Article:

Dodoo-Amoo, NA, Saeed, K, Li, LH et al. (2013) The quantum percolation model of the scaling theory of the quantum Hall effect: a unifying model for plateau-to-plateau transitions. Journal of Physics: Conference Series, 456. 012007. ISSN: 1742-6588

https://doi.org/10.1088/1742-6596/456/1/012007

Journal of Physics: Conference Series **456** (2013) 012007 doi:10.1088/1742-6596/456/1/012007

The quantum percolation model of the scaling theory of the quantum Hall effect: a unifying model for plateau-to-plateau transitions.

N. A. Dodoo-Amoo, K. Saeed, L. H. Li, E. H. Linfield, A. G. Davies, J. E. Cunningham

School of Electronic and Electrical Engineering, University of Leeds, Leeds, LS2 9JT, UK

E-mail: el06nada@leeds.ac.uk

Abstract. We present a unifying model of plateau-to-plateau transitions in the quantum Hall effect based on results from high resolution frequency scaling experiments. We show that as the frequency or quantum coherence length of the two-dimensional electron system is varied, one observes a crossover between classical percolation and quantum percolation in the measured values of the critical scaling exponents of the plateau-to-plateau transitions. This crossover is dependent on the relationship between certain relevant length scales of a two-dimensional system and can be explained using a quantum percolation model. The model explains why quantum criticality can be observed in some systems, but is absent from others.

1. Introduction

The scaling theory of plateau-to-plateau transitions (PPTs) in the quantum Hall effect (QHE) has been of great interest ever since its discovery [1]. The scaling theory has its origins in the early work of Abrahams *et al*. [2], and is based on the observation that, as the centre of a Landau level (LL) is approached, the localization length ξ diverges to infinity, forming a singularity at the centre of the LL. This singularity which can be described as

$$\xi \propto |E - E_c|^{-\gamma}, \tag{1}$$

indicates the existence of a second order quantum phase transition between the two insulating phases of a plateau-to-plateau transition (PPT) [3]. As expected with all phase transitions, this singularity is associated with certain critical universal scaling exponents, such as the critical exponent of the localization length γ in Eq. (1). If this exponent is a quantum critical universal constant it should be independent of heterostructure, LL or microscopic sample details, such as the mobility and carrier concentration.

The value of γ in two-dimensional electron systems (2DESs) has been the subject of sustained theoretical interest [4, 5]. The most numerically accurate value so far as pointed out in a recent review by Kramer *et al*. [4] is $\gamma = 2.33 \pm 0.03$ from a Monte Carlo calculation for chiral fermions [5].

Though the value of γ has been theoretically established and refined over many decades of analytical and numerical investigations, the experimentally determined value of γ has been in dispute [5-7] ever since the first reported experimental studies by Wei *et al*. [8].

The aim of the work presented here is to clarify and illuminate the reasons for the apparent contradiction between the experimentally determined value of γ and its theoretically value which is expected to be $\gamma \approx 2.3$. We achieve this by considering electron transport within the bulk of the 2DES as being through networks of percolating clusters of electrons. We will show that using this percolation framework, one can describe the crossover between regions of universal quantum criticality, and regions where this quantum criticality is absent.

2. Classical and quantum critical percolation of plateau-to-plateau transitions

In 2DESs formed in heterostructures, there are two main sources of disorder which are both connected with impurities. The first is generated by the ionized dopants, which in some heterostructures are separated from the 2DES by a spacer layer. The second is caused by the residual ions located in the vicinity of the 2DES. These two sources of disorder both form a background random potential fluctuation, which in a linear screening regime can be effectively screened by small redistributions of the electron density [9]. Close to integer filling factors, however, screening becomes non-linear and the electron density exhibits strong inhomogeneities across the surface of the 2DES [10]. The background potential fluctuations tear apart the Fermi liquid into isolated clusters of electrons and, as a result, at sufficiently low temperatures the dissipative conductivity tends to zero.

