Monograph:

Banks, S.P. (1990) Infinite- Dimensional Carleman Linearization, the Lie Series and Optimal Control of Nonlinear PDE's. Research Report. Acse Report 392 . Dept of Automatic Control and System Engineering. University of Sheffield

Infinite-Dimensional Carleman Linearization, the Lie Series and Optimal Control of Nonlinear PDEs

by

S. P. Banks

Department of Control Engineering
University of Sheffield
Mappin Street
SHEFFIELD S1 3JD

Research Report No 392

March 1990

Abstract

The Carleman linearization and Lie series techniques are generalized to nonlinear PDEs and applied to nonlinear optimal control theory.

Keywords

Carleman linearization, Lie series, Nonlinear PDEs, Optimal Control.

1 Introduction

In this paper we shall generalize the Carleman linearization and Lie series techniques ([1],[2],[3]) for nonlinear differential equations to nonlinear partial differential (evolution) equations and we shall show that the two methods are essentially equivalent. In the finite-dimensional case of an equation of the form

$$\dot{x} = f(x) \tag{1.1}$$

we linearize the system by introducing the Taylor polynomials

$$\phi_{i_1 \cdots i_n} = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \tag{1.2}$$

and writing (1.1) in the form

$$\dot{\Phi} = A\Phi \tag{1.3}$$

where Φ is the tensor $(\phi_{i_1 \cdots i_n})$ and A is a tensor operator (see [5]). In the distributed case we introduce a graded tensor algebra which is formed from the prolongation spaces of the dependent variable in a similar way to (1.2). It is shown that a partial differential equation of the form

$$\frac{\partial \phi}{\partial t} = f(\phi, \phi_x, \phi_{xx}) \tag{1.4}$$

can be written in the form (1.3) in this tensor space.

In section 3 we obtain a similar equation to (1.3) using Lie series methods, but this time A is the left-shift operator (on a certain space) which is independent of the dynamics of (1.4); all the dynamics are transformed into the initial condition for (1.3). In section 4 this will enable us to obtain explicit solutions for the optimal control of nonlinear PDEs.

2 Infinite-Dimensional Tensor Representation

Consider the partial differential equation

$$\frac{\partial\phi}{\partial t} = f(\phi, \phi_x, \phi_{xx}), x \in [0,1] \ , \ t \geq 0 \tag{2.1}$$

$$\phi(0,t) = \phi(1,t) = 0 \ , \ \phi(x,0) = \phi_0(x) \in \mathcal{D}[0,1]$$

where $f : R^3 \longrightarrow R$ is analytic. We shall generalize the well-known Carleman linearization technique for finite-dimensional systems. In order to do this we introduce the 'variables '

$$\psi^n_{i_0\cdots i_n} = \phi^{i_0}\phi_x^{i_1}\phi_{xx}^{i_2}\cdots\phi^{i_n}_{\underbrace{x\cdots x}_{n}} \tag{2.2}$$

Then $\psi^n_{i_0\cdots i_n} \in \otimes_{|i(n)|}C^\infty[0,1]$, the $|i(n)|^{th}$- order tensor product of $C^\infty[0,1]$ ($|i(n)| = \sum_{\ell=0}^n i_\ell$, where $i(n) = (i_0,\ldots,i_n)$). Let

$$\mathcal{C}(i_0,\ldots,i_n) = \otimes_{|i(n)|}C^\infty[0,1] \ , \ i_n > 0$$

and consider the graded tensor algebra

$$\mathcal{C} \triangleq \otimes_{n=0}^{\infty} \otimes_{i_0,\ldots,i_n} \mathcal{C}(i_0,\ldots,i_n) \ ,$$

where multiplication is defined in the obvious way and the inner sum is over all $(i_0,\ldots,i_n)$ with $i_n > 0$. Then we shall write

$$\Psi = (\psi^0, \psi^1, \psi^2, \ldots) \in \mathcal{C},$$

where $\psi^n = (\psi^n_{i_0,\ldots,i_n}) \in \otimes_{i_0,\ldots,i_n} \mathcal{C}(i_0,\ldots,i_n)$. Clearly, for $k \geq \ell$, we have

$$\psi^k_{i_0,\ldots,i_k}\psi^\ell_{j_0,\ldots,j_\ell} \in \mathcal{C}(i_0+j_0,\ldots,i_\ell+j_\ell,i_{\ell+1},\ldots,i_k)$$

so that

$$\mathcal{C}(i_0,\ldots,i_k)\otimes\mathcal{C}(j_0,\ldots,j_\ell) \subseteq \mathcal{C}(i_0+j_0,\ldots,i_\ell+j_\ell,i_{\ell+1},\ldots,i_k)$$

