longrightarrow E^r_{i-r,j+r-1},$$

and it abuts to the graded algebra $H_* \mathcal{E}$. There is a free resolution of \mathbb{Z}/p over Λ

$$\cdots \xrightarrow{x} \Sigma^{-2} \Lambda \xrightarrow{x} \Sigma^{-1} \Lambda \xrightarrow{x} \Lambda \longrightarrow \mathbb{Z}/p \longrightarrow 0,$$

which leads by calculation to the conclusion that the bigraded algebra $\text{Ext}^i_{\Lambda}(\Sigma^j \mathbb{Z}/p, \mathbb{Z}/p)$ is a polynomial algebra on the extension class

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \Sigma \Lambda \longrightarrow \Sigma \mathbb{Z}/p \longrightarrow 0,$$

in $\text{Ext}^1_{\Lambda}(\Sigma \mathbb{Z}/p, \mathbb{Z}/p)$. It follows that $E^2_{i,j}$ vanishes in the above spectral sequence for $i \neq -j$, the spectral sequence collapses, and $E^{\infty} = E^2$ is concentrated in total degree 0. Hence, $H_i \mathcal{E}$ vanishes for $i \neq 0$, and $H_0 \mathcal{E}$ is a ring \mathcal{O} with a decreasing sequence of ideals

$$\cdots \subset \mathfrak{m}_k \subset \cdots \subset \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset \mathcal{O},$$

such that $\mathfrak{m}_k \mathfrak{m}_\ell \subset \mathfrak{m}_{k+\ell}$, $\text{Gr}(\mathcal{O}) \cong \mathbb{F}_p[t]$, and $\mathcal{O} \cong \lim_k \mathcal{O}/\mathfrak{m}_k$.

Let $\pi \in \mathfrak{m}_1 \subset \mathcal{O}$ be an element that projects to a generator of $\mathfrak{m}_1/\mathfrak{m}_2 \cong \mathbb{Z}/p$, and map $\mathbb{Z}[s]$ to \mathcal{O} by sending s to π . It is easy to argue by induction on k that the composite $\mathbb{Z}[s] \rightarrow \mathcal{O} \rightarrow \mathcal{O}/\mathfrak{m}_k$ is surjective, so that $\mathcal{O}/\mathfrak{m}_k$ is commutative and $\mathcal{O} \cong \lim_k \mathcal{O}/\mathfrak{m}_k$ is commutative as well. The ring \mathcal{O} is a noetherian domain because it has a complete filtration $\{\mathfrak{m}_k\}$ such that $\text{Gr}(\mathcal{O})$ is a noetherian domain [2, Chapter III, Section 2, Corollary 2]. The ideal $\mathfrak{m}_1/\mathfrak{m}_k$ is nilpotent in $\mathcal{O}/\mathfrak{m}_k$, and hence an element $x \in \mathcal{O}/\mathfrak{m}_k$ is a unit if and only the image of x in $\mathcal{O}/\mathfrak{m}_1 \cong \mathbb{F}_p$ is nonzero. It follows directly that $x \in \mathcal{O}$ is a unit if and only if the image of x in $\mathcal{O}/\mathfrak{m}_1$ is nonzero, and so \mathcal{O} is a local ring with maximal ideal $\mathfrak{m} = \mathfrak{m}_1$. Finally, by induction on k there are exact sequences

$$\mathcal{O}/\mathfrak{m}_k \xrightarrow{\pi} \mathcal{O}/\mathfrak{m}_k \longrightarrow \mathcal{O}/\mathfrak{m} \longrightarrow 0.$$

The groups involved are finite, so passing to the limit in k gives an exact sequence

$$\mathcal{O} \xrightarrow{\pi} \mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{m} \longrightarrow 0,$$

expressing the fact that π generates the ideal \mathfrak{m} . The element π is not nilpotent (because its image in $\text{Gr}(\mathcal{O}) \cong \mathbb{F}_p[t]$ is t) and so by Serre [8, I. Section 2] \mathcal{O} is a discrete valuation ring. \square

Proof of Lemma 3.1(2). By Subsection 3.1 and Lemma 3.1(1), $\text{End}_{\mathcal{E}}(X)$ is a DGA of type $\beta_{\mathbb{F}_p}$. It is thus enough to show that the map $H_{-1}A \rightarrow H_{-1}\text{End}_{\mathcal{E}}(X)$ is an isomorphism, or even that this map is nonzero. We will use the notation of Section 2. Note that the mapping cone of the A -module map $\epsilon : A \rightarrow JA$ is again A , and that the mapping cone of any nontrivial map $A \rightarrow X$ is again equivalent to X , this last by a homology calculation and the uniqueness result of Section 2. Consider the following diagram, in which the rows are exact triangles and the middle vertical map is provided by Section 2.