According to the percolation model of the quantum Hall effect, in a 2DES with a slow varying background random potential, these isolated clusters consist of contours of closed equipotential lines. In a sufficiently strong magnetic field, electrons perform small oscillatory orbits while the centre of these orbits propagate along these equipotential lines [11]. Since the electrons are effectively localized within these clusters, the localization length can be defined as the correlation length (the width) of the electron clusters ξ_p. As the critical energy E_c found at the centre of a LL band is approached, percolation theory states that ξ_p increases according to a power law:

$$\xi_p(E) \propto |E - E_c|^{-\gamma_p}, \tag{2}$$

with critical universal exponent $\gamma_p = 4/3$ [11]. In other words, as the critical point or percolation threshold is approached the clusters grow in size, merging together, and then begin to form extended clusters of larger sizes, until at the percolation threshold enough clusters merge to form a percolating equipotential which extends across the entire system. This point determines the onset of dissipative conductivity.

Chalker and Codington modified this quasiclassical view of the QHE to describe universal quantum criticality in 2DESs, by considering the effects of quantum tunnelling and interference [12]. They showed that just before two neighbouring clusters coalesced, points of minimum potential between the two clusters are formed (termed saddle points). These points which occur between the outmost equipotential of the clusters, allow electrons to quantum tunnel between neighbouring clusters as shown in Figure 1. Quantum tunnelling therefore introduces a delocalization mechanism that, when considered within the quasiclassical model, modifies the critical exponent to the value expected in a quantum critical system of $\gamma = 7/3$ or ≈ 2.3 [12]. It is argued [12-14] that it is the inclusion of saddle point tunnelling that distinguishes the two universality classes of quantum critical and classical percolation. Tunnelling is thus crucial to the experimental observation of quantum criticality within a 2DES. A detailed treatment of the model can be found in a recent report by Kramer *et al*. [4]. We argue further, that by taking this model into account, one can explain many of the disagreements found between the quantum critical exponents obtained using the scaling theory of plateau-to-plateau transitions, and experimental observations.

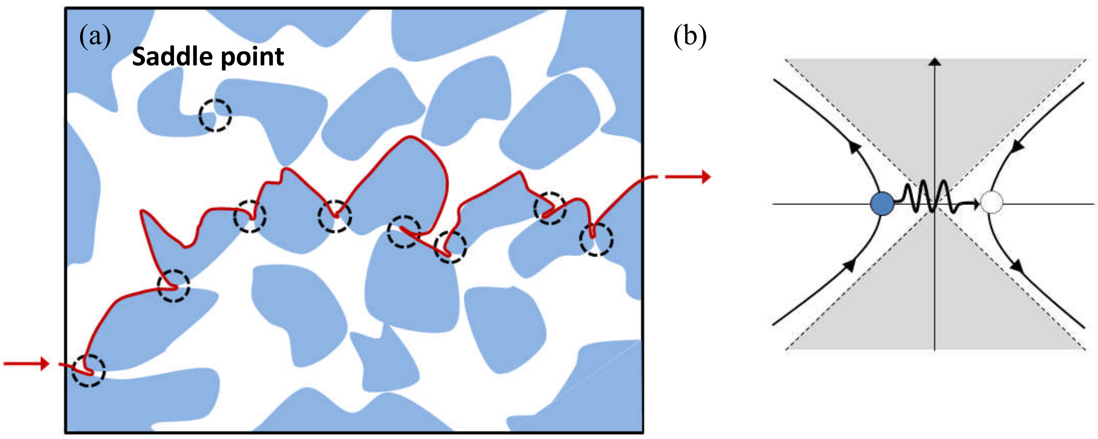

Figure 1. (a) A percolation view of dissipative transport in the bulk of a 2DES, electrons percolate through interlinked networks of saddle points. (b) Saddle point tunnelling of an electron between the outermost trajectories (equipotentials) of two isolated clusters, the shaded area denotes regions of high potential.