Now, from (2.2) we have

$$\begin{aligned}
\frac{\partial}{\partial t}\psi^n_{i_0\cdots i_n} = \quad & i_0\phi^{i_0-1}\phi_x^{i_1}\phi_{xx}^{i_2}\cdots\phi_{x\cdots x}^{i_n}\frac{\partial\phi}{\partial t} \\
&+\ i_1\phi^{i_0}\phi_x^{i_1-1}\phi_{xx}^{i_2}\cdots\phi_{x\cdots x}^{i_n}\left(\frac{\partial\phi}{\partial t}\right)_x \\
&+\ \cdots \\
&+\ i_n\phi^{i_0}\phi_x^{i_1}\phi_{xx}^{i_2}\cdots\phi_{x\cdots x}^{i_n-1}\left(\frac{\partial\phi}{\partial t}\right)_{x\cdots x} \\
= \quad & \sum_{k=0}^{n} i_k\phi^{i_0}\cdots\phi^{i_k-1}_{\underbrace{x\cdots x}_{k}}\cdots\phi^{i_n}_{x\cdots x}\cdot f_{\underbrace{x\cdots x}_{k}}(\phi,\phi_x,\phi_{xx}) \qquad (2.3)
\end{aligned}$$

Since f is analytic we can write

$$f(\phi,\phi_x,\phi_{xx}) = \sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha_{rpq}\phi^r\phi_x^p\phi_{xx}^q$$

for some constants α_{rpq}. In order to evaluate $f_{\underbrace{x\cdots x}_{k}}(\phi,\phi_x,\phi_{xx})$ we first state the following simple lemma:

Lemma 2.1 If $g_1(x),\ldots,g_\ell(x)$ are differentiable functions and $\mathcal{S} \subseteq R^\ell$ is the set $\{\sigma^1,\cdots,\sigma^\ell\}$ where

$$\sigma^j = (0,0,\cdots,\underbrace{1}_{j},0,\cdots,0)$$

i.e. σ^j is the j^{th} unit vector of R^ℓ, then

$$\frac{d^k}{dx^k}(g_1 g_2 \cdots g_\ell) = \sum_{\sigma(1)\in\mathcal{S}} \cdots \sum_{\sigma(k)\in\mathcal{S}} \prod_{i=1}^{\ell} \left(d^{\sigma_i(1)} \cdots d^{\sigma_i(k)} g_i\right)$$

$$= \sum_{\sigma(1)\in\mathcal{S}} \cdots \sum_{\sigma(k)\in\mathcal{S}} \prod_{i=1}^{\ell} \left(d^{\sum_{j=1}^{k} \sigma_i(j)} g_i\right) \qquad (2.4)$$

where $d = d/dx$. □

Remark For $k = 1$ this is merely the formula for differentiation by parts; thus

$$\frac{d}{dx}(g_1 \cdots g_\ell) = \sum_{\sigma\in\mathcal{S}} \prod_{i=1}^{\ell} (d^{\sigma_i} g_i)$$

$$= \sum_{j=1}^{\ell} \prod_{i=1}^{\ell} \left(d^{\sigma_i^j} g_i\right)$$

$$= \sum_{j=1}^{\ell} g_1 \cdots \frac{dg_i}{dx} \cdots g_\ell$$

Note that we could also use Leibnitz' formula

$$\partial^p(g_1 g_2) = \sum_{q\le p} \binom{p}{q} \partial^p g_1 \partial^{p-q} g_2$$

recursively to find $d^k(g_1 g_2 \cdots g_\ell)$. However, the resulting expression is much more complex than (2.4). □

Corollary 2.2 $(\partial_x)^k(\phi^r \phi_x^p \phi_{xx}^q) =$

$$\sum_{\sigma(1)\in\mathcal{S}} \cdots \sum_{\sigma(k)\in\mathcal{S}} \prod_{i=1}^{r} \left(\partial_x^{\sum_{j=1}^{k} \sigma_i(j)} \phi\right) \cdot \prod_{i=r+1}^{r+p} \left(\partial_x^{\sum_{j=1}^{k} \sigma_i(j)+1} \phi\right) \cdot \prod_{i=r+p+1}^{\ell} \left(\partial_x^{\sum_{j=1}^{k} \sigma_i(j)+2} \phi\right).$$

where $\ell = r + p + q$. □

Let the component of $(\partial_x)^k(\phi^r \phi_x^p \phi_{xx}^q)$ in $C(i_0, \cdots, i_\ell)$ be denoted by $c_{i_0,\cdots,i_n}^{\ell,krpq}$.