$$\begin{array}{ccccc} A & \longrightarrow & JA & \longrightarrow & A - \overset{a}{} \triangleright \Sigma A \\ \downarrow = & & \downarrow & & \downarrow \epsilon' \\ A & \xrightarrow{\epsilon} & X & \xrightarrow{\pi} & X \end{array} \tag{3.2}$$

Here a denotes (right) multiplication by a generator of $H_{-1}A$. (In the language of [5], the lower row here shows that X is proxy-small over A , and Lemma 3.1(2) now follows fairly

directly from [5, Proof of 4.10]. For convenience, we continue with a direct argument.) After a suitable identification of the mapping cone of ϵ with X , a homology calculation gives that the right vertical map is homotopic to ϵ . The right lower map is entitled to be labeled π , as in Subsection 3.1, because the homology of its mapping cone (namely $H_*\Sigma A = \Sigma H_*A$) evidently represents a nonzero element of

$$\text{Ext}_{H_*A}^1(\Sigma\mathbb{Z}/p, \mathbb{Z}/p).$$

Note that the element π of \mathcal{O} is determined only up to multiplication by a unit, and this is reflected above in the fact that there are various ways to identify the mapping cone of ϵ with X . Applying $\text{Hom}_A(-, X)$ to (3.2) gives

$$\begin{array}{ccccccc} X & \longleftarrow & \text{Hom}_A(JA, X) & \longleftarrow & X & \xleftarrow{a^*} & \Sigma^{-1}X \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \epsilon^* \\ X & \xleftarrow{\epsilon^*} & \mathcal{E} & \xleftarrow{\pi^*} & \mathcal{E} & & \end{array} \tag{3.3}$$

The map ϵ^* is surjective on homology, since by the argument of Section 2 any A -map $A \rightarrow X$ extends over ϵ to a map $X \rightarrow X$. Let M be the \mathcal{O} -module $H_0 \text{Hom}_A(JA, X)$. Applying H_0 to the solid arrows in diagram (3.3) gives a diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longleftarrow & \mathbb{Z}/p & \longleftarrow & M & \longleftarrow & \mathbb{Z}/p \longleftarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \text{onto} \\ & & = & & & & \\ 0 & \longleftarrow & \mathbb{Z}/p & \longleftarrow & \mathcal{O} & \xleftarrow{\pi} & \mathcal{O} \longleftarrow 0 \end{array}$$

The diagram implies that as an \mathcal{O} -module, M is $\mathcal{O}/(\pi^2)$. In particular, the extension on the top line is nontrivial over \mathcal{O} , which, by backing up the exact triangle as in (3.3), implies that $a^*: \Sigma^{-1}X \rightarrow X$ represents a nonzero element of $H_{-1} \text{End}_{\mathcal{E}}(X)$. But, by construction, a^* is given by left multiplication with a generator of $H_{-1}A$. \square

3.2. Eilenberg–Moore spectral sequence

If A is a DGA and X, Y are modules over A , then there is an Eilenberg–Moore spectral sequence

$$E_{-i,j}^2 = \text{Ext}_{H_*A}^i(\Sigma^j H_*X, H_*Y) \Rightarrow H_{j-i} \text{Hom}_A(X, Y).$$

In a homotopy context, this can be constructed in precisely the same way as an Adams spectral sequence. In an algebraic context, it is constructed by inductively building an exact sequence of A -modules

$$0 \longleftarrow X \longleftarrow F(0) \longleftarrow F(1) \longleftarrow \dots \longleftarrow F(i) \longleftarrow \dots, \tag{3.4}$$

such that

- (i) each $F(i)$ is a sum of shifts of copies of A and
- (ii) applying H_* to (3.4) produces a free resolution of H_*X over H_*A .

See, for instance, [1, 9.11]. The totalization tF of the double complex F is then a projective (or cofibrant) model for X ; filtering tF by $\{tF(\leq n)\}_n$ and applying $\text{Hom}_A(-, Y)$ gives a filtration of $\text{Hom}_A(X, Y)$ that yields the spectral sequence.

4. Exterior algebras over \mathbb{Z}

In this section, we prove the first claim of Theorem 1.3. To be specific, we show that any DGA A of type $\beta_{\mathbb{Z}}$ is equivalent to the formal DGA \mathcal{F} given by the following chain complex concentrated in degrees $0, -1$.

$$\begin{array}{ccc} 0 & \mathbb{Z} & \\ & \downarrow \partial=0 & \\ -1 & \mathbb{Z} & \end{array}$$

The multiplication on \mathcal{F} is the only possible one consistent with the requirement that $1 \in \mathbb{Z} = \mathcal{F}_0$ act as a unit.