3. Quantum verses classical crossover

In this section we present experimental results that determine γ based on the change of the width of the conductivity peak of a LL band as a function of frequency. It was shown by Pruisken [15] that the width of the dissipative conductivity peak vanishes according to a temperature dependent power law of the form

$$\Delta B \propto T^{1/\gamma}. \tag{3}$$

The critical exponent γ can therefore be determined by measuring the temperature dependence of the dissipative conductivity according to Eq. (3).

Using a high resolution frequency technique [16] to investigate the quantum criticality of PPTs, we have been able to find a clear crossover from the expected quantum critical behaviour, to a classical behaviour in the critical exponent. Just as described above for temperature, the LL conductivity peak also has a similar frequency dependence [3],

$$\Delta B \propto f^{1/\gamma}. \tag{4}$$

The overwhelming majority of experimental investigations on the quantum criticality of PPTs have been conducted using temperature dependent experiments [6, 8, 17, 18]. Here, we use a frequency dependent experiment employing a coplanar waveguide which is formed on top of the 2DES and attached to a microwave network analyzer. Details of the experimental technique used in this work is reported in Ref. [16]. This experimental setup provides an extremely high degree of precision in varying the coherence length through the frequency parameter – a degree of precision that cannot be achieved in temperature dependent experiments owing to the limitations of cryogenic temperature controllers. The high resolution frequency dependent data we obtain provides a highly accurate description of the evolution of the width of a LL conductivity peak.

Figure 2 shows frequency dependent results obtained from a sample with carrier density $n_e \approx 2.89 \times 10^{11}/\mathrm{cm}^2$ and mobility $\mu \approx 38 \times 10^4 \mathrm{cm/Vs}$. All results reported here are obtained at a temperature of 35 mK where it was experimental verified that the temperature was sufficiently low for temperature dependent effects to be negligible and for frequency to be the dominant scaling parameter

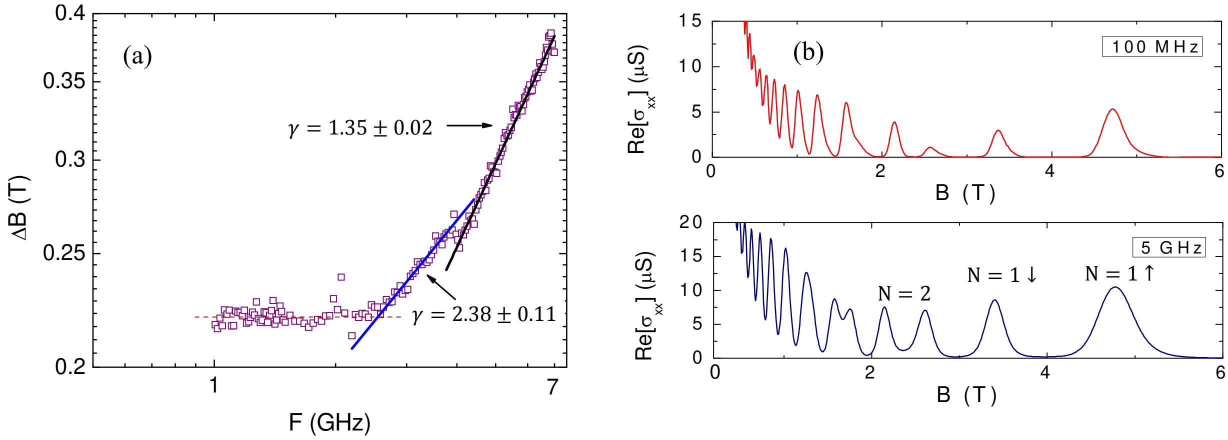

Figure 2. Plot (a) shows a high resolution frequency plot showing the quantum criticality observed in a 2DES. A crossover between a quantum and classical behaviour is observed as a function of frequency on the $N = 1\downarrow$ LL. Plot (b) shows the magnetic field dependence of the real part of conductivity obtained from the high frequency response of the 2DES.

of PPTs. The plot in Figure 2(a) shows the dependence of the width of the conductivity peak of a LL band according Eq. (4). The plot consists of 3 parts: a saturated region, followed by a slope with a quantum critical gradient of $\gamma = 2.38$, then a crossover to a gradient in good agreement with the classically expected value of $\gamma \approx 1.33$. The result shows a clear crossover from a quantum critical behaviour at lower frequencies to a classical behaviour at higher frequencies. The significance of this result to the contradictions reported in the literature is discussed below.