Then

$$f_{\underbrace{x\cdots x}_{k}}(\phi,\phi_x,\phi_{xx}) = \sum_{\ell=0}^{\infty}\sum_{i_0,\cdots,i_\ell}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha_{rpq}c^{\ell,krpq}_{i_0,\cdots,i_n}\phi^{i_0}\phi_x^{i_1}\cdots\phi^{i_\ell}_{\underbrace{x\cdots x}_{\ell}}$$

Hence, from (2.3) we have

$$\begin{aligned}\frac{\partial\psi^n_{i_0\cdots i_n}}{\partial t} &= \sum_{\ell=0}^{\infty}\sum_{i'_0,\cdots,i'_\ell}\sum_{k=0}^{n}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha_{rpq}c^{\ell,krpq}_{i'_0,\cdots,i'_\ell}i_k\phi^{i'_0}\phi_x^{i'_1}\cdots\phi^{i'_\ell}_{\underbrace{x\cdots x}_{\ell}}\phi^{i_0}\cdots\phi^{i_k-1}_{\underbrace{x\cdots x}_{k}}\cdots\phi^{i_n}_{\underbrace{x\cdots x}_{n}} \\ &= \sum_{\lambda=n}^{\infty}\sum_{i'_0,\cdots,i'_\lambda}{}^{n\lambda}a^{i'_0\cdots i'_\lambda}_{i_0\cdots i_n}\psi^{\lambda}_{i'_0\cdots i'_\lambda}\end{aligned}$$

where

$${}^{n\lambda}a^{i'_0\cdots i'_\lambda}_{i_0\cdots i_n} = \sum_{k=0}^{n}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha_{rpq}c^{\lambda,krpq}_{i'_0-i_0\cdots i'_k-i_k-1\cdots i'_n-i_n\cdots i_{\lambda'}}$$

Let $\mathcal{A}:\mathcal{C}\longrightarrow\mathcal{C}$ denote the operator defined by

$$\left(\mathcal{A}(\psi^0,\psi^1,\psi^2,\cdots)\right)^n_{i_0\cdots i_n} = \sum_{\lambda=n}^{\infty}\sum_{i'_0,\cdots,i'_\lambda}{}^{n\lambda}a^{i'_0\cdots i'_\lambda}_{i_0\cdots i_n}\psi^{\lambda}_{i'_0\cdots i'_\lambda}$$

Then equation (2.1) can be written in the form

$$\frac{d\Psi}{dt} = \mathcal{A}\Psi \tag{2.5}$$

where

$$\Psi = (\psi^0,\psi^1,\psi^2,\cdots).$$

Remark For computation, it is convenient to use the infinite product

$$\psi_{i_0i_1i_2\cdots} = \prod_{k=0}^{\infty}\phi^{i_k}_{\underbrace{x\cdots x}_{k}}$$

Then $\psi^n_{i_0\cdots i_n} = \psi_{i_0\cdots i_n00\cdots}$.

Remark 2 In the finite-dimensional case

$$\dot{x} = f(x)$$

we use the functions

$$\phi_{i_0 \cdots i_n} = x_1^{i_1} \cdots x_n^{i_n}$$

(see [5]), based on the Taylor polynomials in R^n. The expression (2.2) is essentially the same, but based on the n^{th} prolongation space ([3]) of ϕ, namely $(\phi, \phi_x, \cdots, \phi_{x \cdots x})$.

Example 2.1 As an illustrative example of the above technique we shall consider the lineqar heat equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \ , \ \ \phi(0,t) = \phi(1,t) = 0 \ , \ \phi(x,0) = \xi_0(x)$$

Then, with

$$\psi_{i_0 \cdots i_n \cdots} = \phi^{i_0} \phi_x^{i_1} \phi_{xx}^{i_2} \cdots \phi_{x \cdots x}^{i_n} \cdots$$

(sunig the convention in the above remark), we have

$$\begin{aligned}
\frac{d\psi_{i_0 \cdots i_n \cdots}}{dt} &= \sum_{k=0}^{\infty} i_k \phi^{i_0} \cdots \phi^{i_k - 1}_{\underbrace{x \cdots x}_{k}} \cdots \phi^{i_n}_{x \cdots x} \cdots \phi_{\underbrace{x \cdots x}_{k+2}} \\
&= \sum_{k=0}^{\infty} i_k \phi^{i_0} \cdots \phi^{i_k - 1}_{\underbrace{x \cdots x}_{k}} \phi^{i_{k+1}}_{\underbrace{x \cdots x}_{k+1}} \phi^{i_{k+2}+1}_{\underbrace{x \cdots x}_{k+2}} \cdots \\
&= \sum_{k=0}^{\infty} i_k \psi_{i_0, \cdots, i_k - 1, i_{k+1}, i_{k+2}+1, i_{k+3}, \cdots}
\end{aligned}$$

Thus,

$$(\mathcal{A}\Psi)_{i_0 i_1 i_2 \cdots} = \sum_{k=0}^{\infty} i_k \psi_{i_0, \cdots, i_k - 1, i_{k+1}, i_{k+2}+1, i_{k+3}, \cdots}$$

$$= \sum_{k=0}^{\infty} i_k S_{i_k} \psi_{i_0,\cdots,i_k,\cdots} \tag{2.6}$$

where S_{i_k} is the operator

$$S_{i_k}\psi_{i_0\cdots i_k} = \psi_{i_0,\cdots,i_k-1,i_{k+1},i_{k+2}+1,i_{k+3},\cdots}$$