We will only sketch the line of reasoning, since it is similar to that in sections Sections 2 and 3, although the conclusion is very different. Along the lines of Section 2 there exists a module X of type $(\mathbb{Z}, 0)$ over A , and this is unique up to noncanonical equivalence. Let $\mathcal{E} = \text{End}_A(X)$. The argument in the proof of Lemma 3.1(1) shows that $H_i\mathcal{E}$ vanishes for $i \neq 0$, and that $H_0\mathcal{E}$ is a ring R with a decreasing sequence of ideals

$$\cdots \subset \mathfrak{m}_k \subset \cdots \subset \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset R,$$

such that $\mathfrak{m}_k\mathfrak{m}_\ell \subset \mathfrak{m}_{k+\ell}$, $\text{Gr}(R) \cong \mathbb{Z}[t]$, and $R \cong \lim_k R/\mathfrak{m}_k$. Let $\sigma \in \mathfrak{m}_1$ be an element that projects to a generator of $\mathfrak{m}_1/\mathfrak{m}_2 \cong \mathbb{Z}$, and map $\mathbb{Z}[s]$ to R by sending s to σ . It is easy to show by induction on k that this map induces isomorphisms $\mathbb{Z}[s]/(s^{k+1}) \rightarrow R/\mathfrak{m}_k$, and so induces an isomorphism $\mathbb{Z}[[s]] \rightarrow R$. Using the free resolution

$$0 \longrightarrow \mathbb{Z}[[s]] \xrightarrow{s} \mathbb{Z}[[s]] \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{4.1}$$

of \mathbb{Z} over $\mathbb{Z}[[s]]$ (s acting by zero on \mathbb{Z}), one argues as in the proof of Lemma 3.2(1) that $\text{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$ is a DGA of type $\beta_{\mathbb{Z}}$. As in the proof of Lemma 3.1(2), the natural map $A \rightarrow \text{End}_{\mathcal{E}}(X) \sim \text{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$ is an equivalence. This reasoning applies in particular to \mathcal{F} , giving $\mathcal{F} \sim \text{End}_{\mathbb{Z}[[s]]}(\mathbb{Z})$. Hence, A is equivalent to \mathcal{F} .

5. Noncanonical modules of type $(\mathbb{Z}/p, 0)$

In this section, we complete the proof of Theorem 1.3. The uniqueness statement for A was proved in Section 4, so we only have to handle the classification of modules of type $(\mathbb{Z}/p, 0)$.

Let $R = \mathbb{Z}[s]$ and let R act on \mathbb{Z} via $\mathbb{Z} \cong R/(s)$. Clearly, $\text{Ext}_R^i(\mathbb{Z}, \mathbb{Z})$ is \mathbb{Z} for $i = 0, 1$ and zero otherwise, so that $\mathcal{E} = \text{End}_R(\mathbb{Z})$ is a DGA of type $\beta_{\mathbb{Z}}$. By uniqueness, we may as well take $A = \mathcal{E}$; the following proposition explains why this is useful.

Say that a chain complex N of R -modules is s -torsion if for each $i \in \mathbb{Z}$ and each $x \in H_i N$ there is an integer $k(x) > 0$ such that $s^{k(x)}x = 0$.

PROPOSITION 5.1 [4, 2.1]. *The assignment $N \mapsto \text{Hom}_R(\mathbb{Z}, N)$ restricts to a bijection between equivalence classes of s -torsion chain complexes over R and equivalence classes of right \mathcal{E} -modules.*

REMARK 5.2. Here \mathcal{E} acts on $\text{Hom}_R(\mathbb{Z}, N)$ through its action on \mathbb{Z} . The inverse to this bijection assigns to a right \mathcal{E} -module X the derived tensor product $X \otimes_{\mathcal{E}} \mathbb{Z}$.

