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GLOBAL STABILIZATION OF LINEAR ANALYTIC SYSTEMS

by

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Abstract

A technique is given for the choice of a stable, and globally-attracting switching surface for a linear-analytic system, assuming discontinuous feedback is implemented. The method is based on the solution of a pair of coupled partial differential equations, and a simple geometrical alternative is derived. Only mild conditions are required to be satisfied by the system, and a generalization to systems expressed as polynomials in the control, is also considered.
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equivalent to the two PDE's will be derived in section 5, and a brief discussion of nonlinear control systems will be given in section 6.

2 Notation

The notation in this paper will be standard. Two points only need to be specified. First we shall denote the inner product of two vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$< \mathbf{x}, \mathbf{y} >$$

rather than the usual $\mathbf{x} \cdot \mathbf{y}$ or $\mathbf{x}^T\mathbf{y}$. Secondly, in section 6 we shall use a number of vector fields $\mathbf{g}^0, \mathbf{g}^1, \ldots, \mathbf{g}^n$. The superscript index notation is needed to distinguish a vector field from its components, and should not be confused with a power.

A final point to note is that $||\cdot||$ denotes the standard Euclidean norm.

3 Choice of Switching Manifold

Consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{3.1}$$

We shall assume that the set

$$\Omega = \{\mathbf{x} : \mathbf{g}(\mathbf{x}) = 0\}$$

is a smooth submanifold of $\mathbf{R}^n$. We shall also assume that the vector field defined by $\mathbf{f}(\mathbf{x})$ is nowhere tangent to Ω.

In this section we shall consider the existence of a switching manifold, and prove its global attractivity. The discussion of the stability of the equivalent system on the manifold will be left to section 4. In order to determine a switching manifold we use an idea from (Banks 1986).

THEOREM 1 *Let $\alpha(\mathbf{x})$ be a nonnegative real-valued function such that*

$$(i) \quad \alpha(\mathbf{x}) \neq 0 \;\; for \;\; all \;\; \mathbf{x} \;\; if \;\; g(0) \neq 0$$

$$(ii) \quad \alpha(\mathbf{x}) \neq 0\,, \;\; \mathbf{x} \neq 0\,, \;\; \alpha(0) = 0 \;\; if \;\; g(0) = 0$$

and assume that the partial differential equation

$$\frac{\partial \sigma(\mathbf{x})}{\partial x_1} g_1(\mathbf{x}) + \ldots + \frac{\partial \sigma(\mathbf{x})}{\partial x_n} g_n(\mathbf{x}) \;=\; \alpha(\mathbf{x}) \tag{3.2}$$

has a solution $\sigma(\mathbf{x})$ such that $\sigma(0) = 0$ and the level curves $\sigma(\mathbf{x}) = const.$ are (n-1)-dimensional smooth submanifolds (or algebraic submanifolds) of $\mathbb{R}^n$. Let $S = \{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$ and suppose that $S \cap \Omega$ is an (n-1-(n-m))-dimensional submanifold of $\mathbb{R}^n$, and that S and Ω intersect properly, where $n = dim\Omega$. Then S is a globally attracting switching manifold for the system (1) if the control is defined by

$$u = \begin{cases} \frac{-<\mathbf{f},\, grad\ \sigma> - c}{<\mathbf{g},\, grad\ \sigma>}, & if \;\; \mathbf{x} \in S^+ \backslash \Omega \\ \\ \frac{-<\mathbf{f},\, grad\ \sigma> + c}{<\mathbf{g},\, grad\ \sigma>}, & if \;\; \mathbf{x} \in S^- \backslash \Omega \end{cases} \tag{3.3}$$

and

$$u = 0\,, \quad \mathbf{x} \in \Omega \tag{3.4}$$

where $S^+ = \{\mathbf{x} : \sigma(\mathbf{x}) > 0\}$, $S^- = \{\mathbf{x} : \sigma(\mathbf{x}) < 0\}$, and $c > 0$ is a constant.