Similarly,

$$(\mathcal{A}^2\Psi)_{i_0 i_1 i_2\cdots} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} i_{k_1} i_{k_2} S_{i_{k_1}} S_{i_{k_2}} \psi_{i_0\cdots[i_{k_1}\cdots i_{k_2}]\cdots}$$

where

$$\psi_{i_0\cdots[i_{k_1}\cdots i_{k_2}]\cdots} = \begin{cases} \psi_{i_0\cdots i_{k_1}\cdots i_{k_2}\cdots} & \text{if } i_{k_1} < i_{k_2} \\ \psi_{i_0\cdots i_{k_1}\cdots} & \text{if } i_{k_1} = i_{k_2} \\ \psi_{i_0\cdots i_{k_2}\cdots i_{k_1}\cdots} & \text{if } i_{k_1} > i_{k_2} \end{cases}$$

In general,

$$(\mathcal{A}^n\Psi)_{i_0 i_1 i_2\cdots} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} i_{k_1}\cdots i_{k_n} S_{i_{k_1}}\cdots S_{i_{k_n}} \psi_{i_0\cdots[i_{k_1}\cdots i_{k_2}\cdots i_{k_n}]\cdots}$$

where the 'time-ordered 'product $\psi_{i_0\cdots[i_{k_1}\cdots i_{k_2}]\cdots}$ is defined in the obvious way.

Since the solution of (2.1) is $\psi_{100\cdots} = \phi$ we have

$$\begin{aligned} \phi(t) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}(\mathcal{A}^n\Psi(0))_{100\cdots} \\ &= \xi(x) + t\xi_{xx}(x) + \frac{t^2}{2!}\xi_{xxxx}(x) + \cdots \end{aligned} \tag{2.7}$$

Example 2.2 Consider the nonlinear reaction-diffusion equation

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} - \phi^2 \ , \ \ \phi(0,t) = \phi(1,t) = 0 \ , \ \phi(x,0) = \xi(x)$$

This equation has a solution for all t (see [4]). As before, if $\psi_{i_0 i_1 i_2\cdots} = \phi^{i_0}\phi_x^{i_1}\phi_{xx}^{i_2}\cdots$, we have

$$\frac{d\psi_{i_0 i_1 i_2\cdots}}{dt} = \sum_{k=0}^{\infty} i_k \phi^{i_0}\cdots\phi^{i_k-1}_{\underbrace{x\cdots x}_{k}}(\dot\phi)_{\underbrace{x\cdots x}_{k}}\phi^{i_{k+1}}_{\underbrace{x\cdots x}_{k+1}}\cdots$$

However,

$$\begin{aligned}\dot\phi_{\underbrace{x\cdots x}_{k}} &= \phi_{\underbrace{x\cdots x}_{k+2}} - (\phi^2)_{\underbrace{x\cdots x}_{k}}\\ &= \phi_{\underbrace{x\cdots x}_{k+2}} - \sum_{q\le k}\phi_{\underbrace{x\cdots x}_{q}}\phi_{\underbrace{x\cdots x}_{k-q}}\end{aligned}$$

and so

$$\begin{aligned}\frac{d\psi_{i_0 i_1 i_2\cdots}}{dt} = & \sum_{k=0}^{\infty} i_k \psi_{i_0,\cdots,i_k-1,i_{k+1},i_{k+2}+1,\cdots}\\ &+ \sum_{k=0}^{\infty} i_k \sum_{q\le k}\psi_{i_0,\cdots,i_q+1,\cdots,i_{k-q}+1,\cdots,i_k-1,i_{k+1},\cdots}\end{aligned}$$

As in example 2.1 we see that the first few terms of the series expansion of the solution are given by

$$\phi(t) = \xi(x) - t\xi^2(x) + t\xi_{xx}(x) + \frac{t^2}{2}(\xi_{xxxx}(x) - 2\xi_x^2(x) - 4\xi(x)\xi_{xx}(x) + 2\xi^3(x)) + \cdots \quad (2.8)$$

3 The Lie Series

In this section we shall generalize the Lie series ([2], [3]) to partial differential equations. Recall that, if v is a vector field defined on a finite-dimensional manifold, then the Lie series is an expression for the flow of v, namely,

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}v^k(f)(x)$$

In local x-coordinates we can write this in the form

$$x(t, x_0) = \left\{ exp \left(t \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i} \right) x \right\} \Bigg|_{x=x_0} \tag{3.1}$$

where f is a local expression for v and $x(t, x_0)$ is the local flow of v through x_0.