4. Impact of disorder

It has long been argued that quantum criticality within a 2DES is affected by the correlation length of disorder found within the 2DES [5]. For example, it has been observed that quantum criticality is generally found within InGaAs/InP heterostructures [8], where decoherence is predominantly caused by short-range alloy-disorder scattering within the vicinity of the 2DES, whereas the critical exponent in GaAs/AlGaAs heterostructures in general deviates from quantum criticality [5, 6]. Scattering in GaAs/AlGaAs heterostructures are dominated by long-range disorder potentials originating from remote ions in the spacer layer. This relation between criticality and the correlation length of disorder, though apparently generally accepted [5, 8] and empirically observed [18], has yet to be properly explained.

Using the same frequency dependent experiment described above [16], we measure quantum criticality in two different types of 2DESs [16]. We used the device discussed in Figure 2(a) as a reference, while in a second device a modulation doped GaAs/AlGaAs heterostructure was grown with Al impurities introduced into the GaAs layer. This results in a $Al_xGa_{1-x}As/Al_{0.33}Ga_{0.67}As$ heterostructure, where x refers to the concentration of the introduced Al impurities, which in our sample was x = 0.15%. It has been shown by Li *et al.* [19] that this level of impurity doping is sufficient to change what would otherwise have been a sample exhibiting long-range scattering into a sample with short-range alloy scattering. The resulting heterostructure had carrier density $n_e \approx 2.57 \times 10^{11}/\mathrm{cm}^2$ and mobility $\mu \approx 30 \times 10^4 \mathrm{cm/Vs}$.

The results are shown in Figure 3. Modifying the scattering mechanism in the 2DES changes the gradient of the slope as determined by Eq. (4) and results in a value of $\gamma = 2.38 \pm 0.34$ measured over two decades of frequency, in agreement with the expected quantum critical value. The value of γ determined in the long-range system deviates from the expected value of $\gamma \approx 2.3$, and tends towards a value of $\gamma = 1.67 \pm 0.14$, which is closer to the classically expected value of the critical exponent which is $\gamma \approx 1.3$.

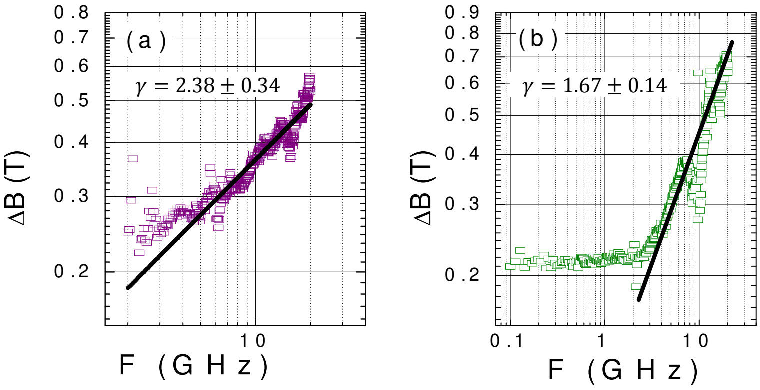

Figure 3. The two plots show results from the $N = 1 \downarrow$ LL from a short-range disordered and a long-range disordered sample. In plot (a), γ is found to be quantum critical when disorder was increased by introducing Al impurities in the vicinity of the 2DES. The plot in (b) shows the result from the reference long-range disordered GaAs/AlGaAs sample (with no added impurities), where the critical exponent tends towards a classical value. The result from this plot is obtained from the extended frequency range (up to 20 GHz) of the sample measured in Figure 2(a).