Proof: Note first that if $\mathbf{x} \neq 0$, then $< \mathbf{g},\ grad\ \sigma >= \alpha(\mathbf{x}) \neq 0$ by (2), and so u is well-defined. The proof that S is a globally attracting switching manifold is now easy. For, we have

$$\dot{\sigma} \ = < grad\ \sigma\ ,\ \dot{\mathbf{x}} > = < grad\ \sigma\ ,\ \mathbf{f} > \ + u < grad\ \sigma,\ \mathbf{g} >$$

and if, for example $\mathbf{x} \in S^+ \setminus \Omega$, then by (3),

$$\dot{\sigma} \ = \ -c$$

and so $\sigma \longrightarrow 0$ in finite time. Similarly, if $\mathbf{x} \in S^- \setminus \Omega$

$$\dot{\sigma} \ = \ c$$

and so $\sigma \longrightarrow 0$ again in finite time. If $\mathbf{x} \in \Omega$ then by assumption (2) if $u = 0$ the trajectories leave Ω and enter either $S^+ \setminus \Omega$ or $S^- \setminus \Omega$ and the above reasoning holds. □

Example 1: Consider the bilinear system with $n = 3$

$$\dot{\mathbf{x}} \ = \ \bar{\mathbf{A}}\mathbf{x} + u\bar{\mathbf{B}}\mathbf{x} \qquad (3.5)$$

with scalar control. For simplicity we shall assume that $\bar{\mathbf{B}}$ is diagonalizable with eigenvalues $\lambda_1 < 0,\ \lambda_2 > 0,\ \lambda_3 = 0$. By changing coordinates we can write (4) in the form

$$\dot{\mathbf{x}} \ = \ \mathbf{A}x + u\boldsymbol{\Lambda}\mathbf{x} \qquad (3.6)$$

where $\boldsymbol{\Lambda} = diag\,(\lambda_1,\ \lambda_2,\ \lambda_3)$

Then equation (2) becomes

$$\frac{\partial \sigma}{\partial x_1}\lambda_1 x_1 + \frac{\partial \sigma}{\partial x_2}\lambda_2 x_2 \;=\; \alpha(\mathbf{x}) \tag{3.7}$$

A solution of this equation can be found by solving the ordinary differential equations

$$\frac{dx_1}{\lambda_1 x_1} \;=\; \frac{dx_2}{\lambda_2 x_2} \;=\; \frac{d\sigma}{\alpha} \tag{3.8}$$

However, in this case, if we take

$$\alpha(\mathbf{x}) \;=\; x_1^2 + x_2^2$$

Then a solution to (8) can be obtained directly as

$$\sigma(\mathbf{x}) \;=\; \frac{x_1^2}{2\lambda_1} + \frac{x_2^2}{2\lambda_2}$$

If we assume for simplicity that $\lambda_1 = -\lambda_2 = \lambda$, then

$$\sigma(\mathbf{x}) \;=\; \frac{x_1^2}{2\lambda} - \frac{x_2^2}{2\lambda}$$

and $\sigma(0) = 0$ when $x_1 = \pm x_2$

In this case $\Omega = \{\mathbf{x} : x_1 = x_2 = 0\}$, and the control (3)-(4) is now

$$u = \begin{cases} \frac{(\sum_k a_{1k}x_k)\frac{x_1}{\lambda} + (\sum_k a_{2k}x_k)\frac{x_2}{\lambda} - c}{x_1^2 + x_2^2}, & if\ \ \sigma > 0 \\ \\ \frac{(\sum_k a_{1k}x_k)\frac{x_1}{\lambda} + (\sum_k a_{2k}x_k)\frac{x_2}{\lambda} + c}{x_1^2 + x_2^2}, & if\ \ \sigma < 0 \end{cases}$$

Remark 1: The ideal control in (3) is unbounded near Ω. This can be overcome in practice by making $u = 0$ in a neighbourhood of Ω.

Remark 2: We can also allow $\alpha(\mathbf{x})$ to be zero away from the origin, provided

$$\{\mathbf{x} : \alpha(\mathbf{x}) = 0\} \cap \{\mathbf{x} : \sigma(\mathbf{x}) = 0\} \;=\; \{0\}$$

and again that $\mathbf{f}$ is nowhere tangent to $\{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$.

This allow us to extend Example 1 to the case where $\lambda_1,\ \lambda_2 > 0$ by taking $\alpha(\mathbf{x}) = x_1^2 - x_2^2$ if $\lambda_1 \neq \lambda_2$ and $\alpha(\mathbf{x}) = x_1^2 + x_1 - x_2^2$ if $\lambda_1 = \lambda_2$.