Equation (3.1) can be derived simply in the following way (which will generalize easily to partial differential equations):

Define

$$g_1(x) = x \in R^n$$

and

$$g_k(x) = \frac{\partial g_{k-1}}{\partial x}(x) f(x) \ , \ k \geq 2$$

Then we have

$$\begin{aligned} \dot{g}_k(x) &= \frac{\partial g_k}{\partial x}(x) \frac{dx}{dt} = \frac{\partial g_k}{\partial x}(x) f(x) \\ &= g_{k+1}(x) \end{aligned}$$

Hence the local differential equation

$$\dot{x} = f(x)$$

is equivalent to the infinite-dimensional system

$$\dot{G} = AG \ , \ G(0) = G_0$$

where A is the left-shift operator, $G = (g_1, g_2, \cdots)^T$ and

$$G_0 = (x_0, f(x_0), (\partial(f)f)(x_0), (\partial(\partial(f)f)f)(x_0), \cdots)^T$$

(see [6]).

Now consider the partial differential equation

$$\frac{\partial \phi}{dt} = f(\phi, \phi_x, \phi_{xx}) \tag{3.2}$$

where

$$\begin{aligned} \phi(0,t) &= \phi(1,t) = 0 \\ \phi(x,0) &= \xi(x), \end{aligned}$$

defined on the one-dimensional spatial interval $[0,1]$. We shall seek solutions in $\mathcal{D}([0,1])$, the space of infinitely differentiable functions with compact support; f will be assumed to be analytic.

Define

$$\gamma_1 = \phi$$

and for $k \geq 2$,

$$\begin{aligned} \gamma_k = \quad & \gamma_{k-1,1}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{2(k-1)}}) f(\phi, \phi_x, \phi_{xx}) \\ + \quad & \gamma_{k-1,2}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{2(k-1)}}) \frac{\partial f}{\partial x}(\phi, \phi_x, \phi_{xx}) \\ + \quad & \cdots \\ + \quad & \gamma_{k-1,2(k-1)+1}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{2(k-1)}}) \frac{\partial^{2(k-1)}}{\partial x^{2(k-1)}} f(\phi, \phi_x, \phi_{xx}) \end{aligned}$$

where $\gamma_k(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{2k}})$ is a function of $\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{2k}})$ and $\gamma_{k,i}$ is the partial derivative of γ_k with respect to its i^{th} argument. Then, we have

$$\frac{d\gamma_1}{dt} = \frac{\partial \phi}{\partial t} = f(\phi, \phi_x, \phi_{xx}) = \gamma_2$$

$$
\begin{aligned}
\frac{d\gamma_k}{dt} &= \gamma_{k,1}\frac{\partial \phi}{\partial t} + \gamma_{k,2}\frac{\partial \phi_x}{\partial t} + \cdots + \gamma_{k,2k+1}\frac{\partial \phi_{x\cdots x}}{\partial t} \\
&= \gamma_{k,1} f + \gamma_{k,2} f_x + \cdots + \gamma_{k,2k+1} f_{\underbrace{x\cdots x}_{2k}} \\
&= \gamma_{k+1}
\end{aligned}
$$

Hence, equation (3.2) is equivalent to the infinite-dimensional system

$$
\frac{d\Gamma}{dt} = A\Gamma \ , \ \Gamma(0) = \Gamma_0
$$

where A is again the left-shift operator on the space $\Delta = \oplus\mathcal{D}([0,1])$ (an infinite number of copies of $\mathcal{D}[0,1]$).

Example 3.1 As a simple illustration of the above technique, consider the heat equation

$$
\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \ , \ x \in [0,1] \ , \tag{3.3}
$$

$$
\phi(0,t) = \phi(1,t) = 0 \ , \ \phi(x,0) = \xi(x) \in \mathcal{D}[0,1]
$$

As above, define

$$
\begin{aligned}
\gamma_1 &= \phi \\
\gamma_2 &= \frac{\partial^2}{\partial x^2} \\
\gamma_k &= \frac{\partial^{2(k-1)}}{\partial x^{2(k-1)}} \ , \ k \geq 1
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{\partial \gamma_1}{\partial t} &= \frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} = \gamma_2 \ , \\
\frac{\partial \gamma_2}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\frac{\partial \phi}{\partial t} = \frac{\partial^4 \phi}{\partial x^4} = \gamma_3
\end{aligned}
$$

$$\begin{aligned} \frac{\partial \gamma_3}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial^4 \phi}{\partial x^4} = \frac{\partial^6 \phi}{\partial x^6} = \gamma_4 \\ \cdots & \quad \cdots \end{aligned} \tag{3.4}$$