5. Discussion

We explain the results presented above using a percolation model which allows us to form a unifying description of PPTs within 2DESs. It is important to first define the three key length scales that form the basis of this model. These are the correlation length of the system ξ_p, which describes the typical width of electron clusters within the system, the localization length ξ, which is the length that defines the quantum coherent displacement of electrons within the system, and the phase coherent length l_φ, which represents the dephasing length of the system. ξ and ξ_p are Fermi energy or magnetic field dependent, as described by Eqs. (1) and (2) respectively. As the magnetic field approaches the critical point at the centre of a LL band ξ_p grows, describing larger cluster sizes close to the critical point. Similarly, ξ diverges, signifying longer extensions of quantum coherent displacements of electrons through interlinked networks of saddle points within the system. It is intuitive that ξ_p depends on the random background potential, or disorder within system. The greater the disorder, the more fragmented the electron clusters, and thus the smaller the cluster widths. l_φ, on the other hand, is a measure of inelastic scattering within the system, and is therefore dependent on the temperature or frequency of the system. An increase in frequency causes a decrease in l_φ and an increase in the inelastic scattering rate. We will henceforth describe increases in frequency of the 2DES simply as a decrease in $l_\varphi(f)$.

As discussed above, according to the percolation theory of the quantum Hall effect, saddle point tunnelling between adjacent electron clusters is the factor which distinguishes quantum critical behaviour from classical percolation. Without quantum tunnelling, ξ is limited to ξ_p, and thus such a system behaves like any classical fluid percolating through a random landscape. Quantum critical percolation can then be simply described as $\xi > \xi_p$, which is enabled by saddle point tunnelling.

Let us consider a system close to the percolation threshold (i.e. near the centre of the LL band) and at a temperature approaching absolute zero. Dissipative conductivity is possible through tunnelling of electrons between clusters since $\xi > \xi_p$. For any practical size of 2DES, l_φ will be very large compared to the sample size L_{2D}, and will theoretically extend beyond the boundaries of the sample. As l_φ is decreased (i.e. as the frequency is increased) dissipative conductivity within the system will remain unchanged as long as $l_\varphi > L_{2D}$ since on average an electron is able to propagate across the entire sample before it is scattered. This region can be seen as the saturated section observed in Figure 2(a).

Once $l_\varphi < L_{2D}$, dissipative conductivity will be influenced by scattering. Though tunnelling is still dominant, dissipative conduction no longer requires electrons to propagate through lengthy interlinked networks of saddle points, but rather electrons can be displace to remote clusters through phonon induced scattering in-between saddle point tunnelling events. The coexistence of tunnelling and scattering enhances dissipative conductivity which increases as l_φ is decreased and allows changes in conductivity to be experimentally observed as a function of l_φ (frequency). Since tunnelling still occurs, crucially $\xi > \xi_p$ and quantum critically is still persistent. This region is depicted in the low-frequency region in Figure 2(a), which shows a slope consistent with a quantum critical exponent of $\gamma \approx 2.38$. The relationship between the onset of inelastic scattering within the system and the sample size means that the saturation region can be eliminated or minimized with larger size samples, as shown experimentally by Koch *et al*. [20]. It is also noted that in this region ξ is now limited by l_φ as electrons will lose coherence beyond l_φ. The effect of a finite sample size is also evident in results reported by Li *et al*. [21], where the occurrence of the saturation region due the finite-size effect is discussed in some detail.

As l_φ (and as a result ξ) is further decreased the condition $l_\varphi, \xi < \xi_p$ will be reached. In other words, ξ will be smaller than the width of a cluster and an electron will on average be scattered before it is able to escape its cluster through a saddle point. The absence of tunnelling reduces the system to a purely classical fluid in a random potential exhibiting no quantum mechanical behaviour. Quantum critically is lost and thus the critical exponent in this region is determined to be classical. This can be seen in Figure 2(a) as a slope with $\gamma \approx 1.35$ in accordance with expectations of classical percolation [22].