4 Stability of the Switching Manifold

Having obtained conditions for the existence of a switching manifold we must now consider the stability of the dynamics on this manifold. By Filippov's result (Filippov 1960) the equivalent dynamics on the manifold are given by the equation

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}) - \frac{<\mathbf{f},\, grad\,\sigma>}{<\mathbf{g},\, grad\,\sigma>}\mathbf{g}(\mathbf{x}) \tag{4.1}$$

where $<\mathbf{g},\, grad\,\sigma> = \alpha > 0$ on the switching manifold (and possibly at $\mathbf{x} = \mathbf{0}$). The next result gives a condition on $\mathbf{f}$ and $\mathbf{g}$ for the switching manifold to be stable.

THEOREM 2 *Suppose that the switching manifold σ satisfies (in addition to (2)) the partial differential equation*

$$<\mathbf{f},\mathbf{x}> - \frac{<\mathbf{f},\, grad\,\sigma><\mathbf{g},\mathbf{x}>}{\alpha} \;=\; -\beta(\mathbf{x}) \tag{4.2}$$

for some function β satisfying

$$\beta(\mathbf{x}) > 0\,,\quad \mathbf{0} \neq \mathbf{x} \in S$$

then the dynamics (9) are stable.

Proof: This follows from (9) by taking the inner product of the equation with $\mathbf{x}$. Thus, on S,

$$\frac{1}{2}\frac{d}{dt} \| \mathbf{x} \|^2 = < \dot{\mathbf{x}}, \mathbf{x} > = < \mathbf{f}, \mathbf{x} > - \frac{< \mathbf{f}, \, grad\, \sigma >}{\alpha} < \mathbf{g}, \mathbf{x} >$$

$$= -\beta(\mathbf{x}) < 0 \qquad \square$$

Equation (10) may be written in the form

$$\frac{\partial \sigma}{\partial x_1} f_1(\mathbf{x})\gamma + \frac{\partial \sigma}{\partial x_2} f_2(\mathbf{x})\gamma + \ldots + \frac{\partial \sigma}{\partial x_n} f_n(\mathbf{x})\gamma \;=\; \eta \tag{4.3}$$

where

$$\gamma(\mathbf{x}) \;= < \mathrm{g}, \mathbf{x} >$$

$$\eta(\mathbf{x}) \;= < \mathbf{f}, \mathbf{x} > \alpha + \alpha\beta$$

Example 2: Consider the bilinear system given by (6) with $\lambda_1, \lambda_2 > 0$ as in Remark 2. Then

$$\sigma \;=\; \frac{x_1^2}{2\lambda_1} - \frac{x_2^2}{2\lambda_2}, \quad \alpha = x_1^2 - x_2^2$$

(from $\lambda_1 \neq \lambda_2$). Then (10) becomes

$$\frac{x_1}{\lambda_1} f_1 - \frac{x_2}{\lambda_2} f_2 \;=\; \frac{(f_1 x_1 + f_2 x_2 + f_3 x_3 + \beta)(x_1^2 - x_2^2)}{\lambda_1 x_1^2 + \lambda_2 x_2^2}$$

where $f_i = \sum a_{ij} x_j$, with $\mathbf{A} = [a_{ij}]$.

On $\sigma = 0$ we have

$$x_1^2 \;=\; \frac{\lambda_2}{\lambda_1} x_1^2$$

and so, taking $\lambda_1 = 2$, $\lambda_2 = 1$ for simplicity, gives

$$a_{13} = 0\,, \quad a_{23} = 0\,, \quad a_{33} < 0$$

and $\frac{3}{2}a_{11}x_1^2 + \frac{3}{2}a_{12}x_2x_1 - 6a_{21}x_1x_2 + 6a_{22}x_2^2 > 0$ when $x_1 = \pm\sqrt{2x_2}$. From this one can obtain, for example, the conditions

$$a_{11} > 0\,, \quad \tfrac{3\sqrt{2}}{2}a_{12} - 6\sqrt{2}a_{21} + 6a_{22} > 0$$

$$\tfrac{3}{2}a_{11} - \tfrac{3}{2\sqrt{2}}a_{12} + \tfrac{6}{\sqrt{2}}a_{21} > 0\,, \quad a_{22} > 0$$

Note that choosing a different function α will give rise to different conditions and the choice of α is difficult in practice. Therefore, in the next section we shall derive a simple geometrical condition for the existence of stable sliding modes.