This gives the system

$$\frac{d\Gamma}{dt} = A\Gamma \ , \tag{3.5}$$

with initial condition

$$\Gamma(0) = (\xi(x), \xi_{xx}(x), \xi_{xxxx}(x), \cdots)$$

Since $\xi(x) \in L^2[0,1]$ and $\xi(0) = \xi(1) = 0$, we can write

$$\xi(x) = \sum_{n=1}^{\infty} \xi_n \sin(n\pi x)$$

where

$$\xi_n = 2\int_0^1 \xi(x)\sin(n\pi x)dx.$$

Thus,

$$\xi_{\underbrace{x \cdots x}_{2k}}(x) = \sum_{n=1}^{\infty} \xi_n (-1)^k (n\pi)^{2k} \sin(n\pi x).$$

Also,

$$(e^{At})_{ij} = \sum_{k=0}^{\infty} \delta_{i,j-k} \frac{t^k}{k!}$$

where $(.)_{ij}$ is the ij^{th} element of an infinite matrix. It follows that the solution of (3.3) is given by

$$\phi(t,x) \quad = \quad \gamma_1(x) = \Gamma_1 = (e^{At}\Gamma(0))$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\delta_{i,j-k}\frac{t^k}{k!}\xi\underbrace{x\cdots x}_{2k}(x) \\
&= \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{n=1}^{\infty}\xi_n(-1)^k(n\pi)^{2k}\sin(n\pi x) \\
&= \sum_{n=1}^{\infty}\xi_n e^{-n^2\pi^2 t}\sin(n\pi x)
\end{aligned}
$$

This is the usual expression for the semigroup $T(t)$ generated by the operator $\mathcal{A} = \partial^2/\partial x^2$ with domain

$$
\mathcal{D}(\mathcal{A}) = \{\phi \in H^2[0,1] : \phi(0) = \phi(1) = 0\}
$$

(see [7]).

Example 3.2 Consider the scalar reaction-diffusion equation

$$
\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} - \phi^2
$$

where f is analytic. Then

$$
\begin{aligned}
\gamma_1 &= \phi \\
\gamma_2 &= \frac{\partial^2\phi}{\partial x^2} - \phi^2 \\
\gamma_3 &= \frac{\partial\gamma_2}{\partial t} \\
&= \frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\phi}{\partial x^2} - \phi^2\right) - 2\phi\left(\frac{\partial^2\phi}{\partial x^2} - \phi^2\right) \\
&= \frac{\partial^4\phi}{\partial x^4} - 4\phi\frac{\partial^2\phi}{\partial x^2} - 2\left(\frac{\partial\phi}{\partial x}\right)^2 + 2\phi^3
\end{aligned}
$$

leading to the solution which was obtained in (2.7).

4 Application to Control Theory

In this section we shall study the nonlinear optimal control problem

$$\frac{\partial \phi}{\partial t} = f(\phi, \phi_x, \phi_{xx}) + u(t) g(x, \phi, \phi_x, \phi_{xx}) \tag{4.1}$$

where

$$\phi(0,t) = \phi(1,t) = 0 \ , \ \phi(x,0) = \xi(x)$$

with cost functional

$$J = \langle \phi(\cdot, t_f), \phi(\cdot, t_f) \rangle_{L^2[0,1]} + \int_0^{t_f} \left\{ \langle \phi(\cdot, t), Q\phi(\cdot, t) \rangle_{L^2[0,1]} + r u^2(t) \right\} dt$$

where $F, Q \in \mathcal{L}(L^2[0,1])$, F, Q are nonnegative and $r > 0$.

We have shown that the Carleman linearization and Lie series techniques are essentially the same and so we shall apply the latter method here. Therefore, as in section 3, put

$$\begin{aligned}
\gamma_1 &= \phi \\
\gamma_i &= \gamma_{i/2,1}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{i}}) f(\phi, \phi_x, \phi_{xx}) \\
&\quad + \gamma_{i/2,2}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{i}}) \frac{\partial f}{\partial x}(\phi, \phi_x, \phi_{xx}) \\
&\quad + \cdots \\
&\quad + \gamma_{i/2,i+1}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{i}}) \frac{\partial^i f}{\partial x^i}(\phi, \phi_x, \phi_{xx}) \ , \ if \ i \ even \\
\gamma_i &= \gamma_{(i-1)/2,1}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{i}}) g(\phi, \phi_x, \phi_{xx}) \\
&\quad + \gamma_{(i-1)/2,2}(\phi, \phi_x, \cdots, \phi_{\underbrace{x \cdots x}_{i}}) \frac{\partial g}{\partial x}(\phi, \phi_x, \phi_{xx})
\end{aligned}$$

$$+\cdots$$

$$+\gamma_{(i-1)/2,i+1}(\phi,\phi_x,\cdots,\phi_{\underbrace{x\cdots x}_{i}})\frac{\partial^i g}{\partial x^i}(\phi,\phi_x,\phi_{xx})\ ,\ if\ i\ odd$$