It is clear from Section 4 that $\mathcal{E} \sim \mathcal{F}$ is equivalent as a DGA to its opposite algebra, so we can pass over the distinction between right and left \mathcal{E} -modules. The question of studying

\mathcal{E} -modules of type $(\mathbb{Z}/p, 0)$ thus becomes one of classifying s -torsion chain complexes N over R such that

$$H_i \operatorname{Hom}_R(\mathbb{Z}, N) \cong \begin{cases} \mathbb{Z}/p, & i = 0, \\ 0, & i \neq 0. \end{cases} \tag{5.1}$$

Suppose that N is such a chain complex. Applying $\operatorname{Hom}_R(-, N)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{s} R \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and taking homology gives a long exact sequence

$$\cdots \longrightarrow H_i \operatorname{Hom}_R(\mathbb{Z}, N) \longrightarrow H_i N \xrightarrow{s} H_i N \longrightarrow \cdots,$$

in which the arrow labeled ‘ s ’ cannot be injective unless $H_i N = 0$. It follows that $H_i N$ vanishes unless $i = 0$, and that $H_0 N$ is an abelian group D on which the operator s acts with the following properties:

- (1) $\ker(s)$ is isomorphic to \mathbb{Z}/p ;
- (2) s is surjective; and
- (3) $\bigcup_{k \geq 0} \ker(s^k) = D$.

Call such a pair (D, s) *distinguished*. In fact, any distinguished pair (D, s) gives a chain complex N over R (D itself concentrated in degree 0) which has property (5.1). Combining this observation with Remark 1.5 and Proposition 5.1 thus provides a bijection between

- (i) equivalence classes of modules over \mathcal{E} of type $(\mathbb{Z}/p, 0)$ and
- (ii) isomorphism classes of distinguished pairs (D, s) .

The proof of Theorem 1.3 is completed by the following two routine lemmas. Recall that $R = \mathbb{Z}[s]$. If \mathcal{O} is a discrete valuation ring with uniformizer π , write \mathcal{O}/π^∞ for the quotient $\mathcal{O}[1/\pi]/\mathcal{O}$. (This quotient is the injective hull of \mathcal{O}/π as an ordinary module over \mathcal{O} .)

LEMMA 5.3. *Suppose that (D, s) is a distinguished pair. Then $\mathcal{O} = \operatorname{Ext}_R^0(D, D)$ is a complete discrete valuation ring with residue field \mathbb{F}_p and uniformizer $\pi = s$. The pair (D, s) is naturally isomorphic to $(\mathcal{O}/\pi^\infty, \pi)$.*

LEMMA 5.4. *Suppose that \mathcal{O} is a complete discrete valuation ring with residue field \mathbb{F}_p and uniformizer π . Then $(D, s) = (\mathcal{O}/\pi^\infty, \pi)$ is a distinguished pair, and the natural map $\mathcal{O} \rightarrow \operatorname{Ext}_R^0(D, D)$ is an isomorphism.*

REMARK 5.5. According to Theorem 1.3, any complete discrete valuation ring \mathcal{O} with residue field \mathbb{F}_p and uniformizer π should give rise to a module X of type $(\mathbb{Z}/p, 0)$ over the formal DGA \mathcal{F} of Section 4. Observe that a module over \mathcal{F} is just a chain complex with a self-map f of degree -1 and square 0. Let $(D, s) = (\mathcal{O}/\pi^\infty, \pi)$ be the distinguished pair associated to \mathcal{O} and π . Tracing through the above arguments shows that X can be taken to be the following object

$$\begin{array}{ccc} 0 & & D \\ & \searrow^{f=\text{id}} & \downarrow \partial=s \\ -1 & & D \end{array}$$

concentrated in degrees 0 and -1 .

6. Other exterior algebras

Suppose that R is a commutative ring, and say that a DGA A is of type $\beta_R(n)$ if H_*A is an exterior algebra over R on a class of degree n . In this section, we briefly consider the problem of classifying such DGAs if $n \neq -1$. If $n = 0$, then A is formal (Subsection 1.1) and hence determined up to equivalence by the ring H_0A , so we may as well also assume $n \neq 0$.

Say that a DGA is of type $P_R(n)$ if H_*A is isomorphic to a polynomial algebra over R on a class of degree n .

PROPOSITION 6.1. *If $n \notin \{0, -1\}$, then there is a natural bijection between equivalence classes of DGAs of type $\beta_R(n)$ and equivalence classes of DGAs of type $P_R(-n-1)$.*

Proof (Sketch). If A is of type $\beta_R(n)$, then the inductive technique of Section 2 produces a left module X_A over A of type $(R, 0)$, that is, a module X_A such that $H_0(X_A)$ is a free (ordinary) module of rank 1 over $R = H_0A$ and $H_i(X_A) = 0$ for $i \neq 0$. If B is of type $P_R(n)$, then it is even easier to produce a right module X_B over B of type $(R, 0)$: just take the mapping cone of $f: \Sigma^n B \rightarrow B$, where f represents left multiplication by a generator of $H_n B$. In both cases, the modules are unique up to possibly noncanonical equivalence. Along the lines of Section 3 (cf. Subsection 1.2), calculating with appropriate collapsing Eilenberg–Moore spectral sequences now gives the desired bijection.