5 Existence of Stable Sliding Modes

We have shown that for a stable sliding mode to exist it is sufficient for the partial differential equations (2) and (11), i.e.

$$\frac{\partial \sigma}{\partial x_1} g_1(\mathbf{x}) + \ldots + \frac{\partial \sigma}{\partial x_n} g_n(\mathbf{x}) \;=\; \alpha(\mathbf{x}) \tag{5.1}$$

and

$$\frac{\partial \sigma}{\partial x_1} f_1(\mathbf{x}) < \mathbf{g}, \mathbf{x} > + \ldots + \frac{\partial \sigma}{\partial x_n} f_n(\mathbf{x}) < \mathbf{g}, \mathbf{x} > \,=\, < \mathbf{f}, \mathbf{x} > \alpha + \alpha\beta \tag{5.2}$$

to have a solution such that $S = \{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$ is an (n-1)-dimensional algebraic submanifold of $\mathbf{R}^n$ with $\mathbf{0} \in S$, and $\beta(\mathbf{x}) > 0$ for $\mathbf{0} \neq \mathbf{x} \in S$. Moreover we should have

$$S \cap \{\mathbf{x} : \alpha(\mathbf{x}) = 0\} = \{0\}$$

In general we could solve (12) and (13) under suitable compatibility assumptions for various functions α and β. However, the choice of such functions α and β is far from obvious, and the solutions of (12), (13) is not an easy matter for highly nonlinear systems. We shall therefore give a simple geometrical criterion equivalent to (12) and (13) which may be used to construct a switching manifold directly.

THEOREM 3 *If $\theta_1, \theta_2, \psi_1, \psi_2$ are given by*

$$cos\ \theta_1 = \frac{< \mathbf{f}, \mathbf{x} >}{\| \mathbf{f} \| \ \| \mathbf{x} \|}, \quad cos\ \theta_2 = \frac{< \mathbf{g}, \mathbf{x} >}{\| \mathbf{g} \| \ \| \mathbf{x} \|}$$

$$cos\ \psi_1 = \frac{< grad\ \sigma, \mathbf{f} >}{\| grad\ \sigma \| \ \| \mathbf{f} \|}, \quad cos\ \psi_2 = \frac{< grad\ \sigma, \mathbf{g} >}{\| grad\ \sigma \| \ \| \mathbf{g} \|}$$

then a necessary and sufficient condition for (12), (13) to hold is that

$$cos\ \psi_2 > 0 \tag{5.3}$$

and

$$cos\ \theta_1\ cos\ \psi_2 - cos\ \theta_2\ cos\ \psi_1 > 0 \tag{5.4}$$

Proof: Since $\alpha > 0$ for all $\mathbf{x} \neq \mathbf{0}$, we have

$$cos\ \psi_2 = \left(\frac{\partial \sigma}{\partial x_1} g_1 + \ldots + \frac{\partial \sigma}{\partial x_n} g_n \right) / (\| grad\ \sigma \| \ \| \mathbf{g} \|) > 0$$

Hence, by (13)

$$\frac{cos\ \psi_1 \ \| \mathbf{f} \| < \mathbf{g}, \mathbf{x} > \ - \ cos\ \psi_2 \ \| \mathbf{g} \| < \mathbf{f}, \mathbf{x} >}{cos\ \psi_2 \ \| \mathbf{g} \|} = \beta > 0 \qquad \square$$

Example 3: Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \\ x_1 \end{bmatrix} u \qquad (5.5)$$

The conditions (14) and (15) can easily be implemented numerically and give rise to a switching surface of the form shown in fig.1. Note, however, that this is by no means unique.

6 Higher Order Systems

In this section we shall consider the more general system

$$\dot{\mathbf{x}} = \mathbf{g}^0(\mathbf{x}) + u\mathbf{g}^1(\mathbf{x}) + \ldots + u^m\mathbf{g}^m(\mathbf{x}) \qquad (6.1)$$

A simple generalization of Theorem 1 gives

THEOREM 4 *Suppose that there exists a differentiable function σ such that the equations*

$$\sum_{i=0}^{m} u^i < grad\,\sigma, \mathbf{g}^i(\mathbf{x}) > = -c \qquad (6.2)$$

for $\mathbf{x} \in S^+$, and

$$\sum_{i=0}^{m} u^i < grad\,\sigma, \mathbf{g}^i(\mathbf{x}) > = +c \qquad (6.3)$$

for $\mathbf{x} \in S^-$, have a real solution u for all $\mathbf{x}$, where

$$S = \{\mathbf{x} : \sigma(\mathbf{x}) = 0\}$$

and S^+, S^- are defined as before, then the hypersurface S is globally attracting for the system (17).