Thus,

$$\begin{aligned}
\frac{d\gamma_i}{dt} &= \gamma_{i,1}(\phi,\cdots,\phi_{x\cdots x})(f(\phi,\phi_x,\phi_{xx})+ug(x,\phi,\phi_x,\phi_{xx})) \\
&\quad \gamma_{i,2}(\phi,\cdots,\phi_{x\cdots x})(f_x(\phi,\phi_x,\phi_{xx})+ug_x(x,\phi,\phi_x,\phi_{xx})) \\
&\quad \gamma_{i,3}(\phi,\cdots,\phi_{x\cdots x})(f_{xx}(\phi,\phi_x,\phi_{xx})+ug_{xx}(x,\phi,\phi_x,\phi_{xx})) \\
&\quad \cdots \\
&\quad \gamma_{i,i+1}(\phi,\cdots,\phi_{x\cdots x})(f_{\underbrace{x\cdots x}_{i}}(\phi,\phi_x,\phi_{xx})+ug_{\underbrace{x\cdots x}_{i}}(x,\phi,\phi_x,\phi_{xx})) \\
&= \gamma_{2i}+u\gamma_{2i+1}
\end{aligned}$$

Hence we can write (4.1) in the form

$$\dot{\Gamma}=A\Gamma+uB\Gamma\ ,\ \Gamma(0)=\Gamma_0 \tag{4.2}$$

where

$$A=(\delta_{2i,j})\ ,\ B=(\delta_{2i+1,j})$$

are the obvious matrix operators on $\oplus_{k=1}^{\infty}\mathcal{D}[0,1]$, and

$$\Gamma_0=(\gamma_{10},\gamma_{20},\gamma_{30},\cdots)$$

with

$$\begin{aligned}
\gamma_{10} &= \phi(x,0)=\xi(x) \\
\gamma_{20} &= f(\xi,\xi_x,\xi_{xx})
\end{aligned}$$

$$\gamma_{30} = g(x, \xi, \xi_x, \xi_{xx})$$

$$\dots$$

Define the operators $\mathcal{F}$ and $\mathcal{Q}$ on $\mathcal{L}(\oplus_{k=1}^{\infty}\mathcal{D}[0,1], \oplus_{k=1}^{\infty}\mathcal{D}[0,1])$ by

$$\langle \Gamma, \mathcal{F}\Gamma \rangle = \langle \phi, F\phi \rangle$$

and

$$\langle \Gamma, \mathcal{Q}\Gamma \rangle = \langle \phi, Q\phi \rangle$$

Then J can be written in the form

$$J(u) = \langle \Gamma(t_f), \mathcal{F}\Gamma(t_f) \rangle + \int_0^{t_f} (\langle \Gamma(t), \mathcal{Q}\Gamma(t) \rangle + ru^2(t))dt \tag{4.3}$$

Infinite-dimensional bilinear-quadratic problems have been completely solved ([8] and [9]) and we may write the optimal control in the form:

$$u(t) = -\frac{1}{2}R^{-1}\sum_{i=1}^{\infty}\langle \otimes_i g, (P_i B) \otimes_i g \rangle \tag{4.4}$$

where $P_i \in \mathcal{L}(\otimes_i H)$ is given recursively by

$$\begin{aligned}
P_1(t) &= e^{\mathcal{A}_1(t_f - t)}\mathcal{F}e^{\mathcal{A}_1^T(t_f - t)} + \int_0^{t_f - t} e^{\mathcal{A}_1(t_f - t - s)}\mathcal{Q}e^{\mathcal{A}_1^T(t_f - t - s)}ds \\
P_m(t) &= -\frac{r^{-1}}{2}\sum_{\substack{i+j=m \\ i,j\geq 1}}\int_0^{t_f - t} e^{\mathcal{A}_m(t_f - t - s)}P_i(t_f - s)B \otimes P_j(t_f - s)Be^{\mathcal{A}_m^T(t_f - t - s)}ds \\
&\quad -\frac{r^{-1}}{2}\sum_{\substack{i+j=m \\ i,j\geq 1}}\int_0^{t_f - t} e^{\mathcal{A}_m(t_f - t - s)}P_i(t_f - s)B^T \otimes P_j(t_f - s)B^T e^{\mathcal{A}_m^T(t_f - t - s)}ds
\end{aligned} \tag{4.5}$$

Here, P is a tensor operator and if C is an infinite matrix, PC is defined by

$$(PC)_{k_1 \dots k_i}^{\kappa_1 \dots \kappa_i} = \sum_{j=1}^{i}\left(\sum_{\ell_j = 1}^{\infty} P_{k_1 \dots \ell_j \dots k_i}^{\kappa_1 \dots \kappa_i} C_{\ell_j k_j}\right)$$