The discussion above highlights the difficulties in the experimental observation of quantum criticality. When using relatively low resolution temperature dependent techniques, it is possible for the quantum critical region to be hidden, especially if the region is very narrow. The possibility of observing quantum critically can however be significantly increased by using samples of larger sizes, which reduces the range in frequency of the saturation region and brings about an earlier onset (in frequency) of the quantum critical region. On the other hand, the observation of quantum criticality can also be enhanced by delaying the quantum-classical crossover point, i.e. by pushing it to higher frequencies. As described above this crossover point occurs when the ξ is equal to the cluster size or $l_\varphi, \xi = \xi_p$. In a practical sense, this can be enhanced by having clusters of smaller sizes (reducing ξ_p), which as we have shown can be achieved by increasing the disorder within the system (as shown in Figure 3, where changing the disorder mechanism from long-range correlation to short-range alloy scattering increased the disorder within the system, and as a result the exponent changed to the expected quantum critical value). This may explain why quantum critically is easily observed systems with short-range scattering, such as those formed in InGaAs/InP heterostructures, but rarely observed in long-range GaAs/AlGaAs heterostructures. A convincing investigation of the dependence of the crossover point on disorder has been report by Li *et al*. [23] in further support of this theory.

Experimental investigations that do not observe this crossover run the risk of applying a fit that covers both the quantum and the classical region when determining γ, and will thus measure a value which is between the quantum and classical values, ie. $2.3 \geq \gamma \geq 1.3$.

Acknowledgements
We are grateful for the support of the EPSRC, the Royal Society, the Wolfson Foundation and the ERC programme, *NOTES*.

References

[1] K. v. Klitzing, G. Dorda, and M. Pepper, Physical Review Letters **45**, 494 (1980).
[2] E. Abrahams *et al.*, Physical Review Letters **42**, 673 (1979).
[3] S. L. Sondhi *et al.*, Reviews of Modern Physics **69**, 315 (1997).
[4] B. Kramer, T. Ohtsuki, and S. Kettemann, Physics Reports **417**, 211 (2005).
[5] B. Huckestein, Reviews of Modern Physics **67**, 357 (1995).
[6] S. Koch *et al.*, Physical Review B **43**, 6828 (1991).
[7] N. Q. Balaban, U. Meirav, and I. Bar-Joseph, Physical Review Letters **81**, 4967 (1998).
[8] H. P. Wei *et al.*, Physical Review Letters **61**, 1294 (1988).
[9] A. L. Efros, Solid State Communications **65**, 1281 (1988).
[10] A. L. Efros, Solid State Communications **67**, 1019 (1988).
[11] S. A. Trugman, Physical Review B **27**, 7539 (1983).
[12] J. Chalker, and P. Coddington, Journal of Physics C: Solid State Physics **21**, 2665 (1988).
[13] Y. Dubi, Y. Meir, and Y. Avishai, Physical Review Letters **94**, 156406 (2005).
[14] Y. Dubi, Y. Meir, and Y. Avishai, Physical Review B **71**, 125311 (2005).
[15] A. M. M. Pruisken, Physical Review Letters **61**, 1297 (1988).
[16] K. Saeed *et al.*, Physical Review B **84**, 155324 (2011).
[17] H. P. Wei *et al.*, Physical Review B **45**, 3926 (1992).
[18] W. Li *et al.*, Physical Review Letters **94**, 206807 (2005).
[19] W. Li *et al.*, Applied Physics Letters **83**, 2832 (2003).
[20] S. Koch *et al.*, Physical Review Letters **67**, 883 (1991).
[21] W. Li *et al.*, Physical Review Letters **102**, 216801 (2009).
[22] D. Stauffer, Physics Reports **54**, 1 (1979).
[23] W. Li *et al.*, Physical Review B **81**, 033305 (2010).