$$\{DGAs A \text{ of type } \beta_R(n)\} \begin{array}{c} \xrightarrow{A \mapsto \text{End}_A(X_A)} \\ \xleftarrow{B \mapsto \text{End}_B(X_B)} \end{array} \{DGAs B \text{ of type } P_R(-n-1)\}. \quad \square$$

REMARK 6.2. It is clear from Theorem 1.1 that Proposition 6.1 fails for $n = -1$ and $R = \mathbb{F}_p$, essentially because if A is of type $\beta_{\mathbb{F}_p}(-1)$, then the nonvanishing entries of the Eilenberg–Moore spectral sequence for $H_* \text{End}_A(X_A)$ accumulate in degree 0. This accumulation creates extension possibilities that allow for a profusion of complete discrete valuation rings in the abutment. Similarly, Proposition 6.1 fails for $n = 0$ and $R = \mathbb{F}_p$, because if B is of type $P_{\mathbb{F}_p}(-1)$, then the Eilenberg–Moore spectral sequence for $H_* \text{End}_B(X_B)$ accumulates in degree 0. In this case, though, the accumulation has consequences which are less drastic, because up to isomorphism there are only two possibilities for a ring R with an ideal I such that $I^2 = 0$ and such that the associated graded ring $\{R/I, I\}$ is an exterior algebra on one generator over \mathbb{F}_p . The conclusion is that up to equivalence there are only two DGAs of type $P_{\mathbb{F}_p}(-1)$, one corresponding to the true exterior algebra $\mathbb{F}_p[t]/t^2$, and the other to the fake exterior algebra \mathbb{Z}/p^2 .

6.1. DGAs of type $\beta_{\mathbb{Z}}(n)$, all n

Up to equivalence, there is only one of these for each n . The case $n = 0$ is trivial (Subsection 1.1), while $n = -1$ is Theorem 1.3. By Proposition 6.1, for other n these correspond to DGAs of type $P_{\mathbb{Z}}(n)$, but there is only one of these for each n , because they are all formal (Subsection 1.1).

6.2. DGAs of type $\beta_{\mathbb{F}_p}(n)$, $n \geq 0$

As usual, the case $n = 0$ is trivial: up to equivalence there is only one example. We sketch an argument that if $n > 0$ is odd, then there is only one example, while if $n > 0$ is even, then there are two. In [3], Dugger and Shipley describe a Postnikov approach to constructing a connective (Remark 1.5) DGA A ; the technique involves starting with the ring H_0A (considered as a DGA with trivial higher homology) and attaching one homology group at a time, working from low dimensions to high. If A is of type $\beta_{\mathbb{F}_p}(n)$, then there is only a single homology group

to deal with, namely \mathbb{Z}/p in degree n . By Dugger and Shipley [3, Theorem 8] and a theorem of Mandell [3, Remark 8.7] the choices involved in attaching $H_n A$ can be identified with the group $HH_{\mathbb{Z}}^{n+2}(\mathbb{F}_p, \mathbb{Z}/p)$; this is Shukla cohomology of \mathbb{F}_p with coefficients in the \mathbb{F}_p -bimodule \mathbb{Z}/p . In our notation, this group might be written

$$H_{-n-2} \operatorname{Hom}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p^{\text{op}}}(\mathbb{Z}/p, \mathbb{Z}/p), \tag{6.1}$$

where the indicated tensor product over \mathbb{Z} is derived. (This is the appropriate variant of Hochschild cohomology when the ring involved, here \mathbb{F}_p , is not flat over the ground ring, here \mathbb{Z} .) The ring $H_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p^{\text{op}})$ is an exterior algebra over \mathbb{F}_p on a class of degree 1, so the Eilenberg–Moore spectral sequence computes

$$H_* \operatorname{Hom}_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p^{\text{op}}}(\mathbb{Z}/p, \mathbb{Z}/p) \cong \mathbb{F}_p[u],$$

where the degree of u is -2 . This immediately shows that the group (6.1) of gluing choices is trivial if n is odd. If n is even, then there are p gluing choices, but $p - 1$ of them are identified by the automorphisms of \mathbb{Z}/p as an \mathbb{F}_p -bimodule. The conclusion is that if n is even, then there are p gluing choices, but that up to equivalence only two DGAs emerge.

By Proposition 6.1, this also gives a classification of DGAs of type $P_{\mathbb{F}_p}(n)$ for $n \leq -2$.

We do not know how to classify DGAs of type $\beta_{\mathbb{F}_p}(n)$ for $n \leq -2$.

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