Proof: As before we simply differentiate σ along the trajectories of (17). □

Conditions for the existence of solutions of the polynomials (18) and (19) in terms of the discriminant variety can be found in (Banks 1986). We shall merely state two obvious special cases.

Case 1. m odd. In this case equations (18) and (19) always have a real solution and we can bound the size of the control by solving the equation

$$< grad\,\sigma, \mathbf{g}^m(\mathbf{x}) > = \alpha(\mathbf{x}) \tag{6.4}$$

for σ just as with linear control.

Case 2. m even. and $\mathbf{g}^{m-1}(\mathbf{x})$ is never parallel to $\mathbf{g}^m(\mathbf{x})$. In this case we can solve the equation

$$< grad\,\sigma, \mathbf{g}^m(\mathbf{x}) > = 0 \tag{6.5}$$

for σ and then we shall have

$$< grad\,\sigma, \mathbf{g}^{m-1}(\mathbf{x}) > \neq 0$$

for all $\mathbf{x}$, and this reduces to case 1 with m replaced by $m-1$.

7 Conclusions

In this paper we have given a technique for the automatic calculation of a stable, globally attracting switching manifold for a linear-analytic system (and more general systems in section 6).

The method can be implemented by solving a pair of coupled PDE's or by satisfying the geometrical conditions (14) and (15). As we have seen, even in simple examples, this can lead to strange switching surfaces which could not be predicted by other methods.

Further work remains to be done in the study of the PDE's (existence and uniqueness theory, for example) and to obtain explicit solutions. Identification of appropriate functions σ and β must also be considered in more detail. We shall address these issues in a future paper.

8 References

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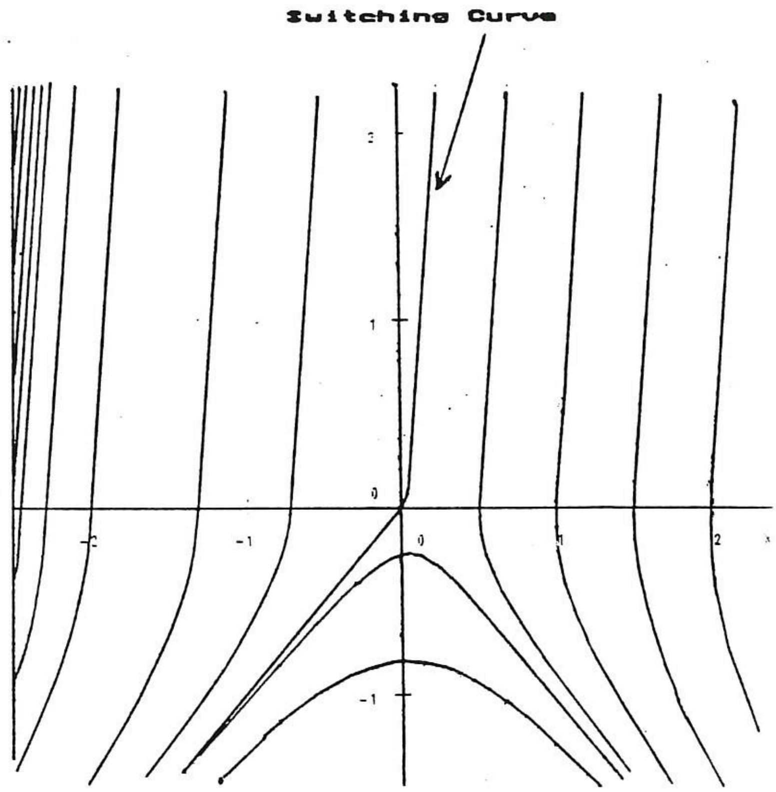

fig. 1

Computer Generated Level Surfaces for Example 3 .
Obtained with $|\text{grad}\,\sigma| = 1$.