Furthermore, $\mathcal{A}_i \in \mathcal{L}(\mathcal{L}(\otimes_i H))$ is defined by

$$\mathcal{A}_i P_i = P_i A \quad , \quad P_i \in \mathcal{L}(\otimes_i H) \quad , \quad i \geq 1 \ .$$

We have

$$||\mathcal{A}_i||_{\mathcal{L}(\mathcal{L}(\otimes_i H))} \leq i||A||_{\mathcal{L}(H)} \quad ,$$

and so $e^{\mathcal{A}_i t}$ is defined by the usual series. It can be shown ([8]) that $e^{\mathcal{A}_i t}$ is given by

$$(e^{\mathcal{A}_i t}Q)^{\kappa_1 \ldots \kappa_i}_{\ell_1 \ldots \ell_i} = \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} Q^{\kappa_1 \ldots \kappa_i}_{k_1 \ldots k_i} (e^{At})_{k_1 \ell_1} (e^{At})_{k_2 \ell_2} \cdots (e^{At})_{k_i \ell_i}$$

for any Q. Now,we have

$$(e^{At})_{ij} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta_{2^n i, j} \quad ,$$

and so

$$\begin{aligned} (e^{\mathcal{A}_i t}Q)^{\kappa_1 \ldots \kappa_i}_{\ell_1 \ldots \ell_i} &= \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_i=0}^{\infty} Q^{\kappa_1 \ldots \kappa_i}_{k_1 \ldots k_i} \frac{t^{n_1}}{n_1!} \frac{t^{n_2}}{n_2!} \cdots \frac{t^{n_i}}{n_i!} \delta_{2^{n_1} k_1, \ell_1} \delta_{2^{n_2} k_1, \ell_2} \cdots \delta_{2^{n_i} k_1, \ell_i} \\ &= \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} P^{\kappa_1 \ldots \kappa_i}_{k_1 \ldots k_i} \frac{t^{n_1}}{n_1!} \frac{t^{n_2}}{n_2!} \cdots \frac{t^{n_i}}{n_i!} \end{aligned}$$

where

$$n_p = log_2 \left(\frac{\ell_p}{k_p} \right)$$

and the sums in the last expression are over k's for which each n_p is a natural number.

Now,

$$\begin{aligned}
(P_i(t_f - s)B)^{\kappa_1\ldots\kappa_i}_{k_1\ldots k_i} &= \sum_{j=1}^{\ell}\left(\sum_{\ell_j=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\ell_j\ldots k_i} B_{\ell_j k_j}\right) \\
&= \sum_{j=1}^{\ell}\left(\sum_{\ell_j=1}^{\infty} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\ell_j\ldots k_i} \delta_{2\ell_j+1,k_j}\right) \\
&= \sum_{j=1}^{\ell} P^{\kappa_1\ldots\kappa_i}_{k_1\ldots\frac{(k_j-1)}{2}\ldots k_i}
\end{aligned}$$

where the element of P in the last sum is zero if $(k_j - 1)/2$ is not a nonnegative integer. The above expressions for $e^{\mathcal{A}_i t}Q$ and $P_i B$ are now sufficient to be able to evaluate $P_m(t)$ from (4.4).

5 Conclusions

In this paper we have generalized the Carleman linearization and Lie series techniques for the solution of nonlinear systems in terms of linear ones to distributed systems, by means of a new infinite-dimensional representation defined in terms of the prolongation space coordinates. Moreover, we have shown that the two methods are essentially equivalent and have given two examples in each case to illustrate the theory. Finally, it has been shown that explicit solutions to nonlinear optimal control problems can be obtained by using these expansions.

References

[1] T.Carleman, 'Application de la

theorie des equations integrales singuliere aux equations differentiales de la dynamique ', T.Ark.Mat.Astron.Fys.,**22B**(1),1932.

[2] S.Steinberg, 'Lie Series and Nonlinear Ordinary Differential Equations ', J. Math. Anal. Apps.,**101**,39-63,1984.

[3] P.J.Olver, 'Applications of Lie groups to differential equations ',Springer-Verlag,New York,1986.

[4] D.Henry, 'Geometric theory of semilinear parabolic equations ', Sringer-Verlag,New York,1981.

[5] S.P.Banks, 'Mathematical Theories of Nonlinear Systems ', Prentice-Hall,1988.

[6] S.P.Banks, 'Global Linear Representations of Nonlinear Systems and the Adjoint Map ', to appear in Int. J. Control, 1989.

[7] S.P.Banks, 'State space and frequency-domain methods in the control of distributed parameter systems ',Peter Peregrinus,London, 1983.

[8] S.P.Banks, 'Optimal Control of Distributed Bilinear Systems ', Sys. Control Letts.,**19**,629-636,1988.

[9] S.P.Banks and M.K.Yew, 'On a Class of Suboptimal Controls for Infinite-Dimensional Bilinear Systems, Sys. Control Letts.,**5**,327-333,1985.